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Influence of heterogeneity on second-kind self-similar solutions for viscous gravity currents

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We report experimental, theoretical and numerical results on the effects of horizontal heterogeneities on the propagation of viscous gravity currents. We use two geometries to highlight these effects: (*a*) a horizontal channel (or crack) whose gap thickness varies as a power-law function of the streamwise coordinate; (*b*) a heterogeneous porous medium whose permeability and porosity have power-law variations. We demonstrate that two types of self-similar behaviours emerge as a result of horizontal heterogeneity: (*a*) a first-kind self-similar solution is found using dimensional analysis (scaling) for viscous gravity currents that propagate away from the origin (a point of zero permeability); (*b*) a second-kind self-similar solution is found using a phase-plane analysis for viscous gravity currents that propagate toward the origin. These theoretical predictions, obtained using the ideas of self-similar intermediate asymptotics, are compared with experimental results and numerical solutions of the governing partial differential equation developed under the lubrication approximation. All three results are found to be in good agreement.

Key words: gravity currents, Hele-Shaw flows, porous media

1. Introduction

The gravitationally driven spreading of viscous fluids, also known as viscous gravity currents, is a process common to a large number of industrial and geological phenomena (Hoult 1972; Simpson 1982; Huppert 2000, 2006). One aspect of their propagation has been particularly attractive, namely the ability, after a sufficient amount of time has elapsed, to obtain the spatial distribution, or shape, of the current at different times through a transformation of the independent variables. In other words, once the initial condition is 'forgotten', gravity currents evolve in a *self-similar* fashion (Barenblatt 1996; Sedov 1993). As a result, the self-similar propagation of gravity currents, which can be understood as intermediate asymptotics in the sense of Barenblatt & Zel'dovich (1972), has been studied in detail; see, e.g., the mathematical framework for problems of this type discussed by Gratton & Minotti (1990) and the references therein.

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One-dimensional gravity currents in homogeneous porous media have been of particular interest (see, e.g., Huppert 2006). In unconfined porous media, self-similar solutions of the first kind, whose scaling exponents can be determined by simple balances of terms in the governing equations, were obtained by Barenblatt (1952) and Pattle (1959) to describe the spread of a finite mass of fluid. The latter solutions were extended to apply to propagation due to fluid injection and shown to agree with laboratory experiments in both Hele-Shaw cells (see, e.g., Huppert & Woods 1995) and systems of packed beads (see, e.g., Lyle *et al.* 2005). In soil mechanics, the propagation of a gravity current into an unsaturated soil is known as infiltration (Philip 1970; Bear 1972), and self-similar solutions of the first kind have also been adapted as approximations for infiltration into homogeneous layers with a pre-existing moisture distribution (Witelski 1998). A wealth of first-kind self-similar solutions also describe the early and late time propagation regimes through a homogeneous porous medium lying above an inclined impermeable boundary (see, e.g., Vella & Huppert 2006), over fractured horizontal substrates (see, e.g., Pritchard 2007) and for drainage from the edge of a finite porous reservoir (Zheng *et al.* 2013). In confined porous media, on the other hand, a rarefaction wave self-similar solution was derived, and verified by numerical simulations, to describe the evolution of the immiscible interface between a fluid being injected at a constant rate and the more viscous fluid being displaced (Nordbotten & Celia 2006); seepage through the confining layer has also been considered (Woods & Farcas 2009). Further work revealed that, after injection ceases, there is a transition from an early time self-similar behaviour to a late time self-similar behaviour, and again the theoretical results were supported by numerical simulations (Hesse *et al.* 2007). Direct numerical simulations have shown that the presence of transverse sidewalls can affect the selection of a self-similar propagation regime or the transition from one to the next (Hallez & Magnaudet 2009).

Two-dimensional (2D) and axisymmetric propagation of viscous gravity currents has also been studied, given the possible practical applications to fluid injection into wells and subsurface reservoirs. For a viscous current propagating away from the centre of a horizontal plate, a first-kind self-similar solution was found in an unconfined system and in a porous medium (Barenblatt 1952; Pattle 1959; Kochina, Mikhailov & Filinov 1983), verified experimentally for spreading into an inviscid fluid (e.g. air) (Didden & Maxworthy 1982; Huppert 1982), and in a system of packed beads (see, e.g., Lyle *et al.* 2005). The axisymmetric propagation of a power-law non-Newtonian gravity current into an inviscid fluid (e.g. air) has also been studied (Gratton, Mahajan & Minotti 1999). For non-axisymmetric configurations, gravity currents in unconfined systems arising from both point and line sources (Lister 1992), converging flows (Diez *et al.* 1998) and currents in porous media (Vella & Huppert 2006; De Loubens & Ramakrishnan 2011) have been considered.

An interesting feature of the axisymmetric problem is that self-similar solutions of the *second kind* exist for the case of propagation toward the centre of a horizontal plate (Gratton & Minotti 1990; Angenent & Aronson 1995), and these were found to agree with experiments (Diez, Gratton & Gratton 1992). Second-kind self-similar solutions, unlike those of the first kind, cannot be obtained by a scaling analysis of the governing partial differential equation (PDE) (Barenblatt & Zel'dovich 1972; Barenblatt 1996; Eggers & Fontelos 2009) but rather are constructed by a more involved mathematical analysis. The significance of this step is that there are propagation modes for viscous gravity currents that can only be obtained via a phase plane analysis (Sedov 1993; Courant & Friedrichs 1999) of the governing PDE, which requires the numerical solution of a nonlinear eigenvalue problem for the scaling exponent in the similarity variable (Barenblatt & Zel'dovich 1972; Barenblatt 1996).

Heterogeneity always exists in real-world porous media, and it may significantly affect the propagation regimes of viscous gravity currents. When there exists a power-law permeability gradient in the vertical direction, the propagation and drainage processes in a porous medium are modified from the homogeneous case (Huppert & Woods 1995; Takagi & Huppert 2007; Ciriello *et al.* 2013; Zheng *et al.* 2013). In vertically layered systems with permeability jumps across layers, flow focusing can occur and may even dominate over buoyancy as the key physical transport mechanism (Huppert, Neufeld & Strandkvist 2013). In general, homogenization techniques must be employed in such cases to derive a depth-averaged governing equation for the current (Anderson, McLaughlin & Miller 2003). It should be noted that heterogeneity exists not only in the vertical but also in the horizontal direction in real-world systems. For example, porous media may have horizontal permeability and porosity gradients on the reservoir scale (Bear 1972; Class *et al.* 2009), while individual channels and cracks in the porous medium can have varying gap thickness in the streamwise flow direction (Spence & Sharp 1985; Detournay 2004). To the best of the authors' knowledge, the propagation of viscous gravity currents into media with horizontal heterogeneity has only been studied previously by Ciriello *et al.* (2013), however, they only considered first-kind self-similar solutions and assumed the porosity of the medium to be constant.

The goal of this paper is to study the effects of horizontal heterogeneity on the propagation of viscous gravity currents. As we document below, horizontal heterogeneity allows for both first- and second-kind self-similar currents. Figure 1 shows a summary of the physical situations in which we expect self-similarity of the second kind: (*a*) converging currents on a horizontal substrate (Gratton & Minotti 1990; Diez *et al.* 1992; Angenent & Aronson 1995); (*b*) converging currents in an unconfined homogeneous porous medium; (*c*) currents propagating in horizontal channels or cracks with varying gap thickness of power-law form; (*d*) currents propagating in a homogeneous porous medium with converging boundaries of power-law shape in the horizontal direction; and (*e*) currents propagating in heterogeneous porous media with power-law permeability and porosity gradients in the horizontal direction.

