*Bull. Aust. Math. Soc.* (First published online 2025), page 1 of 11*
doi:10.1017/S0004972724001217
*Provisional—final page numbers to be inserted when paper edition is published

# ON THE BURES METRIC, $C^*$-NORM AND QUANTUM METRIC

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(Received 2 October 2024; accepted 5 November 2024)

**Abstract**

Given a unital $C^*$-algebra and a faithful trace, we prove that the topology on the associated density space induced by the $C^*$-norm is finer than the Bures metric topology. We also provide an example when this containment is strict. Next, we provide a metric on the density space induced by a quantum metric in the sense of Rieffel and prove that the induced topology is the same as the topology induced by the Bures metric and $C^*$-norm when the $C^*$-algebra is assumed to be finite dimensional. Finally, we provide an example where the Bures metric and induced quantum metric are not metric equivalent. Thus, we provide a bridge between these aspects of quantum information theory and noncommutative metric geometry.

2020 *Mathematics subject classification*: primary 46L52; secondary 46L85, 81P15.

*Keywords and phrases*: quantum metric spaces, Bures metric.

## 1. Introduction and background

The Bures metric, which was introduced by Bures in [2], is a vital tool in quantum information theory (for example, the Bures metric is the quantum version of the Hellinger distance; see [6, Section 3.1.2]). Recently, the Bures metric has been adapted to von Neumann algebras and $C^*$-algebras by Farenick and Rahaman [4]. The quantum metric, which was introduced by Rieffel in [11], allows one to prove powerful results about convergence of quantum spaces including those arising from high energy physics (see [10], where Rieffel proved that matrix algebras converge to the sphere and continuity of quantum tori) and many more continuity results related to the $C^*$-algebraic structure and noncommutative geometric structure (see [1, 5, 7–9]). This paper brings these two important metrics together in a natural way to make comparisons between their topological and geometric properties in the hope of introducing these methods of measurement to the other field.

In Section 2, we provide comparisons of the topological structure of the Bures metric and a third metric, the one induced by the $C^*$-norm on the density space. The purpose of bringing this third metric into the picture is not just to provide another interesting comparison, but also to provide a method of comparing the Bures metric

This work is partially supported by the first author's NSF grant DMS-2316892.

and the quantum metric. In Section 3, we show how one can place a metric on the density space using quantum metrics, and then show that this induced quantum metric is topologically equivalent to the Bures metric. Our approach uses the results in Section 2. Finally, in Section 4, we provide a case when the Bures metric and induced quantum metric are not metric equivalent, which is of interest since contractivity of quantum channels is a main aspect of quantum information theory [4]. The quantum metric approach offers a truly new avenue to study contractivity of quantum channels and fixed points while still agreeing with the Bures metric topology. For the remainder of this section, we provide some necessary background for the rest of the paper.

CONVENTION 1.1. Given a unital $C^*$-algebra $\mathcal{A}$, we denote its unit by $1_{\mathcal{A}}$, its norm by $\|\cdot\|_{\mathcal{A}}$, its self-adjoint elements by $\mathrm{sa}(\mathcal{A})$ and positive elements by $\mathcal{A}_+$. We denote its state space by $S(\mathcal{A})$. We also denote the metric induced by $\|\cdot\|_{\mathcal{A}}$ by $d_{\mathcal{A}}$.

However, given a compact Hausdorff space $X$, we denote the $C^*$-norm on $C(X)$ by $\|\cdot\|_X$ and the unit (the constant 1 function) by $\mathbb{1}$.

DEFINITION 1.2 [11]. Let $\mathcal{A}$ be a unital $C^*$-algebra. If $L : \mathcal{A} \to [0, \infty]$ is a seminorm on $\mathcal{A}$ such that:

(1) $\mathrm{dom}(L) = \{a \in \mathcal{A} : L(a) < \infty\}$ is dense in $\mathcal{A}$;
(2) $L(a) = 0$ if and only if $a \in \mathbb{C}1_{\mathcal{A}}$;
(3) $L(a) = L(a^*)$ for every $a \in \mathcal{A}$; and
(4) the Monge–Kantorovich metric, defined for any two states $\varphi, \psi \in S(\mathcal{A})$ by

$$\mathrm{mk}_L(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathcal{A}, L(a) \leqslant 1\},$$

metrises the weak* topology on $S(\mathcal{A})$,

then $(\mathcal{A}, L)$ is a *compact quantum metric space*, and we call $L$ an $L$-seminorm.

