Journal of the Inst. of Math. Jussieu (2002) **1**(1), 125–143
DOI:10.1017/S1474748002000038 Printed in the United Kingdom

REMARKS ON A CONJECTURE OF FONTAINE AND MAZUR

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(Received 1 May 2001; accepted 30 May 2001)

Abstract We show that a continuous, odd, regular (non-exceptional), ordinary, irreducible, two-dimensional, l-adic representation of the absolute Galois group of the rational numbers is modular over some totally real field. We deduce that it occurs in the l-adic cohomology of some variety over the rationals and that its L-function has meromorphic continuation to the whole complex plane and satisfies the expected functional equation.

Keywords: Galois representation; modularity; Fontaine–Mazur

AMS 2000 *Mathematics subject classification:* Primary 11F80
Secondary 11G40

Introduction

Fontaine and Mazur have made the following extremely influential conjecture [**8**, **9**].

Conjecture A. *Suppose that*

$$\rho : \mathrm{Gal}(\mathbb{Q}^{\mathrm{ac}}/\mathbb{Q}) \to GL_n(\mathbb{Q}_l^{\mathrm{ac}})$$

is a continuous irreducible representation such that

(1) *ρ is ramified at only finitely many primes, and*

(2) *the restriction of ρ to the decomposition group at l is potentially semi-stable in the sense of Fontaine.*

Then ρ occurs in the l-adic cohomology (with respect to a Tate twist of the constant sheaf) of some variety defined over $\mathbb{Q}$.

We remark that it is now known that if ρ does occur in the l-adic cohomology of some variety defined over $\mathbb{Q}$ then (1) and (2) must hold. We also remark that it would follow from this conjecture that there is an integer w (depending on ρ) such that for almost all p the eigenvalues of $\rho(\mathrm{Frob}_p)$ are algebraic and for each embedding into $\mathbb{C}$ have absolute value $p^{w/2}$. Finally we remark that, combining this conjecture with conjectures of Langlands, one further expects that ρ has the same L-series as a cuspidal automorphic representation of $GL_n(\mathbb{A})$, and so in particular its L-series has holomorphic continuation to $\mathbb{C}$ (except for a possible pole when $n = 1$) and satisfies a functional equation (which can be made precise).

The case $n = 1$ of the conjecture has been known to be true for some time. Besides this the only known cases are for $n = 2$ where the methods of Wiles [**27**], Taylor–Wiles [**25**] and Skinner–Wiles [**20**] have been used to verify some cases of the conjecture. Except for a couple of isolated examples (see [**19**] and [**6**]) these methods have been restricted to the case where ρ has pro-soluble image. The purpose of this paper is to verify the Fontaine–Mazur conjecture in a significant number of cases where the image of ρ is not pro-soluble. More precisely we prove the following theorem and its corollaries. (Here, and in the rest of this paper, c denotes complex conjugation and ϵ the l-adic cyclotomic character.)

Theorem B. *Let l be an odd prime and*

$$\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_l^{\mathrm{ac}})$$

a continuous irreducible representation such that

- *ρ is unramified at all but finitely many primes,*
- *$\det \rho(c) = -1$, and*
- $$\rho|_{G_l} \sim \begin{pmatrix} \epsilon^n \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

  for some $n \in \mathbb{Z}_{>0}$ and some finitely ramified characters χ_1, χ_2 for which $(\epsilon^n \chi_1 \chi_2^{-1})(I_l)$ is not pro-l.

Then there is a totally real field E, a regular algebraic cuspidal automorphic representation π of $GL_2(\mathbb{A}_E)$ and a place λ of the field of coefficients of π above l such that $\rho_{\pi,\lambda}$ (the λ-adic representation associated to π) is equivalent to ρ.

Combining this with a result of Brylinski–Labesse [**2**], Langlands's cyclic base change [**12**] and a theorem of Brauer we obtain the following corollary.

Corollary C. *Keep the assumptions of Theorem B and choose an isomorphism $i : \mathbb{Q}_l^{\mathrm{ac}} \xrightarrow{\sim} \mathbb{C}$. For all but finitely many primes p the trace and determinant of $\rho(\mathrm{Frob}_p)$ lie in $\mathbb{Q}^{\mathrm{ac}}$ and we have*

$$|i(\mathrm{tr}\, \rho(\mathrm{Frob}_p))| \leqslant 2p^{n/2}.$$

We define the L-function of ρ with respect to i to be

$$L(i\rho, s) = (1 - i\chi_{1,I_l}(\mathrm{Frob}_l)/l^{s-n})^{-1}(1 - i\chi_{2,I_l}(\mathrm{Frob}_l)/l^s)^{-1} \times \prod_{p \neq l} \det(1 - i\rho_{I_p}(\mathrm{Frob}_p)/p^s)^{-1},$$

https://doi.org/10.1017/S1474748002000038 Published online by Cambridge University Press

except we drop the factor $(1 - i\chi_{1,I_l}(\mathrm{Frob}_l)/l^{s-n})^{-1}$ *if* $n = 1$ *and* $\chi_1 = \chi_2$. *(Here the subscript* I_l *denotes* I_l*-coinvariants.) This converges uniformly absolutely for the real part of* s *sufficiently large. We also define the conductor* $N(\rho)$ *to be the product*

$$N(\chi_1)N(\chi_2)\prod_{p\neq l} N(\rho|_{G_p}),$$

except in the case $n = 1$ *and* $\chi_1 = \chi_2$ *is unramified when we replace* $N(\chi_1)$ *by* l. *(Here* $N(\rho|_{G_p})$ *(respectively* $N(\chi_i)$*) is the usual conductor of* $\rho|_{G_p}$ *(respectively* χ_i*).)*

The function $L(i\rho, s)$ *has unique meromorphic continuation to the whole complex plane and satisfies a functional equation*

$$N(\rho)^{s/2}(2\pi)^{-s}\Gamma(s)L(i\rho, s) = WN(\rho)^{(n+1-s)/2}(2\pi)^{s-1-n}\Gamma(n+1-s) \times L(i(\rho\otimes\epsilon^n(\det\rho)^{-1}), n+1-s),$$

where $|W| = 1$.

In particular this has the following consequence.

Corollary D. *Suppose that* $A/\mathbb{Q}$ *is an abelian variety,* M *is a number field with* $[M : \mathbb{Q}] = \dim A$ *and that* $j : \mathcal{O}_M \hookrightarrow \mathrm{End}(A/\mathbb{Q})$. *Then the* L*-function of* A *(relative to an embedding* $M \hookrightarrow \mathbb{C}$*) has meromorphic continuation to the whole complex plane and satisfies a functional equation*

$$N(A)^{s/2}(2\pi)^{-s}\Gamma(s)L(A, s) = WN(A)^{(2-s)/2}(2\pi)^{s-2}\Gamma(2-s)L(A^\vee, 2-s),$$

where $N(A)$ *denotes the conductor of* A *(relative to the endomorphisms* M*) and where* $|W| = 1$.

Alternatively combining Theorem B with a result of Blasius and Rogawski [**1**] and restriction of scalars, we obtain the following corollary.