In this work, we analyse in detail the effects of heterogeneity, specifically case (*c*) is studied in § 2.1 and case (*e*) is studied in § 2.2. In addition, we briefly discuss cases (*b*) and (*d*) in appendices A.1 and A.2, respectively. We show that viscous gravity currents propagating away from the origin (a point of zero permeability) are described by first-kind self-similar solutions, while currents propagating toward the origin are described by second-kind self-similar solutions. A novel conservative implicit finite-difference scheme is developed (appendix B) to numerically solve the governing PDE. In § 3, the results from the analysis based on self-similar intermediate asymptotics are shown to agree well with the numerical solutions and laboratory experiments.

2. Formulation of the mathematical model

2.1. *Gravity currents in channels*

Hele-Shaw cells, i.e. channels created by two closely spaced plates, are an ideal system in which to study the propagation of viscous gravity currents spreading into an inviscid fluid (e.g. air) or into a porous medium. To formulate a mathematical model of these processes, we assume that the invading and displacing fluids are

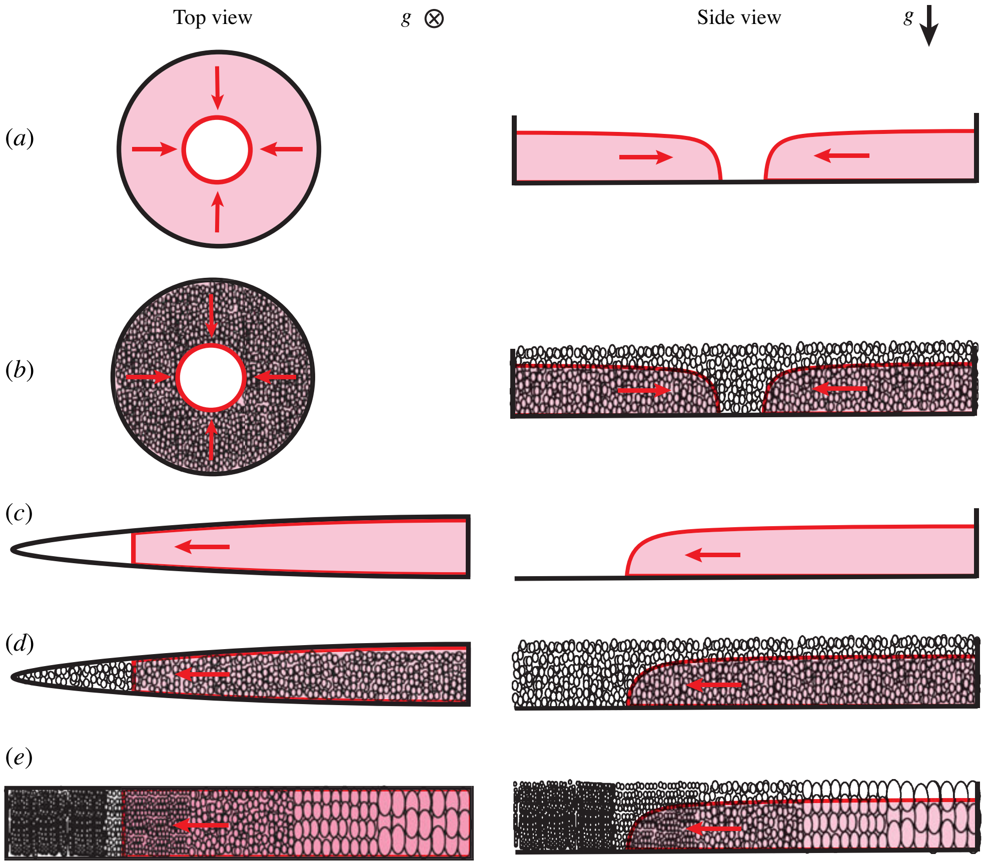

FIGURE 1. (Colour online) Different flows of viscous gravity currents, propagating toward a coordinate system's origin, that may be described by self-similar solutions of the second kind. (*a*) Converging viscous gravity currents on a horizontal substrate (Gratton & Minotti 1990; Diez *et al.* 1992; Angenent & Aronson 1995). (*b*) Converging currents in homogeneous porous media. (*c*) Viscous gravity currents propagating in horizontal channels with varying gap thickness of power-law form. (*d*) Currents propagating horizontally in homogeneous porous media with converging boundaries of power-law form. (*e*) Currents propagating in heterogeneous porous media with permeability and porosity gradients in the horizontal direction.

immiscible, separated by a sharp interface, and we only consider the limit of negligible surface tension. In addition, in the present work, the invading fluid always has greater viscosity than the displaced fluid, so that the interface is not susceptible to the Saffman–Taylor viscous fingering instability (Saffman & Taylor 1958; Homsy 1987). The viscous gravity current in the channel is long and thin, so the lubrication approximation applies, and the flow is horizontal to leading order in the aspect ratio; velocity components in the vertical and cross-stream direction are neglected. Given the unconfined nature of this flow problem, in the following, we also assume motion of the displaced fluid is negligible since the upper layer is sufficiently deep.

2.1.1. *Flow away from the origin*

Consider a viscous gravity current propagating in a horizontal channel in the direction of increasing gap thickness, away from the origin of the chosen coordinate

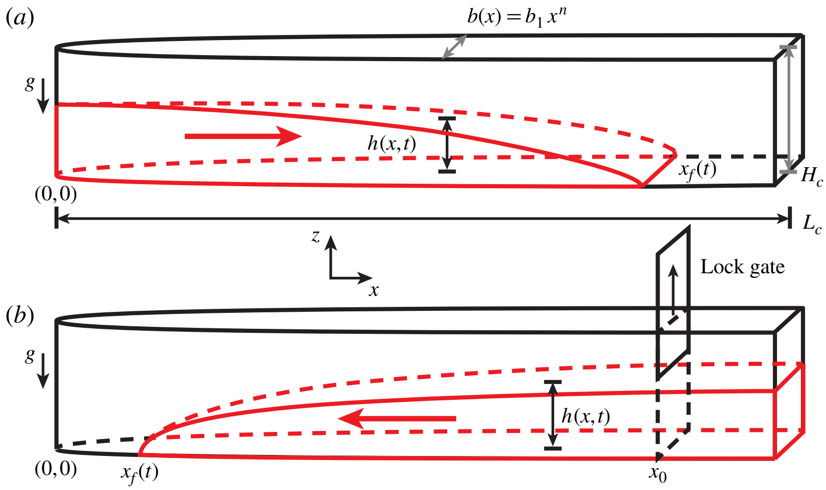

FIGURE 2. (Colour online) Diagram of the geometry of a horizontal channel with varying gap thickness $b(x)$ in which a viscous gravity current is propagating. The shape of the current and the position of the front are denoted by $h(x, t)$ and $x_f(t)$, respectively. (*a*) Propagation away from the origin in the direction of increasing gap thickness. (*b*) Propagation toward the origin, from far downstream, in the direction of decreasing gap thickness. Gravity is directed in the negative vertical (i.e. $-z$) direction.

system (see figure 2*a*). We denote by $h(x, t)$ the shape of the current and by $x_f(t)$ the location of the moving front (nose of the current) such that $h(x_f(t), t) = 0$. We assume that the gap thickness of the channel has a power-law form: $b(x) = b_1 x^n$, where b_1 and n are non-negative real numbers.