THEOREM-DEFINITION 1.3 [4, Theorem 2.6]. *Let $\mathcal{A}$ be a unital $C^*$-algebra. Let $\tau$ be a faithful trace. Define the density space with respect to $\tau$ to be*

$$D_\tau(\mathcal{A}) = \{a \in \mathcal{A}_+ : \tau(a) = 1\}.$$

*Define the Bures metric with respect to $\tau$ for every $x, y \in D_\tau(\mathcal{A})$ by*

$$d_B^\tau(x, y) = \sqrt{1 - \tau(|\sqrt{x}\sqrt{y}|)}.$$

Here, by a faithful trace, we mean a positive linear functional (not necessarily of norm one) such that, for every $a, b \in \mathcal{A}$, we have $\tau(ab) = \tau(ba)$ and $\tau(a^*a) = 0$ implies that $a = 0$. We call the statement above a 'Theorem-Definition' since the proof that the Bures metric is indeed a metric in this general setting of unital $C^*$-algebras equipped with a faithful trace is a nontrivial result of [4].

We also formally state what we mean by topological equivalence and metric equivalence so that there is no confusion.

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DEFINITION 1.4. Let $X$ be a nonempty set and let $d$ and $d'$ be two metrics on $X$.

(1) We say that $d$ and $d'$ are *topologically equivalent* if they induce the same topologies.
(2) We say that $d$ and $d'$ are *metric equivalent* if there exist $\alpha, \beta > 0$ such that

$$\alpha d(x,y) \leqslant d'(x,y) \leqslant \beta d(x,y) \quad \text{for every } x, y \in X,$$

or, equivalently,

$$\alpha \leqslant \frac{d'(x,y)}{d(x,y)} \leqslant \beta \quad \text{for every } x, y \in X \text{ such that } x \neq y.$$

## 2. Comparison of the $C^*$-metric and Bures metric topologies

In this section, we show that the $C^*$-metric topology is finer than the topology induced by the Bures metric. We also show that this containment can be strict by providing an explicit example in the $C^*$-algebra of complex-valued continuous functions on $[0, 1]$ with the trace given by Lebesgue integration. First, we prove two lemmas that are likely to be well known, but we provide their proofs for convenience.

LEMMA 2.1. *Let $\mathcal{A}$ be a unital $C^*$-algebra. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_+$ that converges in the $C^*$-norm to $x \in \mathcal{A}_+$. Let $r \geqslant 0$ be such that, for any $n \in \mathbb{N}$, $\|x_n\|_{\mathcal{A}}, \|x\|_{\mathcal{A}} \leqslant r$. If $f : [0, r] \to \mathbb{R}$ is continuous, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$ in the $C^*$-norm.*

PROOF. By the Weierstrass approximation theorem, there exist polynomials $(p_n)_{n\in\mathbb{N}}$ that converge uniformly to $f$ on $[0, r]$.

Let $\varepsilon > 0$. By uniform convergence of $(p_n)_{n\in\mathbb{N}}$ to $f$, there exists $N \in \mathbb{N}$ such that, for any $n \geqslant N$, we have $\|p_n - f\|_{[0,r]} < \varepsilon/3$, and, in particular, $\|p_N - f\|_{[0,r]} < \varepsilon/3$. Since $p_N$ is a polynomial, there exists $N' \geqslant N$ such that, for any $n \geqslant N'$,

$$\|p_N(x_n) - P_N(x)\|_{\mathcal{A}} < \frac{\varepsilon}{3}.$$

Let $n \geqslant N'$. By functional calculus,

$$\|f(x_n) - p_N(x_n)\|_{\mathcal{A}} = \|f - p_N\|_{\sigma(x_n)} \leqslant \|f - p_N\|_{[0,r]} < \frac{\varepsilon}{3}$$

since $\sigma(x_n) \subseteq [0, r]$. Similarly,

$$\|f(x) - p_N(x)\|_{\mathcal{A}} < \frac{\varepsilon}{3}$$

since $\sigma(x) \subseteq [0, r]$. By the triangle inequality for the norm,

$$\begin{aligned} \|f(x_n) - f(x)\|_{\mathcal{A}} &\leqslant \|f(x_n) - p_N(x_n)\|_{\mathcal{A}} + \|p_N(x_n) - p(x)\|_{\mathcal{A}} + \|p_N(x) - f(x)\|_{\mathcal{A}} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus, for any $\varepsilon > 0$, we can find $N'$ such that $\|f(x_n) - f(x)\|_{\mathcal{A}} \leqslant \varepsilon$ for any $n \geqslant N'$. Thus, $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$ in the $C^*$-norm. □

