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Universal Alternating Semiregular Polytopes

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Abstract. In the classical setting, a convex polytope is said to be semiregular if its facets are regular and its symmetry group is transitive on vertices. This paper continues our study of alternating semiregular abstract polytopes, which have abstract regular facets, still with combinatorial automorphism group transitive on vertices and with two kinds of regular facets occurring in an alternating fashion.

Our main concern here is the universal polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$, an alternating semiregular $(n+1)$-polytope defined for any pair of regular n-polytopes $\mathcal{P}$, $\mathcal{Q}$ with isomorphic facets. After a careful look at the local structure of these objects, we develop the combinatorial machinery needed to explain how $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ can be constructed by "freely assembling" unlimited copies of $\mathcal{P}$, $\mathcal{Q}$ along their facets in alternating fashion. We then examine the connection group of $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$, and from that prove that $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ covers any $(n+1)$-polytope $\mathcal{B}$ whose facets alternate in any way between various quotients of $\mathcal{P}$ or $\mathcal{Q}$.

1 Introduction

The *cuboctahedron* is a familar example of what we will call an *alternating semiregular 3-polytope*: its facets are squares and equilateral triangles, two of each occuring in alternate fashion around each vertex. If one takes those words as instructions for assembling a convex model, then up to similarity there can be only one end result $\mathcal{S}$ (the cuboctahedron). It is a little remarkable that the polyhedron $\mathcal{S}$ then has a symmetry group that is transitive on vertices. Because the facets are regular, this means that $\mathcal{S}$ also belongs to the even more general class of uniform polytopes.

This sort of alternating behaviour also appears in the tiling of Euclidean 3-space by regular octahedra and tetrahedra. The tiling is therefore an infinite alternating semiregular 4-polytope.

Our main concern in this paper will be abstract semiregular polytopes like this, with two kinds of regular facets occurring in an "alternating" fashion.[1] In [12], we established a basic construction for such polytopes as a kind of coset geometry over a tail-triangle C-group. (See also Theorem 2.7.)

Here we investigate in a deeper way the structure of the universal alternating semiregular polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ along with its covering properties. This $(n+1)$-polytope is defined for any pair of regular n-polytopes $\mathcal{P}$, $\mathcal{Q}$ with isomorphic facets. In Section 3,

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[1]We thank Peter McMullen for reminding us that "alternating semiregular polytopes" have usually been called *quasi-regular* in the literature, at least in classical discrete geometry; see [3, pp. 18 and 69], for example. But we prefer to maintain our terminology if only because it emphasizes the local structure more explicitly.

we recall the construction and take a closer look at the local structure of $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$. This is followed in Section 4 by a careful description of what must be involved in attaching an abstract polytope (really a very special sort of flagged poset) to another abstract polytope, along some pair of isomorphic facets. Then in Theorem 4.10 we can explain how it is that $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ can be constructed by "freely assembling" unlimited copies of $\mathcal{P}$, $\mathcal{Q}$ along their facets in alternating fashion. Finally, in Theorem 5.7 and Corollary 5.9, we prove that $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ covers any $(n+1)$-polytope $\mathcal{B}$ whose facets alternate in any way between any sorts of "nice" quotients of $\mathcal{P}$ or $\mathcal{Q}$.

In separate papers, we will explore a parameter called the *interlacing number* (see Section 2). First, in [14], we examine general covering properties of the universal polytope $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ attached to a specific interlacing number k. (With this modified notation, the polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ studied in this paper becomes $\mathcal{U}^\infty_{\mathcal{P},\mathcal{Q}}$.) Second, in [13], we consider situations in which $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ does not even exist, even when $\mathcal{P}$ and $\mathcal{Q}$ have isomorphic facets.

2 Flagged Posets and Polytopes

An abstract n-polytope $\mathcal{P}$ has certain key combinatorial properties of the face lattice of a convex n-polytope; in general, however, $\mathcal{P}$ need not be a lattice, need not be finite, need not have any familiar geometric realization. For a detailed look at the theory of abstract polytopes we refer to [9].

Later on we will frequently encounter structures that "aren't quite" polytopes. Because of this, we find it useful in the following overview to rearrange properties P1, P2, P3, and P4 of [9, §2A] and relabel them below as B, A, D, and C, respectively. (Our description of these in [12] is different again.)

Definition 2.1 A *flagged poset* is a partially ordered set $\mathcal{P}$ with property *A* below. Such a poset is *closed* and has *rank n* if it also satisfies property *B* (taking $n = m-2$). A pre-polytope of rank n is a closed, flagged poset (of rank n), that also satisfies property *C*. An abstract n-polytope $\mathcal{P}$ is a pre-polytope of rank n which also satisfies *D*.

The elements F of $\mathcal{P}$ are called its *faces*. A maximal chain in $\mathcal{P}$ is called a *flag*. We let $\mathcal{F}(\mathcal{P})$ be the set of all flags in $\mathcal{P}$.

We now examine more carefully the key properties mentioned above.

A. Every flag in $\mathcal{P}$ contains a fixed (finite) number (say m) of faces.

B. $\mathcal{P}$ has two faces, denoted F_{-1} and F_n, such that $F_{-1} \leqslant F \leqslant F_n$, for all $F \in \mathcal{P}$. Thus, F_{-1} is the unique *minimal* face in $\mathcal{P}$ and F_n is the unique *maximal* face.

Remark 2.2 It is easy to see that a flagged poset $\mathcal{P}$ has a strictly monotone *rank function* rk. Indeed, for any face $F \in \mathcal{P}$, we will let $\mathrm{rk}(F) := j$ if there are $j+1$ faces strictly below F in any flag Φ containing F. The range of rk is therefore $\{-1, \dots, m-2\}$. We also say that $\mathcal{P}$ itself has rank $m-2$. (This way of defining rank is motivated by the notion of dimension for the faces of a convex n-polytope and will seem more natural in the presence of condition B, with $m = n+2$.) An element $F \in \mathcal{P}$ with $\mathrm{rk}(F) = j$ is called a *j-face*. Faces of rank $0, 1, n-2$ or $n-1$ in a flagged poset of rank n are called *vertices*, *edges*, *ridges*, or *facets*, respectively.

Whenever $F \leqslant G$ are incident faces in $\mathcal{P}$, with $\mathrm{rk}(F) = j \leqslant k = \mathrm{rk}(G)$, the *section* G/F is defined by

$$G/F := \{H \in \mathcal{P} \mid F \leqslant H \leqslant G\}.$$

It is easy to check that the section G/F of the flagged poset $\mathcal{P}$ is itself a flagged poset (of rank $k - j - 1$).

If $\mathcal{P}$ is closed and $m = n + 2$, then the rank function has range $\{-1, 0, \dots, n\}$. The *improper* faces F_{-1} and F_n have ranks -1 and n, respectively. If, in this case, F is a vertex of $\mathcal{P}$, then the section F_n/F is called the *vertex-figure* at F. More generally, if F is a j-face, then F is said to have *co-rank* $n - j - 1$, that being the rank of its *co-face* F_n/F.

Each face of a convex polytope is itself a convex polytope with its own internal structure. Extending that point of view to general closed, flagged posets, we can casually treat any face F as if it were the section $F/. := F/F_{-1}$ below it. (We use "**.**" as an unobtrusive place-holder for minimal elements.) See also Remark 3.1.

According to the terminology of [16, p. 244], a poset that satisfies properties A and B is a *graded poset of rank* $n+1$. However, we prefer to say that the rank is n, this being the more natural geometric parameter. Note that n is the number of *proper* faces in any flag of a closed, flagged poset of rank n.

C. Whenever $F < G$ with $\mathrm{rk}(F) = j-1$ and $\mathrm{rk}(G) = j+1$, there are exactly two j-faces H with $F < H < G$.

Remark 2.3 For $0 \leqslant j \leqslant n - 1$ and any flag $\Phi \in \mathcal{F}(\mathcal{P})$, there thus exists a unique *j-adjacent* flag Φ^j, differing from Φ in just the face of rank j. With this notion of adjacency, $\mathcal{F}(\mathcal{P})$ becomes the *flag graph* for $\mathcal{P}$. ∎

D. $\mathcal{P}$ is *strongly flag–connected*; that is, the flag graph for each section is connected.

Remark 2.4 Observe that every section of a (pre-) polytope $\mathcal{P}$ is itself a (pre-) polytope (of suitable rank).

The *automorphism group* $\Gamma(\mathcal{P})$ of a flagged poset $\mathcal{P}$ consists of all order-preserving bijections on $\mathcal{P}$. Now suppose for the moment that $\mathcal{P}$ is an n-polytope. We say $\mathcal{P}$ is *regular* if $\Gamma(\mathcal{P})$ is transitive on the flag set $\mathcal{F}(\mathcal{P})$. In this case, we can choose any one flag $\Phi \in \mathcal{F}(\mathcal{P})$ as *base flag*, then define ρ_j to be the (unique) automorphism mapping Φ to Φ^j, for each j in the index set $N := \{0, \dots, n - 1\}$. From [9, 2B], we recall that $\Gamma(\mathcal{P})$ is then a *string C-group*. This means first that $\Gamma(\mathcal{P})$ is generated by $\{\rho_j : j \in N\}$. Second, these involutory generators satisfy the commutativity relations typical of a Coxeter group with string diagram, namely,

$$(\rho_j \rho_k)^{p_{jk}} = 1, \text{ for } 0 \leqslant j \leqslant k \leqslant n-1, \tag{2.1}$$

where $p_{jj} = 1$ and $p_{jk} = 2$ whenever $|j - k| > 1$. Finally, $\Gamma(\mathcal{P})$ is a *C-group*, meaning that it satisfies the *intersection condition*

$$\langle \rho_k : k \in I\rangle \cap \langle \rho_k : k \in J\rangle = \langle \rho_k : k \in I \cap J\rangle, \text{ for any } I, J \subseteq N. \tag{2.2}$$

The fact that one can reconstruct a regular polytope in a canonical way from any string C-group Γ is at the heart of the theory [9, 2E].

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The periods $p_j := p_{j-1,j}$ in (2.1) satisfy $2 \leqslant p_j \leqslant \infty$ and are assembled into the *Schläfli symbol* $\{p_1, \dots, p_{n-1}\}$ for the regular polytope $\mathcal{P}$. As a familiar example, we recall that every 2-polytope is automatically abstractly regular. Indeed, a p-gon has Schläfli symbol $\{p\}$, and its automorphism group is the dihedral group $\mathbb{D}_p$ of order $2p$. Usually, however, $\Gamma(\mathcal{P})$ is not a Coxeter group. This happens when the relations in (2.1) do not suffice for a presentation.

There are many ways to relax symmetry and thereby broaden the class of groups $\Gamma(\mathcal{P})$ [8, p. 77]. In [12], we restricted our considerations to the kind of polytopes described in the following definition.

Definition 2.5 An abstract polytope $\mathcal{S}$ is *semiregular* if it has regular facets and its automorphism group $\Gamma(\mathcal{S})$ is transitive on vertices. The semiregular polytope $\mathcal{S}$ is *alternating* if its facets (all regular) are of two kinds $\mathcal{P}$ and $\mathcal{Q}$, which further occur in alternating fashion around any face in $\mathcal{S}$ of co-rank 2. (We allow $\mathcal{P} \simeq \mathcal{Q}$.)

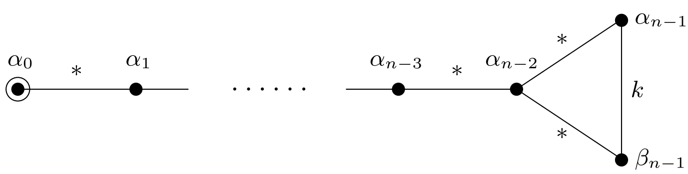
Every regular polytope is clearly semiregular. Notice that in an alternating semiregular polytope $\mathcal{S}$, there is, for each face F of co-rank 2, some $k \geqslant 1$ such that F is surrounded by $2k$ facets. Usually for us, k will be constant from one such F to another. This certainly happens when $\mathcal{S}$ has rank 3 (by vertex transitivity) and also, for instance, if $\mathcal{S}$ is *hereditary* [10]. In a hereditary polytope, every automorphism of every facet extends to an automorphism of the whole polytope. However, such nice conditions might fail.

