Proceedings of the Royal Society of Edinburgh, **149**, 1627–1635, *2019*
DOI:10.1017/prm.2018.127

Non-collapsing in homogeneity greater than one via a two-point method for a special case

Heiko Kröner
University of Hamburg, Department of Mathematics,
Bundesstraße 55, 20146 Hamburg, Germany
(heiko.kroener@math.uni-freiburg.de)

(MS received 9 September 2017; accepted 29 May 2018)

We study the mechanism of proving non-collapsing in the context of extrinsic curvature flows via the maximum principle in combination with a suitable two-point function in homogeneity greater than one. Our paper serves as the first step in this direction and we consider the case of a curve which is C^2-close to a circle initially and which flows by a power greater than one of the curvature along its normal vector.

Keywords: geometric evolution equation, maximum principle

2010 *Mathematics subject classification:* 53C44; 58J35, 35B50

We begin this short note with the definition of δ-non-collapsing, cf. for example, [**4**].

Definition 1.1. A mean-convex hypersurface M bounding an open region Ω in $\mathbb{R}^{n+1}$ is δ-non-collapsed (on the scale of the mean curvature) where $\delta > 0$ is a constant if for every $x \in M$ there is an open ball B of radius $((\delta)/(H(x)))$ contained in Ω with $x \in \partial B$.

Note that the relevant case here is $\delta \leqslant 1$, e.g. the unit circle is δ-non-collapsed only for these values of δ. Sheng and Wang [**24**] showed that any compact mean-convex solution of the mean curvature flow is δ-non-collapsed for some $\delta > 0$. Related statements have been proven earlier by White [**25**]. The proofs in [**24**, **25**] are quite long compared with Andrews' method [**4**] which uses a cleverly chosen ansatz function of two variables and the maximum principle to show that a mean-convex, closed, embedded and δ-non-collapsed initial hypersurface remains δ-non-collapsed under the mean curvature flow. Note that by compactness any mean-convex, closed, embedded hypersurface is δ-non-collapsed for some $\delta > 0$. The challenge was to show that such a δ is preserved under the mean curvature flow. Andrews' new, short and nice method [**4**] was generalized by Andrews, Langford and McCoy [**7**] to flows with fully nonlinear, homogeneous degree one, concave or convex, normal speeds. Later, Brendle [**10**] improved the non-collapsing estimate [**4**] and gave a sharp bound for the inscribed radius under mean curvature flow. Brendle [**11**] also showed an inscribed radius estimate for mean curvature flow in Riemannian manifolds.

A modification of Andrews' ansatz function from [4] served as a crucial tool in Brendle's proof [8] of the Lawson conjecture. Andrews method [4], furthermore, led to a streamlined and new theory of mean-convex mean curvature flow (with surgery), see [12–14].

Andrews and Li [5] applied a similar argument as in [8] to embedded constant mean curvature tori and proved a conjecture of Pinkall and Sterling of the year 1989.

Summarized Andrews' method [4] turned out to be quite rich in content as a tool for further interesting applications, cf. also [9] for an overview about two-point functions.

All the above-mentioned cases use Andrews' tool [4] only in the case that the flow speed is homogeneous of degree one in the principal curvatures. We are aware of only two papers which treat non-collapsing in combination with a different homogeneity of the flow speed, namely Ju and Liu [19] and Liu [20]. They consider in both papers the case of homogeneity -1, that is, expanding curvature flows. While the assertion of [20] is rather in the spirit of [10] the proof in the paper [20] is along Andrews' method [4].

It seems to be tantalizing to show non-collapsing by using the maximum principle in combination with a cleverly chosen ansatz function for extrinsic curvature flows with a general degree of homogeneity greater than one. We are not able to do the latter in full generality.