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LEMMA 2.2. *Let $\mathcal{A}$ be a unital $C^*$-algebra. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$ that converges in the $C^*$-norm to some $x \in \mathcal{A}$, then $(|x_n|)_{n\in\mathbb{N}}$ converges in the $C^*$-norm to $|x|$.*

PROOF. Note that $|x| = \sqrt{x^*x}$ and $|x_n| = \sqrt{x_n^*x_n}$ for every $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, let $a_n = x_n^*x_n$ and $a = x^*x$. Since multiplication and the adjoint are continuous in the $C^*$-norm, $(a_n)_{n\in\mathbb{N}}$ converges to $a$ in the $C^*$-norm. For any $n \in \mathbb{N}$, $|x| = \sqrt{a}$ and $|x_n| = \sqrt{a_n}$. Therefore, by Lemma 2.1 and continuity of the square root, we know that $(|x_n|)_{n\in\mathbb{N}}$ converges to $|x|$ in the $C^*$-norm. □

We now prove our main theorem that allows us to compare the $C^*$-metric topology and the Bures metric topology.

THEOREM 2.3. *Let $\mathcal{A}$ be a unital $C^*$-algebra and let $\tau$ be a faithful trace. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $D_\tau(\mathcal{A})$ and let $x \in D_\tau(\mathcal{A})$. If $(x_n)_{n\in\mathbb{N}}$ converges to $x$ in the $C^*$-norm, then $(x_n)_{n\in\mathbb{N}}$ converges to $x$ with respect to the Bures metric $d_B^\tau$.*

PROOF. If $(x_n)_{n\in\mathbb{N}}$ converges to $x$ in the $C^*$-norm, then, since $g(x) = \sqrt{x}$ is continuous, $(\sqrt{x_n})_{n\in\mathbb{N}}$ converges to $\sqrt{x}$ in the $C^*$-norm by Lemma 2.1.

Since $x$ is positive and fixed, $(\sqrt{x_n} \cdot \sqrt{x})_{n\in\mathbb{N}}$ converges to $\sqrt{x} \cdot \sqrt{x}$ in the $C^*$-norm. By definition, $\sqrt{x} \cdot \sqrt{x} = x$, so $(\sqrt{x_n} \cdot \sqrt{x})_{n\in\mathbb{N}}$ converges in the $C^*$-norm to $x$. By Lemma 2.2, this implies that $(|\sqrt{x_n} \cdot \sqrt{x}|)_{n\in\mathbb{N}}$ converges in the $C^*$-norm to $|x| = x$.

Since $\tau$ is $C^*$-norm continuous, $(|\sqrt{x_n} \cdot \sqrt{x}|)_{n\in\mathbb{N}}$ converging to $x$ in the $C^*$-norm implies that $(\tau(|\sqrt{x_n} \cdot \sqrt{x}|))_{n\in\mathbb{N}}$ converges to $\tau(x)$. Since $x \in D_\tau(A)$, we have $\tau(x) = 1$. Hence, $(\tau(|\sqrt{x_n} \cdot \sqrt{x}|))_{n\in\mathbb{N}}$ converges to 1.

Consider

$$(d_B^\tau(x_n, x))_{n\in\mathbb{N}} = \Big(\sqrt{1 - \tau(|\sqrt{x_n} \cdot \sqrt{x}|)}\Big)_{n\in\mathbb{N}}.$$

Since $(\tau(|\sqrt{x_n} \cdot \sqrt{x}|))_{n\in\mathbb{N}}$ converges to 1, it follows that $(d_B^\tau(x_n, x))_{n\in\mathbb{N}}$ converges to $\sqrt{1-1} = 0$. Hence, $(x_n)_{n\in\mathbb{N}}$ converges to $x$ with respect to the Bures metric $d_B^\tau$. □

COROLLARY 2.4. *Let $\mathcal{A}$ be a unital C*-algebra and let $\tau$ be a faithful trace. Then the topology induced by $d_{\mathcal{A}}$ is finer than the topology induced by $d_B^\tau$.*

This raises the question of whether these topologies are the same in general. The answer is no, and the rest of this section is devoted to providing an example on which these topologies disagree.