Corollary E. *Keep the assumptions of Theorem B and if* $n = 1$ *further assume that*

- *for some prime* $p \neq l$ *we have*

$$\rho|_{G_p} \sim \begin{pmatrix} \epsilon\chi & * \\ 0 & \chi \end{pmatrix}.$$

Then ρ *occurs in the* l*-adic cohomology (with coefficients in some Tate twist of the constant sheaf) of some variety over* $\mathbb{Q}$. *If* $n = 1$ *then there exists a number field* M, *a prime* λ *of* M *above* l, *an abelian variety* $A/\mathbb{Q}$ *of dimension* $[M : \mathbb{Q}]$ *and an embedding* $\mathcal{O}_M \hookrightarrow \mathrm{End}(A/\mathbb{Q})$ *such that* ρ *is equivalent to the representation on the* λ*-adic Tate module of* A.

Combining this corollary with (a slight generalization of) the main result of [**14**] we get the following theorem, which may lend some support to a very important conjecture of Serre (see [**18**]).

Theorem F. *Suppose that l is an odd prime and that*

$$\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_l^{\mathrm{ac}})$$

is a continuous irreducible representation such that $\det \bar{\rho}(c) = -1$ *and*

$$\bar{\rho}|_{G_l} \sim \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

with $\chi_1|_{I_l} \neq \chi_2|_{I_l}$. Then there exists a number field M, a prime λ of M above l, an abelian variety $A/\mathbb{Q}$ of dimension $[M : \mathbb{Q}]$ and an embedding $\mathcal{O}_M \hookrightarrow \mathrm{End}(A/\mathbb{Q})$ such that $\bar{\rho}$ is equivalent to the representation of $G_{\mathbb{Q}}$ on $A[\lambda]$.

We remark that we have not tried to optimize the conditions in these results and some improvement is certainly possible.

Let us briefly describe the proof of Theorem B. Let $\bar{\rho}$ denote a reduction of ρ. The case where $\bar{\rho}$ is reducible is the main result of [**20**]. The case where $\bar{\rho}$ is irreducible but soluble follows from the results of [**4**, **12**, **25**–**27**] and [**22**]. In this paper we treat the case where $\bar{\rho}$ has insoluble image. By the methods of [**27**] and [**25**] and their extension to totally real fields by Diamond [**5**], Fujiwara [**10**] and Skinner and Wiles [**21**] and [**22**], the key point here is to prove that $\bar{\rho}|_{\mathrm{Gal}(E^{\mathrm{ac}}/E)}$ is modular for some totally real field E.

To describe how we do this, let us for simplicity assume that $\bar{\rho}$ has cyclotomic determinant. We find totally real fields E and M, a rational prime p and an abelian variety A/E such that

- p and l are unramified in E,
- $\dim A = [M : \mathbb{Q}]$,
- there is an embedding $i : \mathcal{O}_M \hookrightarrow \mathrm{End}(A/E)$,
- there is a prime $\lambda | l$ of $\mathcal{O}_M$ such that $A[\lambda](E^{\mathrm{ac}})$ is equivalent to $\bar{\rho}|_{\mathrm{Gal}(E^{\mathrm{ac}}/E)}$ as a $\mathrm{Gal}(E^{\mathrm{ac}}/E)$-module,
- A has good ordinary reduction at all primes of $\mathcal{O}_E$ above p,
- there is a prime $\wp$ of $\mathcal{O}_M$ above p such that the action of $\mathrm{Gal}(E^{\mathrm{ac}}/E)$ on $A[\wp](E^{\mathrm{ac}})$ is of the form
$$\mathrm{Ind}_{\mathrm{Gal}(L^{\mathrm{ac}}/L)}^{\mathrm{Gal}(E^{\mathrm{ac}}/E)} \theta$$
for some totally imaginary quadratic extension L/E not contained in $E(\zeta_p)$ and some character θ of $\mathrm{Gal}(L^{\mathrm{ac}}/L)$.

Given such E, M, p and A the above mentioned results of Diamond, Fujiwara and Skinner and Wiles show that the $\wp$-adic Tate module of A is modular and hence that $\bar{\rho}$ is modular.

Having made a suitable choice for M and p the problem of finding a suitable E and A comes down to a problem of constructing points on certain Hilbert–Blumenthal modular varieties over totally real fields in which p and l are unramified. To this end we employ the following general criterion of Moret-Bailly [**13**] which reduces the problem to local problems at ∞, l and p.

Theorem G (Moret-Bailly). *Let K be a number field and S a finite set of places of K. There is a unique maximal extension K_S/K (inside a given algebraic closure of K) in which all places of S split completely. (For example, $\mathbb{Q}_{\{\infty\}}$ is the maximal totally real field.) Suppose that $X/\operatorname{Spec} K$ is a geometrically irreducible smooth quasi-projective scheme and that, for all $v \in S$, $X(K_v)$ is non-empty. Then $X(K_S)$ is Zariski dense in X.*

We would like to ask whether one can replace K_S by K_S^{sol} in this theorem, where K_S^{sol} denotes the maximal soluble extension of K in which all elements of S split completely. An affirmative answer to this question would, by the methods of this paper, have important implications for Serre's conjecture.

Notation

Throughout this paper l will be an odd rational prime.

If K is a perfect field we will let K^{ac} denote its algebraic closure and G_K denote its absolute Galois group $\mathrm{Gal}(K^{\mathrm{ac}}/K)$. If moreover p is a prime number different from the characteristic of K then we will let $\epsilon_p : G_K \to \mathbb{Z}_p^\times$ denote the p-adic cyclotomic character and ω_p the Teichmüller lift of $\epsilon_p \bmod p$. In the case $p = l$ we will drop the subscripts and write simply $\epsilon = \epsilon_l$ and $\omega = \omega_l$. If K is a local field we will let W_K denote the Weil group of K. If K is a number field and x is a finite place of K we will write G_x for a decomposition group above x, I_x for the inertia subgroup of G_x and Frob_x for an arithmetic Frobenius element in G_x/I_x. We will also let $\mathcal{O}_K$ denote the integers of K, $\mathfrak{d}_K$ the different of K and $k(x)$ denote the residue field of $\mathcal{O}_K$ at x. We will let c denote complex conjugation on $\mathbb{C}$.

We will write $\boldsymbol{\mu}_N$ for the group scheme of Nth roots of unity. We will write $W(k)$ for the Witt vectors of k. If G is a group, H a normal subgroup of G and ρ a representation of G, then we will let ρ^H (respectively ρ_H) denote the representation of G/H on the H-invariants (respectively H-coinvariants) of ρ.

Suppose that A/K is an abelian variety with an action of $\mathcal{O}_M$ for some number field M over a perfect field K. Suppose also that X is a finite index $\mathcal{O}_M$-submodule of a free $\mathcal{O}_M$-module. If X is free with basis $e_1, \ldots, e_r$, then by $A \otimes_{\mathcal{O}_M} X$ we shall simply mean A^r. Note that for any ideal $\mathfrak{a}$ of $\mathcal{O}_M$ we have a canonical isomorphism

$$(A \otimes_{\mathcal{O}_M} X)[\mathfrak{a}] \cong A[\mathfrak{a}] \otimes_{\mathcal{O}_M} X.$$

In general if $Y \supset X \supset \mathfrak{a}Y$ with Y free and $\mathfrak{a}$ a non-zero ideal of $\mathcal{O}_M$ then we will set

$$(A \otimes_{\mathcal{O}_M} X) = (A \otimes_{\mathcal{O}_M} \mathfrak{a}Y)/(A[\mathfrak{a}] \otimes_{\mathcal{O}_M} X/\mathfrak{a}Y).$$

This is canonically independent of the choice of $Y \supset X$ and again we get an identification

$$(A \otimes_{\mathcal{O}_M} X)[\mathfrak{a}] \cong A[\mathfrak{a}] \otimes_{\mathcal{O}_M} X.$$

If X has an action of some $\mathcal{O}_M$ algebra then $A \otimes_{\mathcal{O}_M} X$ canonically inherits such an action. We also get a canonical identification $(A \otimes_{\mathcal{O}_M} X)^\vee \cong A^\vee \otimes_{\mathcal{O}_M} \mathrm{Hom}(X, \mathbb{Z})$. Suppose that

$\mu : A \to A^\vee$ is a polarization which induces an involution c on M. Note that c equals complex conjugation for every embedding $M \hookrightarrow \mathbb{C}$. Suppose also that $f : X \to \mathrm{Hom}(X, \mathbb{Z})$ is c-semilinear for the action of $\mathcal{O}_M$. If for all $x \in X - \{0\}$, the totally real number $f(x)(x)$ is totally strictly positive then $\lambda \otimes f : A \otimes_{\mathcal{O}_M} X \to (A \otimes_{\mathcal{O}_M} X)^\vee$ is again a polarization.