The lubrication approximation holds, if $|\mathrm{d}b/\mathrm{d}x| \ll 1$ and $|\partial h/\partial x| \ll 1$, except perhaps at the nose of the current. However, note that for the assumed shape $b(x)$ and $n > 1$, the requirement that $|\mathrm{d}b/\mathrm{d}x| \ll 1$ is violated when the nose of the current $x_f \approx (nb_1)^{1/(1-n)}$. Also, the lubrication approximation fails when the aspect ratio $b(x_f)/x_f \approx 1$ (for propagation away from the origin), or $x_f \approx b_1^{1/(1-n)}$; specifically, the cross-stream velocity is no longer negligible, and the propagation of the gravity current is not described by a one-dimensional model. Moreover, we see that for $n > 1$, $b(x)/x$ is an increasing function of x, hence there is a maximal propagation distance, $x_f = O(b_1^{1/(1-n)})$, beyond which the one-dimensional lubrication model fails. Meanwhile, for $n < 1$, $b(x)/x$ is a decreasing function of x, hence $b(x) < x$ holds for all $x > O(b_1^{1/(1-n)})$; although $b(x_f)/x_f \approx 1$ near the origin for $n < 1$, this in an initial, localized effect that does not affect the overall applicability of the lubrication approximation for the gravity current's propagation.

Under the above assumptions, Darcy's law in the horizontal direction and the one-dimensional form of the continuity equation, respectively, take the form

$$u = -\frac{k(x)}{\mu}\frac{\partial p}{\partial x}, \tag{2.1a}$$

$$\frac{\partial}{\partial t}(hb) + \frac{\partial}{\partial x}(hub) = 0, \tag{2.1b}$$

where u is the transversely averaged horizontal velocity, k is the effective permeability of the medium, p is the fluid pressure and μ is the viscosity of the invading fluid.

In the vertical direction, the pressure field is hydrostatic: $p(x, z, t) = p_0 + \Delta\rho g (h(x, t) - z)$, where p_0 is the ambient pressure in the displaced fluid, and $\Delta\rho$ is the density difference between the two fluids (Huppert & Woods 1995).

Following standard steps (Huppert & Woods 1995; Barenblatt 1996), we substitute the hydrostatic pressure distribution and the gap thickness $b(x) = b_1 x^n$ into (2.1); we neglect drag on the bottom plate and assume h is sufficiently greater than b so that the permeability is given by $k(x) = b(x)^2/12$. Then, we obtain a nonlinear diffusion equation for the current height:

$$\frac{\partial h}{\partial t} - \frac{A}{x^n}\frac{\partial}{\partial x}\left(x^{3n} h \frac{\partial h}{\partial x}\right) = 0, \tag{2.2}$$

where $A = (\Delta\rho g b_1^2)/(12\mu)$. The case of a channel with constant gap thickness is recovered by setting $n = 0$.

To maintain generality, we define the injection rate as $\gamma Q t^{\gamma-1}$ so that, at any time, the total volume of injected liquid is Qt^γ, where $\gamma \geqslant 0$. Since the gravity current is moving away from the origin $x = 0$, the global mass conservation constraint has the form

$$\int_0^{x_f(t)} x^n h(x, t)\,\mathrm{d}x = Bt^\gamma, \tag{2.3}$$

where $B = Q/b_1$. We expect that the details of the solution will depend on n and γ.

From a scaling analysis of (2.2) and (2.3), we find that the appropriate similarity variable is

$$\xi = \frac{x}{(AB)^{1/(3-n)} t^{(\gamma+1)/(3-n)}}, \tag{2.4}$$

from which it immediately follows that the front of the current moves according to

$$x_f(t) = \xi_f(n, \gamma)(AB)^{1/(3-n)} t^{\gamma+1/(3-n)}, \tag{2.5}$$

where the prefactor $\xi_f(n, \gamma)$ is a constant that depends only on the values of n and γ. We define $y \equiv x/x_f(t) = \xi(x, t)/\xi_f(n, \gamma)$, after which the shape of the current can be written as

$$h(x, t) = \xi_f^{2(1-n)} A^{(n+1)/(n-3)} B^{(2(n-1))/(n-3)} t^{(2(n-1)\gamma+(n+1))/(n-3)} f(y), \tag{2.6}$$

where $f(y)$ and $\xi_f(n, \gamma)$ are solutions to the following system:

$$(y^{3n} f f')' - \left(\frac{\gamma+1}{n-3}\right) y^{n+1} f' - \left(\frac{2(n-1)\gamma + (n+1)}{n-3}\right) y^n f = 0, \tag{2.7a}$$

$$f(1) = 0, \tag{2.7b}$$

$$\xi_f(n, \gamma) = \left(\int_0^1 y^n f(y)\,\mathrm{d}y\right)^{1/(n-3)}, \tag{2.7c}$$

and primes denote differentiation with respect to y. For $n = 0$, equation (2.7) reduces to the homogeneous porous medium case in Cartesian coordinates (Huppert & Woods 1995). In this class of problems, one boundary condition is sufficient to uniquely determine the solution, owing to the form of the singularity at the front.

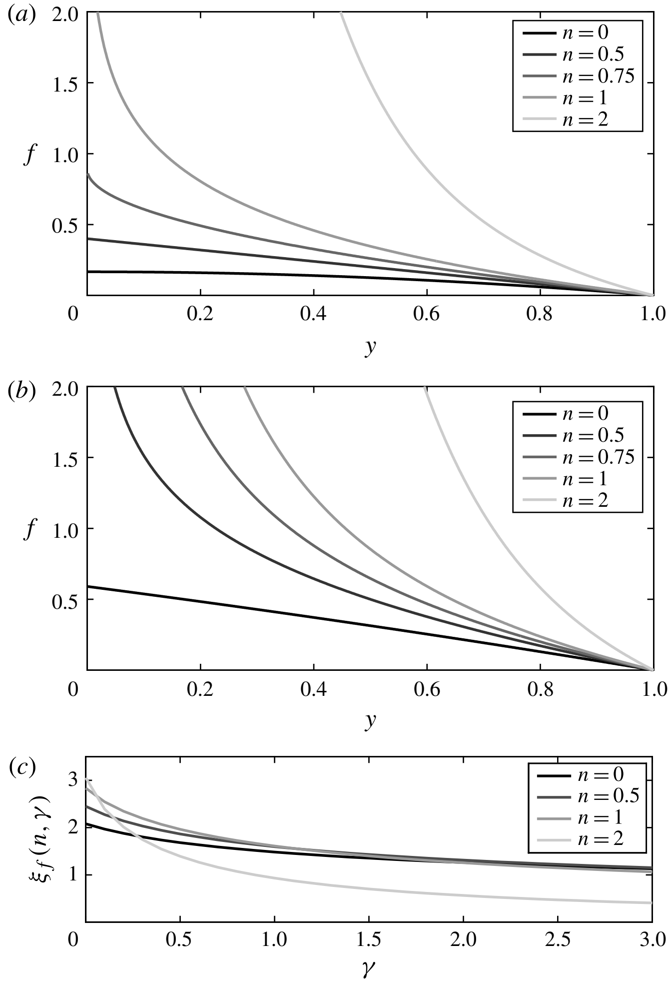

FIGURE 3. Self-similar shape of a viscous gravity current propagating away from the origin in a channel with variable gap thickness of the form $b(x) = b_1 x^n$ for (*a*) constant volume ($\gamma = 0$) and (*b*) constant injection rate ($\gamma = 1$), and various values of n; the prefactor $\xi_f(n, \gamma)$ is shown in (*c*). The solutions for f were found by integrating (2.7*a*) numerically by shooting backwards from $y = 1 - \nu$ ($\nu = 10^{-4}$), where boundary conditions on f and f' are imposed using the asymptotic form from (2.8).

