Math. Struct. in Comp. Science (2018), *vol.* 28, *pp.* 548–561.
doi:10.1017/S0960129517000032 First published online 28 February 2017

Yoneda completeness

TRISTAN BICE

Instytut Matematyczny Polskiej Akademii Nauk, Warszawa, Poland
Email: Tristan.Bice@gmail.com

Received 14 March 2016; revised 4 December 2016

We characterize Yoneda completeness for non-symmetric distances by combinations of metric and directed completeness. One of these generalizes the Kostanek–Waszkiewicz theorem on formal balls.

1. Motivation

Yoneda completeness was introduced in Wagner (1997) and Bonsangue et al. (1998) to unify metric and order theoretic notions of completeness. More precisely, the goal was to find a natural notion of completeness for non-symmetric distances that reduces to Cauchy completeness in the metric case and directed completeness in the partial order case. We aim to take this further by showing that, even in more general distance spaces, Yoneda completeness can still be characterized by several different combinations of metric and directed completeness.

We draw our inspiration from a perhaps surprising source, namely C*-algebra semicontinuity theory – see Akemann and Pedersen (1973), Brown (1988). Various order relations in C*-algebras can be composed with the metric to form non-symmetric distances, although they are never mentioned explicitly in the C*-algebra literature. This is unfortunate, as non-symmetric distances could simplify and generalize certain aspects of C*-algebra theory. In particular, this rings true for C*-algebra semicontinuity theory, where some sophisticated C*-algebraic machinery can be replaced by the elementary net manipulations that we describe here – see Bice (2016). This will also no doubt have applications to distance spaces that arise in other areas of algebra and analysis.

2. Outline

In Section 3, we give the basic definitions and theory of (pre-)Cauchy nets, ball and hole topologies, non-symmetric distances and supremums. We take Wagner (1997) and Bonsangue et al. (1998) as our primary references although our approach is slightly more general, e.g. we deal with distances rather than hemimetrics and nets rather than sequences. Although to keep things simple, the range of our distance functions will always be the positive extended real line $[0,\infty]$ as in Bonsangue et al. (1998), rather than the more general quantales considered in Wagner (1997). For the completeness notions we consider, see Definitions 3.2 and 3.6.

In Section 4, we construct several closely related sequences and subsets from a given Cauchy net (x_λ). Their consequences regarding completeness are collected at the end in

Corollary 4.6. We finish with a simple application to formal balls in Theorem 4.7, showing that Corollary 4.6 1. generalizes the Kostanek–Waszkiewicz theorem.

3. Preliminaries

We make the following standing assumption.

d and e are functions from $X \times X$ to $[0, \infty]$.

3.1. *Nets*

The nets $(x_\lambda) \subseteq X$ we will be concerned with are defined as follows:

$$\lim_{\gamma} \liminf_{\delta} \mathbf{d}(x_\gamma, x_\delta) = 0 \quad \Leftrightarrow \quad (x_\lambda) \text{ is } \mathbf{d}\text{-reflexive.} \tag{1}$$

$$\lim_{\gamma} \limsup_{\delta} \mathbf{d}(x_\gamma, x_\delta) = 0 \quad \Leftrightarrow \quad (x_\lambda) \text{ is } \mathbf{d}\text{-pre-Cauchy.} \tag{2}$$

$$\lim_{\gamma \prec \delta} \mathbf{d}(x_\gamma, x_\delta) = 0 \quad \Leftrightarrow \quad (x_\lambda) \text{ is } \mathbf{d}\text{-Cauchy.} \tag{3}$$

Just to be clear, by a net we mean a set indexed by a directed set Λ, i.e. there is a (possibly non-reflexive) transitive relation $\prec \subseteq \Lambda \times \Lambda$ satisfying $\forall \gamma, \delta\ \exists \lambda\ (\gamma, \delta \prec \lambda)$, with limits inferior and superior defined by

$$\liminf_{\lambda} r_\lambda = \lim_{\gamma} \inf_{\gamma \prec \lambda} r_\lambda,$$

$$\limsup_{\lambda} r_\lambda = \lim_{\gamma} \sup_{\gamma \prec \lambda} r_\lambda.$$

And in Equation (3), we consider $\prec$ itself as a directed subset of $\Lambda \times \Lambda$ ordered by $\prec \times \prec$.

The above nets can also be characterized by a filter $\Phi^{\mathbf{d}} \subseteq \mathcal{P}(X \times X)$ defined from $\mathbf{d}$. Specifically, for $\prec \subseteq [0, \infty] \times [0, \infty]$ and $\epsilon \in [0, \infty]$, define $\prec^{\mathbf{d}}_{\epsilon} \subseteq X \times X$ by

$$x \prec^{\mathbf{d}}_{\epsilon} y \quad \Leftrightarrow \quad \mathbf{d}(x, y) \prec \epsilon.$$

Taking the usual $<$ on $[0, \infty]$ for $\prec$, we define

$$\Phi^{\mathbf{d}} = \{\sqsubset \subseteq X \times X : \epsilon > 0 \text{ and } <^{\mathbf{d}}_{\epsilon} \subseteq \sqsubset\}.$$

So $\leqslant^{\mathbf{d}} = \leqslant^{\mathbf{d}}_0 = \bigcap \Phi^{\mathbf{d}}$ and

$$\forall \sqsubset \in \Phi^{\mathbf{d}}\ \exists \alpha\ \forall \gamma \succ \alpha\ \forall \beta\ \exists \delta \succ \beta\ (x_\gamma \sqsubset x_\delta) \quad \Leftrightarrow \quad (x_\lambda) \text{ is } \mathbf{d}\text{-reflexive.}$$

$$\forall \sqsubset \in \Phi^{\mathbf{d}}\ \exists \alpha\ \forall \gamma \succ \alpha\ \exists \beta\ \forall \delta \succ \beta\ (x_\gamma \sqsubset x_\delta) \quad \Leftrightarrow \quad (x_\lambda) \text{ is } \mathbf{d}\text{-pre-Cauchy.}$$

$$\forall \sqsubset \in \Phi^{\mathbf{d}}\ \exists \alpha\ \forall \gamma \succ \alpha \qquad \forall \delta \succ \gamma\ (x_\gamma \sqsubset x_\delta) \quad \Leftrightarrow \quad (x_\lambda) \text{ is } \mathbf{d}\text{-Cauchy.}$$

We immediately see that

$$\mathbf{d}\text{-Cauchy} \quad \Rightarrow \quad \mathbf{d}\text{-pre-Cauchy} \quad \Rightarrow \quad \mathbf{d}\text{-reflexive.}$$

Conversely, any **d**-reflexive sequence has a **d**-Cauchy subsequence. But for nets something stronger is needed, as in the following result.

Proposition 3.1. Any **d**-pre-Cauchy net $(x_\lambda) \subseteq X$ has a **d**-Cauchy subnet.

https://doi.org/10.1017/S0960129517000032 Published online by Cambridge University Press

Proof. If Λ is finite, then it has a maximum γ, which means the single element net x_γ is a $\mathbf{d}$-Cauchy subnet. Otherwise, consider the finite subsets $[\Lambda]^{<\omega}$ of Λ directed by $\subsetneq$, i.e. with $|F|$ denoting F's cardinality,

$$[\Lambda]^{<\omega} \;=\; \{F \subseteq \Lambda : |F| < \omega\}.$$

Define a map $f : [\Lambda]^{<\omega} \to \Lambda$ recursively as follows. Let $f(\{\lambda\}) = \lambda$, for all $\lambda \in \Lambda$. Given $F \in [\Lambda]^{<\omega}$, take $f(F) \in \Lambda$ such that, for all $E \subsetneq F$, $f(E) \prec f(F)$ and

$$\mathbf{d}(x_{f(E)}, x_{f(F)}) \leqslant \limsup_{\lambda} \mathbf{d}(x_{f(E)}, x_\lambda) + 2^{-|F|}.$$

As (x_λ) is $\mathbf{d}$-pre-Cauchy and $\lambda \prec f(F)$, for all $\lambda \in F$,

$$\lim_{E \in [\Lambda]^{<\omega}} \limsup_{\lambda} \mathbf{d}(x_{f(E)}, x_\lambda) + 2^{-|E|} = 0.$$

Thus, $(x_{f(F)})$ is a $\mathbf{d}$-Cauchy subnet of (x_λ). □

When $\mathbf{d}$ is a metric, there is usually only one type of net of interest as

$$\begin{array}{rcl} \mathbf{d}\text{-Cauchy} & \Leftrightarrow & \mathbf{d}\text{-pre-Cauchy}. \\ \mathbf{d}\text{-reflexive} & \Leftrightarrow & \text{arbitrary}, \quad \text{if } X \text{ is totally bounded.} \end{array}$$

On the other hand, for any partial order $\preceq \subseteq X \times X$,

$$\preceq\text{-Cauchy} \quad \Leftrightarrow \quad \text{eventually increasing},$$

when we identify $\preceq$ with its characteristic function (as we do from now on)

$$\preceq(x, y) = \begin{cases} 0 & \text{if } x \preceq y \\ \infty & \text{otherwise} \end{cases}$$

(in particular, we identify $\leqslant^{\mathbf{d}}$ with $\infty\mathbf{d}$, the function $(x, y) \mapsto \infty\mathbf{d}(x, y)$ where $\infty 0 = 0$ and $\infty r = \infty$ if $r \neq 0$). In this case, there are simple examples of non-Cauchy pre-Cauchy sequences – see Wagner (1997, Remark 2.11).

