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Wave action flux: a physical interpretation

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Steady, gravity water waves on a constant-over-depth current, progressing over a slowly varying bed, are studied with the purpose of connecting the wave action flux concept with conventional energy flux considerations. The analysis is two-dimensional and dissipation is neglected. A new relation between integral properties containing the energy flux referred to a 'global' level, the so-called mean energy level, gives the surprising result that this flux is simply the product of absolute angular frequency and wave action flux. An alternative, less physical, proof of this result is also presented. A general equation for the action velocity is set out and for linear waves shown to equal a well-known expression. Also presented are new expressions for relative phase velocity in terms of kinetic energy and mean momentum for the wave, and the kinetic energy in terms of the characteristic velocities for the combined wave and current motion. In the Appendix a simple relation between energy and action fluxes for small-amplitude waves on a linear shear current is found which resembles the irrotational theory, finite-height result. A possible extension of this relation to finite-height waves on a general shear current is discussed.

1. Introduction

Bretherton & Garrett (1968) introduced the wave action conservation principle for a rather general class of small-amplitude waves in slowly varying media. The approach was based on Whitham's averaged Lagrangian introduced in 1965 (see Whitham 1974). This action principle has been found to be a very useful concept for water waves on currents, see for instance Christoffersen (1982) and Jonsson (1990). The action conservation equation for water waves derived from a Lagrangian has for many years been thought of as being indispensable for the calculation of the evolution of wave heights. We will set out to show, however, that for steady gravity water waves on a current (irrotational and plane flow assumed), we need introduce neither wave action flux nor Lagrangian theory; using the conventional expression for the mean energy flux with a 'global' reference level for flow of potential energy, we will show that this energy flux is in fact proportional to action flux. Thus wave action conservation for steady water waves on a current is nothing but simple energy conservation. The result is general, in that it applies to any phase-function-related water wave theory and there is no restriction on the degree of nonlinearity.

In §2 a simple expression for the set-down Δh, i.e. the depression of the mean water surface due to a change in water depth (and thus also current velocity) is rederived. The energy and energy flux are calculated in §3, the latter with reference to the so-called mean energy level, which is situated a distance Δh above the mean water surface, giving a new relation between integral properties for periodic gravity waves. In §4 we introduce the averaged Lagrangian from Crapper (1979), and the ensuing expressions for action density and action flux. The latter is compared with the energy flux from §3, and proportionality is demonstrated. An alternative proof of this is given. New expressions for relative phase velocity and kinetic wave energy are also introduced. Finally, the action velocity is found and discussed in §5. In the Appendix attempts are made to relate the wave energy flux to the wave action flux for waves on a rotational current.

Dissipation is neglected throughout. (For small-amplitude waves on a constant-over-depth current it is quite easy to include the bed shear and ensuing dissipation in a wave action formulation, see the 'wave action dissipation' term in Christoffersen & Jonsson 1980, steady case, Christoffersen 1982 and Jonsson 1990, unsteady case).

2. Mean energy level and set-down

When water waves propagate over a sloping bottom, the mean water surface (MWS) is not horizontal, and so there is a difference between the still water, or undisturbed, depth D and the actual water depth h (figure 1), the difference representing the set-down Δh. The level from which depth D is measured is termed the mean energy level (MEL), for pure waves first introduced by Lundgren (1963).

The set-down is calculated in Jonsson, Skougaard & Wang (1971) and Jonsson & Wang (1980), see also Jonsson & Arneborg (1995). The method will be briefly outlined here. It is based on the time-mean of the Bernoulli equation which reads

$$z+\left\langle\frac{p}{\rho g}\right\rangle+\frac{1}{2g}(\langle u_1^2\rangle+\langle u_2^2\rangle+\langle w^2\rangle)=-\frac{1}{g}\left\langle\frac{\partial\phi}{\partial t}\right\rangle, \tag{1}$$

symbol $\langle\ \rangle$ denoting time averaging over an absolute wave period. In (1), z is the height over the datum, p the pressure, ρ the density, g the gravity acceleration, u_1 and u_2 the horizontal velocity components, w the vertical velocity component, ϕ the velocity potential, and t the time. Note that u and w are total quantities (wave plus current).