The aim of our paper is to study the mechanism of applying the maximum principle to obtain non-collapsing in the well-understood setting of flowing convex curves (with curvature being initially sufficiently close to a constant) by powers greater than one of the curvature. This setting can be seen as a fully nonlinear variant of the classical curve shortening flow, cf. [15], [16], and [17] for the higher dimensional case, has been treated in [1, 3] without the assumption on the curvature of the initial curve and the second named paper observes convergence to the round circle after proper rescaling with other methods. In this sense, our non-collapsing result is judged from the point of its assertion far behind of what is already known but interesting from the point of view to see how the two-point method [4] based on the maximum principle works in this case of higher homogeneity. To the best knowledge of the author, this is the first paper in this direction and it would be interesting to generalize our assumptions further. We remark that in order to obtain convergence on the C^2-level after suitable rescaling pinching of the principal curvatures (which corresponds to the fact that the curvature is closed to a constant in the case of curves) is a natural assumption in the case of surfaces when being evolved under flows by higher degrees of homogeneity in the curvature, see for example, [6, 22]. On the other hand in the special case of the Gauss curvature flow of surfaces such an assumption is not necessary [2].

We consider the so-called power curvature flow (PCF) for closed, strictly convex curves in $\mathbb{R}^2$. This is a family of immersions $x(t) = x(t, \cdot)$, $x : [0, T) \times S^1 \to \mathbb{R}^2$, T a positive time which solves the equation

$$\dot{x} = -\kappa^p \nu, \quad p > 1, \tag{1.1}$$

where κ is the curvature of the evolving curve and ν its outer unit normal. As mentioned in the introduction this flow has been studied in [1, 3].

PCF is a special case of the power mean curvature flow (PMCF). The definition of PMCF is analogous to the definition of PCF and can be obtained from the latter by considering now $x : [0, T) \times S^n \to \mathbb{R}^{n+1}$ and replacing 'strictly convex curve' by 'mean-convex hypersurface' and, furthermore, by replacing the curvature κ by the mean curvature H. It was proved in [**23**] that PMCF shrinks (for all n) a convex initial hypersurface to a point and in [**22**] that after a proper rescaling a sufficiently pinched (i.e. the ratio of the biggest and the smallest principal curvature is sufficiently close to 1) initial hypersurface converges in C^∞ to a unit sphere.

Our aim is to prove the following theorem 1.2 using the two-point-method from [**7**]. For technical reasons we denote the constant which is involved in the formulation of this theorem by μ and remark that it plays the role of $\frac{1}{\delta}$ when δ is as in definition 1.1. Clearly, the natural range for μ is then $\mu \geqslant 1$.

THEOREM 1.2. *Let $X : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^{n+1}$ be a family of smooth, strictly convex embeddings evolving by PCF (1.1). There is $\mu_0 = \mu_0(p) > 1$ so that if $M_0 = X(\mathbb{S}^1, 0)$ is μ-non-collapsed with some $1 \leqslant \mu \leqslant \mu_0$, then $M_t = X(\mathbb{S}^1, t)$ is μ-non-collapsed for all $t \in [0, T)$.*

Proof. W.l.o.g. we can assume $1 < \mu \leqslant \mu_0$, otherwise consider a sequence $1 < \mu_k \to 1$. For example, from [**21**] we get the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\kappa^p) - p\kappa^{p-1}\Delta\kappa^p = p\kappa^{p-1}\kappa^{2+p} \tag{1.2}$$

and after a short calculation

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa - p\kappa^{p-1}\Delta\kappa = \kappa^{2+p} + p(p-1)\kappa^{p-2}\|D\kappa\|^2. \tag{1.3}$$

Following [**7**] we define

$$Z(x, y, t) := 2\frac{\langle X(x,t) - X(y,t), \nu(x,t)\rangle}{\|X(x,t) - X(y,t)\|^2} = 2\,\langle w, \nu_x\rangle\, d^{-1} \tag{1.4}$$

for $t \in [0, T)$ and $(x, y) \in (\mathbb{S}^1 \times \mathbb{S}^1)\backslash D$ where $D = \{(x, x) : x \in \mathbb{S}^1\}$; we use the abbreviations $d = \|X(x,t) - X(y,t)\|$, $w = d^{-1}(X(x,t) - X(y,t))$, $\partial^x = ((\partial X)/(\partial x))(x,t)$, $\nu_x = \nu(x,t)$, $\kappa_x = \kappa(x,t)$, and so on; the sub- or superscript x (in contrast to y) will be omitted sometimes. The supremum of Z with respect to y gives the curvature of the largest interior sphere which touches at x.

We show that

$$w(x, y, t) = Z(x, y, t) - \mu\kappa(x, t) \leqslant 0 \tag{1.5}$$

for $0 \leqslant t < T$.