3.2. *Topology*

For any $\prec \subseteq X \times X$, we define

$$\begin{array}{rcl} x\prec & = & \{y \in X : x \prec y\}, \\ \prec x & = & \{y \in X : y \prec x\}. \end{array}$$

Define the open upper/lower balls/holes with centre $c \in X$ and radius ϵ by

$$\begin{array}{rcccl} c^\bullet_\epsilon & = & c <^{\mathbf{d}}_\epsilon & = & \{x \in X : \mathbf{d}(c, x) < \epsilon\}, \\ c^\epsilon_\bullet & = & <^{\mathbf{d}}_\epsilon c & = & \{x \in X : \mathbf{d}(x, c) < \epsilon\}, \\ c^\circ_\epsilon & = & >^{\mathbf{d}}_\epsilon c & = & \{x \in X : \mathbf{d}(x, c) > \epsilon\}, \\ c^\epsilon_\circ & = & c >^{\mathbf{d}}_\epsilon & = & \{x \in X : \mathbf{d}(c, x) > \epsilon\}. \end{array}$$

Let $X^\bullet$, $X_\bullet$, X°, $X_\circ$, $X^\bullet_\bullet$, $X^\bullet_\circ$, $X^\circ_\bullet$ and $X^\circ_\circ$ denote the topologies generated by the corresponding balls and holes, i.e. by arbitrary unions of finite intersections. Denote

https://doi.org/10.1017/S0960129517000032 Published online by Cambridge University Press

convergence by $\xrightarrow{\bullet}$, $\xrightarrow[\bullet]{}$, $\xrightarrow{\circ}$, $\xrightarrow[\circ]{}$, etc. so, for any net $(x_\lambda) \subseteq X$,

$$\begin{aligned}
x_\lambda \xrightarrow{\bullet} x \quad &\Leftrightarrow \quad \forall c \in X \;\; \limsup \mathbf{d}(c, x_\lambda) \leqslant \mathbf{d}(c, x),\\
x_\lambda \xrightarrow[\bullet]{} x \quad &\Leftrightarrow \quad \forall c \in X \;\; \limsup \mathbf{d}(x_\lambda, c) \leqslant \mathbf{d}(x, c),\\
x_\lambda \xrightarrow{\circ} x \quad &\Leftrightarrow \quad \forall c \in X \;\; \liminf \;\mathbf{d}(x_\lambda, c) \geqslant \mathbf{d}(x, c),\\
x_\lambda \xrightarrow[\circ]{} x \quad &\Leftrightarrow \quad \forall c \in X \;\; \liminf \;\mathbf{d}(c, x_\lambda) \geqslant \mathbf{d}(c, x).
\end{aligned}$$

Most of the literature on non-symmetric distances has focussed on ball topologies (one of the few places hole topologies are mentioned is Goubault-Larrecq (2013, Exercise 6.2.11)). However, it is really the hole topologies that are more intimately connected to the $\leqslant^{\mathbf{d}}$ order structure. The double hole topology also defines our central concept.

Definition 3.2. X is **d**-complete if every **d**-Cauchy net has a $X_\circ^\circ$-limit.

This was called lim inf-completeness in Wagner (1997) and just completeness in Bonsangue et al. (1998), although the original formulations differ somewhat from Definition 3.2 – see the comments after Corollary 3.4. These days it is usually called Yoneda completeness to distinguish it from other similar notions (e.g. Smyth completeness where $X_\bullet^\bullet$ is considered instead of $X_\circ^\circ$ – see Smyth (1988)) but these will not be discussed here.

Let us point out that, while **d**-Cauchy nets depend only on $\Phi^{\mathbf{d}}$, the double hole topology $X_\circ^\circ$ depends crucially on **d**, i.e. **d**-completeness is not a 'uniform property.' Below we will use uniform concepts where possible, but the inherent non-uniform nature of **d**-completeness means there is a limit to how much this can be done.

3.3. *Distances*

For $x \in X$, define $x\mathbf{d}, \mathbf{d}x : X \to [0, \infty]$ by

$$\begin{aligned}
x\mathbf{d}(y) &= \mathbf{d}(x, y),\\
\mathbf{d}x(y) &= \mathbf{d}(y, x).
\end{aligned}$$

The composition of **d** and **e** is defined by

$$\mathbf{d} \circ \mathbf{e}(x, y) = \inf_{z \in X} (x\mathbf{d} + \mathbf{e}y)(z).$$

We call **d** a *distance* if

$$\mathbf{d} \leqslant \mathbf{d} \circ \mathbf{d}. \qquad (\triangle)$$

This implies $\leqslant^{\mathbf{d}} \circ \leqslant^{\mathbf{d}} \subseteq \leqslant^{\mathbf{d}}$, i.e. $\leqslant^{\mathbf{d}}$ is transitive. As in Goubault-Larrecq (2013, Definition 6.1.1), we call **d** a *hemimetric* if $\leqslant^{\mathbf{d}}$ is also reflexive, i.e. a preorder.

Non-hemimetric distances have rarely been considered until now. Requiring $\leqslant^{\mathbf{d}}$ to be reflexive may seem harmless, but there are indeed natural distances for which this fails, e.g. $\mathbf{d}(x, y) = x(1 - y)$ on $[0, 1]$ or its extension to the positive unit ball of an arbitrary C*-algebra. There are also natural constructions which preserve $(\triangle)$ but not $\leqslant^{\mathbf{d}}$-reflexivity. For example, just as one extends **d** to a distance on subsets of X in the Hausdorf–Hoare construction – see Goubault-Larrecq (2013, Lemma 7.5.1) – one can extend **d** to a distance on nets in X by

$$\mathbf{d}((x_\lambda),(y_\gamma)) = \limsup_\lambda \liminf_\gamma \mathbf{d}(x_\lambda, y_\gamma). \tag{4}$$

However, even if $\leqslant^{\mathbf{d}}$ is reflexive on X, $\leqslant^{\mathbf{d}}$ may not be reflexive on all nets. Indeed

$$(x_\lambda) \text{ is } \mathbf{d}\text{-reflexive} \quad \Leftrightarrow \quad (x_\lambda) \leqslant^{\mathbf{d}} (x_\lambda).$$

Moreover, the extra generality comes at little cost. So let us now on assume that

d and e are arbitrary distances on X.

Now hole limits of **d**-reflexive $(x_\lambda) \subseteq X$ can be characterized as follows:

$$x_\lambda \underset{\circ}{\to} x \quad \Leftrightarrow \quad \mathbf{d}(x_\lambda, x) \to 0, \tag{5}$$

$$x_\lambda \overset{\circ}{\underset{\circ}{\to}} x \quad \Leftrightarrow \quad x_\lambda \overset{\circ}{\underset{\bullet}{\to}} x \leqslant^{\mathbf{d}} x. \tag{6}$$

Proof.

(5) If $x_\lambda \underset{\circ}{\to} x$, then $\lim_\gamma \mathbf{d}(x_\gamma, x) \leqslant \lim_\gamma \liminf_\lambda \mathbf{d}(x_\gamma, x_\lambda) = 0$. If $\mathbf{d}(x_\lambda, x) \to 0$, then $\mathbf{d}(c, x) \leqslant \liminf \mathbf{d}(c, x_\lambda) + \mathbf{d}(x_\lambda, x) = \liminf \mathbf{d}(c, x_\lambda)$, for any $c \in X$.

(6) If $\mathbf{d}(x_\lambda, x) \to 0$, then $\mathbf{d}(x_\lambda, c) \leqslant \mathbf{d}(x_\lambda, x) + \mathbf{d}(x, c) \to \mathbf{d}(x, c)$ so $x_\lambda \underset{\bullet}{\to} x$. If $x_\lambda \overset{\circ}{\to} x$ too, then $\mathbf{d}(x, x) \leqslant \liminf \mathbf{d}(x_\lambda, x) = 0$, i.e. $x \leqslant^{\mathbf{d}} x$. Conversely, if $x_\lambda \underset{\bullet}{\to} x \leqslant^{\mathbf{d}} x$, then $\limsup \mathbf{d}(x_\lambda, x) \leqslant \mathbf{d}(x, x) = 0$, i.e. $\mathbf{d}(x_\lambda, x) \to 0$. □

For an example of **d**-reflexive $x_\lambda \overset{\circ}{\underset{\bullet}{\to}} x \not\leqslant^{\mathbf{d}} x$, take any $x_\lambda \to 0 < x$ in $[0, \infty)$ where, for the distance **d**, we simply consider the coordinate projection $\mathbf{d}(y, z) = z$.