Following Huppert (1982), we can determine the asymptotic behaviour of the current near the nose from (2.7*a,b*):

$$f(y) \sim \left(\frac{\gamma + 1}{3 - n}\right)(1 - y) \quad \text{as } y \to 1^-, \tag{2.8}$$

which can, in turn, be used to provide two boundary conditions near $y = 1$, i.e. $f(1 - \nu)$ and $f'(1 - \nu)$, $\nu \ll 1$, in a shooting procedure for solving (2.7*a*) numerically. Typical numerically computed current shapes $f(y)$ are shown in figure 3 for (*a*) constant volume ($\gamma = 0$) and (*b*) constant injection rate ($\gamma = 1$) and various choices of n. The dependence of ξ_f on γ is illustrated in figure 3(*c*) for various choices of n.

For $\gamma = 0$ and $0 \leqslant n < 3$, (2.7) has the exact solution

$$f(y) = \begin{cases} \dfrac{1}{2(n-1)(n-3)}\left(1 - y^{2(1-n)}\right), & n \neq 1, \\ -\frac{1}{2}\ln y, & n = 1, \end{cases} \tag{2.9a}$$

$$\xi_f(n, 0) = \left[(n+1)(n-3)^2\right]^{1/(3-n)}. \tag{2.9b}$$

For $n = 0$, this solution reduces to a special case of the Barenblatt–Pattle point-source solution (Barenblatt 1952; Pattle 1959). For $n \geqslant 3$, the self-similar solution represents a receding front because $(\gamma + 1)/(3 - n) < 0$, which has also been observed in higher-order lubrication models (King & Bowen 2001; Flitton & King 2004). However, in the present case, the first-kind self-similar solution for $n \geqslant 3$ is unphysical because $f < 0$, which means the height of the current is negative.

2.1.2. *Flow toward the origin*

In this section, we consider the opposite case of a viscous gravity current propagating in the direction of decreasing gap thickness, i.e. toward the origin (see figure 2*b*). Since it is now possible for the gravity current to reach the origin ($x = 0$), we define a critical time t_c at which this occurs; in the present work, t_c is determined from numerical simulations and/or experiments and, in general, could be infinite. Note that, the global mass conservation constraint, as expressed in (2.3), no longer holds. Any modification of the latter for flow toward the origin requires the introduction of another length scale x_0 and another time scale t_c into the problem. Therefore, in this physical situation, it is not *a priori* clear how to obtain a self-similar solution by scaling arguments alone. Moreover, we do not expect complete self-similarity with respect to a similarity variable, i.e. even if a self-similar solution can be obtained it will depend on t_c (or x_0) explicitly (Barenblatt 1952, §5.1.1).

Nevertheless, Darcy's law and the continuity equation still hold, so (2.1) remains the governing equation, but the boundary and initial conditions change. In this physical situation, we introduce the phase-plane formalism, following Gratton & Minotti (1990), Sedov (1993) and Courant & Friedrichs (1999), in order to analyse the governing equation. To this end, we substitute the hydrostatic pressure distribution, the gap thickness $b(x) = b_1 x^n$ and the permeability $k(x) = b(x)^2/12$ into (2.1) to obtain:

$$u = -\frac{\Delta\rho g b_1^2}{12\mu} x^{2n} \frac{\partial h}{\partial x}, \tag{2.10a}$$

$$\frac{\partial h}{\partial t} + \frac{1}{x^n}\frac{\partial}{\partial x}(x^n h u) = 0. \tag{2.10b}$$

These two equations are the starting point of the phase-plane analysis.

Restricting to the case when $t_c < \infty$, we first introduce $\tau = t_c - t$ as the time remaining to reach the origin ('touch-down'). (If $t_c = \infty$, then we would not be able to make this transformation and proceed with the phase-plane analysis. Hence, $t_c = \infty$ corresponds to non-self-similar behaviour beyond the scope of the present work.) Then, in order to make (2.10) dimensionless, we let

$$u(x, t) = \frac{x}{\tau} U(x, \tau), \quad h(x, t) = \left(\frac{12\mu}{\Delta\rho g b_1^2}\right)\frac{x^{2(1-n)}}{\tau} H(x, \tau). \tag{2.11a,b}$$

Note that $u \leqslant 0$ for propagation toward the origin and, hence, $U \leqslant 0$ as well; also, $h \geqslant 0$, hence $H \geqslant 0$. Substituting (2.11) into (2.10), and after some algebra, we obtain the governing equations in terms of the dimensionless variables $H(x, \tau)$ and $U(x, \tau)$:

$$U + x\frac{\partial H}{\partial x} + 2(1-n)H = 0, \tag{2.12a}$$

$$\tau\frac{\partial H}{\partial \tau} - H - x\frac{\partial}{\partial x}(HU) - (3-n)HU = 0. \tag{2.12b}$$

By analogy with other gravity current problems where there is propagation toward the origin (Gratton & Minotti 1990; Diez *et al.* 1992; Angenent & Aronson 1995), we anticipate that we can find a self-similar solution of the second kind. If the flow is to be self-similar, we need to identify a similarity variable. Without loss of generality, we define the similarity variable to be $\xi \equiv x/\tau^{\delta}$, where δ is to be determined. Then, H and U become functions of only the similarity variable ξ, i.e. $H = H(\xi)$ and $U = U(\xi)$. Thus, for a self-similar flow, equation (2.12) takes the form

$$U + \xi H' + 2(1-n)H = 0, \tag{2.13a}$$

$$\delta\xi H' + H + \xi(HU)' + (3-n)HU = 0. \tag{2.13b}$$

We can eliminate ξ from (2.13):

$$\frac{\mathrm{d}U}{\mathrm{d}H} = \frac{H[(n+1)U - 2(1-n)\delta + 1] - U(U+\delta)}{H[2(1-n)H + U]}, \tag{2.14a}$$

$$\frac{\mathrm{d}\ln|\xi|}{\mathrm{d}H} = -\frac{1}{U + 2(1-n)H}. \tag{2.14b}$$

In particular, equation (2.14a) is an autonomous differential equation for the shape of the current. Once $U(H)$ is determined from (2.14a), (2.14b) can be solved to obtain the location of the nose of the current. These steps complete the reduction of the self-similar dynamics to a 2D phase plane (H, U). Different solution (integral) curves in the phase plane represent different flow behaviours, i.e. self-similar solutions of the original PDE. Certain distinguished integral curves of the ordinary differential equation (ODE) (2.14a) in the phase plane are those that start or end at a critical point and those that form limit cycles. The critical points can be associated with different boundary and initial conditions of the physical system (Gratton & Minotti 1990). We identify the finite critical points of (2.14a) by setting its numerator and denominator equal to zero simultaneously to find:

$$\mathrm{O}: (H, U) = (0, 0), \tag{2.15a}$$

$$\mathrm{A}: (H, U) = (0, -\delta), \tag{2.15b}$$

$$\mathrm{B}: (H, U) = \left(\frac{1}{2(1-n)(3-n)}, -\frac{1}{3-n}\right). \tag{2.15c}$$

Gratton & Minotti (1990) also considered critical points where $H = \infty$ and/or $U = \infty$, however, this is beyond the scope of the present work.

Point A describes a state of zero current height and finite velocity, which corresponds to the moving front $x_f(t)$ such that $h(x_f(t), t) = 0$ and $\mathrm{d}x_f(t)/\mathrm{d}t \neq 0$. From the definitions in (2.11), we see that point O corresponds to u, h, x and t

such that $U = H = 0$. Suppose, for the sake of argument, that point O does not correspond to the trivial downstream solution: $u = h = 0$ for all x and t. Then, the only way to satisfy (2.11) is to set $\tau = 0$. From our definition of τ, we see that this state corresponds to the touch-down time $t = t_c$, when the nose of the gravity current reaches the origin ($x = 0$). Finally, point B does not have a clear physical interpretation in the present context.