If λ is an ideal of $\mathcal{O}_M$ prime to the characteristic of K we will write $\bar{\rho}_{A,\lambda}$ for the representation of G_K on $A[\lambda](K^{\mathrm{ac}})$. If λ is prime we will write $T_\lambda A$ for the λ-adic Tate module of A, $V_\lambda A$ for $T_\lambda A \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\rho_{A,\lambda}$ for the representation of G_K on $V_\lambda A$. We have a canonical isomorphism $T_\lambda(A \otimes_{\mathcal{O}_M} X) \xrightarrow{\sim} (T_\lambda A) \otimes_{\mathcal{O}_M} X$.

Suppose that F is a totally real number field and that π is an algebraic cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ with field of definition (or coefficients) $M \subset \mathbb{C}$. In some cases, including the cases that π_∞ is regular and the case π_∞ is weight $(1, \ldots, 1)$, then it is known that M is a CM number field and that for each prime λ of $\mathcal{O}_M$ there is a continuous irreducible representation

$$\rho_{\pi,\lambda} : G_F \to GL_2(M_\lambda)$$

canonically associated to π. (See [**23**] for details.) We may always conjugate $\rho_{\pi,\lambda}$ so that it is valued in $GL_2(\mathcal{O}_{M,\lambda})$ and then reduce it to get a continuous representation $G_F \to GL_2(\mathcal{O}_M/\lambda)$. If for one such choice of conjugate the resulting representation is irreducible then it is independent of the choice of conjugate and we will denote it $\bar{\rho}_{\pi,\lambda}$.

1. A potential version of Serre's conjecture

Suppose that l is an odd prime and that $k/\mathbb{F}_l$ is a finite extension. Suppose also that F is a totally real field and that

$$\bar{\rho} : G_F \to GL_2(k)$$

is a continuous representation such that

- $\bar{\rho}$ has insoluble image,
- for every place v of F above l we have
$$\bar{\rho}|_{G_v} \sim \begin{pmatrix} \epsilon\chi_v^{-1} & * \\ 0 & \chi_v \end{pmatrix},$$
and
- for every complex conjugation c, $\det \bar{\rho}(c) = -1$.

For v a prime of F above l let $\tilde{F}_v$ denote the smallest totally tamely ramified extension of F_v over which χ_v becomes unramified.

Let ζ denote a primitive $\#k^\times$ root of unity and let $N_0 = \mathbb{Q}(\zeta, \sqrt{1-4l})$. By replacing k by a larger finite field we may assume that N_0 is ramified (at some finite prime) over its maximal totally real subfield. Note that l is unramified in N_0 and that each prime of

N_0 above l has residue field isomorphic to k. Choose a prime λ_0 of N_0 above l and an isomorphism $\mathcal{O}_{N_0}/\lambda_0 \cong k$. For v a prime of F above l set

$$\beta_v = \zeta^{b_v}((1+\sqrt{1-4l})/2)^{[k(v):\mathbb{F}_l]}$$

with b_v chosen so that $\beta_v \equiv \chi_v(\phi_v) \bmod \lambda_0$ for $\phi_v \in G_{\tilde{F}_v}$ a lift of Frob_v. Let $\tilde{\chi}_v$ denote the unique character from W_{F_v} to $N_0^\times$ which,

- if $\chi_v^2 \neq 1$, takes ϕ_v to β_v and on inertia is the Teichmüller lift (at λ_0) of χ_v and,
- if $\chi_v^2 = 1$, is the Teichmüller lift (at λ_0) of χ_v.

Choose a prime $p \not| 6l$ such that

- at all primes w of F above p, $\bar{\rho}$ is unramified and $\bar{\rho}(\mathrm{Frob}_w)$ has distinct eigenvalues,
- p splits completely in the Hilbert class field of N_0,
- p splits completely in $(F^{\mathrm{ac}})^{\ker(\epsilon^{-1}\det\bar{\rho})}$, and
- p is coprime to $\beta_v - \beta_v^{\mathrm{c}}$ for all places v of F above l.

Also choose a prime $\wp_0$ of N_0 above p. For each place w of F above p choose $\alpha'_w \in \mathbb{Z}[(1+\sqrt{1-4l})/2]$ with norm p (possible because p splits completely in the Hilbert class field of $\mathbb{Q}((1+\sqrt{1-4l})/2)$) and set

$$\alpha_w = \zeta^{a_w}\alpha'_w$$

with a_w chosen so that α_w is congruent modulo λ to an eigenvalue of $\bar{\rho}(\mathrm{Frob}_w)$.

Lemma 1.1. *Let p be a rational prime, $\mathcal{O}$ the integers of a finite extension of $\mathbb{Q}_p$ and $\mathbb{F}$ the residue field of $\mathcal{O}$. Let K be a totally real field and L/K a totally imaginary quadratic extension in which every place of K above p splits. Let S be a finite set of finite places of K which split in L and suppose S contains all places of K above p. Let S_L be a set of places of L above S which contains exactly one place above every element of S.*

Let $\phi : G_K \to \mathcal{O}^\times$ be a continuous homomorphism

- *which takes every complex conjugation to -1, and*
- *which is of the form ϵ_p^n times a finite order character for some $n \in \mathbb{Z}$.*

Also for each $x \in S_L$ let $\bar{\psi}_x : G_{L_x} \to \mathbb{F}^\times$ be a continuous homomorphism.

Then there is a finite extension of the fraction field of $\mathcal{O}$ with integers $\mathcal{O}'$ and residue field $\mathbb{F}'$, and a continuous character $\psi : G_L \to (\mathcal{O}')^\times$ such that

- *for all $x \in S_L$, $\psi|_{G_{L_x}}$ is finitely ramified and reduces to $\bar{\psi}_x$, and*
- *$\det \mathrm{Ind}_{G_L}^{G_K} \psi = \phi$.*

Proof. We can choose a character $\psi_0 : G_L \to (\mathcal{O}')^\times$ such that

- $\epsilon_p^{-1} \det \mathrm{Ind}_{G_L}^{G_K} \psi_0$ has finite order and
- $\psi_0|_{I_x}$ is finite order for all $x \in S_L$.

Looking for ψ of the form $\psi_0^n \psi'$ we reduce to the case that $n = 0$. We may also suppose that S_L generates the class group of L.