The spatial derivatives of the right-hand side of (1) are (ignoring the constant)

$$\frac{\partial}{\partial x_i}\left\langle\frac{\partial\phi}{\partial t}\right\rangle=\left\langle\frac{\partial}{\partial x_i}\frac{\partial\phi}{\partial t}\right\rangle=\left\langle\frac{\partial}{\partial t}\frac{\partial\phi}{\partial x_i}\right\rangle,\quad i=1,2,3. \tag{2}$$

Since the wave motion is assumed periodic, the last term of (2) vanishes, and thus $\langle\partial\phi/\partial t\rangle$ is shown to be a global constant. Therefore in (1) the sum of the geometrical height, the mean pressure height, and the mean velocity height is seen to be a constant for the entire flow. So, denoting this constant C ($=-\langle\partial\phi/\partial t\rangle/g$), we have the MEL situated a distance C above our datum, see figure 1. At the bed (1) reads

$$z_b+\left\langle\frac{p_b}{\rho g}\right\rangle+\frac{1}{2g}(\langle u_{1b}^2\rangle+\langle u_{2b}^2\rangle+\langle w_b^2\rangle)=C. \tag{3}$$

Since for progressive waves over a gently sloping bed we have $h=\langle p_b\rangle/\rho g$, and the set-down from figure 1 equals $C-(z_b+h)$ (z_b representing the position of the bed), we find from (3) that

$$\Delta h=\frac{1}{2g}(\langle u_{1b}^2\rangle+\langle u_{2b}^2\rangle+\langle w_b^2\rangle). \tag{4}$$

So, for a steady, progressive wave on a current over a gently sloping bed, the set-down (irrotational flow assumed) simply equals the mean velocity head at the bed. This was first pointed out by Jonsson *et al.* (1971).

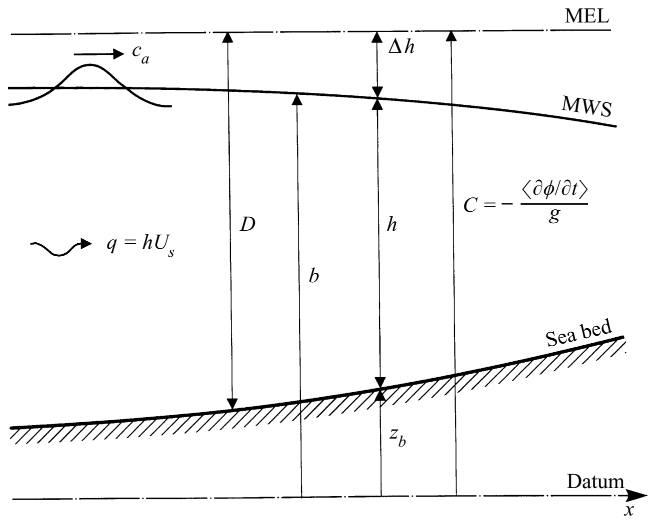

FIGURE 1. The mean water surface MWS, the mean energy level MEL, and the set-down Δh. Quantity D is still water depth, h is actual depth, c_a is absolute wave phase velocity, and q is the mean volume flux.

For plane flow, (4) then gives the following simple relation (ignoring the small last term):

$$\Delta h = \frac{1}{2g}\langle u_b^2\rangle. \tag{5}$$

At infinite water depth the fluid velocity at the bed vanishes, and so by (5) the set-down is zero here. Therefore the MEL in fact equals the 'still water level', and this explains the phrase 'still water depth' for D in figure 1.

Strictly speaking (2) is only valid below wave trough level. However, this does not contest the validity of (5), which is crucial for the calculations in the next section.