Thus, the self-similar solution, which describes a gravity current propagating toward the origin, corresponds to the integral curve connecting O to A in the phase plane. Since δ is still undetermined, it acts as a parameter that allows us to change the phase-plane structure until a heteroclinic trajectory from O to A emerges. In other words, we have to solve a nonlinear eigenvalue problem for δ. We do so numerically by a shooting method that we now describe. First, to compute the heteroclinic trajectory, we employ a fundamental-domain technique (Parker & Chua 1989, Chapter 6). To this end, we first note that the linearization of (2.14a) at point O shows that it is a degenerate saddle point with the eigenspace:

$$\lambda_1 = -\delta, \quad \boldsymbol{e}_1 = (0, \pm 1), \tag{2.16a}$$

$$\lambda_2 = 0, \quad \boldsymbol{e}_2 = \pm\left(\frac{\delta}{1-2(1-n)\delta}, 1\right). \tag{2.16b}$$

Then, we pick two initial conditions (H_i, U_i) that are slight perturbations ($\simeq 10^{-3}$) of point O in the neutral eigendirections given by $\boldsymbol{e}_2$, and we integrate (2.14a), which is rewritten as an autonomous system of two ODEs for convenience, numerically in MATHEMATICA using the built-in subroutine NDSolve. Second, to solve the nonlinear eigenvalue problem, the value of δ is adjusted using a bisection method until the trajectory passes within a prescribed (small $\simeq 10^{-3}$) distance of point A. A visual illustration of this technique is shown in figure 4 for the special case of $n = 0.5$, from which we find that $\delta = 1.54\ldots$. (Note that full double precision, i.e. 16 digits, is required in the value of δ for the heteroclinic trajectory to pass within the prescribed distance of point A and obtain the plot shown in figure 4e.)

Next, we establish the effect of varying the exponent n that controls the gap thickness of the channel. We study the problem in two ways. First, we show the phase portraits for $\delta = 1.54$ and different n in figure 5. Second, we compute the value of δ numerically for a range of n values, and report the resulting $\delta(n)$ curve in figure 6. For $0 \leqslant n < 1$ (regime I), δ is finite, indicating the existence of a self-similar solution of the second kind. Note that this result also corresponds to point B being in the fourth quadrant ($H > 0$, $U < 0$) of the (H, U) plane (see figure 5).

Simultaneously, there exist self-similar solutions of the first kind for propagation away from the origin, for which the exponent is given by $(\gamma+1)/(3-n)$ from (2.4); the cases $\gamma = 0$ and $\gamma = 1$ correspond to the dot-dashed (shown in blue online) and dashed (shown in red online) curves in figure 6, respectively. In regime II ($1 \leqslant n < 3$), there are only self-similar solutions of the first kind, while the second-kind self-similar solution no longer exists for the currents propagating toward the origin. For $n \geqslant 3$ (regime III), we note again that the first-kind self-similar solution is unphysical.

To better understand the apparently singular behaviour of $\delta(n)$ as $n \to 1^-$ in figure 6 for the case of self-similarity of the second kind, we define $\epsilon \equiv 1 - n \to 0^+$. Then, equation (2.14$a$) hints that the eigenvalue $\delta(\epsilon) \sim 1/(2\epsilon)$ as $\epsilon \to 0^+$, which is also supported by the numerics (see figure 6, regime I, dark dashed curve). Meanwhile, we can rescale U and H by the eigenvalue δ, i.e. let $\tilde{H} = H/\delta$ and $\tilde{U} = U/\delta$; for different values of n, the heteroclinic trajectories, which connect point $\tilde{\text{A}}$: $(\tilde{H}, \tilde{U}) = (0, -1)$

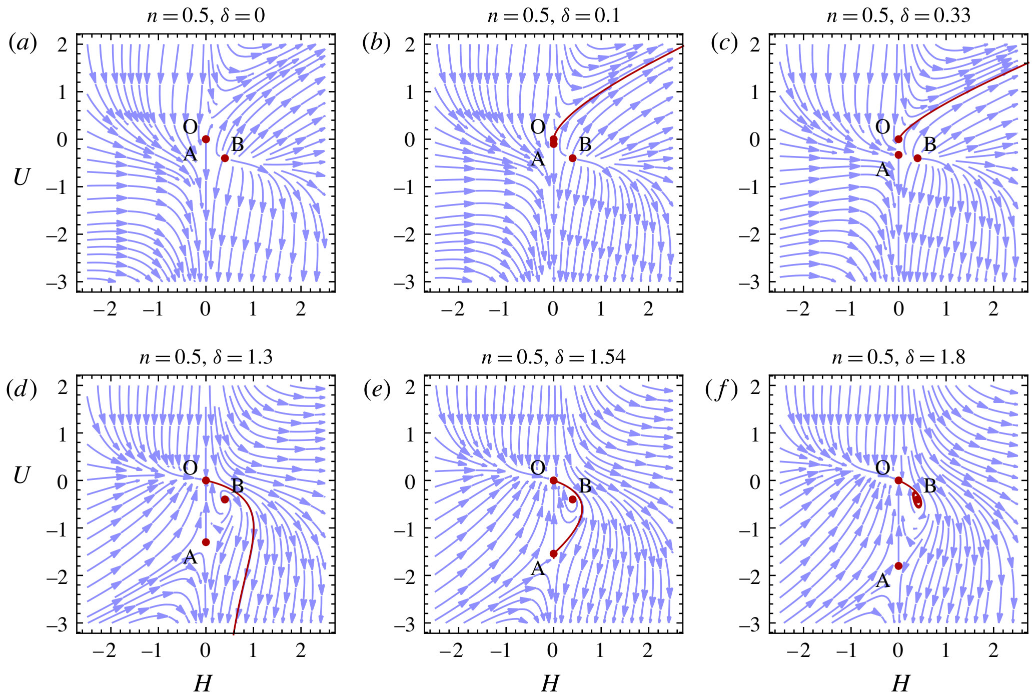

FIGURE 4. (Colour online) Phase portrait of (2.14*a*) with fixed $n = 0.5$ and changing δ: (*a*) $\delta = 0$; (*b*) $\delta = 0.1$; (*c*) $\delta = 0.33$; (*d*) $\delta = 1.3$; (*e*) $\delta = 1.54$; (*f*) $\delta = 1.8$. From (2.16) we see that $\delta = 0$ (*a*) is a degenerate case in which point O is no longer a saddle. In (*b*) and (*c*), the highlighted integral curves (shown in red online) starting at point O in the direction of a neutral eigenvector diverge to infinity. At $\delta = 1/(2-2n)$ $(=1$ for $n = 0.5)$, the neutral eigenvectors flip directions, and now an integral curve starting at point O and passing near point B in the direction of point A exists. At $\delta = \delta_c$, the highlighted integral curve (shown in red online) connects point O to point A as shown in (*e*), which means that $\delta = 1.54\ldots$ is the solution δ_c to the nonlinear eigenvalue problem, and the highlighted integral curve $U(H)$ corresponds to a self-similar solution of the second kind. For $\delta > \delta_c$ (*f*), trajectories starting from point O spiral into point B and, thus, cannot reach point A.