For $x \in S_L$ let $\psi_x : L_x^\times \to \mathcal{O}^\times$ be the character corresponding by class field theory to the Teichmüller lift of $\bar{\psi}_x$. Let ϕ' be the character of $\mathbb{A}_K^\times$ associated by class field theory to ϕ times the quadratic character of G_K with kernel G_L. We must find a character $\psi : \mathbb{A}_L^\times / L^\times \to (\mathcal{O}')^\times$ which restricts to ϕ' on $\mathbb{A}_K^\times$ and to ψ_x on $L_x^\times$ for all $x \in S_L$. Let $L_S^\times$ (respectively $K_S^\times$) denote the subgroup of $L^\times$ (respectively $K^\times$) consisting of elements supported on S_L (respectively S). Let T denote the set of finite places of K which are not in S and at which ϕ' is ramified. For $x \in T$ choose an extension ψ_x of $\phi'|_{\mathcal{O}_{K,x}^\times}$ to $\mathcal{O}_{L,x}^\times$. Let

$$\psi_0 : \left(\prod_{x \in S} L_x^\times \times \prod_{x \in T} \mathcal{O}_{L,x}^\times \right) / K_S^\times \to (\mathcal{O}')^\times$$

denote the unique character which

- coincides with ϕ' on $(\prod_{x \in S} K_x^\times \times \prod_{x \in T} \mathcal{O}_{K,x}^\times)/K_S^\times$,
- coincides with ψ_x on $L_y^\times$ for $y \in S_L$, and
- coincides with ψ_x on $\mathcal{O}_{L,x}^\times$ for $x \in T$.

It suffices to find a continuous character

$$\psi : \left(\prod_{x \in S} L_x^\times \times \prod_{y \in T} \mathcal{O}_{L,y}^\times \times \prod_{y \notin S \cup T} (\mathcal{O}_{L,y}^\times / \mathcal{O}_{K,y}^\times) \right) / L_S^\times \to (\mathcal{O}')^\times$$

which extends ψ_0. Equivalently we must find a continuous character

$$\prod_{y \notin S \cup T} (\mathcal{O}_{L,y}^\times / \mathcal{O}_{K,y}^\times) \to (\mathcal{O}')^\times$$

which coincides with ψ_0^{-1} on $L_S^\times / K_S^\times$.

As ψ_0 has finite order it suffices to show that any finite index subgroup of $L_S^\times / K_S^\times$ contains the preimage of some open subgroup of $\prod_{y \notin S \cup T} \mathcal{O}_{L,y}^\times / \mathcal{O}_{K,y}^\times$. Considering the commutative diagram

$$\begin{array}{ccc} L_S^\times / K_S^\times & \overset{c-1}{\hookrightarrow} & L_S^\times \\ \downarrow & & \downarrow \\ \prod_{y \notin S \cup T} \mathcal{O}_{L,y}^\times / \mathcal{O}_{K,y}^\times & \overset{c-1}{\hookrightarrow} & \prod_{y \notin S \cup T} \mathcal{O}_{L,y}^\times, \end{array}$$

and recalling that $L_S^\times$ is a finitely generated abelian group, we see that we only need prove that for any positive integer n the subgroup $(L_S^\times)^n \subset L_S^\times$ contains the preimage of some open subgroup of

$$\prod_{y \notin S \cup T} \mathcal{O}_{L,y}^\times.$$

This is presumably well known, see for instance Lemma 2.1 of [**24**]. □

Thus we may choose a quadratic extension L/F and a continuous character $\psi : G_L \to (N_{0,\wp_0}^{\mathrm{ac}})^\times$ such that

- L is a totally imaginary field not contained in F adjoin a primitive pth root of 1;
- each place v of F above l splits as $v_1 v_1^c$ in L and $\psi|_{W_{L_{v_1}}} = \tilde{\chi}_v$ in $(\mathcal{O}_{N_0}/\wp_0)^{\mathrm{ac}}$;
- each place w of F above p splits as $w_1 w_1^c$ in L and $\psi|_{G_{w_1}}$ is unramified and takes arithmetic Frobenius to a lift of $\alpha_w \in \mathcal{O}_{N_0}/\wp_0$; and
- $\det \mathrm{Ind}_{G_L}^{G_F} \psi = \epsilon_p$.

Let $\bar{\psi} : G_L \to ((\mathcal{O}_{N_0}/\wp_0)^{\mathrm{ac}})^\times$ denote the reduction of ψ. Note that for any prime v of F above l we have $\bar{\psi}|_{G_{v_1}} \neq \bar{\psi}^c|_{G_{v_1}}$ (as $\beta_v - \beta_v^c$ is coprime to p). Choose N/N_0 be a Galois CM extension such that

- primes above l split in N/N_0,
- primes above p are unramified in N/N_0,
- primes of the maximal totally real subfield of N_0 which ramify in N_0 are unramified in the maximal totally real subfield of N, and
- there is a prime $\wp$ above $\wp_0$ such that $\bar{\psi}$ has image in $\mathcal{O}_N/\wp$.

Let λ denote a prime of $\mathcal{O}_N$ above λ_0 and let M denote the maximal totally real subfield of N.

By an ordered invertible $\mathcal{O}_M$-module we shall mean an invertible $\mathcal{O}_M$-module X together with a choice of connected component X_x^+ of $(X \otimes M_x) - \{0\}$ for each infinite place x of M. By the standard ordered invertible $\mathcal{O}_M$-module $\mathcal{O}_M^+$ we shall mean $(\mathcal{O}_M, \{(M_x^\times)^0\})$, where $(M_x^\times)^0$ denotes the connected component of 1 in $M_x^\times$. By an M-HBAV over a field K we shall mean a triple (A, i, j) where

- A/K is an abelian variety of dimension $[M : \mathbb{Q}]$,
- $i : \mathcal{O}_M \hookrightarrow \mathrm{End}(A/K)$, and
- $j : \mathcal{O}_M^+ \xrightarrow{\sim} \mathcal{P}(A, i)$ is an isomorphism of ordered invertible $\mathcal{O}_M$-modules.

Here $\mathcal{P}(A, i)$ is the invertible $\mathcal{O}_M$-module of symmetric (i.e. $f^\vee = f$) homomorphisms $f : (A, i) \to (A^\vee, i^\vee)$ which is ordered by taking the unique connected component of $(\mathcal{P}(A, i) \otimes M_x)$ which contains the class of a polarization. (See § 1 of [**15**].)

Lemma 1.2. *For each place v of F above l we can find an M-HBAV (A_v, i_v, j_v) over F_v such that*

- *A_v either has potentially good ordinary reduction or potentially multiplicative reduction,*
- *the action of G_v on $A_v[\lambda|_M]$ is equivalent to $\bar{\rho}|_{G_v}$, and*
- *the action of G_v on $A_v[\wp|_M]$ is equivalent to $\bar{\psi}_{v_1} \oplus \bar{\psi}_{v_1^c}$.*

Proof. First suppose that $\chi_v^2 = 1$, so that (by twisting) we may suppose that $\chi_v = 1$. The extension $\bar{\rho}|_{G_v}$ is described by a class

$$\bar{q} \in H^1(G_v, k(\epsilon)) \cong F_v^\times/(F_v^\times)^l \otimes_{\mathbb{F}_l} k \cong F_v^\times \otimes_{\mathbb{Z}} \mathfrak{d}_M^{-1}/\lambda\mathfrak{d}_M^{-1}.$$

We may choose

$$q_0 \in F_v^\times \otimes_{\mathbb{Z}} \wp\mathfrak{d}_M^{-1} \subset F_v^\times \otimes_{\mathbb{Z}} \mathfrak{d}_M^{-1}$$

such that q_0 reduces to $\bar{q} \in F_v^\times \otimes_{\mathbb{Z}} \mathfrak{d}_M^{-1}/\lambda\mathfrak{d}_M^{-1}$. Now set $q = q_0 q_1$, where we choose $q_1 \in F_v^\times \otimes_{\mathbb{Z}} \wp\lambda\mathfrak{d}_M^{-1}$ such that $\mathrm{tr}_{M/\mathbb{Q}}(av(q_1)) > -\mathrm{tr}_{M/\mathbb{Q}}(av(q_0))$ for all totally positive elements $a \in \mathcal{O}_M$. According to § 2 of [**15**] there is an M-HBAV $(A_v, i_v, j_v)/F_v$ such that $A_v(F_v^{\mathrm{ac}}) \cong ((F_v^{\mathrm{ac}})^\times \otimes \mathfrak{d}_M^{-1})/\mathcal{O}_M q$ as an $\mathcal{O}_M[G_v]$-module. This triple suffices to prove the lemma in this case.