3. Energy and energy flux

The energy flux per unit length of the wave front with the MWS as the datum is

$$F_{MWS} = \left\langle \int_{-h}^{\eta} (\rho g z + p + \tfrac{1}{2}\rho(u^2+w^2))\, u\, \mathrm{d}z \right\rangle \tag{6}$$

in which η is the surface elevation measured from the MWS. Sobey *et al.* (1987, equation 93) and Klopman (1990, equation 23) found this to be

$$F_{MWS} = c_a(3T-2V) + \tfrac{1}{2}\langle u_b^2\rangle (I+\rho c_a h) - 2c_a U_E I. \tag{7}$$

In (7), c_a is the absolute wave phase velocity, T and V are kinetic and potential energy per unit horizontal area, I is the mean mass flux

$$I = \left\langle \int_{-h}^{\eta} \rho u\, \mathrm{d}z \right\rangle \equiv \rho U_S h \tag{8}$$

and U_E is the mean current velocity below wave trough level, the mean Eulerian velocity. Quantity U_S in (8) is the mass transport velocity, so that the mean volume flux in figure 1 is $q = hU_S$ ($= I/\rho$).

In the study of wave shoaling on a current, a 'global' datum for the energy flux should be used rather than the MWS. Choosing the MEL as such a datum, we find the energy flux as

$$F_{MEL} = F_{MWS} - \rho g h \Delta h U_S = F_{MWS} - \tfrac{1}{2}\langle u_b^2 \rangle I \tag{9}$$

using (5) and (8). Introducing (7) in (9) leaves us with a new relation between integral properties for periodic gravity waves on a current

$$F_{MEL} = (3T - 2V + \tfrac{1}{2}\rho h \langle u_b^2 \rangle - 2U_E I)\, c_a. \tag{10}$$

It should again be remembered that quantities in (10) are total ones (wave plus current).

Potential energy is as usual

$$V = \tfrac{1}{2}\rho g \langle \eta^2 \rangle \tag{11}$$

with η measured from the MWS. From Longuet-Higgins (1975, equation (2.3)), kinetic energy is found as

$$T = \tfrac{1}{2}(c_a I - U_E Q), \tag{12}$$

where Q is the mean mass flux in the opposite direction to wave propagation in a frame where the wave is stationary, i.e.

$$Q = \rho c_a h - I. \tag{13}$$

4. Action and action flux

For plane flow we have the averaged Lagrangian from equation (35) in Crapper (1979) as

$$\mathscr{L} = \rho(\gamma - \tfrac{1}{2}U_E^2)\, h - \tfrac{1}{2}\rho g(b^2 - d^2) + T^w - V^w, \tag{14}$$

where $-\gamma$ is the factor to time t in the pseudo-phase function in the velocity potential (see (35) later), and b is the height of the MWS over the datum (i.e. $z_b + h$ from figure 1; and $-\Delta h$ with the datum at the MEL). Quantity d is the height of the datum over the sea bed (i.e. $-z_b$ from figure 1; and $h + \Delta h$ with the datum at the MEL). The term $\tfrac{1}{2}\rho g d^2$ does not contribute to any of the Euler–Lagrange equations of the variational principle, and can thus be ignored (Crapper 1979). Superscript 'w' in (14) denotes 'wave', which here means that the energy is related to a reference frame in which the mean Eulerian velocity U_E vanishes.

From (14) we can, following Crapper, define the 'wave Lagrangian' as

$$\mathscr{L}^w = T^w - V^w, \tag{15}$$

where

$$\mathscr{L}^w = \mathscr{L}^w(\omega_r, k, a, h). \tag{16}$$

Here ω_r is the relative angular frequency, k is the wavenumber, and a is a measure of the wave amplitude. Moreover, frequency ω_r is related to the wavenumber through the Doppler relation

$$\omega_r = \omega_a - kU_E, \tag{17}$$

where ω_a is the absolute angular frequency.

Since in this paper we are only interested in action conservation, it suffices when using the Whitham method to look at the variation of $\mathscr{L}$ with respect to the phase function $\theta = kx - \omega_a t$, quantity x being the coordinate in the direction of wave propagation. From (14)–(17) we then get, cf. Crapper (1979, equation (42))