Consider, $C([0, 1])$, the $C^*$-algebra of continuous complex-valued functions on $[0, 1]$. The map

$$\rho : f \in C([0, 1]) \longmapsto \int_0^1 f(x)\, dx,$$

where $\int_0^1 \cdot\, dx$ is the standard Lebesgue integral, is a faithful trace on $C([0, 1])$.

PROPOSITION 2.5. *The topology induced by the metric $d_{C([0,1])}$ induced by the $C^*$-norm on the density space $D_\rho(C([0, 1]))$ is strictly finer than the topology induced by the Bures metric $d_B^\rho$.*

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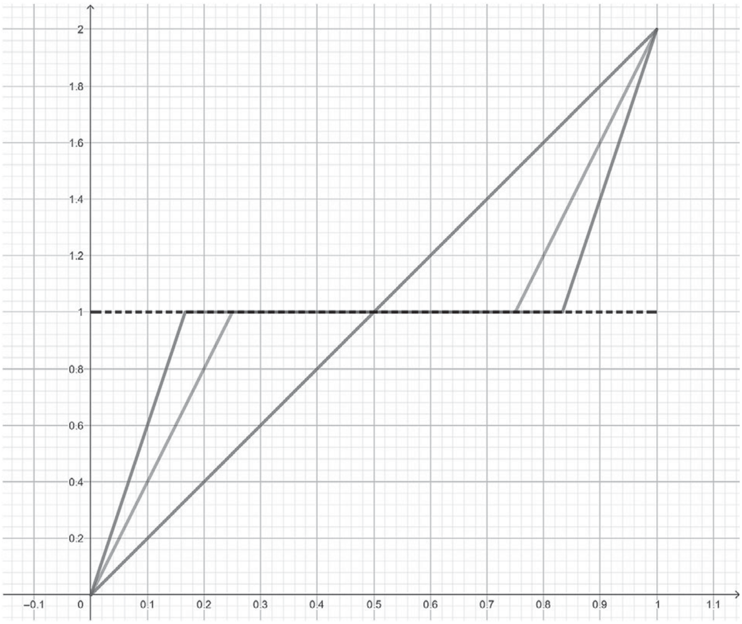


FIGURE 1. $f_1, f_2, f_3$ given by the formula (2.1). (Plotted in GeoGebra).

PROOF. The fact that the topology induced by $d_{C([0,1])}$ is finer is provided by Corollary 2.4. Thus, it remains to show that the topologies are not equal. To accomplish this, we find a sequence in $D_\rho(C([0,1]))$ that converges with respect to $d_B^\rho$, but does not converge uniformly (that is, with respect to $d_{C([0,1])}$).

Let $n \in \mathbb{N}$. Consider

$$f_n(x) = \begin{cases} 2nx & \text{if } x \in [0, 1/2n], \\ 1 & \text{if } x \in (1/2n, 1 - 1/2n), \\ 2nx - 2n + 2 & \text{if } x \in [1 - 1/2n, 1], \end{cases} \tag{2.1}$$

defined for all $x \in [0,1]$ (see Figure 1). Note that $f_n \in C([0,1])$. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = 1$ for all $x \in [0,1]$. Note that $f \in D_\rho(C([0,1]))$.

Next, we check that $f_n \in D_\rho(C([0,1])$ for every $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. First, $f_n \geqslant 0$ by construction. Second,

$$\begin{aligned} \rho(f_n) = \int_0^1 f_n(x)\,dx &= \int_0^{1/2n} 2nx\,dx + \int_{1/2n}^{1-1/2n} 1\,dx + \int_{1-1/2n}^{1} (2nx - 2n + 2)\,dx \\ &= [nx^2]_0^{1/2n} + [x]_{1/2n}^{1-1/2n} + [nx^2 - 2nx + 2x]_{1-1/2n}^{1} \\ &= \frac{1}{4n} + 1 - \frac{1}{n} + \frac{3}{4n} = 1. \end{aligned}$$

Hence, $f_n \in D_\rho(C([0,1]))$.

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We now prove that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges to the function $f$ in the Bures metric, $d_B^\rho$, but fails to converge to $f$ uniformly.