In Goubault-Larrecq (2013, Definition 7.1.15), any x with $\mathbf{d}(x, y) = \limsup \mathbf{d}(x_\lambda, y)$, for all $y \in X$, is called a **d**-*limit* of (x_λ) (these are called *forward limits* in Bonsangue et al. (1998) before Proposition 3.3 and just *limits* in Künzi and Schellekens (2002, Definition 11)). In general, **d**-limits are not true limits in any topological sense, as they are not preserved by taking subnets. But for **d**-pre-Cauchy nets, **d**-limits are $X^\circ_\bullet$-limits, i.e. the limit superior will be a limit, as shown below and in Wagner (1997, Theorem 2.26).

Proposition 3.3. For **d**-pre-Cauchy (x_λ) and $y \in X$, $\mathbf{d}(x_\lambda, y)$ and $\mathbf{d}(y, x_\lambda)$ converge.

Proof. As (x_λ) is **d**-pre-Cauchy,

$$\limsup_\lambda \mathbf{d}(x_\lambda, y) \leqslant \limsup_\lambda \liminf_\gamma \mathbf{d}(x_\lambda, x_\gamma) + \mathbf{d}(x_\gamma, y) = \liminf_\gamma \mathbf{d}(x_\gamma, y).$$

$$\liminf_\lambda \mathbf{d}(y, x_\lambda) \geqslant \liminf_\lambda \limsup_\gamma \mathbf{d}(y, x_\gamma) - \mathbf{d}(x_\lambda, x_\gamma) = \limsup_\gamma \mathbf{d}(y, x_\gamma).$$

□

Corollary 3.4. Any **d**-pre-Cauchy net converges in $X^\bullet$, X°, $X_\bullet$ or $X_\circ$ iff it has a subnet that converges in the same topology.

For distance **d**, we could thus replace **d**-Cauchy nets with **d**-pre-Cauchy nets in Definition 3.2, by Proposition 3.1 and Corollary 3.4. And for hemimetric **d**, Definition 3.2 agrees with the **d**-limit definition of Yoneda completeness in Goubault-Larrecq (2013, Definition 7.4.1). We prefer $X^\circ_\circ$-limits to $X^\circ_\bullet$-limits/**d**-limits for the following reasons:

https://doi.org/10.1017/S0960129517000032 Published online by Cambridge University Press

1. $X^\circ_\circ$ seems more natural for general distances (e.g. $\mathbf{d}(x, y) = y$ noted above).
2. $X^\circ_\circ$ is self-dual, making it clear that the asymmetry in **d**-completeness comes from the nets being considered rather than the topology.
3. $X^\circ_\circ$ already arises naturally in various situations (although this does not appear to be widely recognized), e.g. as the usual product topology for products of bounded intervals, as the Wijsman topology for subsets of X, and as the weak operator topology for projections on a Hilbert space.

If **d** is a metric, then limits of **d**-Cauchy nets are the same in $X^\circ_\circ$ and $X^\bullet = X_\bullet$. Thus, **d**-completeness generalizes the usual notion of metric completeness. If we consider (the characteristic function of) a partial order $\preceq\, \subseteq X \times X$, then $X^\circ_\circ$-limits of increasing nets are precisely their supremums, so **d**-completeness also generalizes directed completeness. Our main thesis is that, even in more general distances spaces, **d**-completeness is a combination of metric and directed completeness. To make this precise, we need to extend the usual order theoretic notion of supremum.

3.4. *Supremums*

For $Y \subseteq X$, define

$$\begin{aligned} Y\mathbf{d} &= \sup y\mathbf{d}. \\ \mathbf{d}Y &= \inf \mathbf{d}y. \\ Y \leqslant^{\mathbf{d}} x &\Leftrightarrow Y \subseteq (\leqslant^{\mathbf{d}} x). \\ x \leqslant^{\mathbf{d}} Y &\Leftrightarrow Y \subseteq (x \leqslant^{\mathbf{d}}). \end{aligned}$$

We define **d**-*supremums* of $Y \subseteq X$ by

$$x = \mathbf{d}\text{-}\sup Y \quad \Leftrightarrow \quad x\mathbf{d} = Y\mathbf{d} \quad \text{and} \quad Y \leqslant^{\mathbf{d}} x.$$

Note $=$ is a slight abuse of notation, as **d**-supremums are only unique up to the equivalence relation $x \leqslant^{\mathbf{d}} y \leqslant^{\mathbf{d}} x$. Also, we could replace $x\mathbf{d} = Y\mathbf{d}$ with $x\mathbf{d} \leqslant Y\mathbf{d}$, as $Y \leqslant^{\mathbf{d}} x \Rightarrow Y\mathbf{d} \leqslant x\mathbf{d}$. Alternatively, we could replace $Y \leqslant^{\mathbf{d}} x$ with $x \leqslant^{\mathbf{d}} x$ as $x\mathbf{d} = Y\mathbf{d}$ implies $x \leqslant^{\mathbf{d}} x \Leftrightarrow x\mathbf{d}(x) = 0 \Leftrightarrow Y\mathbf{d}(x) = 0 \Leftrightarrow Y \leqslant^{\mathbf{d}} x$.

Note **d**-supremums are $\leqslant^{\mathbf{d}}$-supremums, as $x\mathbf{d} = Y\mathbf{d}$ implies $\infty x\mathbf{d} = \infty Y\mathbf{d}$. However, unless we place some extra condition on **d**, the converse can fail, e.g. if $\mathbf{d}(r, s) = (r - s)_+$ (where $r_+ = r \vee 0$) on $X = [0, 1) \cup \{2\}$, then we see that $2 = \leqslant^{\mathbf{d}}\text{-}\sup[0, 1) \neq \mathbf{d}\text{-}\sup[0, 1)$, as $\sup_{x \in [0,1)} \mathbf{d}(x, 0) = 1 \neq 2 = \mathbf{d}(2, 0)$. Indeed, in general **d**-supremums depend crucially on **d**, not just $\leqslant^{\mathbf{d}}$ or even $\Phi^{\mathbf{d}}$.

One such condition would be ‘every closed lower ball has a maximum.’ For example, this holds for $C(X, \mathbb{R})$, the space of real-valued continuous functions on some compact X with $\mathbf{d}(f, g) = \sup_{x \in X}(f(x) - g(x))_+$. More generally, this holds for the self-adjoint elements of any unital C*-algebra with $\mathbf{d}(a, b) = ||(a - b)_+||$. In fact, a weaker condition suffices which we can describe with the following functions on $[0,\infty]$:

$$\mathbf{d}^\bullet(r) = \sup_{x\in X}\inf_{y\leqslant^{\mathbf{d}} x_r^\bullet} \mathbf{d}(x,y),$$
$$\mathbf{d}_\bullet(r) = \sup_{x\in X}\inf_{x_\bullet^r\leqslant^{\mathbf{d}} y} \mathbf{d}(y,x).$$

Also, let $\mathbf{I}$ denote the identity on $[0,\infty]$, so

$$\mathbf{d}_\bullet \leqslant \mathbf{I} \quad\Leftrightarrow\quad \sup_{y\in Y}\mathbf{d}(y,x) = \inf_{Y\leqslant^{\mathbf{d}} y}\mathbf{d}(y,x) \text{ whenever } x\in X\supseteq Y.$$

Note closed lower balls have maximums precisely when the infimum on the right-hand side above is actually attained, i.e. iff this infimum is a minimum.

Proposition 3.5. If $\mathbf{d}_\bullet \leqslant \mathbf{I}$, then $\leqslant^{\mathbf{d}}$-supremums are $\mathbf{d}$-supremums.

Proof. Assume $Y\subseteq X$ and $z = \leqslant^{\mathbf{d}}\text{-}\sup Y \neq \mathbf{d}\text{-}\sup Y$ so $\sup_{y\in Y}\mathbf{d}(y,x) < \mathbf{d}(z,x)$, for some $x\in X$. As $\mathbf{d}_\bullet\leqslant\mathbf{I}$, we have $w\in X$ with $Y\leqslant^{\mathbf{d}} w$ and $\mathbf{d}(w,x)<\mathbf{d}(z,x)$. But then $z = \leqslant^{\mathbf{d}}\text{-}\sup Y \leqslant^{\mathbf{d}} w$ so $\mathbf{d}(z,x)\leqslant\mathbf{d}(w,x)$, a contradiction. □

We also need to generalize directedness. Specifically, for $Y\subseteq X$, we define

$$Y \text{ is } \mathbf{d}\textit{-directed} \quad\Leftrightarrow\quad \forall F\in[Y]^{<\omega}\ \inf_{y\in Y} F\mathbf{d}(y) = 0.$$

By ($\triangle$), $[Y]^{<3}$ suffices. Also, define $Y\leqslant^{\mathbf{d}}(x_\lambda) \Leftrightarrow \mathbf{d}(y,x_\lambda)\to 0$, for all $y\in Y$, so

$$Y\leqslant^{\mathbf{d}}(x_\lambda)\subseteq Y \quad\Rightarrow\quad (x_\lambda) \text{ is } \mathbf{d}\text{-pre-Cauchy}.$$
$$\exists(x_\lambda)\ Y\leqslant^{\mathbf{d}}(x_\lambda)\subseteq Y \quad\Leftrightarrow\quad Y \text{ is } \mathbf{d}\text{-directed}.$$

Indeed, if Y is $\mathbf{d}$-directed then, for $F\in[Y]^{<\omega}$ and $\epsilon>0$, take $y_{F,\epsilon}\in Y$ with $F\mathbf{d}(y_{F,\epsilon})<\epsilon$, so $Y\leqslant^{\mathbf{d}}(y_{F,\epsilon})\subseteq Y$, ordering $[Y]^{<\omega}\times(0,\infty)$ by $\subseteq\times\geqslant$.