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}^w}{\partial \omega_r}\right) + \frac{\partial}{\partial x}\left(U_E \frac{\partial \mathscr{L}^w}{\partial \omega_r} - \frac{\partial \mathscr{L}^w}{\partial k}\right) = 0, \tag{18}$$

noting that $\mathscr{L}$ does not depend explicitly on θ, but only on its derivatives. From (18) we have wave action A and total wave action flux B as

$$A = \frac{\partial \mathscr{L}^w}{\partial \omega_r}, \tag{19}$$

$$B = U_E \frac{\partial \mathscr{L}^w}{\partial \omega_r} - \frac{\partial \mathscr{L}^w}{\partial k}. \tag{20}$$

From mass conservation Crapper finds (his equations (5) and (41), cf. (44)),

$$A = \frac{I^w}{k}, \tag{21}$$

where I^w is the mean wave momentum (from Crapper, equation (7))

$$I^w = \rho h (U_S - U_E). \tag{22}$$

Thus, as noted by Stiassnie & Peregrine (1979), apart from a factor $1/2\pi$ wave action is the momentum per wave relative to the mean current velocity U_E. (Alternatively A can be written as $2T^w/\omega_r$, Crapper 1979, equation (75), top. This gives a new expression for the relative phase velocity, $c_r = c_a - U_E = 2T^w/I^w$ using that $c_r = \omega_r/k$ together with (21)).

Further Crapper finds, comparing two results for the radiation stress (his (47) and (74)) that

$$-\frac{\partial \mathscr{L}^w}{\partial k} = \frac{1}{k}\left(3T^w - 2V^w + \tfrac{1}{2}\rho h \langle (u_b^w)^2 \rangle\right), \tag{23}$$

cf. his (75), middle. Here u_b^w is the wave particle velocity at the bed, i.e.

$$u_b^w = u_b - U_E. \tag{24}$$

From (19), (20), (21), and (23), we finally find

$$B = \frac{1}{k}\left(U_E I^w + 3T^w - 2V^w + \tfrac{1}{2}\rho h \langle (u_b^w)^2 \rangle\right) \tag{25}$$

which agrees with Stiassnie & Peregrine (1980, equation 2); see also the second parenthesis in equation (9) in Stiassnie & Peregrine (1979).

We will now rewrite (25) in terms of 'total' quantities like those appearing in (10). From (8) and (22) we have readily

$$I^w = I - \rho U_E h \tag{26}$$

and from (24), since we consider steady waves, that

$$\langle (u_b^w)^2 \rangle = \langle u_b^2 \rangle - U_E^2. \tag{27}$$

It follows immediately that

$$V^w = V. \tag{28}$$

Finally, from Klopman (1990, equation (17)) we get

$$T^w = T - U_E I + \tfrac{1}{2}\rho h U_E^2. \tag{29}$$

(Alternatively T^w can be written

$$T^w = T - \tfrac{1}{2}\rho h U_S^2 + \tfrac{1}{2}\rho h (U_S - U_E)^2. \tag{30}$$

This form is better suited for physical interpretation. Yet another form can be found by introducing (8), (12), and (13) in (29) giving

$$T^w = \tfrac{1}{2}\rho h(U_S - U_E)(c_a - U_E). \tag{31}$$

This new form is particularly interesting, since the term in the first parentheses is the so-called 'return current velocity' – the difference between Stokes' first and second definition of phase velocity – and the term in the second parentheses is the relative phase velocity.)

Introducing (26), (27), (28) and (29) in (25) yields

$$B = \frac{1}{k}(3T - 2V + \tfrac{1}{2}\rho h\langle u_b^2\rangle - 2U_E I). \tag{32}$$

Comparing (10) and (32), and using that $c_a = \omega_a/k$, we find the following simple, yet fundamental, relation for steady waves on a current:

$$F_{MEL} = \omega_a B. \tag{33}$$

This result is a generalization of the similar relation to second order for Stokes waves on a current found by Jonsson (1978); see also Jonsson *et al.* (1971). From this it can be seen that for steady waves on a current, the more abstract requirement of wave action conservation ($\mathrm{d}B/\mathrm{d}x = 0$) in shoaling water waves is nothing more than stating the simple physical requirement that the total energy flux related to the MEL (or any other fixed datum for that matter) be constant. This is not surprising. But only *one* of these datums yields the simple relation (33).

Note that (33) is a general result for steady waves on a current, independent of wave theory and wave height. An analogy to (33) for rotational flow is discussed in the Appendix.