and point $\tilde{\mathrm{O}} : (\tilde{H}, \tilde{U}) = (0, 0)$ in the rescaled coordinates, approach a limiting shape as $n \to 1^-$, as shown in figure 7. Equation (2.14*a*) can be rewritten in terms of $\tilde{H}$ and $\tilde{U}$; then, near point $\tilde{\mathrm{A}}$, which represents the nose of the current, the behaviour is given by

$$\frac{\mathrm{d}\tilde{U}}{\mathrm{d}\tilde{H}} = \frac{\tilde{H}[(2-\epsilon)\tilde{U} - 2\epsilon + 1/\delta] - \tilde{U}(\tilde{U}+1)}{\tilde{H}[2\epsilon\tilde{H} + \tilde{U}]} \sim \frac{2\tilde{H} - \tilde{U} - 1}{\tilde{H}} \quad \text{as } \epsilon \to 0^+. \tag{2.17}$$

The latter captures the asymptotic behaviour of the heteroclinic trajectory near the nose of the current (point $\tilde{\mathrm{A}}$), namely $\tilde{H} \sim \tilde{U} + 1$, shown as the dotted line in figure 7. However, the $\epsilon \to 0^+$ behaviour of the ODE does not satisfy the boundary condition near point $\tilde{\mathrm{O}}$. A more complete analysis would provide the solution's behaviour near point $\tilde{\mathrm{O}}$.

It should be noted that when the current is moving toward the origin, the formulation of the nonlinear eigenvalue problem and phase-plane analysis does not

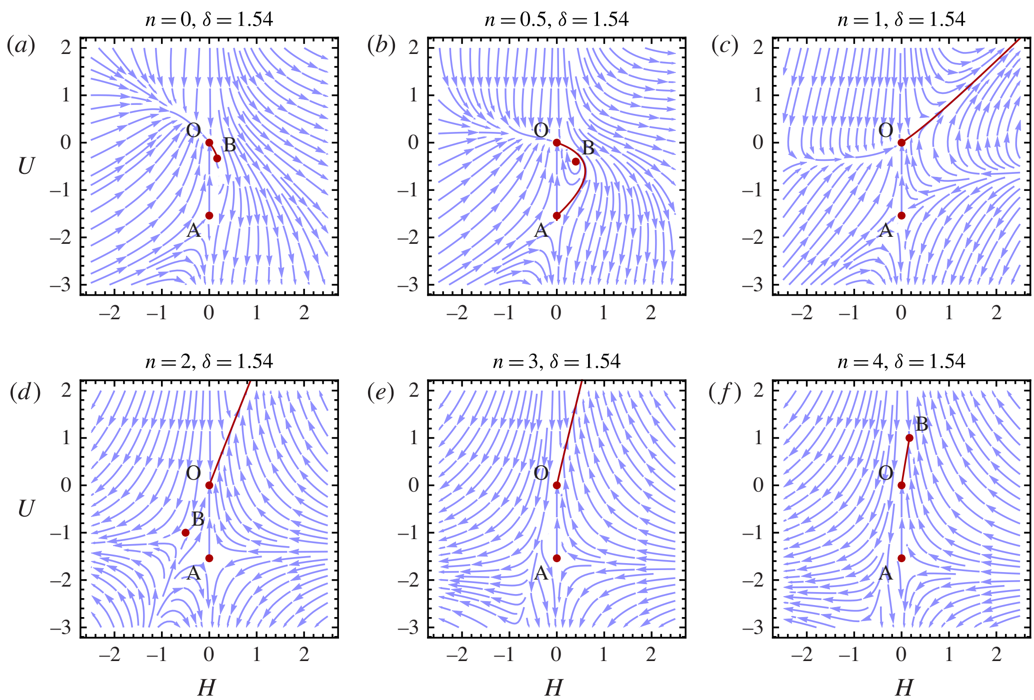

FIGURE 5. (Colour online) Phase portrait with different values of n and $\delta = 1.54$: (*a*) $n=0$; (*b*) $n=0.5$; (*c*) $n=1.8$; (*d*) $n=2$; (*e*) $n=3$; (*f*) $n=4$. Changing the value of the parameter n changes the location of the fixed point B, which can lead to the nonlinear eigenvalue problem for δ having no solution. As seen in (*c*)–(*f*), a self-similar solution of the second kind, which corresponds to the heteroclinic trajectory from O to A, does not exist for $n \geqslant 1$. Note that, for $n=1$ and $n=3$, point B is at infinity, thus it is not shown in (*c*) and (*e*). The highlighted integral curves (shown in red online) start at point O in a neutral eigendirection, $\boldsymbol{e}_2$ from (2.16).

make use of the global mass conservation equation (2.3). Therefore, the injection rate $\gamma Q t^{\gamma-1}$ does not affect the existence of second-kind self-similar solutions. However, Q and γ may affect the validity of the model assumptions by changing the aspect ratio of the gravity current; in particular, $|\partial h/\partial x| \ll 1$ (lubrication) may fail to hold or the drag at the bottom plate may become important if $h = O(b)$ throughout most of the current. In either case, the values of Q and γ will change the initial transition period before the gravity current's propagation becomes self-similar in this case.

2.2. *Gravity currents in heterogeneous porous media*

In this subsection, we consider a related problem: the propagation of gravity currents in heterogeneous porous media in which the permeability and porosity follow power laws in the horizontal direction. We place the origin at the location where the permeability and porosity vanish, as shown in figure 8. Then, the permeability and porosity are given by $k_p(x) = k_1 x^n$ and $\phi_p(x) = \phi_1 x^m$, respectively, where k_1, ϕ_1, n and m are non-negative real numbers, and x is the streamwise coordinate as before. In general, n and m are not independent of each other. In practice, $2 < n/m < 3$, where $n/m = 2$ corresponds to the extreme case of a porous medium composed of tubular

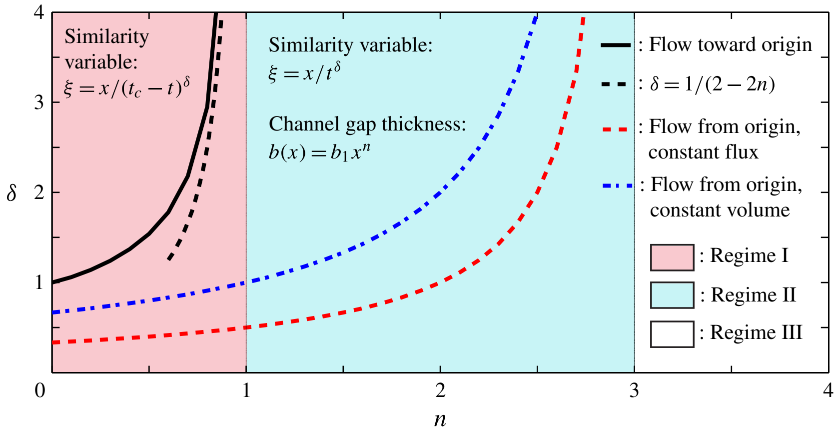

FIGURE 6. (Colour online) Dependence of the exponent of the similarity variable on the gap thickness power-law exponent for various self-similar solutions for viscous gravity currents propagating in a horizontal channel with gap thickness variations in power-law form. For flow from the origin, first-kind self-similar solutions exist in both regime I and regime II; for flow toward the origin, second-kind self-similar solutions exist only in regime I. No self-similar solutions exist in regime III. The exponent δ for the first-kind self-similar solutions was calculated from a scaling argument, see (2.4); the exponent δ for the second-kind self-similar solutions was calculated by solving numerically the nonlinear eigenvalue problem described below (2.16); the dashed black curve represents the asymptotic behaviour of the second-kind self-similar solution's exponent $\delta \sim 1/(2-2n)$ as $n \to 1^-$.

pores, while $n/m = 3$ corresponds to a porous medium composed of a network of intersecting fissures (Phillips 1991; Dullien 1992).