Let $n \in \mathbb{N}$. Then

$$\begin{aligned}\rho(\sqrt{f_n}\sqrt{f}) &= \int_0^{1/2n} (\sqrt{2nx}\cdot 1)\,dx + \int_{1/2n}^{1-1/2n} (\sqrt{1}\cdot 1)\,dx \\ &\quad + \int_{1-1/2n}^{1} (\sqrt{2nx-2n+2}\cdot 1)\,dx \\ &= \int_0^{1/2n} \sqrt{2nx}\,dx + \int_{1/2n}^{1-1/2n} 1\,dx + \int_{1-1/2n}^{1} \sqrt{2nx-2n+2}\,dx \\ &= \sqrt{2n}\cdot\Big[\frac{2}{3}x^{3/2}\Big]_0^{1/2n} + [x]_{1/2n}^{1-1/2n} + \sqrt{2}\cdot\Big[\frac{2}{3n}\cdot(nx-n+1)^{3/2}\Big]_{1-1/2n}^{1} \\ &= \frac{1}{3n} + 1 - \frac{1}{n} + \frac{2\sqrt{2}}{3n} - \frac{1}{3n} = 1 - \frac{3-2\sqrt{2}}{3n}.\end{aligned}$$

Therefore,

$$d_B^\rho(f_n, f) = \sqrt{1-\rho(\sqrt{f_n}\sqrt{f})} = \sqrt{1-\Big(1-\frac{3-2\sqrt{2}}{3n}\Big)} = \sqrt{\frac{3-2\sqrt{2}}{3n}}$$

and $\lim_{n\to\infty} d_B^\rho(f_n, f) = 0$. Therefore, we have shown that $(f_n)_{n\in\mathbb{N}}$ converges to $f$ in the $\rho$-Bures metric.

Finally, suppose, by way of contradiction, that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f$. Then, for any $\delta > 0$, there exists $M \in \mathbb{N}$ such that, for any $n > M$ and any $x \in [0, 1]$, we have $|f_n(x) - f| < \delta$. Therefore, for some $\delta_0 \in (0, 1)$, there exists $M_0 \in \mathbb{N}$ such that, for any $n > M_0$ and any $x \in [0, 1]$, we have $|f_n(x) - f| < \delta_0$. However, let $x = 1$. Then $|f_n(1) - f(1)| = |2 - 1| = 1 > \delta_0$, which is a contradiction. Hence, our assumption is false. □

## 3. Topological equivalence of Bures $C^*$- and quantum metrics for finite-dimensional $C^*$-algebras

Although we just saw an example where the $C^*$-metric topology and Bures metric topology do not agree on the density space, in this section, we will see that if the $C^*$-algebra is assumed to be finite dimensional, then these topologies agree. Moreover, we will also show that in the finite-dimensional case, the Bures metric agrees with a metric on the density space induced by the quantum metric, which thus brings these two important metrics together. Some of the results in this section are similar to some of those of [4] or can be obtained from finite-dimensionality arguments without much further effort. We include the results and proofs of this section to create a bridge between the previous section and the next, while also introducing the third metric that we study in this article.

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We first show how one can induce a metric on the density space using a quantum metric in a natural way. Let $\mathcal{A}$ be a unital $C^*$-algebra. Let $\tau$ be a faithful trace on $\mathcal{A}$. For each $a \in D_\tau(\mathcal{A})$, define

$$\varphi_a^\tau(b) = \tau(ab) \qquad \text{for every } b \in \mathcal{A}.$$

The next result might be well known, but we provide a proof here for convenience.

Proposition 3.1. *Let $\mathcal{A}$ be a unital $C^*$-algebra and let $\tau$ be a faithful trace on $\mathcal{A}$. The map*

$$\Phi_\tau : a \in D_\tau(\mathcal{A}) \longmapsto \varphi_a^\tau \in S(\mathcal{A})$$

*is well defined and injective.*

Proof. For $a \in \mathcal{A}_+$, there exists $b \in \mathcal{A}$ such that $a = b^*b$ by [3, Lemma I.4.3]. Let $c \in \mathcal{A}$. Since $\tau$ is a trace,

$$\varphi_a^\tau(c^*c) = \tau(b^*bc^*c) = \tau(cb^*bc^*) = \tau(cb^*(cb^*)^*) \geqslant 0.$$

Thus, $\varphi$ is a positive linear functional, and, in particular, $\varphi_a^\tau(1_{\mathcal{A}}) = \|\varphi_a^\tau\|_{\mathrm{op}}$ by [3, Lemma I.9.5]. Since $a \in D_\tau(\mathcal{A})$,

$$\|\varphi_a^\tau\|_{\mathrm{op}} = \tau(a1_{\mathcal{A}}) = \tau(a) = 1.$$

Thus, $\varphi_a^\tau$ is a state, and therefore, $\Phi_\tau$ is well defined.