Definition 3.6. X is $\mathbf{e}$-$\mathbf{d}$-*complete* if every $\mathbf{e}$-directed $Y\subseteq X$ has a $\mathbf{d}$-supremum.

If $\preceq\ \subseteq X\times X$ is a partial order, $\preceq$-$\preceq$-completeness is directed completeness. Thus, both $\mathbf{d}$-$\mathbf{d}$-completeness and $\leqslant^{\mathbf{d}}$-$\mathbf{d}$-completeness are valid generalizations. But if $\mathbf{d}$ is a metric, then every $\mathbf{d}$-directed subset contains at most one element, which makes X trivially $\mathbf{d}$-$\mathbf{d}$-complete. So, unlike $\mathbf{d}$-completeness, $\mathbf{d}$-$\mathbf{d}$-completeness does not generalize metric completeness. In general, $\mathbf{d}$-completeness is a stronger notion, as we now show.

Proposition 3.7. If $Y\leqslant^{\mathbf{d}}(x_\lambda)\subseteq Y$ and $x\in X$, then

$$x_\lambda \underset{\circ}{\to} x \quad\Leftrightarrow\quad Y\leqslant^{\mathbf{d}} x, \tag{7}$$
$$x_\lambda \overset{\circ}{\to} x \quad\Leftrightarrow\quad x\mathbf{d}\leqslant Y\mathbf{d}, \tag{8}$$
$$x_\lambda \overset{\circ}{\underset{\circ}{\to}} x \quad\Leftrightarrow\quad x = \mathbf{d}\text{-}\sup Y. \tag{9}$$

Proof.

Equation (7): If $Y\leqslant^{\mathbf{d}} x$, then $\mathbf{d}(x_\lambda,x)=0$, for all λ, so $x_\lambda\underset{\circ}{\to} x$, by Equation (5). While if $x_\lambda\underset{\circ}{\to} x$ and $y\in Y$, then $\mathbf{d}(y,x)\leqslant\liminf\mathbf{d}(y,x_\lambda)=0$, as $Y\leqslant^{\mathbf{d}}(x_\lambda)$, i.e. $Y\leqslant^{\mathbf{d}} x$.

https://doi.org/10.1017/S0960129517000032 Published online by Cambridge University Press

Equation (8): If $x\mathbf{d} \leqslant Y\mathbf{d}$ then, as $Y \leqslant^{\mathbf{d}} (x_\lambda)$, for any $z \in X$, we have

$$\mathbf{d}(x,z) \leqslant \sup_{y\in Y} \mathbf{d}(y,z) \leqslant \sup_{y\in Y} \liminf(\mathbf{d}(y,x_\lambda) + \mathbf{d}(x_\lambda,z)) = \liminf \mathbf{d}(x_\lambda,z).$$

While if $x_\lambda \overset{\circ}{\to} x$, then $x\mathbf{d}(z) = \mathbf{d}(x,z) \leqslant \liminf \mathbf{d}(x_\lambda,z) \leqslant Y\mathbf{d}(z)$, for all $z \in X$.
Equation (9): See equations (7) and (8). □

Corollary 3.8. If X is $\mathbf{d}$-complete, then X is $\mathbf{d}$-$\mathbf{d}$-complete.

Proof. For any $\mathbf{d}$-directed $Y \subseteq X$, take (x_λ) with $Y \leqslant^{\mathbf{d}} (x_\lambda) \subseteq Y$. By Proposition 3.1, we may revert to a Cauchy subnet (which still satisfies $Y \leqslant^{\mathbf{d}} (x_\lambda)$). As X is $\mathbf{d}$-complete, $x_\lambda \overset{\circ}{\underset{\circ}{\to}} x$, for some $x \in X$. By Equation (9), x is a $\mathbf{d}$-supremum of Y. □

For $\mathbf{d}$-pre-Cauchy $(x_\lambda) \subseteq X$, it will also be convenient to define

$$(x_\lambda)\mathbf{d} = \lim x_\lambda\mathbf{d},$$
$$\mathbf{d}(x_\lambda) = \lim \mathbf{d}x_\lambda.$$

It then follows immediately from the definitions that

$$x_\lambda \underset{\circ}{\to} x \quad\Leftrightarrow\quad \mathbf{d}x \leqslant \mathbf{d}(x_\lambda),$$
$$x_\lambda \overset{\circ}{\to} x \quad\Leftrightarrow\quad x\mathbf{d} \leqslant (x_\lambda)\mathbf{d},$$
$$x_\lambda \overset{\circ}{\underset{\bullet}{\to}} x \quad\Leftrightarrow\quad x\mathbf{d} = (x_\lambda)\mathbf{d}.$$

Proposition 3.9. If Y is $\mathbf{d}$-directed and $(x_\lambda) \subseteq X$ is $\mathbf{d}$-pre-Cauchy, then

$$Y \leqslant^{\mathbf{d}} (x_\lambda) \quad\Leftrightarrow\quad \mathbf{d}(x_\lambda) \leqslant \mathbf{d}Y.$$

Proof. If $y \in Y \leqslant^{\mathbf{d}} (x_\lambda)$ and $x \in X$, then $\mathbf{d}(x,x_\lambda) \leqslant \mathbf{d}(x,y) + \mathbf{d}(y,x_\lambda) \to \mathbf{d}(x,y)$, i.e. $\mathbf{d}(x_\lambda) \leqslant \mathbf{d}y$, for all $y \in Y$, so $\mathbf{d}(x_\lambda) \leqslant \mathbf{d}Y$. While if $\mathbf{d}(x_\lambda) \leqslant \mathbf{d}Y$ and $y \in Y$, then $\lim \mathbf{d}(y,x_\lambda) \leqslant \mathbf{d}Y(y) = 0$, as Y is $\mathbf{d}$-directed, i.e. $Y \leqslant^{\mathbf{d}} (x_\lambda)$. □

4. Cauchy nets

In this section, we make the following standing assumption:

$$(x_\lambda) \subseteq X \textbf{ is d-Cauchy}.$$

For our first result, we could assume 'every closed upper ball has a minimum.' As in Proposition 3.5, we can weaken this to $\mathbf{d}^\bullet \leqslant \mathbf{I}$, but here even $\mathbf{d}^\bullet \lessapprox \mathbf{I}$ suffices, where $\lessapprox$ is 'uniform subequivalence.' Specifically, for $f, g : X \to [0,\infty]$, define

$$\sup_{g(x)\leqslant r} f(x) = {}^{f}/_{g}(r),$$
$$f \lessapprox g \;\Leftrightarrow\; {}^{f}/_{g}(r) \to 0.$$

So $\mathbf{d}^\bullet \lessapprox \mathbf{I} \;\Leftrightarrow\; \lim_{r\to 0} \mathbf{d}^\bullet(r) = 0 \;\Leftrightarrow\; \forall \prec \in \Phi^{\mathbf{d}}\ \exists \sqsubset \in \Phi^{\mathbf{d}}\ \forall x \in X\ \exists y \leqslant^{\mathbf{d}}(x \sqsubset)\ x \prec y.$

https://doi.org/10.1017/S0960129517000032 Published online by Cambridge University Press

Theorem 4.1. If $\mathbf{d}^\bullet \lessapprox \mathbf{I}$, then we have $\leqslant^{\mathbf{d}}$-directed $Y \subseteq X$ with

$$Y\mathbf{d} = (x_\lambda)\mathbf{d} \qquad \text{and} \qquad \mathbf{d}Y = \mathbf{d}(x_\lambda).$$

Proof. As $\mathbf{d}^\bullet \lessapprox \mathbf{I}$, i.e. $\lim_{r\to 0}\mathbf{d}^\bullet(r) = 0$, we can define $r_n \downarrow 0$ with $\mathbf{d}^\bullet(2r_{n+1}) < r_n$.