It turns out that the mean energy level concept can be combined directly with Whitham's method to provide an alternative – yet less physical – proof of (33). Going back to Whitham (1965) we find the energy equation from his equation (39) in the steady state and for plane flow as

$$\frac{\mathrm{d}}{\mathrm{d}x}(-\omega_a \mathscr{L}_k - \gamma \mathscr{L}_{U_E}) = O, \tag{34}$$

where γ comes from the velocity potential

$$\phi = \Phi(\theta) + U_E x - \gamma t. \tag{35}$$

Here $\Phi(\theta)$ represents the periodic part (θ is still the phase function), and $U_E x - \gamma t$ is the so-called pseudo-phase function ψ.

Since the term in parentheses in (34) is the energy flux we have

$$F = -\omega_a \mathscr{L}_k - \gamma \mathscr{L}_{U_E}. \tag{36}$$

Bearing (33) in mind we shall first inspect the second term on the right-hand side of (36). From (iii) on p. 19 in Crapper (1979) (variation with respect to ψ) we have immediately that for steady flow $\mathscr{L}_{U_E}$ is a constant, which we will find here for completeness. From (41) in Crapper we have

$$-\mathscr{L}_{U_E} = \rho h U_E + k\frac{\partial \mathscr{L}^w}{\partial \omega_r} \tag{37}$$

which also can be seen to stem from (14), (15) and (17) in the present paper. Then from (19), (21) and (22) we readily get

$$-\mathscr{L}_{U_E} = \rho h U_S, \tag{38}$$

i.e. the mean mass flux, which indeed is a constant in our case.

The quantity γ in (36) is determined by again looking at the time-averaged Bernoulli equation. For plane flow and a small bed slope we have from (1), evaluated at the bed (with $\langle p_b \rangle / \rho g = h$), and (5) that

$$-\frac{1}{g}\left\langle \frac{\partial \phi}{\partial t} \right\rangle = z_b + h + \Delta h \tag{39}$$

omitting suffix 'b' on ϕ, since $\langle \partial\phi/\partial t \rangle$ is the same everywhere. This is illustrated in figure 1. From (35) we have further

$$\left\langle \frac{\partial \phi}{\partial t} \right\rangle = -\gamma, \tag{40}$$

which with (39) yields

$$\gamma = g(z_b + h + \Delta h). \tag{41}$$

Thus we have a simple relation between quantity γ in the pseudo-phase function and position of datum through z_b. From (41) (see also figure 1) it follows that with the datum situated at the mean energy level, γ is zero, and so the second term on the right-hand side of (36) vanishes. As a result (33) follows, since $\mathscr{L}_k$ equals $-B$, see (20).

In analogy to the comment at the end of §2 it can be stated that while (39) and (40) are only valid below the wave trough level, (41) is general since γ must be the same constant everywhere in the fluid.

It may prove useful to round off the discussion with a few general remarks. Wave action conservation is an elegant and extremely useful tool for the study of waves progressing over varying depths and currents. By comparing with a 'classical' approach, where one deals with conservation equations for mass, momentum and energy (see e.g. equations 3.6.4, 3.6.11 and 3.6.18 in Phillips 1977), it is seen that radiation stress and position of mean water surface are both eliminated in this formulation. (How this is done to second order for Stokes waves on a current can be studied in detail in Christoffersen 1982; see also Jonsson 1990). And further, in more general situations (unsteady flow), the energy formulation of the present paper naturally cannot replace wave action conservation. However, the apparent simplicity of this principle is at the expense of a higher level of abstraction; how can one get a physical 'feel' of what are wave action and wave action flux? The present paper represents an attempt to improve on this by presenting a direct interpretation of the latter concept for steady waves; this is believed to give some new insight and understanding, especially to those who are not familiar with Lagrangians and the wave action concept.