In view of the assumptions (1.5) holds for $t = 0$. We argue by contradiction. Let $\delta > 0$ be small, assume $\sup w(\cdot, \cdot, t) = \delta$ for a $0 < t < T$ and choose t minimal with these properties. Let $x, y \in \mathbb{S}^1$, so that $w(x, y, t) = \delta$, then $x \neq y$. We choose normal coordinates (x) and (y) around x and y, respectively, and obtain in (x, y, t)

by adapting the equations [7, (8)–(11)]

$$
\begin{aligned}
0 \leqslant \dot{w} - p\kappa^{p-1}\left(\frac{\partial^2}{\partial x^2}w + 2\frac{\partial^2}{\partial x\partial y}w + \frac{\partial^2}{\partial y^2}w\right) \\
= -\mu\frac{\partial}{\partial t}\kappa - \frac{2}{d^2}\kappa^p + \frac{2}{d^2}\kappa_y^p\left\langle\nu_y, \nu - dZw\right\rangle + \frac{2}{d}\left\langle w, \nabla(\kappa^p)\right\rangle \\
+ Z^2\kappa^p + \frac{2p}{d^2}\kappa^{p-1}(Z-\kappa) + p\kappa^{p+1}Z \\
- \frac{2p}{d}\kappa^{p-1}\nabla\kappa\left\langle w, \partial^x\right\rangle - p\kappa^p Z^2 \\
+ \frac{4p\mu}{d}\kappa^{p-1}\nabla\kappa\left\langle w, \partial^x\right\rangle + \mu p\kappa^{p-1}\frac{\partial^2\kappa}{\partial x^2} \\
- \frac{4p}{d^2}\kappa^{p-1}(Z-\kappa)\left\langle\partial^y, \partial^x\right\rangle - \frac{4p\mu}{d}\kappa^{p-1}\nabla\kappa\left\langle w, \partial^y\right\rangle \\
+ \frac{2p}{d^2}\kappa^{p-1}(Z-\kappa_y) \\
= -\mu\kappa^{p+2} - \mu p(p-1)\kappa^{p-2}\|D\kappa\|^2 + p\kappa^{p+1}Z \\
- \frac{2(1+p)}{d^2}\kappa^p + \frac{4p}{d^2}\kappa^p\left\langle\partial^y, \partial^x\right\rangle \\
+ \frac{2}{d^2}\kappa_y^p - \frac{2p}{d^2}\kappa^{p-1}\kappa_y \\
+ \frac{4p}{d^2}\kappa^{p-1}Z - \frac{4p}{d^2}\kappa^{p-1}Z\left\langle\partial^y, \partial^x\right\rangle \\
+ (1-p)\kappa^p Z^2 \\
+ \frac{4p\mu}{d}\kappa^{p-1}\nabla\kappa\left\langle w, \partial^x - \partial^y\right\rangle.
\end{aligned} \tag{1.6}
$$

The second line of the right-hand side of inequality (1.6) can be rewritten as

$$ -\frac{4p}{d^2}\kappa^p\left(\frac{1+p}{2p} - \left\langle\partial^y, \partial^x\right\rangle\right) \tag{1.7} $$

and the fourth line as

$$ \frac{4p}{d^2}Z\kappa^{p-1}\left(1 - \left\langle\partial^y, \partial^x\right\rangle\right). \tag{1.8} $$

From

$$ \frac{\partial w}{\partial x}(x, y, t) = 0 \tag{1.9} $$

we conclude

$$ \nabla\kappa = \frac{2}{\mu \mathrm{d}}(\kappa - Z)\left\langle w, \partial^x\right\rangle. \tag{1.10} $$

so that inequality (1.6) can be written as

$$
\begin{aligned}
0 \leqslant & -(\mu\kappa - pZ)\kappa^{p+1} - \mu p(p-1)\kappa^{p-2}\|D\kappa\|^2 \\
& + \frac{2}{d^2}\kappa_y^p - \frac{2p}{d^2}\kappa^{p-1}\kappa_y \\
& + (1-p)\kappa^p Z^2 \\
& + \frac{4p}{d^2}\kappa^{p-1}(Z-\kappa) \\
& \quad \left(1 - \langle \partial^y, \partial^x \rangle + 2\langle w, \partial^y - \partial^x \rangle \langle w, \partial^x \rangle + \frac{1-p}{2p}\right) \\
& + \frac{2(p-1)}{d^2} Z\kappa^{p-1}.
\end{aligned} \tag{1.11}
$$