As (x_λ) is $\mathbf{d}$-Cauchy, we can define $f : [\Lambda]^{<\omega} \to \Lambda$ as follows. Let $f(\{\lambda\}) = \lambda$ and, given $F \in [\Lambda]^{<\omega}$ with $|F| > 1$, take $f(F) \succ f(E)$, for all $E \subsetneq F$, such that

$$\sup_{f(F)<\lambda} \mathbf{d}(x_{f(F)}, x_\lambda) < r_{|F|}.$$

As $\mathbf{d}^\bullet(2r_{|F|}) < r_{|F|-1}$, we can take $y_F \leqslant^{\mathbf{d}} (x_{f(F)})^\bullet_{2r_{|F|}}$ with $\mathbf{d}(x_{f(F)}, y_F) < r_{|F|-1}$. If $F \subsetneq G$, then $\mathbf{d}(x_{f(F)}, y_G) \leqslant \mathbf{d}(x_{f(F)}, x_{f(G)}) + \mathbf{d}(x_{f(G)}, y_G) < 2r_{|F|}$ and hence $y_G \in (x_{f(F)})^\bullet_{2r_{|F|}}$ so $y_F \leqslant^{\mathbf{d}} y_G$. Thus, $Y = \{y_F : F \in [\Lambda]^{<\omega}\}$ is $\leqslant^{\mathbf{d}}$-directed. For $\lambda \succ f(F)$, $x_\lambda \in (x_{f(F)})^\bullet_{r_{|F|}} \subseteq (x_{f(F)})^\bullet_{2r_{|F|}}$ so $y_F \leqslant^{\mathbf{d}} x_\lambda$. Thus, $Y \leqslant^{\mathbf{d}} (x_\lambda)$ so

$$Y\mathbf{d} \leqslant (x_\lambda)\mathbf{d} \qquad \text{and} \qquad \mathbf{d}Y \geqslant \mathbf{d}(x_\lambda).$$

Also, $\mathbf{d}(x_{f(F)}, y_F) \leqslant r_{|F|-1} \to 0$ so

$$Y\mathbf{d} \geqslant (x_\lambda)\mathbf{d} \qquad \text{and} \qquad \mathbf{d}Y \leqslant \mathbf{d}(x_\lambda).$$

□

Thus, $\mathbf{d}$-completeness follows from $\leqslant^{\mathbf{d}}$-$\mathbf{d}$-completeness when $\mathbf{d}^\bullet \lessapprox \mathbf{I}$. As noted after Definition 3.6, consideration of metric $\mathbf{d}$ shows we can not drop the condition $\mathbf{d}^\bullet \lessapprox \mathbf{I}$. But it does suggest we might replace $\mathbf{d}^\bullet \lessapprox \mathbf{I}$ with metric completeness. More precisely, letting $\mathbf{d}^{op}(x, y) = (y, x)$ and $\mathbf{d}^\vee = \mathbf{d} \vee \mathbf{d}^{op}$, we might ask if

$$\mathbf{d}\text{-complete} \quad \Leftrightarrow \quad \leqslant^{\mathbf{d}}\text{-}\mathbf{d}\text{-complete} \quad \text{and} \quad \mathbf{d}^\vee\text{-complete?} \tag{10}$$

In general, the answer is no, as the following simple example shows.

Consider the sequence (f_m) in $[0,\infty]^{\mathbb{N}}$ defined by

$$f_m(n) = \begin{cases} \infty & \text{if } n < m, \\ 0 & \text{if } n = m, \\ 1/n & \text{if } n > m. \end{cases}$$

Set $X = \{f_m : m \in \mathbb{N}\}$ and $\mathbf{d}(f, g) = \sup(f(n) - g(n))_+$. Then, $\leqslant^{\mathbf{d}}$ and $\mathbf{d}^\vee$ become identified with $=$ so X is trivially $\leqslant^{\mathbf{d}}$-$\mathbf{d}$-complete and $\mathbf{d}^\vee$-complete, even though (f_m) is $\mathbf{d}$-Cauchy with no $X^\circ_\circ$-limit in X ($f_m \overset{\circ}{\underset{\circ}{\to}} f_\infty$ in $[0,\infty]^{\mathbb{N}}$ but $f_\infty \notin X$).

Thus, if we are to have any hope of proving Equation (10), we need some extra condition. We could use $\mathbf{d}_\bullet \leqslant \mathbf{I}$ as in Proposition 3.5 or the significantly weaker assumption 'every open lower ball is directed.' Again, we can even describe slightly weaker conditions that suffice if we consider the following functions on $[0,\infty]$:

$$\mathbf{d}_{\mathbf{F}}(r) = \sup_{x\in X} \sup_{F\in[x^r_\bullet]^{<\omega}} \inf_{F\leqslant^{\mathbf{d}} y} \mathbf{d}(y, x),$$

$$\mathbf{d}_\Phi(r) = \sup_{x\in X} \sup_{F\in[x^r_\bullet]^{<\omega}} \sup_{\preceq\in\Phi^{\mathbf{d}}} \inf_{F\preceq y} \mathbf{d}(y, x).$$

https://doi.org/10.1017/S0960129517000032 Published online by Cambridge University Press

$$\begin{aligned} \text{So} \quad \mathbf{d_F} \leqslant \mathbf{I} \quad &\Leftrightarrow \quad \mathbf{d_F}[0,r) \subseteq [0,r), \text{ for all } r \in (0,\infty). \\ &\Leftrightarrow \quad x^r_\bullet \text{ is } \leqslant^{\mathbf{d}}\text{-directed, for all } x \in X \text{ and } r \in (0,\infty). \\ &\Leftrightarrow \quad \sup_{y \in F} \mathbf{d}(y,x) = \inf_{F \leqslant^{\mathbf{d}} y} \mathbf{d}(y,x), \text{ for all } x \in X \text{ and finite } F \subseteq X. \end{aligned} \tag{11}$$

In general, $\mathbf{d}_\Phi \leqslant \mathbf{d_F} \leqslant \mathbf{d}_\bullet$, but $\mathbf{d_F}$ can be much smaller than $\mathbf{d}_\bullet$. For example, consider the continuous functions $C_0(X,\mathbb{R})$ vanishing at ∞ on some locally compact X (i.e. such that $(|f|-\epsilon)_+$ has compact support, for all $\epsilon > 0$) again with $\mathbf{d}(f,g) = \sup_{x\in X}(f(x)-g(x))_+$. As $C_0(X,\mathbb{R})$ is a lattice, closed lower balls are directed and hence $\mathbf{d_F} \leqslant \mathbf{I}$. However, if X itself is not compact, then $\mathbf{d}_\bullet(r) = \infty$, for all $r > 0$. More generally, open lower balls in the self-adjoint part of any C*-algebra are directed (even though there is no lattice structure in the non-commutative case), as can be seen from the construction of the canonical approximate unit – see Pedersen (1979, Theorem 1.4.2).

However, $\mathbf{d_F}$ and $\mathbf{d}_\Phi$ often coincide.

Proposition 4.2. If $\mathbf{d}$ is a hemimetric, X is $\mathbf{d}^\vee$-complete and $\mathbf{d}_\Phi \lessapprox \mathbf{I}$, then $\mathbf{d_F} = \mathbf{d}_\Phi$.

Proof. For any $r \in [0,\infty]$, $x \in X$, finite $F \subseteq x^r_\bullet$ and $\epsilon > 0$, we take (ϵ_n) with $0 < \epsilon_n < 2^{-n}\epsilon$ and $\mathbf{d}_\Phi(\epsilon_n) < 2^{-n}\epsilon$, for all $n \in \mathbb{N}$. Now take $x_1 \in X$ with $\mathbf{d}(x_1,x) < \mathbf{d}_\Phi(r) + \epsilon$ and $\sup_{z\in F\cup\{x\}} \mathbf{d}(z,x_1) < \epsilon_1$. We can then take $x_2 \in X$ with $\mathbf{d}(x_2,x_1) < \frac{1}{2}\epsilon$ and $\sup_{z\in F\cup\{x,x_1\}} \mathbf{d}(z,x_2) < \epsilon_2$, as $\mathbf{d}_\Phi(\epsilon_1) < \frac{1}{2}\epsilon$. Continuing in this way, we obtain (x_n) with $\mathbf{d}^\vee(x_{n+1},x_n) < 2^{-n}\epsilon$, for all $n \in \mathbb{N}$. As X is $\mathbf{d}^\vee$-complete, we have $y \in X$ with $\mathbf{d}^\vee(x_n,y) \to 0$ so $F \leqslant^{\mathbf{d}} y$ and $\mathbf{d}(y,x) < \mathbf{d}_\Phi(r) + 2\epsilon$. As $\epsilon > 0$ was arbitrary, $\mathbf{d_F} \leqslant \mathbf{d}_\Phi$. □

Theorem 4.3. If X is $\leqslant^{\mathbf{d}}$-$\mathbf{d}$-complete and $\mathbf{d_F} \leqslant \mathbf{I}$, then we have $\mathbf{d}^\vee$-Cauchy (y_n) with

$$(x_\lambda)\mathbf{d} = (y_n)\mathbf{d} \qquad \text{and} \qquad \lim_{\lambda,n} \mathbf{d}(x_\lambda, y_n) = 0.$$