Secondly suppose that $\chi_v^2 \neq 1$. By the theory of Honda and Tate we can find a simple ordinary abelian variety $A_0/k(v)$ of dimension $[\mathbb{Q}(\beta_v) : \mathbb{Q}]/2$ and an isomorphism $i_0 : \mathcal{O}_{\mathbb{Q}(\beta_v)} \xrightarrow{\sim} \mathrm{End}(A_0/k(v))$ such that $A_0[l](k(v)^{\mathrm{ac}})$ is isomorphic to $\mathcal{O}_{\mathbb{Q}(\beta_v)}/(\beta_v^{\mathrm{c}})$ and β_v is the Frobenius endomorphism of $A_0/k(v)$. Choose a polarization $\mu_0 : A_0 \to A_0^\vee$. The corresponding Rosati involution must correspond to complex conjugation on $\mathcal{O}_{\mathbb{Q}(\beta_v)}$. Set $A_1 = A_0 \otimes_{\mathcal{O}_{\mathbb{Q}(\beta_v)}} \mathcal{O}_N$, an ordinary abelian variety of dimension $[M : \mathbb{Q}]$ over $k(v)$ with an embedding $i_1 : \mathcal{O}_N \hookrightarrow \mathrm{End}(A_1/k(v))$. Then $A_1[l](k(v)^{\mathrm{ac}}) \cong \mathcal{O}_N/(\beta_v^{\mathrm{c}})$ and β_v is the Frobenius endomorphism. The polarization μ_0 and the pairing

$$\begin{aligned} \mathcal{O}_N \times \mathcal{O}_N &\to \mathbb{Z} \\ (a, b) &\mapsto \mathrm{tr}_{N/\mathbb{Q}}(ab^{\mathrm{c}}) \end{aligned}$$

defines a polarization $\mu_1 : A_1 \to A_1^\vee$ such that the μ_1-Rosati involution acts as c on $\mathcal{O}_N$. The choice of μ_1 makes $\mathrm{Hom}_{\mathcal{O}_M}(A_1, A_1^\vee)$ isomorphic to a fractional ideal $\mathfrak{a}$ of $\mathcal{O}_N$. The symmetric elements then correspond to $\mathfrak{a} \cap M$ with the order structure coming from the subset of totally positive elements. If we replace A_1 by $A_1/A_1[\mathfrak{b}]$ for some ideal $\mathfrak{b}$ of $\mathcal{O}_N$ then $\mathcal{P}(A_1, i_1)$ is replaced by $\mathfrak{a}\mathfrak{b}\mathfrak{b}^{\mathrm{c}} \cap M$ with the order structure coming from the totally positive elements. The norm map from the class group of N to the strict class group of M (i.e. the group of fractional ideals modulo principal ideals generated by a totally positive element of $M^\times$) is surjective, because the strict Hilbert class field of M is contained in the Hilbert class field of N and is disjoint from N over M (because N/M ramifies at some finite prime). Thus replacing A_1 by $A_1/A_1[\mathfrak{b}]$ for a suitable $\mathfrak{b}$ we may suppose that $(A_1, i_1|_{\mathcal{O}_M})$ extends to an M-HBAV $(A_1, i_1|_{\mathcal{O}_M}, j_1)$.

Let $\tilde{\chi}'_v$ denote the unique continuous unramified extension of $\tilde{\chi}_v|_{W_{\tilde{F}_v}}$ to a character

$$G_v \to \mathcal{O}^\times_{N,(\beta_v^c)} \cong \mathcal{O}^\times_{M,l}.$$

Serre–Tate theory tells us that liftings of the triple $(A_1, i_1|_{\mathcal{O}_M}, j_1)$ to $\mathcal{O}_{\tilde{F}_v}$ are parametrized by the extensions of $M_l/\mathcal{O}_{M,l}(\tilde{\chi}'_v)$ by $\boldsymbol{\mu}_{l^\infty} \otimes \mathcal{O}_M((\tilde{\chi}'_v)^{-1})$ as Barsotti–Tate groups over $\mathcal{O}_{\tilde{F}_v}$; that is by

$$H^1(G_{\tilde{F}_v}, \mathcal{O}_{M,l}(\epsilon(\tilde{\chi}'_v)^{-2})).$$

(Note that $\tilde{\chi}_v|^2_{W_{\tilde{F}_v}} \neq 1$.) We will write (A_x, i_x, j_x) for the lift corresponding to an element x of the group $H^1(G_{\tilde{F}_v}, \mathcal{O}_{M,l}(\epsilon(\tilde{\chi}'_v)^{-2}))$. If $\sigma \in \mathrm{Gal}(\tilde{F}_v/F_v)$ then

$$\sigma(A_x, i_x, j_x) = (A_{\sigma x}, i_{\sigma x}, j_{\sigma x}).$$

If $\gamma \in \mathcal{O}_N$ then $i_1(\gamma)$ lifts to a homomorphism from (A_x, i_x, j_x) to (A_y, i_y, j_y) if and only if $\gamma\gamma^c = 1$ and $\gamma^2 x = y$, where we let $\mathcal{O}_N$ acts on $\mathcal{O}_{M,l}$ via the map $\mathcal{O}_N \to \mathcal{O}_{N,\beta_v^c} \cong \mathcal{O}_{M,l}$. Thus to give a triple (A, i, j) over F_v which restricts to some lift of (A_1, i_1, j_1) over $\tilde{F}_v$ is the same as giving a character $\psi : \mathrm{Gal}(\tilde{F}_v/F_v) \to \boldsymbol{\mu}_\infty(N)$ and an element $x \in H^1(G_{\tilde{F}_v}, \mathcal{O}_{M,l}(\epsilon(\tilde{\chi}'_v)^{-2}))$ such that $\sigma x = \psi(\sigma)^2 x$ for all $\sigma \in \mathrm{Gal}(\tilde{F}_v/F_v)$, i.e. by continuous characters $\chi' : G_v \to \mathcal{O}^\times_{M,l}$ with $\chi'|_{W_{\tilde{F}_v}} = \tilde{\chi}_v|_{G_{\tilde{F}_v}}$ and elements

$$x \in H^1(G_{\tilde{F}_v}, \mathcal{O}_{M,l}(\epsilon(\chi')^{-2}))^{\mathrm{Gal}(L/K)} = H^1(G_v, \mathcal{O}_{M,l}(\epsilon(\chi')^{-2})).$$