Next, let $a, a' \in \mathcal{A}_+$ such that $\Phi_\tau(a) = \Phi_\tau(a')$. Hence, $\varphi_a^\tau((a - a')^*) = \varphi_{a'}^\tau((a - a')^*)$. Thus, $\tau(a(a - a')^*) = \tau(a'(a - a')^*)$ and so $\tau((a - a')(a - a')^*) = 0$. Therefore, $a - a' = 0$ as $\tau$ is faithful, that is, $a = a'$. Thus, $\Phi_\tau$ is injective. □

This allows us to define a new metric on $D_\tau(\mathcal{A})$.

Definition 3.2. Let $(\mathcal{A}, L)$ be a compact quantum metric space. Let $\tau$ be a faithful trace on $\mathcal{A}$. For every $x, y \in D_\tau(\mathcal{A})$, define

$$d_L^\tau(x, y) = \mathrm{mk}_L(\Phi_\tau(x), \Phi_\tau(y)) = \mathrm{mk}_L(\varphi_x, \varphi_y),$$

which defines a metric on $D_\tau(\mathcal{A})$ since $\Phi_\tau$ is well defined and injective. We will still call $d_L^\tau$ a quantum metric.

Before moving on to the finite-dimensional setting, we prove that the topology generated by the $C^*$-norm is always finer than the topology induced by a quantum metric.

Proposition 3.3. *Let $(\mathcal{A}, L)$ be a compact quantum metric space and let $\tau$ be a faithful trace on $\mathcal{A}$. Then, on $D_\tau(\mathcal{A})$, the topology induced by $d_{\mathcal{A}}$ is finer than the topology induced by $d_L^\tau$.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $D_\tau(\mathcal{A})$ that converges to $a \in D_\tau(\mathcal{A})$ with respect to $d_{\mathcal{A}}$. Then, for each $b \in \mathcal{A}$,

$$\lim_{n\to\infty} \varphi_{a_n}^\tau(b) = \lim_{n\to\infty} \tau(a_n b) = \tau(ab) = \varphi_a^\tau(b)$$

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since multiplication is continuous and $\tau$ is continuous. Since convergence in $\mathrm{mk}_L$ is equivalent to weak* convergence by definition, the proof is complete. □

For the remainder of this section, $\mathcal{A}$ will be a finite-dimensional $C^*$-algebra. We will now prove that the $C^*$-metric topology and quantum metric topology are the same. First, we establish that $(D_\tau(\mathcal{A}), d_{\mathcal{A}})$ is a compact metric space.

PROPOSITION 3.4. *If $\mathcal{A}$ is a finite-dimensional $C^*$-algebra, then $(D_\tau(\mathcal{A}), d_{\mathcal{A}})$ is a compact metric space.*

PROOF. We only need to show that $D_\tau(\mathcal{A})$ is closed and bounded with respect to the $C^*$-norm. First, it is closed since $\tau$ is continuous with respect to the $C^*$-norm.

Since $\mathcal{A}$ is finite dimensional, the norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_\tau$ are equivalent, where $\|a\|_\tau = \sqrt{\tau(a^*a)}$ for $a \in \mathcal{A}$. Now, let $a \in D_\tau(\mathcal{A})$ and let $\sqrt{a}$ denote its unique positive square root. By the $C^*$-identity,

$$\begin{aligned}\sqrt{\|a\|_{\mathcal{A}}} = \sqrt{\|(\sqrt{a})^*\sqrt{a}\|_{\mathcal{A}}} = \|\sqrt{a}\|_{\mathcal{A}} \\ \leqslant \alpha\|\sqrt{a}\|_\tau = \alpha\sqrt{\tau((\sqrt{a})^*\sqrt{a})} = \alpha\sqrt{\tau(a)} = \alpha,\end{aligned}$$

since $\tau(a) = 1$. So $\|a\|_{\mathcal{A}} \leqslant \alpha^2$. Hence, $D_\tau(\mathcal{A})$ is bounded and thus is compact by the Heine–Borel theorem. □