Proof. Instead of $\mathbf{d_F} \leqslant \mathbf{I}$, we can work with a slightly even weaker condition

$$0 \in \overline{\{r \in (0,\infty) : \mathbf{d_F}[0,r) \subseteq [0,r)\}}, \tag{12}$$

which means we have $r_n \downarrow 0$ with $\mathbf{d_F}[0,r_n) \subseteq [0,r_n)$, for all $n \in \mathbb{N}$. Then, we have (r^m_n) with $\mathbf{d_F}(r^m_n) < r^{m+1}_n < r_n$, for all $m \in \mathbb{N}$ (taking $\mathbf{d_F}(r^0_n) = 0$). Set

$$\epsilon^m_n = \tfrac{1}{2}(r^m_n - \mathbf{d_F}(r^{m-1}_n)).$$

Again define a map $f : [\Lambda]^{<\omega} \to \Lambda$ such that, for all $\lambda \in \Lambda$, $f(\{\lambda\}) = \lambda$, for all $F \in [\Lambda]^{<\omega}$ with $|F| > 1$ and all $E \subsetneqq F$, $f(E) \prec f(F)$ and

$$\sup_{f(F)\prec\lambda} \mathbf{d}(x_{f(F)}, x_\lambda) < \min_{1\leqslant n<|F|} \epsilon^{|F|-n}_n.$$

For any $n \in \mathbb{N}$, let $\Lambda_n = \{F \in [\Lambda]^{<\omega} : |F| > n\}$ and define $(y^n_F)_{F\in\Lambda_n}$ recursively as follows. When $|F| = n+1$, let $y^n_F = x_{f(F)}$ so if $F \subsetneqq G$, then

$$\mathbf{d}(y^n_F, x_{f(G)}) < \epsilon^1_n < r^1_n.$$

When $|G| = n+2$, we take y^n_G with $y^n_F \leqslant^{\mathbf{d}} y^n_G$, for all $F \subsetneqq G$ with $|F| = n+1$, and

$$\mathbf{d}(y^n_G, x_{f(G)}) < \mathbf{d_F}(r^1_n) + \epsilon^2_n.$$

As $\mathbf{d}(x_{f(G)}, x_{f(H)}) < \epsilon_n^2$, whenever $G \subsetneq H$ and $|G| = n+2$,

$$\mathbf{d}(y_G^n, x_{f(H)}) \leqslant \mathbf{d}(y_G^n, x_{f(G)}) + \mathbf{d}(x_{f(G)}, x_{f(H)}) < \mathbf{d_F}(r_n^1) + 2\epsilon_n^2 = r_n^2.$$

For $|H| = n+3$, take y_H^n with $y_G^n, x_{f(G)} \leqslant^{\mathbf{d}} y_H^n$, for $G \subsetneq H$ with $|G| = n+2$, and

$$\mathbf{d}(y_H^n, x_{f(H)}) < \mathbf{d_F}(r_n^2) + \epsilon_n^3.$$

Continuing in this way, we obtain increasing (y_F^n) with $\mathbf{d}(y_F^n, x_{f(F)}) < r_n$ and $x_{f(F)} \leqslant^{\mathbf{d}} y_G^n$, for all $F \in \Lambda_{n+1}$ and $F \subsetneq G$. As X is $\leqslant^{\mathbf{d}}$-$\mathbf{d}$-complete, (y_F^n) has $\mathbf{d}$-supremum y^n. For all $m, n \in \mathbb{N}$ and $F \in \Lambda_{\max(m,n)+1}$, we have

$$\mathbf{d}(y_F^m, y^n) \leqslant \mathbf{d}(y_F^m, y_F^n) \leqslant \mathbf{d}(y_F^m, x_{f(F)}) < r_m$$

and hence $\mathbf{d}(y^m, y^n) \leqslant r_m$, so (y^n) is $\mathbf{d}^\vee$-Cauchy. For any $\epsilon > 0$, we have $r_n < \epsilon$ for all sufficiently large $n \in \mathbb{N}$. Then, for any $z \in X$ and all sufficiently large $F \in [\Lambda]^{<\omega}$,

$$\mathbf{d}(y^n, z) \leqslant \mathbf{d}(y_F^n, z) + \epsilon \leqslant \mathbf{d}(x_{f(F)}, z) + r_n + \epsilon < \mathbf{d}(x_{f(F)}, z) + 2\epsilon,$$

so $(y^n)\mathbf{d} \leqslant (x_\lambda)\mathbf{d}$. For all sufficiently large $F \in [\Lambda]^{<\omega}$, $\sup_{f(F)<\lambda} \mathbf{d}(x_{f(F)}, x_\lambda) < \epsilon$ so, as $x_{f(G)} \leqslant^{\mathbf{d}} y^n$ when $F \subsetneq G \in \Lambda_n$, $\mathbf{d}(x_{f(F)}, y^n) < \epsilon$ and $\lim_{\lambda,n} \mathbf{d}(x_\lambda, y^n) = 0$. □

Above we obtained symmetric $\mathbf{d}^\vee$ and transitive $\leqslant^{\mathbf{d}}$ from $\mathbf{d}$. But in practice, it often happens the other way around, i.e. we compose symmetric $\mathbf{e}$ with transitive $\preceq$ to obtain $\mathbf{d} = \mathbf{e} \circ \preceq$ (($\triangle$) is not automatic but follows from e.g. $\mathbf{e} \circ \preceq \, = \, \preceq \circ \, \mathbf{e}$).

Question 4.4. If $\mathbf{d} = \mathbf{e} \circ \leqslant^{\mathbf{d}}$ for a metric $\mathbf{e}$, then does

$$\leqslant^{\mathbf{d}}\text{-}\mathbf{d}\text{-complete} \quad \text{and} \quad \mathbf{e}\text{-complete} \quad \Rightarrow \quad \mathbf{d}\text{-complete?}$$

Unlike with Equation (10), we do not know of a counterexample. Indeed, an answer to Question 4.4 would likely shed some light on an old problem from Akemann and Pedersen (1973) and Brown (1988) for C*-algebra A, namely whether every strongly lower semicontinuous element of A_{sa}^{**} can be obtained from A_{sa} as a monotone limit. However, we can give a positive answer to Question 4.4 if we assume $\mathbf{e}$-separability, i.e. $\mathbf{e}Y = 0$ for some countable $Y \subseteq X$, or consider $\mathbf{d}$-$\mathbf{d}$-completeness instead of $\leqslant^{\mathbf{d}}$-$\mathbf{d}$-completeness.

Again we work with a weaker assumption than $\mathbf{d} = \mathbf{e} \circ \leqslant^{\mathbf{d}}$ which depends only on $\Phi^{\mathbf{d}}$ and $\Phi^{\mathbf{e}}$. Specifically, note $\mathbf{d} \lessapprox \mathbf{e} \Leftrightarrow \Phi^{\mathbf{d}} \subseteq \Phi^{\mathbf{e}}$ and define

$$\mathbf{e} \circ \Phi^{\mathbf{d}} = \sup_{\preceq \in \Phi^{\mathbf{d}}} \mathbf{e} \circ \preceq \, = \sup_{\preceq \in \Phi^{\mathbf{d}}} \inf_{z \preceq y} \mathbf{e}(x, z) = \sup_{\epsilon > 0} \inf_{z <_\epsilon^{\mathbf{d}} y} \mathbf{e}(x, z).$$

Theorem 4.5. If X is $\mathbf{e}$-complete and $\mathbf{e} \circ \Phi^{\mathbf{d}} \lessapprox \mathbf{d} \lessapprox \mathbf{e} = \mathbf{e}^{op}$, then $\mathbf{e} \circ \Phi^{\mathbf{d}} = \mathbf{e} \circ \leqslant^{\mathbf{d}}$,

$$Y\mathbf{d} = (x_\lambda)\mathbf{d} \qquad \text{and} \qquad \mathbf{d}Y = \mathbf{d}(x_\lambda), \tag{13}$$

for $\mathbf{d}$-directed $Y \subseteq X$. If X is $\mathbf{e}$-separable, then we can choose Y to be $\leqslant^{\mathbf{d}}$-directed.