Example 2.6 Construct a 3-polytope $\mathcal{K}$ by gluing two regular octahedra face-to-face, removing the triangular barrier between. $\mathcal{K}$ has vertices of valency 4 or 6. Now construct the 4-polytope $\mathcal{S} = 2^{\mathcal{K}}$, as described in [15, §3]. From Theorem 3.1 there, we observe that $\mathcal{S}$ is vertex-transitive, with each vertex-figure isomorphic to $\mathcal{K}$. Moreover, all facets of $\mathcal{S}$ are isomorphic to $2^{\{3\}}$, that is, to the cube, which of course is regular. Therefore, $\mathcal{S}$ is a semiregular 4-polytope, but with 4 facets surrounding some edges and 6 facets around certain others.

Evidently, there will be a huge variety of abstract semiregular polytopes. As in [12], we confine our investigation to a very symmetric class of semiregular polytopes in which the parameter k is constant for all faces F of co-rank 2 in $\mathcal{S}$. We call this value of k the *interlacing number* for $\mathcal{S}$.

In fact, we will focus mainly on semiregular polytopes that can be constructed using the combinatorial version of Wythoff's construction described in [12, §4]. Suppose that $\Gamma = \langle \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ is a group generated by involutions that satisfy the commutativity relations implicit in the *tail-triangle diagram*

(2.3)



The label "k" indicates that $\alpha_{n-1}\beta_{n-1}$ has period k, for some $k = 2, \dots, \infty$. However, all other periods of products of two "adjacent" generators are unspecified for the

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moment and indicated by a $*$. The label "2" is possible and indicates the actual absence of the corresponding branch in the diagram. The group Γ is called a *tail-triangle group*. (Anticipating Theorem 2.7, we also say that the group Γ has interlacing number k.) We allow the degenerate (base) case $n = 1$, in which $\Gamma = \langle \alpha_0, \beta_0 \rangle$ is just the dihedral group $\mathbb{D}_k$.

Suppose also that Γ is a C-group, now satisfying the intersection condition (2.2) over a suitable index set N for the $n+1$ generators of Γ. Then Γ is a *tail-triangle C-group*. It follows that the subgroups

$$\Gamma_n^{\mathcal{P}} := \langle \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1} \rangle$$

and

$$\Gamma_n^{\mathcal{Q}} := \langle \alpha_0, \dots, \alpha_{n-2}, \beta_{n-1} \rangle$$

are string C-groups; indeed, automorphism groups for the regular n-polytopes $\mathcal{P}$ and $\mathcal{Q}$, respectively.

As described in [12, Definition 4.2], the ringed node in (2.3) initiates Wythoff's construction for an $(n+1)$-polytope $\mathcal{S} = \mathcal{S}(\Gamma)$ as a coset geometry over Γ. First of all, for $0 \leqslant j \leqslant n-1$, we let

$$(2.4) \qquad \Gamma_j^- := \langle \alpha_0, \dots, \alpha_{j-1} \rangle, \quad \Gamma_j^+ := \langle \alpha_{j+1}, \dots, \alpha_{n-1}, \beta_{n-1} \rangle$$

and

$$(2.5) \qquad \Gamma_j := \langle \alpha_0, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_{n-1}, \beta_{n-1} \rangle = \Gamma_j^- \times \Gamma_j^+.$$

The direct product follows easily from (2.2) and the structure of the diagram in (2.3). When $j = n-1$, we interpret (2.4) and (2.5) as $\Gamma_{n-1} = \Gamma_{n-1}^-$, with $\Gamma_{n-1}^+ = \{1\}$.

The j-faces of $\mathcal{S}$ with $j \leqslant n-1$ can be identified with all right cosets of Γ_j in Γ. Likewise, the n-faces of $\mathcal{S}$ are all right cosets of either $\Gamma_n^{\mathcal{P}}$ or $\Gamma_n^{\mathcal{Q}}$. For the improper faces of $\mathcal{S}$, we can take two distinct copies of Γ. Faces of distinct rank are now incident if and only if the corresponding cosets have non-empty intersection. We find that the polytope $\mathcal{S}$ has two flag orbits under the action of Γ, with *base flags*

$$(2.6) \qquad \Phi := [\Gamma_0, \dots, \Gamma_{n-2}, \Gamma_{n-1}, \Gamma_n^{\mathcal{P}}] \quad \text{and} \quad \Psi := [\Gamma_0, \dots, \Gamma_{n-2}, \Gamma_{n-1}, \Gamma_n^{\mathcal{Q}}].$$

(As usual, we have suppressed improper faces.) Flags in the two orbits are said to have type $\mathcal{P}$, $\mathcal{Q}$, respectively. Such base flags are indicated for the 3-polytope $\mathcal{U}_{\{4\},\{3\}}$ displayed in Figure 3 in Section 5.

We combine the key results of [12, §4] about the structure of $\mathcal{S}$ into the following theorem.

Theorem 2.7 *Suppose Γ is a tail-triangle C-group corresponding to the diagram (2.3) and let $\mathcal{S} = \mathcal{S}(\Gamma)$ be the resulting $(n+1)$-polytope.*

(i) *$\mathcal{S}$ is an alternating semiregular polytope. Its facets are isomorphic to $\mathcal{P}$ or $\mathcal{Q}$, with k of each of these occurring alternately around each face $\mathcal{R}$ of co-rank 2. Each 2-section $\mathcal{S}/\mathcal{R}$ is therefore a $2k$-gon, and $\mathcal{S}$ has interlacing number k.*

(ii) *The face-wise Γ-stabilizer of any ridge of $\mathcal{S}$ is trivial.*

(iii) *Each vertex-figure of $\mathcal{S}$ is isomorphic to the alternating semiregular n-polytope $\widehat{\mathcal{S}}$ defined by the tail-triangle C-group whose diagram is obtained by deleting the node labelled α_0 in the diagram (2.3), then ringing the node labelled α_1.*

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(iv) *$\mathcal{S}$ is a regular polytope if and only if Γ admits a group automorphism induced by the diagram symmetry which swaps α_{n-1} and β_{n-1} in (2.3), while fixing the remaining α_j's. In this case $\mathcal{P} \simeq \mathcal{Q}$, say with Schläfli type $\{p_1, \dots, p_{n-1}\}$, and $\mathcal{S}$ is regular of type $\{p_1, \dots, p_{n-1}, 2k\}$; moreover, $\Gamma(\mathcal{S}) \simeq \Gamma \rtimes C_2$.*

(v) *If $\mathcal{S}$ is not regular, then $\mathcal{S}$ is a 2-orbit polytope and $\Gamma(\mathcal{S}) \simeq \Gamma$. In particular, this is so if the facets $\mathcal{P}$ and $\mathcal{Q}$ are non-isomorphic.*

Recall that an abstract polytope is called a 2-*orbit polytope* if its automorphism group has precisely two flag orbits. (See [6]; for general structural results about the groups of 2-orbit polytopes, see also [7].)

Each ridge $\mathcal{K}$ in the polytope $\mathcal{S}$ of Theorem 2.7 can just as well be viewed as a facet of either $\mathcal{P}$ or of $\mathcal{Q}$. Thus, the regular n-polytopes $\mathcal{P}$ and $\mathcal{Q}$ have all their facets isomorphic to $\mathcal{K}$, which in turn is a regular $(n-1)$-polytope with automorphism group $\Gamma(\mathcal{K}) \simeq \Gamma_{n-1}$.

In fact, whenever $\mathcal{P}$ and $\mathcal{Q}$ have matching facets $\mathcal{K}$, we can amalgamate the groups $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ along the common facet subgroup $\Gamma(\mathcal{K})$. This construction is described in [12, Section 5] and summarized in Section 3. It does indeed yield a tail-triangle C-group, with interlacing number $k = \infty$. The resulting alternating semiregular $(n+1)$-polytope is denoted $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$. The remaining sections describe algebraic and geometric constructions for $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$, as well as its universal property.

3 The Universal Semiregular Polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$

In this and the following section, we set up the machinery needed to understand how we can freely assemble compatible regular polytopes in an alternating way. Referring to [12, §5], let us first recall some features of our construction of the universal, semiregular $(n+1)$-polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$, which for brevity we call $\mathcal{S}$. We begin with regular n-polytopes $\mathcal{P}$ and $\mathcal{Q}$, each with facets isomorphic to a common regular $(n-1)$-polytope $\mathcal{K}$. Thus,

$$\Gamma(\mathcal{P}) = \langle \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1} \rangle, \quad \Gamma(\mathcal{Q}) = \langle \beta_0, \dots, \beta_{n-2}, \beta_{n-1} \rangle, \tag{3.1}$$

and $\Gamma(\mathcal{K}) \simeq \langle \alpha_0, \dots, \alpha_{n-2} \rangle \simeq \langle \beta_0, \dots, \beta_{n-2} \rangle$. The tail-triangle C-group used to manufacture $\mathcal{S}$ is the amalgamated product

$$\Gamma = \Gamma(\mathcal{P}) *_{\Gamma(\mathcal{K})} \Gamma(\mathcal{Q}) = \langle \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$$

obtained by identifying α_j with β_j for $0 \leqslant j \leqslant n-2$. Therefore, with no loss of generality, we can conveniently take

$$\begin{aligned} \Gamma(\mathcal{P}) &= \langle \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1} \rangle, \\ \Gamma(\mathcal{Q}) &= \langle \alpha_0, \dots, \alpha_{n-2}, \beta_{n-1} \rangle, \\ \Gamma(\mathcal{K}) &= \langle \alpha_0, \dots, \alpha_{n-2} \rangle, \end{aligned}$$

all subgroups of Γ.

Every $(n-1)$-face (ridge) in $\mathcal{S}$ is equivalent to $\mathcal{K}$ under Γ. The n-faces (facets) of $\mathcal{S}$ are isomorphic to either $\mathcal{P}$ or $\mathcal{Q}$ (which could themselves be isomorphic). One copy of each lies on each ridge. In particular, $\mathcal{P}$ and $\mathcal{Q}$ can be identified with the n-faces on the base ridge $\mathcal{K}$ of $\mathcal{S}$. Then, for each $\mu \in \Gamma$, the n-faces of $\mathcal{S}$ on $\mathcal{K}\mu$ are given by $\mathcal{P}\mu$

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and $\mathcal{Q}\mu$. Also, the base face $\mathcal{L}$ of rank $n-2$ (*i.e.*, with co-rank 2 in $\mathcal{S}$) is stabilized by the subgroup $\Gamma(\mathcal{L}) = \langle \alpha_0, \dots, \alpha_{n-3}, \alpha_{n-1}, \beta_{n-1} \rangle$. Infinitely many copies of $\mathcal{P}$ and $\mathcal{Q}$ lie alternately on $\mathcal{L}$. Indeed, for each $\mu \in \Gamma$, the n-faces on $\mathcal{L}\mu$ are all $\mathcal{P}\lambda\mu$, $\mathcal{Q}\lambda\mu$, where λ runs through the infinite dihedral group $\langle \alpha_{n-1}, \beta_{n-1} \rangle$.

Remark 3.1 Here, and frequently below, we indulge in an abuse of notation. For example, by $\mathcal{P}\mu$, we really mean a section of $\mathcal{S}$, rather than merely the image $F\mu$ of the base facet F of type $\mathcal{P}$ in $\mathcal{S}$. In particular, $\mathcal{P} \simeq F/.$.

Now suppose that $T_{\mathcal{P}}$ and $T_{\mathcal{Q}}$ are right transversals for $\Gamma(\mathcal{K})$ in $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$, respectively, with $T_{\mathcal{P}} \cap T_{\mathcal{Q}} = \{1\}$ (the identity in Γ). The facets of $\mathcal{P}$ and of $\mathcal{Q}$ are in 1–1 correspondence with the elements of $T_{\mathcal{P}}$ and $T_{\mathcal{Q}}$, respectively. These transversals also enable more explicit calculation in Γ. However, the particular choice of transversals will ultimately not matter for our applications.

Let us say that a product such as $\omega = \lambda_1 \cdots \lambda_m$, with all $\lambda_j \neq 1$ and with λ_j and λ_{j+1} in different transversals $T_{\mathcal{P}}, T_{\mathcal{Q}}$, for $1 \leqslant j < m$, is an *alternating word* of length m. We allow the *empty* word $\omega = 1$ as the only alternating word of length 0. Referring to [2, Chapter 1, §7.3, Proposition 5], we recall that each $\mu \in \Gamma$ has a unique *reduced decomposition*

$$\mu = \kappa\lambda_1 \cdots \lambda_m = \kappa\omega, \tag{3.2}$$

where $\kappa \in \Gamma(\mathcal{K})$ and $\omega = \lambda_1 \cdots \lambda_m$ is an alternating word. Note that the length m of the reduced decomposition of an element $\mu \in \Gamma$ is independent of the particular choice of transversals $\mathcal{T}_{\mathcal{P}}$ and $\mathcal{T}_{\mathcal{Q}}$. (It will not matter to us that a product might be an alternating word with respect to one choice of transversals but not with respect to another.)