Choose $x \in H^1(G_v, \mathcal{O}_{M,l}(\epsilon\tilde{\chi}_v^{-2}))$ so that its λ-component

$$x_\lambda \in H^1(G_v, \mathcal{O}_{M,\lambda}(\epsilon\tilde{\chi}_v^{-2}))$$

maps to the class of the extension $\bar{\rho}|_{G_v}$ in $H^1(G_v, \mathcal{O}_M/\lambda(\epsilon\chi_v^{-2}))$. This is possible as $H^2(G_v, \mathcal{O}_{M,\lambda}(\epsilon\tilde{\chi}_v^{-2})) = (0)$ (as it is dual to $H^0(G_v, M_\lambda/\mathcal{O}_{M,\lambda}(\tilde{\chi}_v^2))$). Finally, let $(A_v, i_v, j_v)/K$ correspond to $(\tilde{\chi}_v, x)$. □

Lemma 1.3. *For each place w of F above p there is an M-HBAV (A_w, i_w, j_w) over F_w such that*

- *A_w has good ordinary reduction,*
- *the action of G_w on $A_w[\lambda|_M]$ is equivalent to $\bar{\rho}|_{G_w}$, and*
- *the action of G_w on both $A_v[\wp|_M]$ is equivalent to $\bar{\psi}_{w_1} \oplus \bar{\psi}_{w_1^c}$.*

Proof. This is proved in the same way as Lemma 1.2 but is much easier so we leave the details to the reader. □

Lemma 1.4. *For each infinite place x of F there is an M-HBAV (A_x, i_x, j_x) over F_x.*

Proof. Choose an elliptic curve E/F_x and set $A_x = E \otimes_{\mathbb{Z}} \mathcal{O}_M$. Let i_x be the canonical action of $\mathcal{O}_M$ on A_x. Finally A_x has a polarization corresponding to the unique principal polarization on E and the pairing $\mathcal{O}_M \times \mathcal{O}_M \to \mathbb{Z}$ which sends $(a, b) \mapsto \mathrm{tr}(ab)$. This shows that $\mathcal{P}(A_x, i_x) \cong \mathcal{O}_M^+$. □

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Let V_λ/F be the two-dimensional $(\mathcal{O}_M/\lambda)$-vector space scheme corresponding to $\bar\rho$ and fix an alternating isomorphism a_λ of V_λ with its Cartier dual. Also let $V_\wp$ be the two-dimensional $(\mathcal{O}_M/\wp)$-vector space scheme corresponding to $\mathrm{Ind}_{G_L}^{G_F}\bar\psi$ and fix an alternating isomorphism $a_\wp$ of $V_\wp$ with its Cartier dual. As in § 1 of [**15**] we see that there is a fine moduli space X/F for quintuples $(A,i,j,m_\lambda,m_\wp)$ where (A,i,j) is an M-HBAV, $m_\lambda : V_\lambda \xrightarrow{\sim} A[\lambda]$ and $m_\wp : V_\wp \xrightarrow{\sim} A[\wp]$ such that a_λ corresponds to the $j(1)$-Weil pairing on $A[\lambda]$ and $a_\wp$ corresponds to the $j(1)$-Weil pairing on $A[\wp]$. (To define the moduli problem over a general base one must proceed as in § 1 of [**15**]. To see that the moduli space is fine note that $\ker(GL_2(\mathcal{O}_{M,\wp}) \twoheadrightarrow GL_2(\mathcal{O}_M/\wp))$ has no element of finite order other than the identity.) As in § 1 of [**15**] one can see that X is smooth and one can describe for any infinite place x of F the complex manifold $X(F\otimes_{F_x}\mathbb{C})$ as a quotient of the product of $[M:\mathbb{Q}]$ copies of the upper half complex plane and deduce that X is geometrically connected.

It follows from Lemmas 1.2–1.4 that for any place x of F above l, p or infinity we have $X(F_v)\neq\emptyset$. (Note that $\rho|_{G_x}$ and $(\mathrm{Ind}_{G_L}^{G_F}\bar\psi)|_{G_x}$ are reducible and so any alternating isomorphisms of $V_\lambda\times F_x$ or $V_\wp\times F_x$ with its Cartier dual are equivalent.) Applying a theorem of Moret-Bailly [**13**], which we recalled in the introduction (Theorem G), we obtain a totally real field E/F in which every place above l and p splits completely and an M-HBAV $(A,i,j)/E$ such that

- the representation of G_E on $A[\lambda]$ is equivalent to $\bar\rho|_{G_E}$, and
- the representation of G_E on $A[\wp]$ is equivalent to $(\mathrm{Ind}_{G_L}^{G_F}\bar\psi)|_{G_E}$.

Note that $(\mathrm{Ind}_{G_L}^{G_F}\bar\psi)|_{G_E}$ is absolutely irreducible, because for any place x of E above p the restriction of $\bar\psi$ to the two places of LE above x are different. Also note that for any place x of E above p there is a finite extension E_x'/E_x with ramification index at most 3 such that A has semi-stable reduction over E_x'. (Because $A[\lambda]$ is unramified at x and the only elements of finite order in $\ker(GL_2(\mathcal{O}_{M,\lambda}) \twoheadrightarrow GL_2(\mathcal{O}_M/\lambda))$ have order dividing 3.) As E_x' has absolute ramification index less than $p-1$ and the I_x-coinvariants of $A[\wp]$ are non-trivial, we see that A has ordinary reduction over E_x' and hence that $T_\wp A$ is ordinary at x.

In some cases we can conclude a little more.

Lemma 1.5. *Suppose that v is an unramified place of F above l and that x is a place of E above v. Suppose also that $\chi_v^2|_{I_v}=\epsilon^n|_{I_v}$ for some integer $0\leqslant n<l-1$. Suppose finally that if $\bar\rho|_{G_v}$ is semisimple then $n\neq 1$. Then the representation of G_x on $T_\lambda A\otimes\mathbb{Q}_l$ has the form*

$$\begin{pmatrix}\epsilon(\chi_v')^{-1} & * \\ 0 & \chi_v'\end{pmatrix},$$

where χ_v' is a tamely ramified lift of χ_v.

Proof. To prove the lemma we may first replace A by $A\otimes_{\mathcal{O}_M}\mathcal{O}_N$ and then replace A by a twist so that

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- the representation of G_x on $A[\lambda]$ has the form

$$\begin{pmatrix} \epsilon\chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

with χ_2 unramified and $\chi_1|_{I_x} \sim \epsilon^{-n}$, and

- the representation of G_x on $A[\wp]$ has the form $\psi_1 \oplus \psi_2$ with ψ_2 unramified and $\psi_1|_{I_x} = \omega^{-n}$.

We must show that the representation of G_x on $T_\lambda A \otimes \mathbb{Q}_l$ has the form

$$\begin{pmatrix} \epsilon\chi_1' & * \\ 0 & \chi_2' \end{pmatrix},$$

where χ_2' is an unramified lift of χ_2.

Looking at the action of G_x on $T_\wp A$ we see that either A has multiplicative reduction over E_x or it has good reduction over $E_x(\zeta_l)$. If it has multiplicative reduction then χ_1 is unramified and the result is clear.