Using

$$
\delta \geqslant \sup Z(y,\cdot) - \mu\kappa_y = Z(x,y) - \mu\kappa_y \tag{1.12}
$$

we get

$$
\kappa_x \leqslant \kappa_y \leqslant Z. \tag{1.13}
$$

Let us write $\kappa_y = (1 + \eta + ((\tilde{\delta})/(\kappa_x)))\kappa_x$ with suitable $0 \leqslant \eta \leqslant \mu - 1$ and $0 \leqslant \tilde{\delta} \leqslant \delta$. By using Taylor's expansion of the function

$$
f(s) = (1+s)^{p-1}, \quad s \geqslant 0, \tag{1.14}
$$

around 0 we obtain at $s = \eta + ((\tilde{\delta})/(\kappa_x))$ the following upper bound for the second line of (1.11)

$$
\begin{aligned}
& \frac{2}{d^2}\kappa^p(1+s)((1+s)^{p-1} - p) \\
& \quad \leqslant \frac{2}{d^2}\kappa^p \left[1 - p + (p-1)\left(1 + \frac{p-2}{2}\right)s^2 \right. \\
& \quad \left. + \frac{1}{2}(p-1)(p-2)\left(1 + \frac{1}{3}(p-3)(1+\xi)^{p-4}(1+s)\right)s^3\right]
\end{aligned} \tag{1.15}
$$

with a suitable $0 \leqslant \xi \leqslant s$. We have

$$
\frac{4p}{d^2}\kappa^{p-1}(Z-\kappa)\frac{1-p}{2p} = \frac{2}{d^2}(1-p)\left(\mu - 1 + \frac{\delta}{\kappa}\right)\kappa^p. \tag{1.16}
$$

The proof of [7, lemma 6] shows that there are $\alpha \in [0, \frac{\pi}{2}]$ and normal coordinates (x) and (y) so that

$$
|\langle w, \nu_x \rangle| = \sin\alpha \quad \wedge \quad \langle \partial^y, \partial^x \rangle = -\cos 2\alpha \tag{1.17}
$$

and

$$
1 - \langle \partial^y, \partial^x \rangle + 2\langle w, \partial^y - \partial^x \rangle \langle w, \partial^x \rangle = -2\cos^2\alpha. \tag{1.18}
$$

https://doi.org/10.1017/prm.2018.127 Published online by Cambridge University Press

Estimating (1.11) using (1.15), (1.16) and (1.18) leads to

$$\begin{aligned}
0 &\leqslant -(\mu\kappa - pZ)\kappa^{p+1} - \mu p(p-1)\kappa^{p-2}\|D\kappa\|^2 \\
&\quad + \frac{2}{d^2}\kappa^p \left[1-p+(p-1)\left(1+\frac{p-2}{2}\right)s^2\right. \\
&\quad \left.+\frac{1}{2}(p-1)(p-2)\left(1+\frac{1}{3}(p-3)(1+\xi)^{p-4}(1+s)\right)s^3\right] \\
&\quad + (1-p)\kappa^p Z^2 - \frac{8p}{d^2}\kappa^{p-1}(Z-\kappa)\cos^2\alpha + 2\frac{p-1}{d^2}Z\kappa^{p-1} \\
&\quad + \frac{2}{d^2}(1-p)\left(\mu - 1 + \frac{\delta}{\kappa}\right)\kappa^p \\
&\leqslant -(\mu\kappa - p\mu\kappa - p\delta)\kappa^{p+1} \\
&\quad + (1-p)\kappa^p(\mu^2\kappa^2 + 2\mu\kappa\delta + \delta^2) \\
&\quad - \frac{8p}{d^2}\cos^2\alpha\kappa^{p-1}((\mu-1)\kappa + \delta) \\
&\quad + \frac{2}{d^2}\kappa^p\left[(p-1)\left(1+\frac{p-2}{2}\right)s^2\right. \\
&\quad \left.+\frac{1}{2}(p-1)(p-2)\left(1+\frac{1}{3}(p-3)(1+\xi)^{p-4}(1+s)\right)s^3\right] \\
&= (1-p)(\mu-1)\mu\kappa^{p+2}(1+O(\delta)) \\
&\quad + (\mu-1)\kappa^p\frac{2}{d^2}[-4p\cos^2\alpha](1+O(\delta)) \\
&\quad + \frac{2}{d^2}\kappa^p\left[(p-1)\left(1+\frac{p-2}{2}\right)s^2\right. \\
&\quad \left.+\frac{1}{2}(p-1)(p-2)\left(1+\frac{1}{3}(p-3)(1+\xi)^{p-4}(1+s)\right)s^3\right] \\
&\equiv (1-p)(\mu-1)\mu\kappa^{p+2}(1+O(\delta)) \\
&\quad + (\mu-1)\kappa^p\frac{2}{d^2}[-4p\cos^2\alpha](1+O(\delta)) \\
&\quad + \frac{2}{d^2}\kappa^p(p-1)c(p)sO(s).
\end{aligned} \tag{1.19}$$