Notice as well that if $\tau \in T_{\mathcal{P}}$ and $\kappa \in \Gamma(\mathcal{K})$, then there exist unique $\tau' \in T_{\mathcal{P}}$, $\kappa' \in \Gamma(\mathcal{K})$ such that $\tau\kappa = \kappa'\tau'$; here, $\tau = 1$ if and only if $\tau' = 1$. A similar calculation works for $\Gamma(\mathcal{K})$ and $T_{\mathcal{Q}}$. It is then easy to see that if ω is an alternating word of length m and $\kappa \in \Gamma(\mathcal{K})$, then $\omega\kappa = \kappa'\omega'$, where $\kappa' \in \Gamma(\mathcal{K})$ and ω' is another alternating word of length m.

To serve Definition 3.2, it is helpful to single out certain $\mu \in \Gamma$. We say that $\mu = \tau_{l+1}\sigma_l\tau_l \cdots \sigma_1\tau_1$, when $k = 2l+1$ is odd (resp. $\mu = \sigma_l\tau_l \cdots \sigma_1\tau_1$, when $k = 2l$ is even) has *type* k. Here, all $\sigma_j \in T_{\mathcal{Q}} \setminus \{1\}$ and $\tau_j \in T_{\mathcal{P}}$, where we allow $\tau_j = 1$ only when $j = 1$. Also, $\mu = 1$ is the unique element of type 0. (Thus, μ is an alternating word, so long as we suppress τ_1 when it equals 1. This slight asymmetry of type is due to our starting the construction below with $\mathcal{P}$ rather than $\mathcal{Q}$.)

Now we can define a certain subposet $\mathcal{S}_k$ of rank n in $\mathcal{S}$. Our goal in Section 4 will be to understand $\mathcal{S}_k$ in a more intuitively geometrical way.

Definition 3.2 Let $\mathcal{S}_0 := \mathcal{P}$ (as before, we really mean the section under the base facet of type $\mathcal{P}$ in $\mathcal{S}$). For $k \geqslant 1$, $\mathcal{S}_k$ is the subposet of $\mathcal{S}$ induced on all faces of the union of $\mathcal{S}_{k-1}$ with all facets $\mathcal{Q}\mu$ (resp. $\mathcal{P}\mu$), where μ is a word of type k, with $k = 2l+1$ odd (resp. $k = 2l$ even).

Remark 3.3 Note that $\mathcal{S}_k$ is independent of our choice of transversals $T_{\mathcal{P}}, T_{\mathcal{Q}}$ (with $T_{\mathcal{P}} \cap T_{\mathcal{Q}} = \{1\}$). This is because the length of a reduced decomposition is independent of this choice. Recall that such transversals merely index the facets of $\mathcal{P}$ and $\mathcal{Q}$.

Now we investigate more carefully the structure of $\mathcal{S}_k$.

Lemma 3.4 *If $\mathcal{P}\mu = \mathcal{P}\gamma$ (resp. $\mathcal{Q}\mu = \mathcal{Q}\gamma$) for two elements μ, γ of even type (resp. odd type), then $\mu = \gamma$. Also, if $\mathcal{K}\mu = \mathcal{K}\gamma$ for two elements μ, γ of any types, then $\mu = \gamma$.*

Proof Suppose $\mathcal{P}\mu = \mathcal{P}\gamma$ for two elements

$$\mu = \sigma_l \tau_l \cdots \sigma_1 \tau_1, \quad \gamma = \sigma'_m \tau'_m \cdots \sigma'_1 \tau'_1$$

of even types. Since $\mu\gamma^{-1}$ stabilizes $\mathcal{P}$, we have $\mu\gamma^{-1} \in \Gamma(\mathcal{P})$, so that $\mu\gamma^{-1} = \kappa\tau$ for some unique $\kappa \in \Gamma(\mathcal{K}), \tau \in T_{\mathcal{P}}$. But then

$$\mu = 1 \cdot \sigma_l \tau_l \cdots \sigma_1 \tau_1 = \kappa\tau \cdot \sigma'_m \tau'_m \cdots \sigma'_1 \tau'_1.$$

By the uniqueness in (3.2), we have $\kappa\tau = 1$ (and $l = m$), so $\mu = \gamma$. The conclusion holds even if, as is possible, $\tau_1 = 1$ or $\tau'_1 = 1$. The calculations for $\mathcal{Q}$ and $\mathcal{K}$ are similar. ∎

Some properties of these posets are summarized in the next result.

Proposition 3.5 (i) *$\mathcal{S}_k$ is a flagged poset of rank n, always with a unique minimal element but not with a unique maximal element when $k \geqslant 1$. For $k \geqslant 1$, we have $\mathcal{S}_{k-1} \subset \mathcal{S}_k$.*

(ii) *Every face of rank n in $\mathcal{S}_k$ is isomorphic to $\mathcal{P}$ or to $\mathcal{Q}$.*

(iii) *Every face of rank $n-1$ in $\mathcal{S}_k$ is isomorphic to $\mathcal{K}$.*

(iv) *Each face of rank $n-1$ in $\mathcal{S}_k$ that also lies in $\mathcal{S}_{k-1}$ is* covered, *that is, lies in two faces of rank n in $\mathcal{S}_k$, namely $\mathcal{P}\mu$ and $\mathcal{Q}\mu$, and coincides with $\mathcal{K}\mu$, for some unique element μ of some type m, with $m \leqslant k$.*

(v) *Each $(n-1)$-face $\widetilde{\mathcal{K}}$ of $\mathcal{S}_k$ that does not lie in $\mathcal{S}_{k-1}$ is* exposed, *that is, lies in just one face of rank n in $\mathcal{S}_k$. If $k = 2l$ is even, this one face is $\mathcal{P}\mu$, for a unique μ of type k; and $\widetilde{\mathcal{K}} = \mathcal{K}\widetilde{\mu}$, for some unique element $\widetilde{\mu} = \tau_{l+1}\mu$ of type $k+1$. On the other hand, if $k = 2l+1$ is odd, the single face is $\mathcal{Q}\mu$, again for a unique element μ of type k; and $\widetilde{\mathcal{K}} = \mathcal{K}\widetilde{\mu}$, for some unique element $\widetilde{\mu} = \sigma_{l+1}\mu$ of type $k+1$.*

(vi) *$\mathcal{S}$ (with its unique maximal element deleted) equals $\cup_{k=0}^{\infty} \mathcal{S}_k$.*

Proof Parts (i) and (ii) are clear from Definition 3.2. Part (iii) follows, since all $(n-1)$-faces in the regular polytopes $\mathcal{P}, \mathcal{Q}$ are isomorphic to $\mathcal{K}$.

Parts (iv) and (v) are proved by induction on k. The base case $k = 0$ is settled easily by inspection (interpret $\mathcal{S}_{-1}$ as $\varnothing$). So assume (iv) and (v) hold for $\mathcal{S}_{k-1}$, with $k \geqslant 1$. Let $\widetilde{\mathcal{K}}$ be an $(n-1)$-face of $\mathcal{S}_k$.

Suppose first that $\widetilde{\mathcal{K}} \in \mathcal{S}_{k-1}$. Then by induction, either $\widetilde{\mathcal{K}} = \mathcal{K}\mu \in \mathcal{S}_{k-2}$ is covered, for some μ of type $m \leqslant k-1$; or $\widetilde{\mathcal{K}} \in \mathcal{S}_{k-1} \backslash \mathcal{S}_{k-2}$ is exposed in $\mathcal{S}_{k-1}$. In this case, there are two subcases.

(a) k is odd and $\widetilde{\mathcal{K}}$ lies on just $\mathcal{P}\mu$ in $\mathcal{S}_{k-1}$, where $k-1 = 2l$, $\mu = \sigma_l \tau_l \cdots \sigma_1 \tau_1$. Thus, $\widetilde{\mathcal{K}}\mu^{-1}$ is a facet of $\mathcal{P}$, say $\widetilde{\mathcal{K}}\mu^{-1} = \mathcal{K}\tau_{l+1}$ for $\tau_{l+1} \in T_{\mathcal{P}}$. But then $\widetilde{\mathcal{K}}$ also lies on $\mathcal{Q}\widetilde{\mu}$ (and $\mathcal{P}\widetilde{\mu} = \mathcal{P}\mu$) in $\mathcal{S}_k$, where $\widetilde{\mu} = \tau_{l+1}\mu$. Notice that $\widetilde{\mu}$ has type k. In fact, if $\tau_{l+1} = 1$, then $\widetilde{\mu} = \mu$, and so $\widetilde{\mathcal{K}}$ lies on both $\mathcal{P}\mu$ and $\mathcal{Q}\mu$; but then

$$\mathcal{Q}\mu = \mathcal{Q}\sigma_l \tau_l \cdots \sigma_1 \tau_1 = \mathcal{Q}\tau_l \cdots \sigma_1 \tau_1 \in \mathcal{S}_{k-2},$$

again by induction. This is a contradiction unless $k = 1$ (and $\tau_{l+1} = \tau_1 = 1$). Thus $\widetilde{\mathcal{K}}$ is covered, as required.

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(b) k is even and $\widetilde{\mathcal{K}}$ lies on just $\mathcal{Q}\mu$ in $\mathcal{S}_{k-1}$, where $k-1 = 2l+1$, $\mu = \tau_{l+1}\sigma_l\tau_l\cdots\sigma_1\tau_1$. Now $\widetilde{\mathcal{K}}$ also lies on $\mathcal{P}\widetilde{\mu}$ in $\mathcal{S}_k$, where $\widetilde{\mu} = \sigma_{l+1}\mu$ has type k. Again $\widetilde{\mathcal{K}}$ is covered.

Uniqueness of μ follows from Lemma 3.4.

To finish part (v) we finally suppose that $\widetilde{\mathcal{K}} \in \mathcal{S}_k\backslash\mathcal{S}_{k-1}$. The proof that $\widetilde{\mathcal{K}}$ is exposed in $\mathcal{S}_k$ is very similar to what we have seen before. For example, if $k = 2l$ is even, then by definition $\widetilde{\mathcal{K}}$ lies on some $\mathcal{P}\mu$, with $\mu = \sigma_l\tau_l\cdots\sigma_1\tau_1$. But $\mathcal{Q}\mu = \mathcal{Q}\tau_l\cdots\sigma_1\tau_1 \in \mathcal{S}_{k-1}$, so $\widetilde{\mathcal{K}}$ cannot lie on $\mathcal{Q}\mu$. Thus, $\widetilde{\mathcal{K}}$ lies only on $\mathcal{P}\mu$, which is indeed a face of $\mathcal{S}_k$. Furthermore, $\widetilde{\mathcal{K}}\mu^{-1}$ lies on $\mathcal{P}$, so $\widetilde{\mathcal{K}}\mu^{-1} = \mathcal{K}\tau_{l+1}$ for some $\tau_{l+1} \in T_{\mathcal{P}}$. Then $\widetilde{\mu} = \tau_{l+1}\mu$ has type $k+1$. Otherwise, we would have $\tau_{l+1} = 1$ and $\widetilde{\mu} = \mu$ of type $k = (k-1)+1$; by induction, this would put $\widetilde{\mathcal{K}}$ in $\mathcal{S}_{k-1}$.

Part (vi) follows from the fact that every $\mu \in \Gamma$ has a reduced decomposition as described in equation (3.2). ∎

Next we show that the "outer" n-faces meet $\mathcal{S}_k$ as we would expect. To prepare the way, we must refine our choice of transversals $T_{\mathcal{P}}, T_{\mathcal{Q}}$ (see Remark 3.3). We simply extract, with a bit of relabelling, what we need from [12, Lemmas 5.1–5.2].