Suppose it has good reduction over $E_x(\zeta_l)$. We will also denote by A the Neron model of A over $W(k(x)^{\mathrm{ac}})[\zeta_l]$. The only possible simple subquotients of the finite flat group scheme $A[\lambda]/W(k(x)^{\mathrm{ac}})[\zeta_l]$ are $\mathbb{Z}/l\mathbb{Z}$ and $\boldsymbol{\mu}_l$. As there are no non-trivial extensions of $\mathbb{Z}/l\mathbb{Z}$ by $\mathbb{Z}/l\mathbb{Z}$ nor of $\boldsymbol{\mu}_l$ by $\boldsymbol{\mu}_l$ over $W(k(x)^{\mathrm{ac}})[\zeta_l]$ we see that there is a short exact sequence

$$(0) \to \boldsymbol{\mu}_l^{[\mathcal{O}_N/\lambda:\mathbb{F}_l]} \to A[\lambda] \to (\mathbb{Z}/l\mathbb{Z})^{[\mathcal{O}_N/\lambda:\mathbb{F}_l]} \to (0)$$

over $W(k(x)^{\mathrm{ac}})[\zeta_l]$. (We are using connected-etale exact sequence and the fact that $\mathrm{Lie}(A[\lambda] \times \mathbb{F}_l^{\mathrm{ac}})$ has dimension $[\mathcal{O}_N/\lambda : \mathbb{F}_l]$.) In particular A has ordinary reduction. If $\bar{\rho}|_{G_v}$ is not semi-simple we are done.

So suppose $\bar{\rho}|_{G_v}$ is semi-simple. Then I_x either acts on $\mathrm{Lie}(A[\lambda] \times \mathbb{F}_l^{\mathrm{ac}})$ by ϵ^{-n} or ϵ^{-1}, according as $A[\lambda]^0 \sim \epsilon\chi_1$ or χ_2 as G_x-modules. (See § 5 of [**7**].) If it acted by ϵ^{-1} then it would also act by ω^{-1} on some subquotient of $A[\wp]$ (see Appendix B of [**3**]). Hence $\chi_1|_{I_x} = \omega$, which we are assuming does not occur when $A[\lambda]$ is semi-simple as a G_x-module. □

Because $(\mathrm{Ind}_{G_L}^{G_F} \psi)|_{G_E}$ is modular we may apply Theorem 5.1 of [**22**] to deduce that $T_\wp A$ is modular and hence that $T_\lambda A$ is modular. Thus we have proved the following theorem in the case that $\bar{\rho}$ has insoluble image. The case that $\bar{\rho}$ has soluble image follows from known cases of the strong Artin conjecture (see [**26**], and [**16**] for how to use congruences to ensure the regularity of π).

Theorem 1.6. *Suppose that l is an odd prime and that $k/\mathbb{F}_l$ is a finite extension. Suppose also that F is a totally real field and that*

$$\bar{\rho} : G_F \to GL_2(k)$$

is a continuous irreducible representation such that

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- *for every place v of F above l we have*

$$\bar{\rho}|_{G_v} \sim \begin{pmatrix} \epsilon\chi_v^{-1} & * \\ 0 & \chi_v \end{pmatrix},$$

and

- *for every complex conjugation c we have* $\det \bar{\rho}(c) = -1$.

Then there is a finite Galois totally real extension E/F in which every prime of F above l splits completely, a regular algebraic cuspidal automorphic representation π of $GL_2(\mathbb{A}_E)$ and a place λ' of the field of coefficients of π above l such that $\bar{\rho}_{\pi,\lambda'} \sim \bar{\rho}|_{G_E}$.

Moreover E, π and λ' may be chosen so that the following holds. If x is an unramified prime of E above l such that

- $\chi_x^2|_{I_x} = \epsilon^n|_{I_x}$, *and*
- $\bar{\rho}(I_x)$ *does not consist of scalar matrices,*

then

$$\rho_{\pi,\lambda'}|_{G_x} \sim \begin{pmatrix} \epsilon(\chi_x')^{-1} & * \\ 0 & \chi_x' \end{pmatrix},$$

where χ_x' is a tamely ramified lift of χ_x.

Corollary 1.7. *Suppose that l is an odd prime and that $k/\mathbb{F}_l$ is a finite extension. Suppose also that F is a totally real field and that*

$$\bar{\rho} : G_F \to GL_2(k)$$

is a continuous irreducible representation such that for every complex conjugation c we have $\det \bar{\rho}(c) = -1$. Then there is a finite Galois totally real extension E/F in which every prime of F above l is unramified with inertial degree at most 2, a regular algebraic cuspidal automorphic representation π of $GL_2(\mathbb{A}_E)$ and a place λ' of the field of coefficients of π above l such that $\rho_{\pi,\lambda'} \sim \bar{\rho}|_{G_E}$.

Moreover let T denote the set of unramified primes v of F above l such that $\bar{\rho}(I_v)$ does not consist of scalar matrices and

$$\bar{\rho}|_{G_v} \sim \begin{pmatrix} \chi_{v,1} & * \\ 0 & \chi_{v,2} \end{pmatrix}$$

with $(\chi_2\chi_1^{-1})|_{I_v} = \epsilon^n|_{I_v}$ for some $n \in \mathbb{Z}/(l-1)\mathbb{Z}$. Then we may choose E, π and λ' so that for any place x of E above a place $v \in T$ we have

$$\rho_{\pi,\lambda'}|_{G_x} \sim \begin{pmatrix} \chi_{x,1}' & * \\ 0 & \chi_{x,2}' \end{pmatrix},$$

where $\chi_{x,2}'$ is a tamely ramified lift of $\chi_{v,2}$.

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2. Applications

Theorem 2.1. *Let l be an odd prime and*

$$\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_l^{\mathrm{ac}})$$

a continuous irreducible representation such that

- *ρ is unramified at all but finitely many primes,*
- *$\det \rho(c) = -1$, and*
- $$\rho|_{G_l} \sim \begin{pmatrix} \epsilon^n \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

  for some $n \in \mathbb{Z}_{>0}$ and some finitely ramified characters χ_1, χ_2 for which $(\epsilon^n \chi_1 \chi_2^{-1})(I_l)$ is not pro-l.

Then there is a finite Galois totally real extension $E/\mathbb{Q}$, a regular algebraic cuspidal automorphic representation π of $GL_2(\mathbb{A}_E)$ and a place λ' of the field of coefficients of π above l such that $\rho_{\pi,\lambda'} \sim \rho$.

Proof. Let $\bar{\rho}$ denote a reduction of some conjugate of ρ which is valued in $GL_2(\mathcal{O}_{\mathbb{Q}_l^{\mathrm{ac}}})$. If $\bar{\rho}$ is reducible then the theorem follows from Theorem A of [**20**]. If $\bar{\rho}$ is induced from a character of a real quadratic field then $\bar{\rho}$ is modular (of weight 1) and so the theorem follows from Theorem 5.1 of [**22**]. Otherwise $\bar{\rho}$ remains irreducible on restriction to any totally real field. In this case the theorem follows from combining Corollary 1.7 with Theorem 5.1 of [**22**]. □

Corollary 2.2. *Keep the assumptions of Theorem 2.1 and choose an isomorphism $i : \mathbb{Q}_l^{\mathrm{ac}} \xrightarrow{\sim} \mathbb{C}$. For all but finitely many primes p the trace and determinant of $\rho(\mathrm{Frob}_p)$ lie in $\mathbb{Q}^{\mathrm{ac}}$ and we have*

$$|i(\mathrm{tr}\, \rho(\mathrm{Frob}_p))| \leqslant 2p^{n/2}.$$

We define the L-function of ρ with respect to i to be

$$L(i\rho, s) = (1 - i\chi_{1,I_l}(\mathrm{Frob}_l)/l^{s-n})^{-1}(1 - i\chi_{2,I_l}(\mathrm{Frob}_l)/l^s)^{-1} \times \prod_{p \neq l} \det(1 - i\rho_{I_p}(\mathrm{Frob}_p)/p^s)^{-1},$$

except we drop the factor $(1 - i\chi_{1,I_l}(\mathrm{Frob}_l)/l^{s-n})^{-1}$ if $n = 1$ and $\chi_1 = \chi_2$. This converges uniformly absolutely for the real part of s sufficiently large. We also define the conductor $N(\rho)$ to be the product

$$N(\chi_1)N(\chi_2)\prod_{p \neq l} N(\rho|_{G_p}),$$

except we replace $N(\chi_1)$ by l if $n = 1$ and $\chi_1 = \chi_2$ is unramified. (Here $N(\rho|_{G_p})$ (respectively $N(\chi_i)$) is the usual conductor of $\rho|_{G_p}$ (respectively χ_i).)