THEOREM 3.5. *Let $(\mathcal{A}, L)$ be a compact quantum metric space and let $\tau$ be a faithful trace on $\mathcal{A}$. If $\mathcal{A}$ is finite dimensional, then $d_L^\tau$ and $d_{\mathcal{A}}$ are topologically equivalent.*

PROOF. The proof that $\Phi_\tau$ is continuous from $(D_\tau(\mathcal{A}), d_{\mathcal{A}})$ to $(D_\tau(\mathcal{A}), d_L^\tau)$ is the same as the proof of Proposition 3.3. Since $(D_\tau(\mathcal{A}), d_{\mathcal{A}})$ is compact by Proposition 3.4, $\Phi_\tau$ is a homeomorphism onto its image with respect to the weak* topology on $S(\mathcal{A})$.

Since convergence in $\mathrm{mk}_L$ is equivalent to weak* convergence by definition, $d_{\mathcal{A}}$ and $d_L^\tau$ are topologically equivalent. □

Now, we establish the topological equivalence of the $C^*$-metric and Bures metric.

THEOREM 3.6. *Let $\mathcal{A}$ be a unital $C^*$-algebra and let $\tau$ be a faithful trace on $\mathcal{A}$. If $\mathcal{A}$ is finite dimensional, then the $C^*$-metric $d_{\mathcal{A}}$ and the Bures metric $d_B^\tau$ are topologically equivalent.*

PROOF. Consider

$$\mathrm{id}_{D_\tau(\mathcal{A})} : (D_\tau(\mathcal{A}), d_{\mathcal{A}}) \longrightarrow (D_\tau(\mathcal{A}), d_B^\tau).$$

By Theorem 2.3, convergence in $d_{\mathcal{A}}$ implies convergence in $d_B^\tau$. Hence, $\mathrm{id}_{D_\tau(\mathcal{A})}$ is continuous, and since $(D_\tau(\mathcal{A}), d_{\mathcal{A}})$ is compact, $\mathrm{id}_{D_\tau(\mathcal{A})}$ is a homeomorphism. Hence, $d_{\mathcal{A}}$ and $d_B^\tau$ are topologically equivalent. □

We conclude this section with the topological equivalence of the Bures metric and quantum metric.

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COROLLARY 3.7. *Let $(\mathcal{A}, L)$ be a compact quantum metric space and let $\tau$ be a faithful trace on $\mathcal{A}$. If $\mathcal{A}$ is finite dimensional, then the Bures metric $d_B^\tau$ and quantum metric $d_L^\tau$ are topologically equivalent.*

PROOF. This follows immediately from Theorems 3.6 and 3.5 and since homeomorphism is an equivalence relation. □

## 4. Metric inequivalence for $\mathbb{C}^2$

In this last section, we see that, although the Bures metric and quantum metric are topologically equivalent for finite-dimensional $C^*$-algebras, there exist finite-dimensional $C^*$-algebras for which they are not equivalent as metric spaces. Our approach also proves that the Bures metric and $C^*$-norm are not metrically equivalent since we find a quantum metric that, in fact, agrees with the metric on the density space induced by the $C^*$-norm.

Consider the unital $C^*$-algebra $\mathbb{C}^2$. For every $x = (x_1, x_2) \in \mathbb{C}^2$, define

$$L^B(x) = |x_1 - x_2|.$$

Then $L^B$ is an $L$-seminorm since $\mathbb{C}^2$ is finite dimensional and by [11, Proposition 1.6]. So $\mathrm{mk}_{L^B}$ is a quantum metric.

Consider the faithful trace $\tau$ on $\mathbb{C}^2$ defined by

$$\tau(x) = x_1 + x_2 \quad \text{for } x = (x_1, x_2) \in \mathbb{C}^2.$$

The Bures metric is given explicitly by the trace, but as the quantum metric is defined by way of a supremum, we first find a formula to explicitly calculate the quantum metric in this case. This recovers the $C^*$-metric.