Lemma 3.6 (i) *There are transversals $T_{\mathcal{P}}$ and $T_{\mathcal{Q}}$ such that for $-1 \leqslant j \leqslant n-2$, $T_{\mathcal{P}}$ contains a transversal $T_{\mathcal{P},j}$ for $\langle\alpha_{j+1},\dots,\alpha_{n-2}\rangle$ in $\langle\alpha_{j+1},\dots,\alpha_{n-2},\alpha_{n-1}\rangle$, and $T_{\mathcal{Q}}$ contains a transversal $T_{\mathcal{Q},j}$ for $\langle\alpha_{j+1},\dots,\alpha_{n-2}\rangle$ in $\langle\alpha_{j+1},\dots,\alpha_{n-2},\beta_{n-1}\rangle$. Moreover,*

$$\{1,\alpha_{n-1}\} = T_{\mathcal{P},n-2} \subseteq T_{\mathcal{P},n-3} \subseteq \cdots \subseteq T_{\mathcal{P},0} \subseteq T_{\mathcal{P},-1} = T_{\mathcal{P}}$$

and

$$\{1,\beta_{n-1}\} = T_{\mathcal{Q},n-2} \subseteq T_{\mathcal{Q},n-3} \subseteq \cdots \subseteq T_{\mathcal{Q},0} \subseteq T_{\mathcal{Q},-1} = T_{\mathcal{Q}}.$$

(ii) *For $-1 \leqslant j \leqslant n-2$, $\Gamma_j^+ = \langle\alpha_{j+1},\dots,\alpha_{n-2},\alpha_{n-1},\beta_{n-1}\rangle$ is isomorphic to the amalgamated product*

$$\langle\alpha_{j+1},\dots,\alpha_{n-2},\alpha_{n-1}\rangle *_{\langle\alpha_{j+1},\dots,\alpha_{n-2}\rangle} \langle\alpha_{j+1},\dots,\alpha_{n-2},\beta_{n-1}\rangle.$$

This tail-triangle C-group is the automorphism group for (the isomorphic) co-faces of co-rank $n-j$ in $\mathcal{S}$. In particular, $\langle\alpha_{n-1},\beta_{n-1}\rangle$ is the infinite dihedral group, and $\langle\alpha_{n-2},\alpha_{n-1},\beta_{n-1}\rangle$ is a Coxeter group (usually with triangular diagram).

Let us work from now on with the transversals described in Lemma 3.6.

Proposition 3.7 *Suppose for some $k \geqslant 0$ that $\widetilde{\mu} \neq \widetilde{\varphi}$ are two elements of type $k+1$ in Γ. Then for k even (resp. k odd)*

(i) *each face G of $\mathcal{S}_k$ that is incident with $\mathcal{Q}\widetilde{\mu}$ (resp. $\mathcal{P}\widetilde{\mu}$) must also be incident with $\mathcal{K}\widetilde{\mu}$.*

(ii) *each face G of $\mathcal{S}_{k+1}$ that is incident with both $\mathcal{Q}\widetilde{\mu}$ and $\mathcal{Q}\widetilde{\varphi}$ (resp. both $\mathcal{P}\widetilde{\mu}$ and $\mathcal{P}\widetilde{\varphi}$) must also be incident with both $\mathcal{K}\widetilde{\mu}$ and $\mathcal{K}\widetilde{\varphi}$. These $(n-1)$-faces, and hence also G, actually lie in $\mathcal{S}_k$.*

Proof Assume $k = 2l$ is even; the calculations for k odd are quite similar. Suppose then for part (i) that G is a j-face incident with both $\mathcal{S}_k$ and $\mathcal{Q}\widetilde{\mu}$. By Definition 3.2 and

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Lemma 3.4, $\mathcal{Q}\widetilde{\mu}$ is not itself an n-face of $\mathcal{S}_k$, so we can assume $j \leqslant n-1$. By Proposition 3.5, we can further assume that G is incident with an $(n-1)$-face $\mathcal{K}\lambda$, for some λ of type $m \leqslant k+1$.

Now recall that faces are right cosets, which here must intersect. For some $\gamma \in \Gamma$ we have $G = \Gamma_j \gamma$, for the subgroup $\Gamma_j = \Gamma_j^- \Gamma_j^+$ described in (2.5). This coset meets $\mathcal{K}\lambda = \Gamma_{n-1}\lambda$. Our goal is to exploit the special structure of the transversals to find an alternative expression for G as a coset of Γ_j from which we can read off the incidence with $\mathcal{K}\widetilde{\mu}$. Now since $\Gamma_j^- \leqslant \Gamma_{n-1}$, we have $\eta_1 \in \Gamma_j^+, \kappa_1 \in \Gamma_{n-1}$ such that

$$\eta_1 \gamma = \kappa_1 \lambda.$$

Likewise, since G is incident with $\mathcal{Q}\widetilde{\mu}$, we have $\eta_2 \in \Gamma_j^+, \kappa_2 \in \Gamma_{n-1}, \sigma \in T_{\mathcal{Q}}$ such that $\eta_2\gamma = \kappa_2 \sigma \widetilde{\mu}$. Now suppose $\widetilde{\mu} = \tau_{l+1}\sigma_l \tau_l \cdots \sigma_1 \tau_1$, and let $\lambda = \cdots \sigma_1' \tau_1'$. A little thought shows that $\widetilde{\mu}\lambda^{-1} = \kappa' \omega'$, where $\kappa' \in \Gamma_{n-1}$ and ω' is an alternating word of length at least $k+1-m$. Thus, $\widetilde{\mu}\lambda^{-1}\kappa_1^{-1} = \kappa'\omega'\kappa_1^{-1} = \kappa_3 \omega$, say, where $\kappa_3 \in \Gamma_{n-1}$ and ω is also an alternating word of length at least $k+1-m$. In fact, if ω is non-empty, then $\omega = \tau \cdots$, for some $\tau \in T_{\mathcal{P}} \backslash \{1\}$. Likewise, $\sigma\kappa_3 = \kappa_4 \widetilde{\sigma}$ for some $\kappa_4 \in \Gamma_{n-1}, \widetilde{\sigma} \in T_{\mathcal{Q}}$. Thus,

$$\eta_2 \eta_1^{-1} = \kappa_2 \sigma \widetilde{\mu} \lambda^{-1} \kappa_1^{-1} = \kappa_2 \sigma \kappa_3 \omega = (\kappa_2 \kappa_4) \widetilde{\sigma} \omega.$$

But $\eta_2\eta_1^{-1} \in \Gamma_j^+$. By the choice of transversals described in Lemma 3.6, it must be that both $\kappa_2\kappa_4$ and $\widetilde{\sigma}$ lie in Γ_j^+; compare [12, Lemma 5.3]. Thus,

$$G = \Gamma_j \gamma = \Gamma_j \eta_2^{-1} \kappa_2 \sigma \widetilde{\mu} = \Gamma_j \kappa_2 \sigma \widetilde{\mu} = \Gamma_j (\kappa_2 \kappa_4) \widetilde{\sigma} \kappa_3^{-1} \widetilde{\mu} = \Gamma_j \kappa_3^{-1} \widetilde{\mu}.$$

Hence, $\kappa_3^{-1}\widetilde{\mu} \in \Gamma_j \gamma \cap \Gamma_{n-1}\widetilde{\mu}$, so that G is incident with $\mathcal{K}\widetilde{\mu} = \Gamma_{n-1}\widetilde{\mu}$. This is part (i).

The calculations for (ii) are much the same. Given $G = \Gamma_j \gamma$, we obtain $\eta_1, \eta_2 \in \Gamma_j^+$, $\kappa_1, \kappa_2 \in \Gamma_{n-1}$, $\sigma_1, \sigma_2 \in T_{\mathcal{Q}}$ such that

$$\eta_1 \gamma = \kappa_1 \sigma_1 \widetilde{\mu}, \quad \eta_2 \gamma = \kappa_2 \sigma_2 \widetilde{\varphi}.$$

Now $\widetilde{\mu}\widetilde{\varphi}^{-1} = \kappa\omega$ for some $\kappa \in \Gamma_{n-1}$ and alternating word ω. But from the uniqueness of the reduced decomposition in Γ, it is easy to check that ω is non-empty, since $\widetilde{\mu} \neq \widetilde{\varphi}$ (and elements of type $k+1$ cannot differ by a nontrivial element from Γ_{n-1}), so that $\omega = \tau \ldots \tau'$ begins and ends with elements of $\mathcal{T}_{\mathcal{P}} \backslash \{1\}$. As before, we obtain certain $\kappa_j \in \Gamma_{n-1}, \sigma_j \in T_{\mathcal{Q}}$ such that $\sigma_2^{-1}\kappa_2^{-1} = \kappa_3\sigma_3$; then $\omega\kappa_3 = \kappa_4\omega_1$, where ω_1 is non-empty of the same structure as ω. Next, we get $\sigma_1(\kappa\kappa_4) = \kappa_5\sigma_4$, so that

$$\begin{aligned} \eta_1\eta_2^{-1} &= \kappa_1\sigma_1(\widetilde{\mu}\widetilde{\varphi}^{-1})\sigma_2^{-1}\kappa_2^{-1} = \kappa_1\sigma_1\kappa\omega\kappa_3\sigma_3 = \kappa_1\sigma_1(\kappa\kappa_4)\omega_1\sigma_3 \\ &= (\kappa_1\kappa_5)(\sigma_4\omega_1\sigma_3). \end{aligned}$$

Since ω_1 is non-empty and alternating, $\sigma_4\omega_1\sigma_3$ is also that way. Thus, we force $\kappa_1\kappa_5$, σ_4, ω_1, σ_3 to lie in Γ_j^+. From this we have

$$\begin{aligned} \Gamma_j\gamma &= \Gamma_j\eta_1^{-1}\kappa_1\sigma_1\widetilde{\mu} = \Gamma_j(\kappa_1\kappa_5)\kappa_5^{-1}\sigma_1\widetilde{\mu} = \Gamma_j\sigma_4\kappa_4^{-1}\kappa^{-1}\widetilde{\mu} = \Gamma_j\kappa_4^{-1}\omega\widetilde{\varphi} \\ &= \Gamma_j\omega_1\kappa_3^{-1}\widetilde{\varphi} = \Gamma_j\kappa_3^{-1}\widetilde{\varphi}. \end{aligned}$$

Thus, $\kappa_3^{-1}\widetilde{\varphi} \in \Gamma_j\gamma \cap \Gamma_{n-1}\widetilde{\varphi}$, so G is incident with $\mathcal{K}\widetilde{\varphi}$. By a symmetrical argument with $\eta_2\eta_1^{-1}$, we get that G is also incident with $\mathcal{K}\widetilde{\varphi}$. ∎

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At this point, it is convenient to define the *facet graph* $\mathcal{G}$ for $\mathcal{S} = \mathcal{U}_{\mathcal{P},\mathcal{Q}}$: its nodes are all facets of $\mathcal{S}$ (*i.e.*, copies of $\mathcal{P}$ or $\mathcal{Q}$) and two distinct facets are adjacent if and only if they share an $(n-1)$-face (*i.e.*, copy of $\mathcal{K}$).

Proposition 3.8 *The facet graph $\mathcal{G}$ for $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ is an infinite tree.*

Proof The $(n-1)$-face $\mathcal{K}\mu$ of $\mathcal{S}$ lies on just the two n-faces $\mathcal{P}\mu$ and $\mathcal{Q}\mu$. If $\mathcal{K}\lambda$ also lies on these two n-faces, then $\mu\lambda^{-1} \in \Gamma_n^{\mathcal{P}} \cap \Gamma_n^{\mathcal{Q}} = \Gamma(\mathcal{K})$. This forces $\mathcal{K}\mu = \mathcal{K}\lambda$ after all. It follows that $\mathcal{G}$ is a simple, bipartite graph. Its branches correspond exactly to the $(n-1)$-faces $\mathcal{K}\mu$ of $\mathcal{S}$, which in turn are in one-to-one correspondence with the special elements μ of all types $k \geqslant 0$ (Lemma 3.4).

Suppose $\mathcal{G}$ has a cycle. Since Γ acts transitively on each facet class, we can assume that the cycle contains the base n-face $\mathcal{P}$ as a node. If $k+1$ is the largest integer m such that $\mathcal{S}_m \backslash \mathcal{S}_{m-1}$ contains a node in the cycle, then clearly the n-face of $\mathcal{S}$ corresponding to this node lies in $\mathcal{S}_{k+1} \backslash \mathcal{S}_k$ and must be adjacent to two different n-faces in $\mathcal{S}_k$. This contradicts Proposition 3.7(i). ∎

We will make use of these results in the next section. But before that we need a careful discussion of certain quotients.

4 Quotients of Flagged Posets

It is intuitively clear what it means to identify faces in a polytope, or to attach one polytope to another along a common facet. However, if we are to prove things, we need to set down some precise terms and constructions. See [11, pp. 2655–2659] for some related ideas.