Proof. For $\mathbf{e} \circ \Phi^{\mathbf{d}} = \mathbf{e} \circ \leqslant^{\mathbf{d}}$, we argue as in the proof of Proposition 4.2. Specifically, for any $x, y \in X$ and $\epsilon > 0$, take $\epsilon_n \downarrow 0$ with ${}^{\mathbf{e} \circ \Phi^{\mathbf{d}}}/_{\mathbf{d}}(\epsilon_n) < 2^{-n}\epsilon$, for all $n \in \mathbb{N}$. Now take $z_1 \in X$ with $\mathbf{e}(x, z_1) < \mathbf{e} \circ \Phi^{\mathbf{d}}(x, y) + \epsilon$ and $\mathbf{d}(z_1, y) < \epsilon_1$. Thus, $\mathbf{e} \circ \Phi^{\mathbf{d}}(z_1, y) < \frac{1}{2}\epsilon$ and we can take

https://doi.org/10.1017/S0960129517000032 Published online by Cambridge University Press

$z_2 \in X$ such that $\mathbf{e}(z_1, z_2) < \frac{1}{2}\epsilon$ and $\mathbf{d}(z_2, y) < \epsilon_2$. Continuing in this way, we obtain a sequence $(z_n) \subseteq X$ such that, for all $n \in \mathbb{N}$,

$$\mathbf{e}(z_n, z_{n+1}) \leqslant 2^{-n}\epsilon \qquad \text{and} \qquad \mathbf{d}(z_n, y) < \epsilon_n \to 0.$$

As X is $\mathbf{e}$-complete, $\mathbf{e}(z_n, z) \to 0$, for some $z \in X$, so

$$\mathbf{e}(x, z) \leqslant \mathbf{e}(x, z_1) + \mathbf{e}(z_1, z) \leqslant \mathbf{e} \circ \Phi^{\mathbf{d}}(x, y) + 2\epsilon.$$

As $\mathbf{e} = \mathbf{e}^{op}$, $\mathbf{e}(z, z_n) \to 0$ so, as $\mathbf{d} \lessapprox \mathbf{e}$, $\mathbf{d}(z, z_n) \to 0$. Then, $z \leqslant^{\mathbf{d}} y$ follows from $\mathbf{d}(z, y) \leqslant \mathbf{d}(z, z_n) + \mathbf{d}(z_n, y) \to 0$. As $\epsilon > 0$ was aribtrary, $\mathbf{e} \circ \leqslant^{\mathbf{d}} = \mathbf{e} \circ \Phi^{\mathbf{d}}$.

As (x_λ) is Cauchy, we can take a subnet and $(s_\lambda), (t_\lambda) \subseteq (0, \infty)$ such that

$$\sup_{\lambda < \delta} \mathbf{d}(x_\lambda, x_\delta) < s_\lambda \to 0,$$
$$^{\mathbf{e}\circ\leqslant^{\mathbf{d}}}/_{\mathbf{d}}(s_\lambda) < t_\lambda \to 0.$$

Define γ^n_λ and $x^n_\lambda \leqslant^{\mathbf{d}} x_{\gamma^n_\lambda}$ recursively as follows. First, set $\gamma^1_\lambda = \lambda$ and $x^1_\lambda = x_\lambda$. Then, for all $n \in \mathbb{N}$, take $\gamma^{n+1}_\lambda > \gamma^n_\lambda$ such that $^{\mathbf{e}\circ\leqslant^{\mathbf{d}}}/_{\mathbf{d}}(s_{\gamma^{n+1}_\lambda}), s_{\gamma^{n+1}_\lambda} < 2^{-n}t_\lambda$. As $\mathbf{d}(x^n_\lambda, x_{\gamma^{n+1}_\lambda}) \leqslant \mathbf{d}(x_{\gamma^n_\lambda}, x_{\gamma^{n+1}_\lambda}) < s_{\gamma^n_\lambda}$ and $^{\mathbf{e}\circ\leqslant^{\mathbf{d}}}/_{\mathbf{d}}(s_{\gamma^n_\lambda}) < 2^{1-n}t_\lambda$, we can take $x^{n+1}_\lambda \leqslant^{\mathbf{d}} x_{\gamma^{n+1}_\lambda}$ such that $\mathbf{e}(x^n_\lambda, x^{n+1}_\lambda) < 2^{1-n}t_\lambda$. For each λ, (x^n_λ) is $\mathbf{e}$-Cauchy so $\mathbf{e}$-completeness implies that $\mathbf{e}(x^n_\lambda, y_\lambda) \to 0$, for some $y_\lambda \in X$.

For any λ and $\epsilon > 0$, we can take n with $2^{1-n}t_\lambda < \epsilon$ so $\mathbf{e}(x^n_\lambda, y_\lambda) < 2\epsilon$ and $\mathbf{d}(x_{\gamma^n_\lambda}, x_\delta) < s_{\gamma^n_\lambda} < \epsilon$, for any $\delta \succ \gamma^n_\lambda$. For all sufficiently large δ, we also have $t_\delta < \epsilon$ so $\mathbf{e}(x_\delta, y_\delta) < 2\epsilon$ and hence

$$\begin{aligned}\mathbf{d}(y_\lambda, y_\delta) &\leqslant \mathbf{d}(y_\lambda, x^n_\lambda) + \mathbf{d}(x^n_\lambda, x_{\gamma^n_\lambda}) + \mathbf{d}(x_{\gamma^n_\lambda}, x_\delta) + \mathbf{d}(x_\delta, y_\delta)\\ &\leqslant {}^{\mathbf{d}}/_{\mathbf{e}}(2\epsilon) + 0 + \epsilon + {}^{\mathbf{d}}/_{\mathbf{e}}(2\epsilon).\end{aligned}$$

As $\mathbf{d} \lessapprox \mathbf{e}$, $Y = \{y_\lambda : \lambda \in \Lambda\}$ is $\mathbf{d}$-directed. As $\mathbf{e}(x_\lambda, y_\lambda) < 2t_\lambda \to 0$, Equation (13) follows.

If X is $\mathbf{e}$-separable, then $\mathbf{e}$ is a pseudometric, as $\mathbf{e} = \mathbf{e}^{op}$. Thus, Y is also $\mathbf{e}$-separable and can be replaced by a countable subset. Then, we can replace (x_λ) with a $\mathbf{d}$-Cauchy sequence $(x_n) \subseteq Y$ with $Y \leqslant^{\mathbf{d}} (x_n)$.

Take $(s^m_n), (t^m_n) \subseteq (0, \infty)$ such that, for all $m, n \in \mathbb{N}$,

$$s^m_n < 2^{-m-n}, \quad {}^{\mathbf{d}}/_{\mathbf{e}}(s^m_n) < t^m_{n-1} \quad \text{and} \quad {}^{\mathbf{e}\circ\leqslant^{\mathbf{d}}}/_{\mathbf{d}}(t^m_n) < s^{m+1}_n$$

(define and $(s^m_1)_{m\in\mathbb{N}}$ first then $(t^m_1)_{m\in\mathbb{N}}$, $(s^m_2)_{m\in\mathbb{N}}$ etc.). Take a subsequence (x_n) with $\mathbf{d}(x_n, x_{n+1}) < t^1_n$, for all n, and define y^m_n with $\mathbf{d}(y^m_n, y^m_{n+1}) < t^m_n$, for all m, recursively as follows. First, let $y^1_n = x_n$, for all n. Assume y^m_n is defined for all n and fixed m. For each n, we can take $y^{m+1}_n \leqslant^{\mathbf{d}} y^m_{n+1}$ with $\mathbf{e}(y^m_n, y^{m+1}_n) < s^{m+1}_n$ as

$$\mathbf{e} \circ \leqslant^{\mathbf{d}}(y^m_n, y^m_{n+1}) \leqslant {}^{\mathbf{e}\circ\leqslant^{\mathbf{d}}}/_{\mathbf{d}}(\mathbf{d}(y^m_n, y^m_{n+1})) \leqslant {}^{\mathbf{e}\circ\leqslant^{\mathbf{d}}}/_{\mathbf{d}}(t^m_n) < s^{m+1}_n.$$

Thus, $\mathbf{d}(y^{m+1}_n, y^{m+1}_{n+1}) \leqslant \mathbf{d}(y^m_{n+1}, y^{m+1}_{n+1}) \leqslant {}^{\mathbf{d}}/_{\mathbf{e}}(\mathbf{e}(y^m_{n+1}, y^{m+1}_{n+1})) \leqslant {}^{\mathbf{d}}/_{\mathbf{e}}(s^{m+1}_{n+1}) < t^{m+1}_n$.

For all $m, n \in \mathbb{N}$, $\mathbf{e}(y^m_n, y^{m+1}_n) < s^{m+1}_n < 2^{-m-n}$ so, as X is $\mathbf{e}$-complete, we have $y_n \in X$ with $\lim_m \mathbf{e}(y^m_n, y_n) = 0$. As $\mathbf{d} \lessapprox \mathbf{e} = \mathbf{e}^{op}$ and $y^{m+1}_n \leqslant^{\mathbf{d}} y^m_{n+1}$,

$$\mathbf{d}(y_n, y_{n+1}) \leqslant \liminf_m (\mathbf{d}(y_n, y^{m+1}_n) + \mathbf{d}(y^{m+1}_n, y^m_{n+1}) + \mathbf{d}(y^m_{n+1}, y_{n+1})) = 0,$$

https://doi.org/10.1017/S0960129517000032 Published online by Cambridge University Press

i.e. $y_n \leqslant^{\mathbf{d}} y_{n+1}$ so $Y = \{y_n : n \in \mathbb{N}\}$ is $\leqslant^{\mathbf{d}}$-directed. Lastly, Equation (13) follows from

$$\mathbf{e}(x_n, y_n) = \lim_m \mathbf{e}(x_n, y_n^m) < \sum_{m=2}^{\infty} s_n^m < \sum_{m=2}^{\infty} 2^{-m-n} < 2^{-n} \to 0.$$

□

Corollary 4.6. X is $\mathbf{d}$-complete if any of the following hold:

1. X is $\leqslant^{\mathbf{d}}$-$\mathbf{d}$-complete and $\mathbf{d}^{\bullet} \lessapprox \mathbf{I}$.
2. X is $\leqslant^{\mathbf{d}}$-$\mathbf{d}$-complete, $\mathbf{d}^{\vee}$-complete and $\mathbf{d}_{\mathbf{F}} \leqslant \mathbf{I}$.
3. X is $\mathbf{d}$-$\mathbf{d}$-complete, $\mathbf{e}$-complete and $\mathbf{e} \circ \Phi^{\mathbf{d}} \lessapprox \mathbf{d} \lessapprox \mathbf{e} = \mathbf{e}^{op}$.
4. X is $\leqslant^{\mathbf{d}}$-$\mathbf{d}$-complete, $\mathbf{e}$-complete, $\mathbf{e}$-separable and $\mathbf{e} \circ \Phi^{\mathbf{d}} \lessapprox \mathbf{d} \lessapprox \mathbf{e} = \mathbf{e}^{op}$.