5. Action velocity

From (21), (26) and (32) we can find the wave action velocity c_A as

$$c_A \equiv \frac{B}{A} \tag{42}$$

$$= \frac{1}{I - \rho U_E h}\left(3T - 2V + \tfrac{1}{2}\rho h \langle u_b^2 \rangle - 2U_E I\right). \tag{43}$$

The third term in the parentheses can by (5) be replaced by $\rho g h \Delta h$. In terms of 'wave quantities' (42) can further be written, using (21) and (25)

$$c_A = U_E + \frac{1}{I^w}(3T^w - 2V^w + \tfrac{1}{2}\rho h \langle (u_b^w)^2 \rangle) \tag{44}$$

thus introducing a 'carrier velocity' U_E. Note that the second term on the right-hand side of (44) is simply

$$-\frac{\partial \mathscr{L}^w}{\partial k} \bigg/ \frac{\partial \mathscr{L}^w}{\partial \omega_r}$$

cf. (19) and (20).

For small-amplitude waves (44) gives

$$c_A = U_E + \tfrac{1}{2} c_r^l \left(1 + \frac{2kh}{\sinh 2kh}\right) \tag{45}$$

as it should, since the second term on the right-hand side equals the linear, relative group velocity, $c_r^l = ((g/k)\tanh kh)^{1/2}$ being linear, relative phase velocity.

From (21) and (42), we can write (33), using also that $\omega_a = kc_a$, as

$$F_{MEL} = c_a c_A I^w. \tag{46}$$

The physical interpretation – if any – of this simple result is yet unclear.

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Appendix. Waves on a rotational current

The preceding deliberations were all founded on the assumption of irrotational flow; without a velocity potential we cannot define a mean energy level and thus introduce an energy flux F_{MEL}, nor can we apply an averaged Lagrangian in the conventional sense. For rotational flow it would therefore immediately seem meaningless to seek a relation in a manner similar to (33). It turns out to be possible to do so, however, for the special case of steady small-amplitude waves on a linear shear current (constant vorticity). Here a wave potential can still be defined, and in the study by Jonsson, Brink-Kjær & Thomas (1978), an (incomplete) averaged Lagrangian obtained by heuristic arguments from Clebsch potentials (see for instance Luke 1967) was found for steady waves to second order in wave steepness, and so also a wave action conservation equation. Comparing this with conventional energy conservation lead – again by heuristic arguments and thereafter indirectly proved – to the following simple result in Jonsson *et al.* (1978) (cf. their equation (37)):

$$F_b = \omega_a B + \text{const.} \tag{A 1}$$

Here F_b is the energy flux for a reference level so defined that the MWS is at a height b above it, see figure 1 (subscript 'b' not to be confused with 'bed' as used previously). For details of action flux, etcetera, see the paper cited.

Now (A 1) has indeed a similarity to (33); and for irrotational flow and the datum situated at the MEL (in which case $b = -\Delta h$), the constant is zero as demonstrated in this paper. However, (A 1) was only shown to be correct at second order, whereas (33) has no such restriction. It remains to be proved whether (A 1) is in fact a general result for finite-height irrotational waves on a linear shear current.

This could perhaps be so, even for an arbitrary current profile. If the appropriate Clebsch potentials can be found, and if the Whitham approach is assumed to be valid here, an averaged Lagrangian can be constructed, and wave action will exist as also an associated conservation principle, including wave action flux. If further the conventional energy flux can be found, the following two 'energy transport equations' in conservation form (steady waves still assumed) can be set up:

$$\frac{\mathrm{d}F_b}{\mathrm{d}x} = 0, \quad \frac{\mathrm{d}B}{\mathrm{d}x} = 0. \qquad \text{(A 2}a, b\text{)}$$

It is hereafter plausible to assume, following Jonsson *et al*. (1978), that the quantities under the differential operators are equal, apart from a constant factor and an arbitrary constant as in (A 1).

Assuming this, it can be seen that the factor to B in such a relation must have the dimension of s^{-1}. And the most obvious constant factor at hand with this characteristic is the absolute angular frequency, leading again to (A 1). This is still speculation, however, and has to be proved rigorously – if possible at all! One might start with finite-height waves on a linear current profile. The speculation is supported by the fact that in the two special cases, irrotational finite-height wave and current flow (present paper), and irrotational small-amplitude waves on a linear shear current (Jonsson *et al*. 1978), the factor in such a relation is indeed ω_a.

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