Let $\mathcal{P}$ be any flagged poset and let $\sim$ be any equivalence relation on the faces of $\mathcal{P}$. The *quotient set* $\mathcal{Q} := \mathcal{P}/\!\sim$ consists of all classes $\widehat{F} := \{G \in \mathcal{P} : G \sim F\}$. Let

$$\begin{aligned} \eta\colon \mathcal{P} &\longrightarrow \mathcal{Q} \\ F &\longmapsto \widehat{F} \end{aligned}$$

be the corresponding natural map. In order that $\mathcal{Q}$ be partially ordered, and more specifically a flagged poset, we need some sensible restrictions on $\sim$. First, we assume:

Q1. The equivalence relation $\sim$ on $\mathcal{P}$ is stratified by rank, that is, for all $F, G \in \mathcal{P}$,

$$F \sim G \Longrightarrow \mathrm{rk}(F) = \mathrm{rk}(G).$$

Then $\mathcal{Q}$ can be ordered by agreeing that $\widehat{F} \leqslant \widehat{G}$ if and only if there exists a finite sequence of faces $F_1, \dots, F_k, G_1, \dots, G_k$ in $\mathcal{P}$ such that

$$F = F_1 \sim G_1 \leqslant F_2 \sim G_2 \leqslant \cdots \leqslant F_{k-1} \sim G_{k-1} \leqslant F_k \sim G_k = G.$$

Certainly "$\leqslant$" is a reflexive and transitive relation on $\mathcal{Q}$. Moreover, if $\mathrm{rk}(F) = \mathrm{rk}(G)$ in such a chain of faces, then all faces must have the same rank and so lie in the same class. Hence, the relation is also antisymmetric, and $\mathcal{Q}$ is partially ordered. (We have defined the transitive closure of the more obvious "order" relation.) In fact, $\mathcal{Q}$ is a flagged poset of the same rank as $\mathcal{P}$ and the natural map η is a rank-preserving, surjective poset homomorphism. Of course, we say that $\mathcal{Q}$ is the *quotient* of $\mathcal{P}$ induced by $\sim$.

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This notion of quotient is still very general. See [11, §2.4] for some counter-intuitive examples.

Soon we will work with the disjoint union of several flagged posets. In order that the union be flagged, we next require the following condition:

Q2. In a disjoint union of flagged posets, all components must share a common rank.

Suppose now that our flagged posets are closed. Perhaps we want the quotient of some disjoint union to be closed. But then, by condition Q1, all maximal (resp. minimal) elements for different components have to be identified. Consider, for example, the two 1-polytopes (segments) ab and cd in Figure 1, in which we want to identify vertices b and c. Now we could, if we wanted, identify just the 0-faces b and c and nothing else. But, conventionally, and certainly here, we really do mean to fully identify along the isomorphic sections $b/\mathbf{.} \simeq c/\mathbf{.}$, so that minimal elements should also be identified under $\sim$.

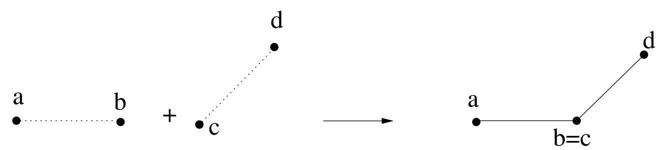

Figure 1: Two attached segments require higher rank.

The resulting closed, flagged poset in the above example is unsatisfactory if we also identify maximal elements, since then we could only interpret the structure on the right in Figure 1 as a single segment with three vertices. This was not our intent.

How then can we distinguish the two segments after attachment? One way is to view them as beginning a polygon, so that the structure really wants to have rank 2. We can achieve this by formally adjoining a new maximal element.

Definition 4.1 The *cap* $\overline{\mathcal{P}}$ of a flagged poset $\mathcal{P}$ is obtained from $\mathcal{P}$ by adjoining a new, unique maximal element.

It is clear then that $\text{rk}(\overline{\mathcal{P}}) = \text{rk}(\mathcal{P}) + 1$. For our two segments, the Hasse diagrams evolve as in Figure 2.

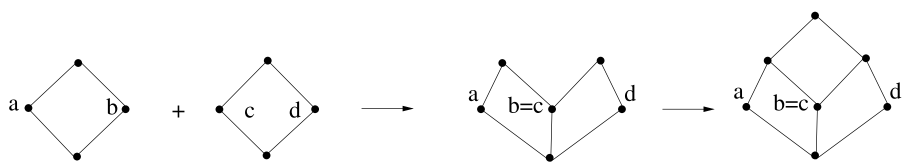

Figure 2: Two disjoint segments joined then capped off.

Clearly, we do want the right-most structure in Figure 2. But now if we must attach a third segment, we run up against requirement Q2. We can readily describe all sorts of contrary situations like this, so that it is awkward to establish general rules. For our purposes, we will make do with a final requirement, which is more a rule of thumb.

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Q3: When making identifications within the disjoint union of flagged posets (of like rank), be sure to identify all separate minimal elements. When finished, cap off the resulting flagged poset, unless it already has a unique maximal element.

Remark 4.2 By judiciously obeying requirements Q1, Q2, Q3, we will clearly produce closed, flagged posets. It is peculiar that a capped segment $\overline{ab}$, considered as a closed, flagged poset of rank 2, is dual as a poset to a single vertex with two *semi-edges*:

We could (somewhat eccentrically) interpret $\overline{ab}$ as having two *semi-vertices*.

For the rest of this section we will be concerned with flagged posets, each having a unique minimal element. (Such a poset is closed if it also has a unique maximal element.) We attempt to identify these objects along isomorphic facets.

Suppose that $\mathcal{P}$ is such a poset of rank n. Say $\{F(j) : j \in J\}$ is a *non-empty* indexed family of distinct facets of $\mathcal{P}$. The indexing set J could be finite or countably infinite (if $\mathcal{P}$ itself is infinite). Notice that all $F(j)$ share the same unique minimum as $\mathcal{P}$. Suppose also that for each $j \in J$ there is a flagged poset $\mathcal{Q}_j$ (of rank n, with unique minimum), with a specified facet $G(j)$ isomorphic to $F(j)$. This more precisely means that there is an isomorphism

$$\varphi_j\colon F(j)/\bullet \longrightarrow G(j)/\bullet.$$

Our goal is to glue each $\mathcal{Q}_j$ to $\mathcal{P}$ along the specified pair of isomorphic facets. We start with the disjoint union

$$\mathcal{D} := \mathcal{P} \sqcup \bigsqcup_{j \in J} \mathcal{Q}_j,$$

which has an obvious structure as a flagged poset of rank n. (There will usually be many distinct maximal or minimal elements.) Notice that we can identify $\mathcal{P}$ and each $\mathcal{Q}_j$, as a poset, with their images in $\mathcal{D}$.

Next define on $\mathcal{D}$ the equivalence relation $\sim$ generated by all pairs (G, H), where $G \leqslant F(j)$ and $H = (G)\varphi_j \in \mathcal{Q}_j$, for some $j \in J$. Notice that minimal elements of $\mathcal{P}$, Q_j are equivalent under $\sim$. And observe that the only other pairs of *distinct* $\sim$-equivalent faces in $\mathcal{D}$ must look like $((G)\varphi_i, (G)\varphi_j)$, for $i \neq j \in J$, when G is incident to both $F(i)$ and $F(j)$ in $\mathcal{P}$.

Clearly, $\sim$ is stratified by rank, so that $\mathcal{G} := \mathcal{D}/{\sim}$ is a flagged poset of rank n, now with a unique minimum. Let

$$\eta\colon \mathcal{D} \longrightarrow \mathcal{G}$$

be the natural map, a rank-preserving epimorphism of posets. It follows from the observations just above that $\mathcal{P}$ and each $\mathcal{Q}_j$ can be identified with their η-images in $\mathcal{G}$.

We now have the following proposition.

Proposition 4.3 *For $\mathcal{P}$, $\mathcal{Q}_j$, φ_j ($j \in J$), as above,*

(i) *the quotient $\mathcal{G}$ is a flagged poset of rank n and with a unique minimal element;*
(ii) *the cap $\overline{\mathcal{G}}$ is a closed, flagged poset of rank $n+1$;*
(iii) *the posets $\mathcal{P}$ and all $\mathcal{Q}_j$ are isomorphic to sections of rank n in $\overline{\mathcal{G}}$.*

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Proof The overriding assumption is that $J \neq \varnothing$ forces $\mathcal{G}$ to have at least two maximal elements (one from $\mathcal{P}$ and from each $\mathcal{Q}_j$). Thus, the closure is required and $\overline{\mathcal{G}}$ has rank $n+1$. ∎

Remark 4.4 Suppose $\mathcal{K}$ is a facet of $\mathcal{P}$ or some $\mathcal{Q}_j$ used in the construction. Thus, $\mathcal{K}$ appears as a facet in $\mathcal{G}$ and so also as a ridge in $\overline{\mathcal{G}}$. Note that ingredients such as $\mathcal{P}$ need not be polytopes. The construction is far more general than our use of it below would suggest.

A more subtle possibility is that the structure of $\mathcal{G}$ can also depend on the chosen isomorphisms φ_j. However, this will not concern us in our main applications, since we will restrict ourselves to more controlled situations involving regular polytopes, as outlined next.

Remark 4.5 Suppose that the flagged posets $\mathcal{Q}_j$ $(j \in J)$ are regular polytopes, and that the family of facets $\{F(j) : j \in J\}$ contains all facets of $\mathcal{P}$. (We are not assuming that $\mathcal{P}$ is a regular polytope.) Then the structure of the quotient $\mathcal{G}$ of Proposition 4.3 does not depend on the particular choice of the facets $G(j)$ of $\mathcal{Q}_j$ $(j \in J)$ nor on the isomorphisms $\varphi_j: F(j)/\mathbf{.} \to G(j)/\mathbf{.}$ $(j \in J)$. In other words, a different choice of $G(j), \varphi_j$ $(j \in J)$ leads to an isomorphic poset. The proof of this rests on the properties of the automorphism groups of regular polytopes. For example, if $G(j)'$ is another facet of $\mathcal{Q}_j$ and $\varphi_j': F(j)/\mathbf{.} \to G(j)'/\mathbf{.}$ is a corresponding isomorphism, then $\kappa := \varphi_j^{-1}\varphi_j': G(j)/\mathbf{.} \to G(j)'/\mathbf{.}$ is an isomorphism between facets of $\mathcal{Q}_j$. But for a regular polytope, any isomorphism between facets can be extended uniquely to the entire polytope. It follows that κ is the restriction to $G(j)/\mathbf{.}$ of an automorphism, again κ (say), of $\mathcal{Q}_j$. Thus, $\varphi_j' = \varphi_j\kappa$ with $\kappa \in \Gamma(\mathcal{Q}_j)$. In other words, changing $G(j), \varphi_j$ to $G(j)', \varphi_j'$ amounts to replacing $\mathcal{Q}_j$ by the isomorphic copy $\mathcal{Q}_j\kappa$. In essence, this says that one can push any change in the $G(j), \varphi_j$ $(j \in J)$ into the components $\mathcal{Q}_j$. As the components are mutually disjoint, and disjoint from $\mathcal{P}$, we can safely manufacture an overall isomorphism of the new resulting poset with the original poset $\mathcal{G}$.

Now we resume our main construction and return to the situation described in Section 3, where we took $\mathcal{P}$ and $\mathcal{Q}$ to be regular n-polytopes, each with facets isomorphic to the regular $(n-1)$-polytope $\mathcal{K}$. Notice that Remark 4.5 applies in this case. Our chief goal with all the above machinery is to rigorously describe the process of freely attaching copies of $\mathcal{Q}$ to all facets of one $\mathcal{P}$, then copies of $\mathcal{P}$ to all remaining facets of all the $\mathcal{Q}$'s, and so forth, alternating between $\mathcal{P}$'s and $\mathcal{Q}$'s without end. As in Proposition 3.5, we say that an $(n-1)$-face in a poset of rank n is exposed (resp. covered) if it lies on one (resp. two) n-faces.

Definition 4.6 Let $\mathcal{G}_0 := \mathcal{P}$. For $k \geqslant 1$, we obtain $\mathcal{G}_k$ as follows: if k is odd (resp. even), then to each exposed facet (isomorphic to $\mathcal{K}$) in $\mathcal{G}_{k-1}$ we attach a distinct copy of $\mathcal{Q}$ (resp. $\mathcal{P}$). We enforce the identifications as summarized in Proposition 4.3(i) to produce $\mathcal{G}_k$.

Of course, we must convince ourselves that this definition passes closer inspection and does what we want.