Proof. If $\mathbf{d}^{\bullet} \lessapprox \mathbf{I}$, then, for any $\mathbf{d}$-Cauchy (x_λ), we have $Y \subseteq X$ with $Y\mathbf{d} = (x_\lambda)\mathbf{d}$, by Theorem 4.1. If X is also $\leqslant^{\mathbf{d}}$-$\mathbf{d}$-complete, then we have $x = \mathbf{d}\text{-}\sup Y$ and hence $x\mathbf{d} = Y\mathbf{d} = (x_\lambda)\mathbf{d}$ so $x_\lambda \overset{\circ}{\underset{\bullet}{\rightarrow}} x \leqslant^{\mathbf{d}} x$, i.e. $x_\lambda \overset{\circ}{\underset{\circ}{\rightarrow}} x$, by Equation (6). This proves 1. and likewise 2. follows from Theorem 4.3, while 3. and 4. follow from Theorem 4.5. □

Note in Corollary 4.6 2., if $\mathbf{d}$ is a hemimetric, then we can replace $\mathbf{d}_{\mathbf{F}}$ with $\mathbf{d}_{\Phi}$ for a formally weaker assumption (even weaker if we consider Equation (12)), by Proposition 4.2.

For a simple application of Corollary 4.6 1., we consider the space of 'generalized formal balls' of X. Specifically, identify X with $X \times \{0\}$ and extend $\mathbf{d}$ to $X \times \mathbb{R}$ by

$$\mathbf{d}((x,r),(y,s)) = (\mathbf{d}(x,y) + r - s)_+.$$

For any $x, y \in X$, $r, s \in \mathbb{R}$ and $t \in [0, \infty)$,

$$\begin{aligned} & \mathbf{d}(x,y) + r - s \leqslant t && \Leftrightarrow && \mathbf{d}((x,r),(y,s)) \leqslant t. \\ \Leftrightarrow\quad & \mathbf{d}(x,y) + r - t - s \leqslant 0 && \Leftrightarrow && (x, r - t) \leqslant^{\mathbf{d}} (y,s). \\ \Leftrightarrow\quad & \mathbf{d}(x,y) + r - (t+s) \leqslant 0 && \Leftrightarrow && (x,r) \leqslant^{\mathbf{d}} (y, t+s). \end{aligned}$$

So finite radius closed upper balls have minimums and likewise for lower balls, i.e.

$$\overline{(x,r)}_t^{\bullet} = (x, r-t)^{\leqslant^{\mathbf{d}}} \qquad \text{and} \qquad {}^{\leqslant^{\mathbf{d}}}(y, t+s) = \overline{(y,s)}^t_{\bullet}.$$

Thus, $\mathbf{d}^{\bullet} \leqslant \mathbf{I}$ and $\mathbf{d}_{\bullet} \leqslant \mathbf{I}$. And $\mathbf{d}^{\bullet} \leqslant \mathbf{I}$ still applies to $X \times \mathbb{R}_-$, where $\mathbb{R}_- = (-\infty, 0]$.

We can now give a concise proof of Kostanek and Waszkiewicz (2011, Theorem 7.1).

Theorem 4.7. The following are equivalent:

1. X is $\mathbf{d}$-complete.
2. $X \times \mathbb{R}_-$ is $\mathbf{d}$-complete.
3. $X \times \mathbb{R}_-$ is $\leqslant^{\mathbf{d}}$-complete.

Proof.

1.⇒2. If (x_λ, r_λ) is $\mathbf{d}$-Cauchy, then, as $(r-s)_+ \leqslant \mathbf{d}((x,r),(y,s))$ and $\mathbb{R}_-$ is bounded above by 0, (r_λ) must be Cauchy (for the usual metric on $\mathbb{R}$). Thus, $r_\lambda \to r$ for some $r \in \mathbb{R}_-$, and hence (x_λ) is $\mathbf{d}$-Cauchy. Thus, $x_\lambda \overset{\circ}{\underset{\circ}{\rightarrow}} x$, for some $x \in X$, and hence $(x_\lambda, r_\lambda) \overset{\circ}{\underset{\circ}{\rightarrow}} (x,r)$ in $X \times \mathbb{R}_-$.

https://doi.org/10.1017/S0960129517000032 Published online by Cambridge University Press

2.⇒1. Identify X with $X \times \{0\}$.

2.⇒3. Immediate.

3.⇒2. We claim that any $\leqslant^{\mathbf{d}}$-supremum (x,r) of $\leqslant^{\mathbf{d}}$-directed (x_λ, r_λ) in $X \times \mathbb{R}_-$ remains a $\leqslant^{\mathbf{d}}$-supremum in $X \times \mathbb{R}$. Indeed, say $(x_\lambda, r_\lambda) \leqslant^{\mathbf{d}} (y,s) \in X \times \mathbb{R}$, for all λ. As $X \times \mathbb{R}_-$ is $\leqslant^{\mathbf{d}}$-complete, we have $(z,t) = \leqslant^{\mathbf{d}}\text{-sup}(x_\lambda, r_\lambda - s)$ in $X \times \mathbb{R}_-$, so $(z, t+s) = \leqslant^{\mathbf{d}}\text{-sup}(x_\lambda, r_\lambda) = (x,r)$ and hence $(x, r-s) = \leqslant^{\mathbf{d}}\text{-sup}(x_\lambda, r_\lambda - s)$. Also, $(x_\lambda, r_\lambda - s) \leqslant^{\mathbf{d}} (y,0)$, for all λ, so $(x, r-s) \leqslant^{\mathbf{d}} (y,0)$ and hence $(x,r) \leqslant^{\mathbf{d}} (y,s)$, proving the claim. Thus, $(x,r) = \mathbf{d}\text{-sup}(x_\lambda, r_\lambda)$ in $X \times \mathbb{R}$, by Proposition 3.5, and hence in $X \times \mathbb{R}_-$. This shows that $X \times \mathbb{R}_-$ is $\leqslant^{\mathbf{d}}$-**d**-complete and hence **d**-complete, by Corollary 4.6 1. □

This research has been supported by IMPAN (Poland), UFBA and IMPA (Brazil).

References

Akemann, C.A. and Pedersen, G.K. (1973). Complications of semicontinuity in C*-algebra theory. *Duke Mathematical Journal* **40** (4) 785–795.

Bice, T. (2016). Semicontinuity in ordered Banach spaces. arXiv:1604.03154v1 [math.FA].

Bonsangue, M., van Breugel, F. and Rutten, J. (1998). Generalized metric spaces: Completion, topology, and powerdomains via the Yoneda embedding. *Theoretical Computer Science* **193** (12) 1–51.

Brown, L.G. (1988). Semicontinuity and multipliers of C*-algebras. *Canadian Journal of Mathematics* **40** (4) 865–988.

Goubault-Larrecq, J. (2013). *Non-Hausdorff Topology and Domain Theory*, New Mathematical Monographs, vol. 22, Cambridge University Press, Cambridge. [On the cover: Selected topics in point-set topology].

Kostanek, M. and Waszkiewicz, P. (2011). The formal ball model for $\mathcal{Q}$-categories. *Mathematical Structures in Computer Science* **21** (1) 41–64.

Künzi, H.P. and Schellekens, M.P. (2002). On the Yoneda completion of a quasi-metric space. *Theoretical Computer Science* **278** (1–2) 159–194. Mathematical foundations of programming semantics (Boulder, CO, 1996).

Pedersen, G.K. (1979). *C^*-Algebras and Their Automorphism Groups*, London Mathematical Society Monographs, vol. 14, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London.

Smyth, M.B. (1988). Quasi-uniformities: Reconciling domains with metric spaces. In: Main, M., Melton, A., Mislove, M. and Schmidt, D. (eds.) *Mathematical Foundations of Programming Language Semantics (New Orleans, LA, 1987)*, Lecture Notes in Computer Science, vol. 298, Springer, Berlin, 236–253.

Wagner, K.R. (1997). Liminf convergence in Ω-categories. *Theoretical Computer Science* **184** (1–2) 61–104.