The function $L(i\rho, s)$ has unique meromorphic continuation to the whole complex plane and satisfies a functional equation

$$N(\rho)^{s/2}(2\pi)^{-s}\Gamma(s)L(i\rho,s) = WN(\rho)^{(n+1-s)/2}(2\pi)^{s-1-n}\Gamma(n+1-s) \times L(i(\rho\otimes\epsilon^n(\det\rho)^{-1}), n+1-s),$$

where $|W| = 1$.

Proof. We will simply sketch the proof. The first assertion follows on combining Theorem 2.1 with Theorem 3.4.6 of [**2**]. This implies the uniform absolute convergence of the L-function in some right half-plane.

By Brauer's Theorem (see for instance [**17**], Theorems 16 and 19 in §§ 8.5 and 10.5, respectively), we may find fields $F_j \subset E$ such that $\mathrm{Gal}(E/F_j)$ is soluble, characters $\chi_j : \mathrm{Gal}(E/F_j) \to (\mathbb{Q}^{\mathrm{ac}})^\times$ and integers n_j such that the trivial representation of $\mathrm{Gal}(E/\mathbb{Q})$ has the form

$$\sum_j n_j \operatorname{Ind}_{G_{F_j}}^{G_\mathbb{Q}} \chi_j.$$

Let χ_j also denote the corresponding character of $\mathbb{A}_{F_j}^\times/F_j^\times$. By the argument of the last paragraph of the proof of Theorem 2.4 of [**24**], we see that there is a regular algebraic cuspidal automorphic representation π_j of $GL_2(\mathbb{A}_{F_j})$ such that $\rho|_{G_{F_j}} \sim \rho_{\pi_j,l}$. Then

$$L(i\rho, s) = \prod_j L(\pi_j \otimes (\chi_j \circ \det), s)^{n_j}.$$

□

Corollary 2.3. *Suppose that $A/\mathbb{Q}$ is an abelian variety, M is a number field with $[M:\mathbb{Q}] = \dim A$ and that $j : \mathcal{O}_M \hookrightarrow \mathrm{End}(A/\mathbb{Q})$. Then the L-function of A (relative to an embedding $M \hookrightarrow \mathbb{C}$) has meromorphic continuation to the whole complex plane and satisfies a functional equation*

$$N(A)^{s/2}(2\pi)^{-s}\Gamma(s)L(A,s) = WN(A)^{(2-s)/2}(2\pi)^{s-2}\Gamma(2-s)L(A^\vee, 2-s),$$

where $N(A)$ denotes the conductor of A (relative to the endomorphisms M) and where $|W| = 1$.

Proof. By the last corollary it suffices to find a prime λ of M such that $T_\lambda A$ is ordinary at l. Fix a prime μ of M. Using the Weil bound, we see that it suffices to find a prime $l > 3$ which is unramified in M, at which A has good reduction, which does not divide the residue characteristic of μ and such that

$$\operatorname{tr}\rho_{A,\mu}(\mathrm{Frob}_l) \neq 0.$$

The construction of such a prime l is a standard application of the Cebotarev Density Theorem. □

Corollary 2.4. *Keep the assumptions of Theorem 2.1 and if $n = 1$ further assume that*

- *for some prime $p \neq l$ we have*

$$\rho|_{G_p} \sim \begin{pmatrix} \epsilon\chi & * \\ 0 & \chi \end{pmatrix}.$$

Then ρ occurs in the l-adic cohomology (with coefficients in some Tate twist of the constant sheaf) of some variety over $\mathbb{Q}$. If $n = 1$ then there exists a number field M, a prime λ of M above l, an abelian variety $A/\mathbb{Q}$ of dimension $[M : \mathbb{Q}]$ and an embedding $\mathcal{O}_M \hookrightarrow \mathrm{End}(A/\mathbb{Q})$ such that

$$\rho_{A,\lambda} \sim \rho.$$

Proof. The first part follows by combining Theorem 2.1 with Theorem 2.5.1 of [**1**] and using restriction of scalars.

By Theorem 2.1 and (for instance) Theorem 4.12 (and Proposition 2.5) of [**11**] there is a totally real field E, a number field N, a prime λ' of N above l, an abelian variety B/E of dimension $[N : \mathbb{Q}]$ and an embedding $\mathcal{O}_N \hookrightarrow \mathrm{End}(B/E)$ such that

$$\rho_{B,\lambda'} \sim \rho|_{G_E}.$$

Let C denote the restriction of scalars from E to $\mathbb{Q}$ of B. Then

$$\mathrm{End}_{\mathcal{O}_N}(C/\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong P \oplus \bigoplus_{i=1}^{r} M_i,$$

where all simple constituents of P are non-abelian and where M_i are finite extensions of N. We have a corresponding decomposition up to isogeny

$$C \sim A_P \oplus \bigoplus_{i=1}^{r} A_i,$$

where $\mathcal{O}_{M_i} \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_N}(A_i/\mathbb{Q})$. Note that

$$V_{\lambda'}C \cong \mathrm{Ind}_{G_E}^{G_{\mathbb{Q}}} V_{\lambda'}B \cong X \oplus Y,$$

where $X \sim \rho$ but X is not equivalent to any subquotient of Y. By Faltings's Theorem $(\mathrm{End}_{\mathcal{O}_N}(C/\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q})_{\lambda'}$ has a corresponding decomposition $P_X \oplus P_Y$ where $P_X \hookrightarrow \mathrm{End}(X)$ and $P_Y \hookrightarrow \mathrm{End}(Y)$. Thus for some choice of $i = 1, \ldots, r$ and some prime λ_i of M_i above λ' we have $V_{\lambda_i}A_1 = X$. Take $M = M_i$, $\lambda = \lambda_i$ and $A = A_i$. □

Our final theorem results by combining the last with a beautiful result of Ramakrishna [**14**] (but see Theorem 1.3 of [**24**] for the precise formulation we are using here).

Theorem 2.5. *Suppose that is an odd prime and that*

$$\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_l^{\mathrm{ac}})$$

is a continuous irreducible representation such that $\det \bar{\rho}(c) = -1$ *and*

$$\bar{\rho}|_{G_l} \sim \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

with $\chi_1|_{I_l} \neq \chi_2|_{I_l}$. *Then there exists a number field* M, *a prime* λ *of* M *above* l, *an abelian variety* $A/\mathbb{Q}$ *of dimension* $[M : \mathbb{Q}]$ *and an embedding* $\mathcal{O}_M \hookrightarrow \mathrm{End}(A/\mathbb{Q})$ *such that* $\bar{\rho}$ *is equivalent to the representation of* $G_{\mathbb{Q}}$ *on* $A[\lambda]$.

Acknowledgements. The author was partly supported by NSF grant no. DMS-9702885.

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