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Proposition 4.7 *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are regular n-polytopes, each with facets isomorphic to the regular $(n-1)$-polytope $\mathcal{K}$. Let $\mathcal{G}_k$ be the poset defined in* Definition 4.6. *Then for each $k \geqslant 0$,*

(i) *$\mathcal{G}_k$ is isomorphic to the flagged poset $\mathcal{S}_k$ described in* Proposition 3.5*;*
(ii) *there is a natural embedding $\varepsilon_k \colon \mathcal{G}_k \to \mathcal{G}_{k+1}$, for $k \geqslant 0$.*

Proof We prove (i) by induction on k. By definition, $\mathcal{G}_0 \simeq \mathcal{P} \simeq \mathcal{S}_0$, so assume that $\mathcal{G}_k \simeq \mathcal{S}_k$, for some $k \geqslant 0$. Let us simply put $\mathcal{G}_k$ aside and work with $\mathcal{S}_k$ in its place. (We comment a little on this "transfer of structure" in Remark 4.11.) Recall from Proposition 3.5 that $\mathcal{S}_k$ is a subposet of $\mathcal{S} = \mathcal{U}_{\mathcal{P},\mathcal{Q}}$. It is flagged with rank n, with n-faces isomorphic to $\mathcal{P}$ or $\mathcal{Q}$ and with certain faces $\widetilde{\mathcal{K}}$ of rank $n-1$ exposed.

Suppose $k = 2l$ is even (the case k odd is very similar). We want to construct $\mathcal{G}_{k+1}$ from $\mathcal{S}_k$ (well, really from $\mathcal{G}_k$). According to Definition 4.6, we must attach disjoint copies of $\mathcal{Q}$ to $\mathcal{S}_k$ along the exposed $(n-1)$-faces $\widetilde{\mathcal{K}}$ of $\mathcal{S}_k$. But by Proposition 3.5(v), these $\widetilde{\mathcal{K}} = \mathcal{K}\widetilde{\mu}$ are indexed by the set J of all elements $\widetilde{\mu} = \tau_{l+1}\mu$ of type $k+1$ in Γ. (Recall that $\mathcal{K}$ is the base $(n-1)$-face in $\mathcal{S}$.)

Now in applying Proposition 4.3, $\mathcal{S}_k$ plays the role of $\mathcal{P}$, and for each $\widetilde{\mu} \in J$, we require a copy of $\mathcal{Q}$. For this copy of $\mathcal{Q}$, let us simply extract from $\mathcal{S}$ the subposet $\mathcal{Q}\widetilde{\mu}$, the "missing" n-face lying on $\mathcal{K}\widetilde{\mu}$. (Recall that $\mathcal{Q}\widetilde{\mu}$ really means a section of rank n in $\mathcal{S}$; this is even a section of $\mathcal{S}_{k+1}$.) The required identification map $\varphi_{\widetilde{\mu}}$ is just the identity map on $\mathcal{K}\widetilde{\mu}$.

The next step is to form the disjoint union

$$\mathcal{D} := \mathcal{S}_k \sqcup \bigsqcup_{\widetilde{\mu} \in J} \mathcal{Q}\widetilde{\mu}\,,$$

each component of which is a subposet of $\mathcal{S}_{k+1}$. Combining several inclusions, we get, in an obvious way, the poset surjection

$$\lambda \colon \mathcal{D} \longrightarrow \mathcal{S}_{k+1}.$$

Now we obtain $\mathcal{G}_{k+1}$ from $\mathcal{D}$ by making identifications of the form $G \sim (G)\varphi_{\widetilde{\mu}}$, whenever G is a face of $\mathcal{K}\widetilde{\mu}$. However, from the point of view of $\mathcal{S}_{k+1}$ this merely requires $(G)\varphi_{\widetilde{\mu}} = G$, which holds by our choice of $\varphi_{\widetilde{\mu}}$. Thus, λ induces a well-defined surjective poset map

$$\lambda_{k+1} \colon \mathcal{G}_{k+1} \longrightarrow \mathcal{S}_{k+1}.$$

Some careful thought shows that Proposition 3.7 is just what we need to show that λ_{k+1} is injective. It is then easy to check that λ_{k+1}^{-1} is order preserving.

For part (ii), the mapping ε_k comes from the remarks preceding Proposition 4.3, whereby the $\mathcal{P}$ there (here $\mathcal{G}_k$) can be identified with its image in the quotient $\mathcal{G}$ (here $\mathcal{G}_{k+1}$). More simply, we have $\varepsilon_k = \lambda_k \iota_k \lambda_{k+1}^{-1}$, where $\iota_k \colon \mathcal{S}_k \to \mathcal{S}_{k+1}$ is inclusion. ■

Our next task is to assemble the $\mathcal{G}_k$ into one package. To do so, we take a *direct limit* [1, pp. E.R.29–30]. Let ε_{kk} be the identity map on $\mathcal{G}_k$; and for $k < l$, let

$$\varepsilon_{kl} = \varepsilon_k \varepsilon_{k+1} \dots \varepsilon_{l-1} \colon \mathcal{G}_k \to \mathcal{G}_l,$$

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with the ε_j as in Proposition 4.7(ii), and composing left to right. Notice that each ε_{kl} is injective. We obtain a directed system (of flagged posets of rank n). Let

$$\mathcal{G} := \varinjlim \mathcal{G}_k.$$

To describe this more explicitly, we first take the disjoint union of all the $\mathcal{G}_k$. In this, we factor by the equivalence relation "≡" under which $F \equiv G$, for $F \in \mathcal{G}_i$ and $G \in \mathcal{G}_j$, whenever there is some $k \geqslant \max\{i, j\}$ such that $(F)\varepsilon_{ik} = (G)\varepsilon_{jk}$. Now in our case, the directed system is indexed by non-negative integers. Thus, more simply, we can say that $F \equiv G$ if $i = j$ and $F = G$, or if $i < j$, say, with $(F)\varepsilon_{i,j} = G$. As part of this construction, we get natural injective maps (still of flagged posets of rank n)

$$\eta_k : \mathcal{G}_k \to \mathcal{G},$$

such that $\varepsilon_{kl}\eta_l = \eta_k$ whenever $k \leqslant l$. The flagged poset $\mathcal{G}$ now has the *universal property* one would expect for a direct limit (of flagged posets). We summarize this in the following proposition.

Proposition 4.8 *If $\mathcal{H}$ is a flagged poset and $\lambda_k : \mathcal{G}_k \to \mathcal{H}$ are morphisms such that $\varepsilon_{kl}\lambda_l = \lambda_k$, for $l \geqslant k$, then there exists a unique morphism $\lambda : \mathcal{G} \to \mathcal{H}$ such that $\eta_k\lambda = \lambda_k$ for all k.*

Remark 4.9 By "morphism" here, we naturally mean "rank preserving homomorphism of posets". It is easy to check that λ is injective if all λ_k are injective. Furthermore, λ is onto if

$$\mathcal{H} = \bigcup_{k \geqslant 0} (\mathcal{G}_k)\lambda_k.$$

Now for formal purposes, let $\mathcal{S}^*$ be $\mathcal{S}$ with its unique maximal element removed. Thus, $\mathcal{S}^*$ has rank n and $\overline{\mathcal{S}^*} \simeq \mathcal{S}$. From Proposition 4.7 we have isomorphisms

$$\lambda_k : \mathcal{G}_k \longrightarrow \mathcal{S}_k \subset \mathcal{S}^*, \quad \text{for } k \geqslant 0.$$

These satisfy $\varepsilon_{kl}\lambda_l = \lambda_k$, for $l \geqslant k$. From Proposition 4.8 we then get a morphism $\lambda : \mathcal{G} \to \mathcal{S}^*$. By Remark 4.9, λ is an isomorphism. Finally, let us reintroduce maximal elements (by capping off). Taking note of Proposition 3.5, we obtain the key result of this section.

Theorem 4.10 *The poset $\mathcal{G}$ described in* Definition 4.6 *can be capped off to give $\overline{\mathcal{G}} \simeq \mathcal{U}_{\mathcal{P},\mathcal{Q}}$.*

Remark 4.11 We have described the universal, alternating, semiregular polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ as the end result of the free assembly described in Definition 4.6. To even more closely mimic that construction, we could, in the proof of Proposition 4.7, refuse to "casually identify" $\mathcal{G}_k$ with $\mathcal{S}_k$. But this would require introducing several more or less obvious layers of isomorphisms; we trust that we have been precise enough in the discussion above.

To convince oneself that this machinery has some power, consider the case where $\mathcal{P} = \{4\}$ and $\mathcal{Q} = \{3\}$ (see Figure 3). Say we have unlimited supplies of (flexible) squares and triangles. Take one square, attach a triangle along each of its edges, then to

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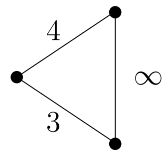
each leftover edge attach a square, then continue without end in this alternating fashion. Theorem 4.10 asserts that the resulting structure is $\mathcal{U}_{\{4\},\{3\}}$. Surely it is not obvious (without having first followed our discussion) that the automorphism group for this intuitively described object is the Coxeter group with diagram

(4.1)



Another non-obvious consequence is that we get the same object starting with the triangle ($\mathcal{Q}$) rather than with the square ($\mathcal{P}$).

5 Coverings by $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$

If the universal polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ is to merit its name, it should have some sort of universal mapping property. To understand this we must first choose a sensible class of morphisms.

Definition 5.1 ([9, Sect. 2D]) Let $\mathcal{A}$ and $\mathcal{B}$ be (pre-) polytopes, both of rank $n+1$. A *rap-map* is a rank and adjacency preserving homomorphism $\eta: \mathcal{A} \to \mathcal{B}$. A surjective rap-map is called a *covering*; we then say $\mathcal{A}$ is a *cover* of $\mathcal{B}$ and write $\mathcal{A} \searrow \mathcal{B}$.

Remark 5.2 The definition requires that η induce a mapping $\mathcal{F}(\mathcal{A}) \to \mathcal{F}(\mathcal{B})$ (of flag sets) that sends any j-adjacent pair of flags in $\mathcal{A}$ to another such pair in $\mathcal{B}$. It is a useful exercise to check that the rap-map $\eta: \mathcal{A} \to \mathcal{B}$ must be a covering if the pre-polytope $\mathcal{B}$ is actually a polytope (or even just flag-connected). In this case, $\mathcal{B}$ is isomorphic in a natural way to a quotient of $\mathcal{A}$ [11, Lemmas 2.5 and 2.10]. Let us say that $\mathcal{B}$ is *induced by the rap-map* η from $\mathcal{A}$. (See [11, Example 2.13] for a quotient of a regular polyhedron that is not induced by a rap-map and which is therefore not suitable for our use below.)

The choice of $n+1$ for the rank here is convenient for the examples that follow.

In order to understand covers of most polytopes, we need some new tools. Suppose $\mathcal{A}$ is a (pre-) polytope of rank $n+1$. For $0 \leqslant j \leqslant n$, let

$$r_j: \mathcal{F}(\mathcal{A}) \longrightarrow \mathcal{F}(\mathcal{A})$$
$$\Lambda \longmapsto \Lambda^j.$$

Thus, r_j maps a flag Λ of $\mathcal{A}$ to its j-adjacent flag Λ^j in $\mathcal{A}$. Note that each r_j is a fixed point free involution on the flag set $\mathcal{F}(\mathcal{A})$.

Definition 5.3 The *connection group* of $\mathcal{A}$ is

$$\mathrm{Mon}(\mathcal{A}) = \langle r_0, \ldots, r_n \rangle.$$

Remark 5.4 The term "connection group" was introduced in [17]. Indeed, one can interpret relations in the group as describing how an abstract set of flags might be connected together to constitute a polytope. We use the notation $\mathrm{Mon}(\mathcal{A})$ as a

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reminder that this group is often called the "monodromy group" in the literature. Etymologically, "monodromy" does make sense; however, in complex analysis, "monodromy group" has come to mean something a bit at odds with our intent here.

Clearly, $\mathrm{Mon}(\mathcal{A})$ is a subgroup of the symmetric group on the flag set. It is always a *string group generated by involutions* (sggi), meaning that relations parallel to those displayed in (2.1) must hold on the specified generators. However, the intersection condition (2.2) can fail, even when $\mathcal{A}$ is a polytope, if the rank $n+1 \geqslant 4$.

Since any automorphism γ of $\mathcal{A}$ preserves adjacency of flags, we have

$$(\Lambda\gamma)^g = (\Lambda^g)\gamma, \tag{5.1}$$

for any flag Λ of $\mathcal{A}$ and $g \in \mathrm{Mon}(\mathcal{A})$.

Many covering properties can be rephrased in terms of connection groups. For this, it is, in turn, crucial to understand the flag stabilizer in $\mathrm{Mon}(\mathcal{A})$. (Recall that when $\mathcal{A}$ is a polytope, $\mathrm{Mon}(\mathcal{A})$ acts transitively on the flag set.) For more details, we refer the reader to [11], in particular to Propositions 3.11 and 3.13, the latter of which we quote and rephrase as the following lemma.