There holds

$$d \leqslant \frac{\sqrt{2}}{Z} \Leftrightarrow \alpha \leqslant \frac{\pi}{4}. \tag{1.20}$$

In order to see the last equivalence we make a short elementary geometric deliberation. Remember that $1/Z$ is the radius of the largest open ball B which is enclosed by $M(t)$ and which satisfies $X(x,t) \in \partial B$. Recall that d is the distance of $X(x,t)$ and $X(y,t)$ from each other and both of them lie on ∂B. Note that $\sqrt{2}1/Z$ is the distance of any point on ∂B from its image under a rotation by $\pi/2$ around the centre of ∂B. Hence for the angle $\varphi \in [0, \pi/2]$ between ν_x and w we conclude in view of

$$\cos\varphi = |\langle w, \nu_x\rangle| = \sin\alpha \tag{1.21}$$

that

$$d \leqslant \frac{\sqrt{2}}{Z} \Leftrightarrow \varphi \geqslant \frac{\pi}{4} \Leftrightarrow \alpha \leqslant \frac{\pi}{4}. \tag{1.22}$$

In the following, we may w.l.o.g. use inequality (1.19) under the assumption that $O(\delta) = 0$, especially we have then $Z = \mu\kappa$. We distinguish cases and show thereby that a sufficiently strong initial pinching quantified by μ implies non-positivity of the RHS of (1.19). Since the sufficient condition for p and μ to obtain non-positivity of the RHS of (1.19) is of polynomial type of higher order we only give an example condition which is sufficient.

(i) We assume that $d \geqslant \sqrt{2}/Z$. Then we can estimate the right-hand side of (1.19) as follows.

$$\begin{aligned}\text{RHS of (1.19)} &\leqslant (1-p)(\mu-1)\mu\kappa^p\frac{2}{\mu^2 d^2}\\ &\quad + (\mu-1)\kappa^p\frac{2}{d^2}[-4p\cos^2\alpha + (p-1)c(p)O(s)]\\ &\leqslant (1-p)(\mu-1)\mu\kappa^p\frac{2}{\mu^2 d^2} + (\mu-1)\kappa^p\frac{2}{d^2}(p-1)c(p)O(s)\\ &< 0,\end{aligned} \tag{1.23}$$

or, equivalently, for the last inequality,

$$\frac{1}{\mu} > c(p)O(s). \tag{1.24}$$

(ii) We assume that $d < \sqrt{2}/Z$. In view of (1.20) we have $\alpha \leqslant \pi/4$. Now we estimate as follows.

$$\begin{aligned}\text{RHS of (1.19)} &\leqslant (1-p)(\mu-1)\mu\kappa^{p+2}\\ &\quad + (\mu-1)\mu^2\kappa^{p+2}[-4p\cos^2\alpha + (p-1)c(p)O(s)]\\ &\leqslant (1-p)(\mu-1)\mu\kappa^{p+2}\\ &\quad + (\mu-1)\mu^2\kappa^{p+2}(-2p + (p-1)c(p)O(s))\\ &\overset{!}{<} 0.\end{aligned} \tag{1.25}$$

Assuming also in the present case (ii) that (1.24) holds we see that $c(p)O(s) < 1$ and the last inequality in (1.25) is valid.

https://doi.org/10.1017/prm.2018.127 Published online by Cambridge University Press

Summarized we get as sufficient condition for the fact that the pinching is preserved for example,

$$\frac{1}{\mu} > c(p)O(s). \tag{1.26}$$

where the quantities $c(p)$ and $O(s)$ are defined in the last line of (1.19). Note that $s \leqslant \mu - 1$ by definition. □

Acknowledgements
Parts of the manuscript were written while the author was supported by the Weierstrass Postdoctoral Fellowship Program of the Weierstrass Institute Berlin (WIAS).