As before, we assume the two fluids are immiscible but we neglect surface tension effects. We also assume the current is long and thin, so the flow is mainly in the horizontal direction, and we neglect any motion in the displaced fluid. For clarity, the variables in this subsection are appended with the subscript 'p'. However, note that n still denotes the exponent in the assumed power-law variation of the permeability, just as in the Hele-Shaw case in § 2.1, but the expressions for $k(x)$ and $k_p(x)$ are not identical.

2.2.1. *Flow away from the origin*

When the gravity current is propagating horizontally from the origin to regions of higher permeability and porosity, we once again start from Darcy's law and the local continuity equation, which now take the form

$$u_p = -\frac{k_p(x)}{\mu}\frac{\partial p}{\partial x}, \tag{2.18a}$$

$$\phi_p(x)\frac{\partial h_p}{\partial t} + \frac{\partial}{\partial x}(h_p u_p) = 0. \tag{2.18b}$$

Substituting the hydrostatic pressure distribution, the posited permeability and porosity expressions, and combining the two equations in (2.18), we obtain a nonlinear

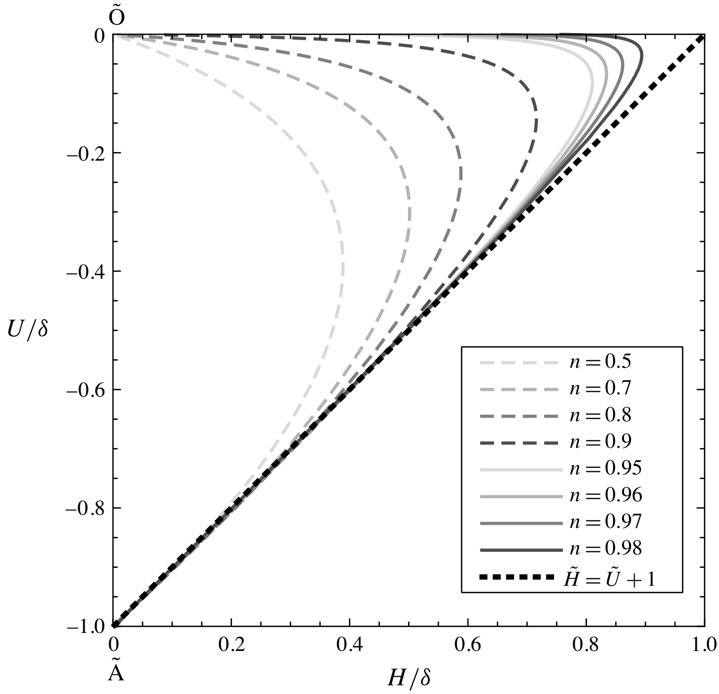

FIGURE 7. Shape of the heteroclinic trajectory of (2.14) as $n \to 1^-$ (regime I), in rescaled coordinates. The dotted line represents the asymptotic behaviour near point A, i.e. $H \sim U + \delta$.

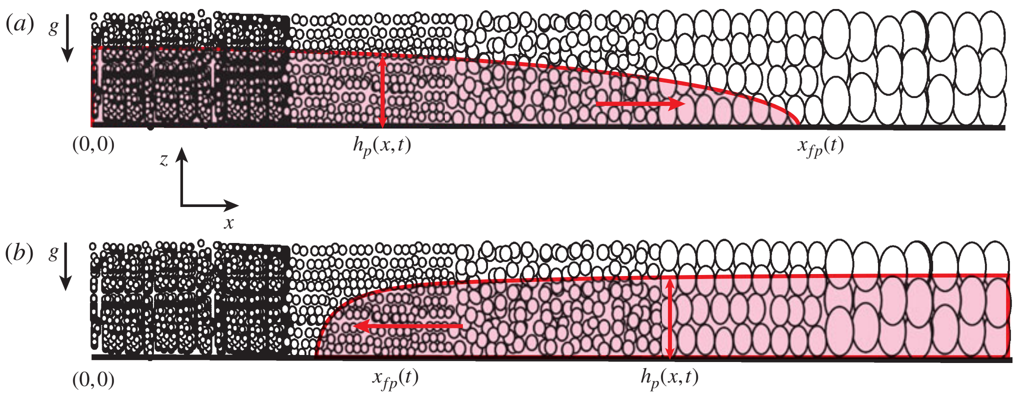

FIGURE 8. (Colour online) Diagram of gravity currents propagating along a horizontal boundary in heterogeneous porous media with power-law permeability $k_p(x) = k_1 x^n$ and porosity $\phi_p(x) = \phi_1 x^m$ variations, which give rise to gradients in the horizontal direction. (*a*) A gravity current propagating away from the origin in the direction of increasing permeability and porosity. (*b*) A gravity current propagating toward the origin from far away. Gravity is directed in the negative z direction as before; $h_p(x, t)$ and $x_{fp}(t)$ represent the shape and location of the nose of the current, respectively.

diffusion equation for the shape $h_p(x, t)$ of the gravity current:

$$\frac{\partial h_p}{\partial t} - \frac{A_p}{x^m}\frac{\partial}{\partial x}\left(x^n h_p \frac{\partial h_p}{\partial x}\right) = 0, \tag{2.19}$$

where $A_p = (\Delta\rho g k_1)/(\mu\phi_1)$. The global mass conservation constraint takes the form

$$\int_0^{x_{fp}(t)} x^m h_p(x, t)\,\mathrm{d}x = B_p t^\gamma, \tag{2.20}$$

where, similar to before, $B_p = Q_p/\phi_1$, and $\gamma Q_p t^{\gamma-1}$ is the injection rate. For clarity, we now denote the front position as $x_{fp}(t)$, where $h_p\left(x_{fp}(t), t\right) = 0$.

As in § 2.1, through a scaling analysis of (2.19) and (2.20), we find a similarity variable

$$\xi = \frac{x}{(AB)^{1/(2m-n+3)} t^{(\gamma+1)/(2m-n+3)}}. \tag{2.21}$$

Then, the location of the nose of the current is given by

$$x_{fp}(t) = \xi_{fp}(m, n, \gamma)(AB)^{1/(2m-n+3)} t^{(\gamma+1)/(2m-n+3)}. \tag{2.22}$$

In terms of the similarity variable, the shape of the current is given by

$$h_p(x, t) = \xi_{fp}^{m-n+2} A^{-(m+1)/(2m-n+3)} B^{(m-n+2)/(2m-n+3)} t^{((m-n+2)\gamma-(m+1))/(2m-n+3)} f_p(y), \tag{2.23}$$

where $y \equiv x/x_{fp}(t) = \xi(x, t)/\xi_{fp}(m, n, \gamma)$. Then, $f_p(y)$ and $\xi_{fp}(m, n, \gamma)$ are obtained by solving the following system:

$$(y^n f_p f_p')' + \left(\frac{\gamma+1}{2m-n+3}\right) y^{m+1} f_p' - \left(\frac{(m-n+2)\gamma-(m+1)}{2m-n+3}\right) y^m f_p = 0, \tag{2.24a}$$

$$f_p(1) = 0, \tag{2.24b}$$

$$\xi_{fp}(m, n, \gamma) = \left(\int_0^1 y^m f_p(y)\,\mathrm{d}y\right)^{1/(n-2m-3)}. \tag{2.24c}$$

For $n = 3m$, equation (2.24) reduces to (2.7) with n replaced by m. Also, when $m = n = 0$, equation (2.24) reduces to the homogeneous porous medium case in Cartesian coordinates, as above. Meanwhile, when $m = n = 1$, equation (2.24a) reduces to the equation for homogeneous case in axisymmetric coordinates (Lyle *et al.* 2005).