Lemma 5.5 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $(n+1)$-polytopes and that*

$$\overline{\eta}\colon \mathrm{Mon}(\mathcal{A}) \to \mathrm{Mon}(\mathcal{B})$$

is an epimorphism of sggis. Suppose also that there are flags Λ' of $\mathcal{A}$ and Λ of $\mathcal{B}$ such that $\overline{\eta}$ maps the stabilizer of Λ' in $\mathrm{Mon}(\mathcal{A})$ into the stabilizer of Λ in $\mathrm{Mon}(\mathcal{B})$. Then there is a covering $\eta : \mathcal{A} \to \mathcal{B}$ that maps Λ' to Λ.

Let us now return our attention to the universal polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$. We carry on with our earlier, abbreviated notation, so that the polytope $\mathcal{S} = \mathcal{U}_{\mathcal{P},\mathcal{Q}}$ has group $\Gamma = \langle \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ (of index at most 2 in the full automorphism group). The $(n+1)$-polytope $\mathcal{S}$ has the two base flags described in (2.6). We also need the connection group

$$M := \mathrm{Mon}(\mathcal{S}) = \langle r_0, \dots, r_{n-1}, r_n \rangle.$$

Let K be the stabilizer in M of some flag, say the base flag Φ of type $\mathcal{P}$. Since $\Psi = \Phi^n = \Phi^{r_n}$, the stabilizer of base flag Ψ of type $\mathcal{Q}$ is $K^{r_n} := r_n K r_n$.

We illustrate this set-up in Figure 3, which shows a fragment of $\mathcal{U}_{\{4\},\{3\}}$. Its group Γ is the Coxeter group whose diagram appears in (4.1).

Observe how g in Figure 3 really does stabilize Φ. Pondering this a little, we see that g corresponds to a closed walk in the flag graph, which is somehow patched together from cycles in copies of $\mathcal{P} = \{4\}$ or $\mathcal{Q} = \{3\}$. We thereby find that

$$\begin{aligned}
g &= r_1 r_0 r_1 r_2 r_1 r_2 (r_1 r_0)^3 r_1 r_2 r_1 r_0 r_1 r_2 (r_1 r_0)^2 \\
&= r_1 r_0 r_1 r_2 r_1 r_2 w_1 r_0 r_2 r_1 r_0 r_1 r_2 (r_1 r_0)^2 && (w_1 = (r_1 r_0)^4) \\
&= w_1^{t_1} r_1 r_0 r_1 r_2 r_1 r_0 r_1 r_0 r_1 r_2 (r_1 r_0)^2 && (t_1 = r_2 r_1 r_2 r_1 r_0 r_1) \\
&= w_1^{t_1} r_1 r_0 r_1 r_2 (r_1 r_0)^3 r_2 r_0 (r_1 r_0)^2 \\
&= w_1^{t_1} w_2^{t_2} r_1 r_0 r_1 r_0 (r_1 r_0)^2 && (w_2 = (r_1 r_0)^3, t_2 = r_2 r_1 r_0 r_1) \\
&= w_1^{t_1} w_2^{t_2} w_3^{t_3} && (w_3 = (r_1 r_0)^4, t_3 = 1).
\end{aligned}$$

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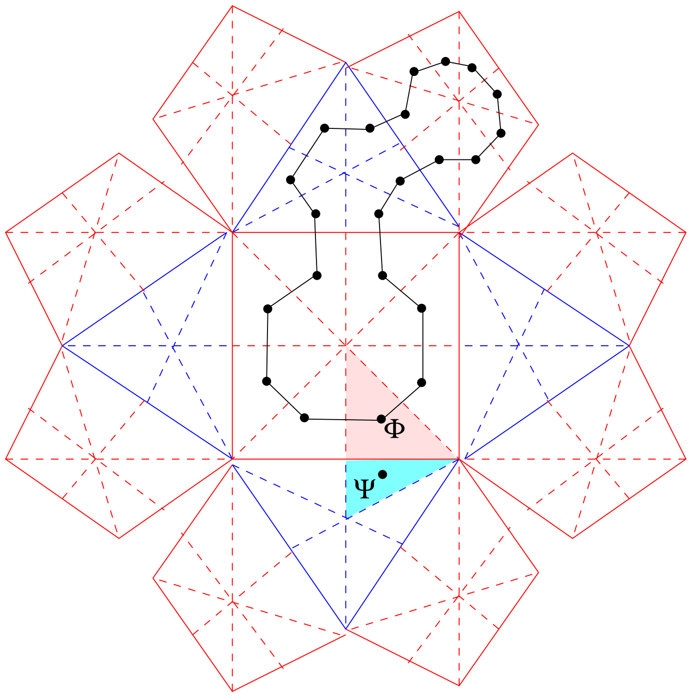

Figure 3: The action of $g = r_1r_0(r_1r_2)^2(r_1r_0)^3r_1r_2r_1r_0r_1r_2(r_1r_0)^2$ on the base flag Φ in $\mathcal{U}_{\{4\},\{3\}}$.

This calculation illustrates the essential idea in the proof of the following theorem.

Theorem 5.6 *The stabilizer K of the base flag Φ for $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ under the action of its connection group M is generated by all conjugates of the form w^t, where $w \in \langle r_0, \ldots, r_{n-1}\rangle$, $t = r_{i_1} \cdots r_{i_m}$ and either*

(i) *w stabilizes Φ and r_n appears an* even *number of times in the word for t; or*

(ii) *w stabilizes Ψ and r_n appears an* odd *number of times in the word for t.*

Proof Notice that each $g \in M_n := \langle r_0, \ldots, r_{n-1}\rangle$ preserves the flag set for each facet. Furthermore, by (5.1), such a g acts in essentially the same way on the flag set for each facet in one of the two symmetry orbits. It is easy then to check that the w^t described in (i) or (ii) really do stabilize Φ.

Now consider a general element $g = r_{j_1} \cdots r_{j_m}$ in the stabilizer K. As suggested by Figure 3, this g defines a closed walk in the flag graph proceeding through the flags

$$\Phi, \Phi^{j_1}, \Phi^{j_1j_2}, \ldots, \Phi^{j_1j_2\cdots j_{m-1}}, \Phi^{j_1j_2\cdots j_{m-1}j_m} = \Phi^g = \Phi.$$

Since r_n always swaps type $\mathcal{P}$ flags with type $\mathcal{Q}$ flags, the word g must contain r_n an even number $2l$ times. If $l = 0$, then $g \in M_n$ is of type (i). So consider two consecutive appearances of r_n in g, say

$$g = g_1 r_n g_2 r_n g_3,$$

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where $g_1, g_3 \in M$ and $g_2 \in M_n$. The flags Φ^{g_1} and $\Phi^{g_1 r_n g_2 r_n}$ might contain different facets, although these facets must be of the same type. By discounting the action of M_n, we see that g also defines a walk in the facet graph of $\mathcal{S}$, which we know to be a tree by Theorem 3.8. Our walk proceeds along a subtree, so we can assume that the facet X on the flag $\Phi^{g_1 r_n}$ is furthest from the base facet $\mathcal{P}$ in this walk. This means, crucially, that flags Φ^{g_1} and $\Phi^{g_1 r_n g_2 r_n}$ contain the *same* facet Y, which is adjacent to X. Since adjacent facets in $\mathcal{S}$ share a unique $(n-1)$-face (of the form $\mathcal{K}\mu$), we see that Φ^{g_1} and $\Phi^{g_1 r_n g_2 r_n}$ share the same faces of ranks n and $n-1$. Since $\mathcal{K}$ is regular, this means that there exists $h \in \langle r_0, \dots, r_{n-2} \rangle$ such that $\Phi^{(g_1 r_n g_2 r_n)h} = \Phi^{g_1}$; see [11, Corollary 3.10]. Since h commutes with r_n, we have $\Lambda^w = \Lambda$, for $w = g_2 h \in M_n$ and the flag $\Lambda = \Phi^{g_1 r_n}$. Also, for $t = r_n g_1^{-1}$, we have

$$g = w^t \cdot (g_1 h^{-1} g_3),$$

where $g_1 h^{-1} g_3$ lies in K and has fewer occurrences of r_n. A little thought shows that w^t must have the form required in (i) or (ii), so the proof ends by induction. ∎

Here is our main application of Theorem 5.6.

Theorem 5.7 *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are regular n-polytopes, each with facets isomorphic to the regular $(n-1)$-polytope $\mathcal{K}$. Let $\mathcal{B}$ be any $(n+1)$-polytope, each of whose facets is isomorphic to either $\mathcal{P}$ or to $\mathcal{Q}$, and where these facets occur in alternating fashion around any face of rank $n-2$ in $\mathcal{B}$. (The number of such $\mathcal{P}$'s and $\mathcal{Q}$'s can here vary from one $(n-2)$-face to another, as in Example 2.6.) Then there exists a covering $\eta\colon \mathcal{U}_{\mathcal{P},\mathcal{Q}} \to \mathcal{B}$, so that $\mathcal{B}$ is a quotient of the universal alternating semiregular polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$.*

Proof Let $\mathrm{Mon}(\mathcal{B}) = \langle r'_0, \dots, r'_n \rangle$. In order to apply Lemma 5.5, we must first show that the mapping $r_j \mapsto r'_j$, $0 \leqslant j \leqslant n$, induces a well-defined epimorphism $\overline{\eta}\colon \mathrm{Mon}(\mathcal{S}) \to \mathrm{Mon}(\mathcal{B})$.

Suppose $r_{j_1} \cdots r_{j_m} = 1$ is any relation in $M = \mathrm{Mon}(\mathcal{S})$. Certainly for $g = r_{j_1} \cdots r_{j_m}$, we have $\Phi^g = \Phi$ and $\Psi^g = \Psi$. (Recall that Φ, Ψ are the base flags of type $\mathcal{P}$, $\mathcal{Q}$, respectively.)

Let $g' = r'_{j_1} \cdots r'_{j_m}$ be the corresponding word in $\mathrm{Mon}(\mathcal{B})$. (We are not assuming that $g \to g'$ is a well-defined mapping; formally, we could work instead in a free group of rank $n+1$.)

But the word g is a product of conjugates of the form w^t, with w, t as described in (i) and (ii) of Theorem 5.6. Now it is clear from the structure of $\mathcal{B}$ that $(w')^{t'}$ fixes any flag containing a copy of $\mathcal{P}$ in $\mathcal{B}$. Thus, g' fixes each such flag. Likewise (by an analogue of Theorem 5.6 for the stabilizer of Ψ), since $\Psi^g = \Psi$, g' also fixes each flag containing a copy of $\mathcal{Q}$. But these are all flags in $\mathcal{B}$, so $g' = 1$ in $\mathrm{Mon}(\mathcal{B})$. We therefore have an epimorphism $\overline{\eta}$ as required.

Finally, choose a fixed (base) flag Λ for $\mathcal{B}$, say, containing a copy of $\mathcal{P}$. We have already observed that $\overline{\eta}$ maps K, the stabilizer of Φ in $\mathrm{Mon}(\mathcal{S})$, into the stabilizer of Λ in $\mathrm{Mon}(\mathcal{B})$. By Lemma 5.5, we have the desired covering $\eta\colon \mathcal{S} \to \mathcal{B}$. ∎

Remark 5.8 The covering map η is uniquely specified if Φ is required to map onto Λ.

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Corollary 5.9 *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are regular n-polytopes, each with facets isomorphic to the regular $(n-1)$-polytope $\mathcal{K}$. Suppose as well that $\mathcal{B}$ is an $(n+1)$-polytope whose facets are, in alternating fashion, any selection of quotients induced by rap-maps from $\mathcal{P}$ or $\mathcal{Q}$, respectively. Then there exists a covering $\eta: \mathcal{U}_{\mathcal{P},\mathcal{Q}} \to \mathcal{B}$.*

Proof The argument used in Theorem 5.7 applies with little change. This is because any flag relation in, say, $\mathcal{P}$ must hold in any quotient via the associated rap-map. ■

At this point, it is useful to introduce a new construction for groups $\Delta = \langle \delta_0, \dots, \delta_m \rangle$ and $\Delta' = \langle \delta'_0, \dots, \delta'_m \rangle$, each with specified generators. (These lists of equal length could include redundant generators.) The *mix* of Δ and Δ' is the subgroup $\Delta \diamond \Delta'$ of the direct product $\Delta \times \Delta'$ that is generated by all (δ_j, δ'_j), $0 \leqslant j \leqslant m$ (see [11, Section 5]).