References

1 B. Andrews. Evolving convex curves. *Calc. Var.* **7** (1998), 315–371.
2 B. Andrews. Gauss curvature flow: the fate of the rolling stones. *Invent. math.* **138** (1999), 151–161.
3 B. Andrews. Classification of limiting shapes for isotropic curve flows. *J. Amer. Math. Soc.* **16** (2003), 443–459.
4 B. Andrews. Non-collapsing in mean-convex mean curvature flow. *Geom. Topol.* **16** (2012), 1413–1418.
5 B. Andrews, H. Li. Embedded constant mean curvature tori in the three-sphere. *J. Differ. Geom.* **99** (2015), 169–189.
6 B. Andrews and J. McCoy. Convex hypersurfaces with pinched principal curvatures and flow of convex hypersurfaces by high powers of curvature. *Trans. Amer. Math. Soc.* **364** (2012), 3427–3447.
7 B. Andrews, M. Langford and J. McCoy. Non-collapsing in fully nonlinear curvature flows. *Annales de I'Institut Henri Poincare/Analyse non lineaire* **30** (2013), 23–32.
8 S. Brendle. Embedded minimal tori in S^3 and the Lawson conjecture. *Acta Math.* **211** (2013), 177–190.
9 S. Brendle. Two-point functions and their applications in geometry. *Bull. Am. Math. Soc., New Ser.* **51** (2014), 581–596.
10 S. Brendle. A sharp bound for the inscribed radius under mean curvature flow. *Invent. Math.* **202** (2015), 217–237.
11 S. Brendle. An inscribed radius estimate for mean curvature flow in Riemannian manifolds. *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)* **16** (2016), 1447–1472.
12 R. Haslhofer and B. Kleiner. On Brendle's estimate for the inscribed radius under mean curvature flow. *Int. Math. Res. Not.* **2015** (2015), 6558–6561.
13 R. Haslhofer and B. Kleiner. Mean curvature flow with surgery. *Duke Math. J.* **166** (2017a), 1591–1626.
14 R. Haslhofer and B. Kleiner. Mean curvature flow of mean-convex hypersurfaces. *Commun. Pure Appl. Math.* **70** (2017b), 511–546.
15 M. E. Gage. Curve shortening makes convex curves circular. *Invent. Math.* **76** (1984), 357–364.
16 M. Gage and R. S. Hamilton. The heat equation shrinking convex plane curves. *J. Diff. Geom.* **23** (1986), 69–96.
17 M. A. Grayson. The heat equation shrinks embedded plane curves to round points. *J. Differ. Geom.* **26** (1987), 285–314.
18 G. Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differ. Geom.* **20** (1984), 117–138.
19 H. Ju and Y. Liu. Non-collapsing for a fully nonlinear inverse curvature flow. *Comm. Pure Appl. Anal.* **16** (2017), 945–952.

20 Y. Liu. Inscribed radius estimates for inverse curvature flow in sphere and hyperbolic space. *Nonlin. Anal.* **155** (2017), 198–206.

21 F. Schulze. Evolution of convex hypersurfaces by powers of the mean curvature. *Math. Z.* **251** (2005), 721–733.

22 F. Schulze. Convexity estimates for flows by powers of the mean curvature. *Ann. Sc. Norm. Super. Pisa Cl. Sci (5)* **5** (2006), 261–277.

23 F. Schulze. Nonlinear evolution by mean curvature and isoperimetric inequalities. *J. Diff. Geom.* **79** (2008), 197–241.

24 W. Sheng and X.-J. Wang. Singularity profile in the mean curvature flow. *Methods Appl. Anal.* **16** (2009), 139–155.

25 B. White. The size of the singular set in mean curvature flow of mean-convex sets. *J. Amer. Math. Soc.* **13** (2000), 665–695 (electronic).