THEOREM 4.1. *With the setting as above, for every $x, y \in D_\tau(\mathbb{C}^2)$,*

$$d_{L^B}^\tau(x, y) = |x_1 - y_1| = \|x - y\|_{\mathbb{C}^2} = d_{\mathcal{A}}(x, y).$$

PROOF. First, let $x \in D_\tau(\mathbb{C}^2)$ and let $x' \in \mathbb{C}^2$. Then

$$\Phi_\tau(x)(x') = \phi_x^\tau(x') = \tau(xx') = x_1x_1' + x_2x_2'.$$

Now, assume that $y \in D_\tau(\mathbb{C}^2)$. Since $\tau(x) = 1 = \tau(y)$, we have $x_1 + x_2 = 1 = y_1 + y_2$ and so $x_1 - y_1 = y_2 - x_2$. Hence, if $L^B(x') \leqslant 1$,

$$\begin{aligned}
|\varphi_x^\tau(x') - \varphi_y^\tau(x')| &= |x_1x_1' + x_2x_2' - y_1x_1' - y_2x_2'| \\
&= |(x_1 - y_1)x_1' + (x_2 - y_2)x_2'| \\
&= |(x_1 - y_1)x_1' - (y_2 - x_2)x_2'| \\
&= |(x_1 - y_1)x_1' - (x_1 - y_1)x_2'| \\
&= |(x_1 - y_1)(x_1' - x_2')| \\
&= |x_1 - y_1| \cdot |x_1' - x_2'| \\
&\leqslant |x_1 - y_1|.
\end{aligned}$$

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Now, consider $x' = (1, 0)$. Then

$$|\varphi_x^\tau(x') - \varphi_y^\tau(x')| = |x_1 - y_1|.$$

Thus,

$$d_{L^B}^\tau(x, y) = \sup\{|\varphi_x^\tau(x') - \varphi_y^\tau(x')| : L^B(x') \leqslant 1\} = |x_1 - y_1|.$$

We also note that

$$\|x - y\|_{\mathbb{C}^2} = \max\{|x_1 - y_1|, |x_2 - y_2|\} = \max\{|x_1 - y_1|, |x_1 - y_1|\} = |x_1 - y_1|,$$

as desired. □

THEOREM 4.2. *Let $d = d_{L^B}^\tau$, the Bures metric, or $d = d_{\mathcal{A}}$. With the setting as above, the quantum metric $d_{L^B}^\tau$ and $d$ are topologically equivalent but not metric equivalent.*

PROOF. Corollary 3.7 provides topological equivalence.

Consider $x = (1, 0) \in D_\tau(\mathbb{C}^2)$. Let $y = (y_1, y_2) \in D_\tau(\mathbb{C}^2)$. By Theorem 4.1,

$$d_{L^B}^\tau(x, y) = |1 - y_1|.$$

Also,

$$d_B^\tau(x, y) = \sqrt{1 - \tau(|\sqrt{x}\sqrt{y}|)} = \sqrt{1 - \sqrt{y_1}}.$$

Now, consider the ratio

$$\frac{d_B^\tau(x, y)}{d_{L^B}^\tau(x, y)} = \frac{\sqrt{1 - \sqrt{y_1}}}{|1 - y_1|}.$$

We have $\lim_{y_1 \to 1^-} \sqrt{1 - \sqrt{y_1}} = 0$ and $\lim_{y_1 \to 1^-} 1 - y_1 = 0$. Further,

$$\frac{d}{dy_1}\Big(\sqrt{1 - \sqrt{y_1}}\Big) = \frac{-1}{4\sqrt{y_1}\sqrt{1 - \sqrt{y_1}}}, \qquad \frac{d}{dy_1}(1 - y_1) = -1$$

and

$$\lim_{y_1 \to 1^-} \frac{\frac{-1}{4\sqrt{y_1}\sqrt{1 - \sqrt{y_1}}}}{-1} = \lim_{y_1 \to 1^-} \frac{1}{4\sqrt{y_1}\sqrt{1 - \sqrt{y_1}}} = \infty.$$

Thus, by L'Hopital's rule,

$$\lim_{y_1 \to 1^-} \frac{d_B^\tau(x, y)}{d_{L^B}^\tau(x, y)} = \infty.$$

Hence, the set

$$\left\{\frac{d_B^\tau(x, y)}{d_{L^B}^\tau(x, y)} : x, y \in D_\tau(\mathbb{C}^2), x \neq y\right\}$$

is unbounded, and so the metrics $d_B^\tau$ and $d_{L^B}^\tau$ are not metric equivalent. □

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## Acknowledgement

The third author would like to thank Erik Bédos for helpful discussions.

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https://doi.org/10.1017/S0004972724001217 Published online by Cambridge University Press