Returning to our customary set-up, let us mix the string C-groups $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$. It can be tricky to understand the mix of two string C-groups like these. If, however, $\mathcal{P}$ and $\mathcal{Q}$ have isomorphic facets $\mathcal{K}$ (as we assume here), then we conclude from [11, Theorem 5.12] that $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$ is itself a string C-group, and so is the automorphism group of some regular n-polytope $\mathcal{R} := \mathcal{P} \diamond \mathcal{Q}$, again with facets $\mathcal{K}$. Here, then

$$\Gamma(\mathcal{R}) = \Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q}) = \langle \rho_0, \dots, \rho_{n-2}, \rho_{n-1} \rangle$$

is the subgroup of $\Gamma \times \Gamma$ generated by $\rho_j := (\alpha_j, \alpha_j)$, for $0 \leqslant j \leqslant n-2$, together with $\rho_{n-1} := (\alpha_{n-1}, \beta_{n-1})$.

Before making use of this mix, it will be useful to take a bit of a detour. Let us recall that a polytope is said to be *flat* if each of its vertices is incident with each of its facets [9, Section 4E]. A familiar flat regular polytope is the dihedron $\{p, 2\}$. We can imagine that its p vertices and edges form a spherical p-gon along a great circle of the sphere $\mathbb{S}^2$. The two hemispheres are then the 2-faces of the polyhedron.

Example 5.10 **Deflating flat polytopes.** Donald Coxeter was fond of describing the quadrangular dihedron $\{4, 2\}$ as an ordinary cushion. This is apt, since we now want to imagine deflating and collapsing the cushion onto the quadrangle $\{4\}$.

In fact, from any regular $(n-1)$-polytope $\mathcal{K}$, we may construct a "trivial" extension $\mathcal{T} = \{\mathcal{K}, 2\}$, itself a regular n-polytope with automorphism group

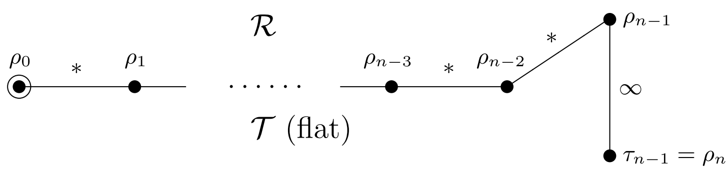
$$\Gamma(\mathcal{T}) = \langle \tau_0, \dots, \tau_{n-2}, \tau_{n-1} \rangle \simeq \Gamma(\mathcal{K}) \times C_2.$$

Here, $\Gamma(\mathcal{K}) \simeq \langle \tau_0, \dots, \tau_{n-2} \rangle$, and the C_2 factor is generated by τ_{n-1}. It is easy to see that $\mathcal{T}$ is flat with each of its two facets a copy of $\mathcal{K}$.

Continuing with this, suppose $\mathcal{R}$ is another regular n-polytope with facets isomorphic to $\mathcal{K}$. (For the moment, $\mathcal{R}$ need not be the mix of $\mathcal{P}$ and $\mathcal{Q}$.) Let $\Gamma(\mathcal{R}) = \langle \rho_0, \dots, \rho_{n-2}, \rho_{n-1} \rangle$ be the automorphism group of $\mathcal{R}$. We can construct the universal, alternating semiregular polytope $\mathcal{U}_{\mathcal{R},\mathcal{T}}$. From Section 3, we see that its automorphism group is the amalgamated product

$$\Gamma(\mathcal{R}) *_{\Gamma(\mathcal{K})} \Gamma(\mathcal{T}) = \langle \rho_0, \dots, \rho_{n-2}, \rho_{n-1}, \tau_{n-1} \rangle,$$

in which τ_j and ρ_j have been identified for $0 \leqslant j \leqslant n-2$. It is also convenient to simply let $\rho_n := \tau_{n-1}$. The corresponding diagram is

(5.2)


But this diagram straightens into a string (and still satisfies (2.2)), so that the group $\langle \rho_0, \dots, \rho_{n-2}, \rho_{n-1}, \rho_n \rangle$ must, from another point of view, be a string C-group. We use $\{\mathcal{R}, \infty\}$ to denote the corresponding regular $(n+1)$-polytope; it is called the *free extension* of $\mathcal{R}$ [9, Theorem 4D4].

Now we come to a subtle point. The alternating semiregular polytope $\mathcal{U}_{\mathcal{R},\mathcal{T}}$ and the regular polytope $\{\mathcal{R}, \infty\}$ have the same group with the same specified generators! However, as coset geometries these $(n+1)$-polytopes are slightly different. Comparing the descriptions in [12, Definition 4.1] (earlier summarized in Section 2) and in [9, Section 2E], we first check that the two polytopes have identical j-faces for $j \leqslant n-2$. Now the n-faces in $\{\mathcal{R}, \infty\}$ are all right cosets of $\langle \rho_0, \dots, \rho_{n-2}, \rho_{n-1} \rangle$ (giving copies of $\mathcal{R}$). In $\mathcal{U}_{\mathcal{R},\mathcal{T}}$ we do have these, as well as all right cosets of $\langle \rho_0, \dots, \rho_{n-2}, \rho_n \rangle$ (giving copies of $\mathcal{T}$). Finally, let us consider $(n-1)$-faces. In $\{\mathcal{R}, \infty\}$, these are all right cosets of

$$\langle \rho_0, \dots, \rho_{n-2}, \rho_n \rangle = \langle \rho_0, \dots, \rho_{n-2} \rangle \cup \langle \rho_0, \dots, \rho_{n-2} \rangle \rho_n .$$

(Notice how we use the fact that $\rho_n = \tau_{n-1}$ and ρ_j commute for $j \leqslant n-2$.) On the other hand, the $(n-1)$-faces in $\mathcal{U}_{\mathcal{R},\mathcal{T}}$ are just all right cosets of $\langle \rho_0, \dots, \rho_{n-2} \rangle$. Each such $(n-1)$-face is a copy of $\mathcal{K}$, which further lies on one copy of $\mathcal{R}$ and on one copy of $\mathcal{T}$. Therefore, we effectively pass from the semiregular polytope to the regular polytope by first collapsing $(n-1)$-faces in pairs then discarding the flat facet (a copy of $\mathcal{T}$) enclosed by each such pair. We might imagine this process as deflating the copies of $\mathcal{T}$ in $\mathcal{U}_{\mathcal{R},\mathcal{T}}$ so as to get $\{\mathcal{R}, \infty\}$.

Remark 5.11 Polytopes of the kind described in Example 5.10 are particular instances of the "bubble polytopes" studied in [5]. In the terminology of that thesis, our $(n+1)$-polytope $\mathcal{U}_{\mathcal{R},\mathcal{T}}$ is the $(n-1)$-bubble associated with the $(n+1)$-polytope $\{\mathcal{R}, \infty\}$. ■

Now let us reconsider the universal alternating semiregular polytope $\mathcal{S} = \mathcal{U}_{\mathcal{P},\mathcal{Q}}$. Backtracking a little, we recall the set-up in equation (3.1). In our construction of the group Γ, we amalgamated $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ along $\Gamma(\mathcal{K})$ by identifying α_j with β_j for $0 \leqslant j \leqslant n-2$. We also constructed the regular n-polytope $\mathcal{R} := \mathcal{P} \diamond \mathcal{Q}$.

Proposition 5.12 *Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are regular n-polytopes, each with facets isomorphic to the regular $(n-1)$-polytope $\mathcal{K}$. Let $\mathcal{R} = \mathcal{P} \diamond \mathcal{Q}$. Then the mapping $\rho_j \mapsto r_j$, $0 \leqslant j \leqslant n$, induces an epimorphism*

$$\lambda \colon \Gamma(\{\mathcal{R}, \infty\}) \longrightarrow \mathrm{Mon}(\mathcal{U}_{\mathcal{P},\mathcal{Q}}),$$

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from the full automorphism group of the free extension of $\mathcal{R}$ to the connection group of $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$. Consequently, we have the cover

$$\{\mathcal{P} \diamond \mathcal{Q}, \infty\} \searrow \mathcal{U}_{\mathcal{P},\mathcal{Q}}.$$

Proof Recall that the mix $\mathcal{R}$ really is a regular n-polytope. Let $\gamma = \rho_{i_1} \cdots \rho_{i_m} = 1$ be any relation holding for the generators of $\Gamma(\{\mathcal{R}, \infty\})$. We must show $g = r_{i_1} \cdots r_{i_m} = 1$ in $M := \mathrm{Mon}(\mathcal{U}_{\mathcal{P},\mathcal{Q}})$.

As defining relations for the amalgamated product underlying the diagram in (5.2), we can take relations $\gamma = \rho_{i_1} \cdots \rho_{i_m} = 1$ of two types. First, we have $(\rho_j \tau_{n-1})^2 = (\rho_j \rho_n)^2 = 1$, for $0 \leqslant j \leqslant n-2$, but notice then that $(r_j r_n)^2 = 1$ in the sggi M.

Second, we have relations that hold in the subgroup $\Gamma(\mathcal{R})$. But

$$\begin{aligned}\langle r_0, \ldots, r_{n-2}, r_{n-1}\rangle &\simeq \mathrm{Mon}(\mathcal{P}) \diamond \mathrm{Mon}(\mathcal{Q}) && \text{([11, Theorem 5.5])}\\ &\simeq \Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q}) && \text{([11, Theorem 3.9])}\\ &\simeq \Gamma(\mathcal{R}).\end{aligned}$$

(The middle isomorphism uses the fact that $\mathcal{P}, \mathcal{Q}$ are regular.) Thus any relation that holds in $\Gamma(\mathcal{R})$ implies the corresponding relation in M, indeed in its subgroup M_n. This means that the epimorphism λ does exist.

The flag stabilizer in the connection group of the regular polytope $\{\mathcal{R}, \infty\}$ is trivial, and, as with any regular polytope, we have $\Gamma(\{\mathcal{R}, \infty\}) \simeq \mathrm{Mon}(\{\mathcal{R}, \infty\})$. By Lemma 5.5, the polytope $\{\mathcal{R}, \infty\}$ covers $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$. ∎

One might reasonably conjecture that the epimorphism λ in Proposition 5.12 is injective. In fact, this is not so. There are already critical counter-examples when $n = 2$.

Example 5.13 Let $\mathcal{P} = \{p\}$, $\mathcal{Q} = \{q\}$. Then $\mathcal{R} = \{p\} \diamond \{q\} = \{r\}$, where $r = \mathrm{lcm}(p, q)$. The group $\Gamma(\{\mathcal{R}, \infty\}) = [r, \infty] = \langle \rho_0, \rho_1, \rho_2\rangle$ of the free extension is a string Coxeter group. The mapping $\rho_j \mapsto r_j$ induces the epimorphism $\lambda\colon \Gamma(\{\mathcal{R}, \infty\}) \to \mathrm{Mon}(\mathcal{U}_{\mathcal{P},\mathcal{Q}})$ of concern to us. Now suppose $p \leqslant q$ and let

$$\omega = \rho_2(\rho_0\rho_1)^p \rho_2(\rho_0\rho_1)^p \rho_2(\rho_1\rho_0)^p \rho_2(\rho_1\rho_0)^p.$$

A look at a simple figure containing a few facets of $\mathcal{U}_{\{p\},\{q\}}$ will convince one that ω acts trivially on flags of both symmetry types in $\mathcal{U}_{\{p\},\{q\}}$. Thus, ω is in the kernel of the action. (The same will certainly be true if we replace the exponent p by q.)

In the regular case, with $p = q = r$, we have $\omega = 1$, as we would expect. But if $p < q$ (so $p < r$), then we just have to convince ourselves that $\omega \neq 1$ in the Coxeter group $[r, \infty]$. This will become clear from a picture of the corresponding hyperbolic tessellation $\{r, \infty\}$. The skeptic can compute concretely with the standard real representation of $[r, \infty]$.

This sort of behaviour is actually commonplace. Following [4] or [11, Section 4], we say that the mapping λ in Proposition 5.12 induces the *flag action* of the string C-group $\Gamma(\{\mathcal{R}, \infty\})$ on the flag set of $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$. Using (5.1), it is easy to see that if, under this action, $\mu, \nu \in \Gamma(\{\mathcal{R}, \infty\})$ stabilize the two base flags Φ and Ψ, respectively, then the commutator $\mu\nu\mu^{-1}\nu^{-1}$ acts trivially on all flags and thus lies in $\ker \lambda$. Typically,

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this kernel will not be trivial, and so the structure of $\mathrm{Mon}(\mathcal{U}_{\mathcal{P},\mathcal{Q}})$ is more elusive than we might have hoped.

We hope to pursue the questions that arise here elsewhere. In fact, one can broaden the investigation to include all kinds of 2-orbit polytopes besides $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$, including those without any sort of universal property.

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