Again, we can determine the asymptotic behaviour of the current near the nose from (2.24a,b):

$$f_p(y) \sim \left(\frac{\gamma+1}{2m-n+3}\right)(1-y) \quad \text{as } y \to 1^-, \tag{2.25}$$

which provides two boundary conditions near $y = 1$, i.e. $f_p(1-\nu)$ and $f_p'(1-\nu)$, $\nu \ll 1$, in a shooting procedure for solving (2.24a) numerically. Typical shapes, $f_p(y)$, of the current are show in figure 9 for (*a*) constant volume ($\gamma = 0$) and (*b*) constant injection rate ($\gamma = 1$) and various choices of m and n. The dependence of ξ_{fp} on γ is illustrated in figure 9(*c*) for various choices of m and n.

For $\gamma = 0$ and $0 \leqslant n < 2m + 3$, it can be shown that (2.24) possesses the exact solution

$$f_p(y) = \begin{cases} \dfrac{1}{(m-n+2)(2m-n+3)}\left(1 - y^{m-n+2}\right), & n \neq m+2, \\ -\dfrac{1}{m+1}\ln y, & n = m+2, \end{cases} \tag{2.26a}$$

$$\xi_{fp}(m, n, 0) = \left[(m+1)(2m-n+3)^2\right]^{1/(2m-n+3)}. \tag{2.26b}$$

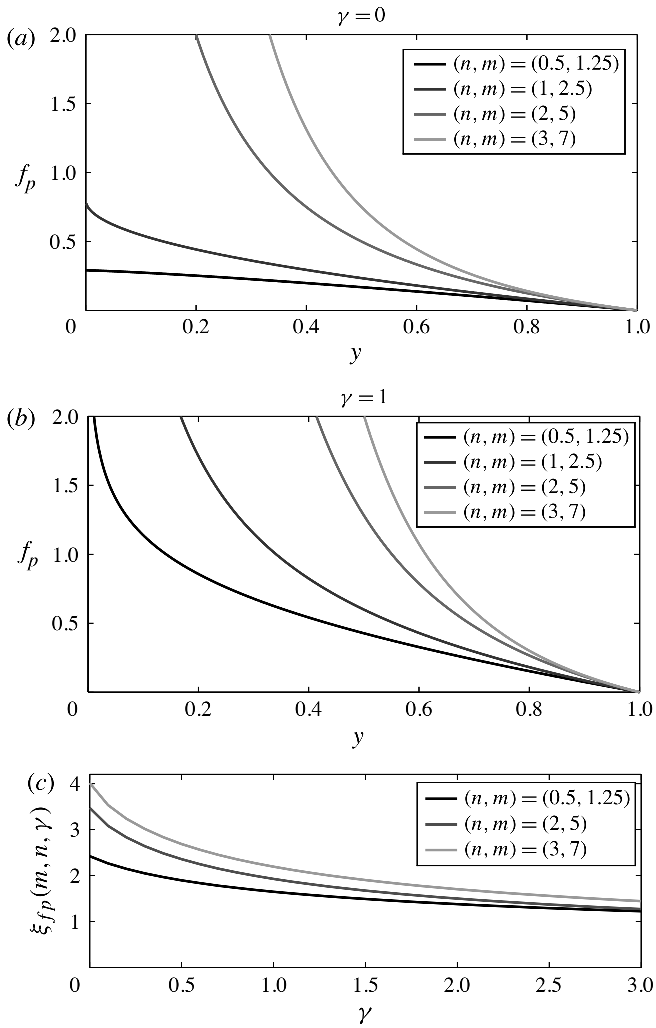

FIGURE 9. Self-similar shape of a viscous gravity current propagating away from the origin in a heterogeneous porous medium with variable permeability $k_p(x) = k_1 x^n$ and porosity $\phi_p(x) = \phi_1 x^m$ for (*a*) constant volume, $\gamma = 0$, and (*b*) constant injection rate, $\gamma = 1$, and various choices of n and m; the prefactor $\xi_{fp}(m, n, \gamma)$ is shown in (*c*). The solutions for f_p were found by integrating (2.24) numerically by shooting backwards from $y = 1 - \nu$ ($\nu = 10^{-4}$), where boundary conditions on f_p and f_p' are imposed using the asymptotic form given from (2.25) .

2.2.2. *Flow toward the origin*

When the gravity current is moving horizontally toward the origin, i.e. in the direction from higher to lower permeability, in a porous medium, equation (2.18) still holds. However, as above, we cannot determine the self-similar solution by scaling alone, so we introduce a phase-plane analysis to seek a second-kind similarity solution. We begin with (2.18), i.e. the coupled equations for the transversely averaged fluid velocity and continuity. First, we substitute in the hydrostatic pressure distribution, the permeability and porosity expressions into (2.18) to obtain

$$u_p = -\frac{\Delta\rho g k_1 x^n}{\mu}\frac{\partial h_p}{\partial x}, \tag{2.27a}$$

$$\frac{\partial h_p}{\partial t} + \frac{1}{\phi_1 x^m}\frac{\partial (h_p u_p)}{\partial x} = 0. \tag{2.27b}$$

Again, assuming $t_c < \infty$, we let $\tau = t_c - t$ and, then, introduce dimensionless variables via

$$u_p(x, t) = \phi_1 \frac{x^{m+1}}{\tau} U_p(x, \tau), \quad h_p(x, t) = \left(\frac{\mu\phi_1}{\Delta\rho g k_1}\right)\frac{x^{m-n+2}}{\tau} H_p(x, \tau). \tag{2.28a,b}$$

Substituting the latter into (2.27), we obtain

$$U_p + x\frac{\partial H_p}{\partial x} + (m-n+2)H_p = 0, \tag{2.29a}$$

$$\tau\frac{\partial H_p}{\partial \tau} - H_p - x\frac{\partial (H_p U_p)}{\partial x} - (2m-n+3)H_p U_p = 0. \tag{2.29b}$$

Now, if the flow is to be self-similar, we should be able to reduce the x and t dependence to dependence on a single similarity variable $\xi \equiv x/\tau^\delta$, where δ remains to be determined, so that H_p and U_p become functions of ξ alone. Under this assumption, equation (2.29) becomes

$$U_p + \xi H_p' + (m-n+2)H_p = 0, \tag{2.30a}$$

$$\delta\xi H_p' + H_p + \xi(H_p U_p)' + (2m-n+3)H_p U_p = 0. \tag{2.30b}$$

Finally, we eliminate ξ from (2.30) to find

$$\frac{\mathrm{d}U_p}{\mathrm{d}H_p} = \frac{H_p[(m+1)U_p - (m-n+2)\delta + 1] - U_p(U_p+\delta)}{H_p[(m-n+2)H_p + U_p]}, \tag{2.31a}$$

$$\frac{\mathrm{d}\ln|\xi|}{\mathrm{d}H_p} = -\frac{1}{U_p + (m-n+2)H_p}. \tag{2.31b}$$

Thus, we have constructed the phase plane (H_p, U_p) for a gravity current propagating toward the origin in a heterogeneous porous medium. Equation (2.31a) has three finite critical points:

$$\mathrm{O}: (H_p, U_p) = (0, 0), \tag{2.32a}$$

$$\mathrm{A}: (H_p, U_p) = (0, -\delta), \tag{2.32b}$$

$$\mathrm{B}: (H_p, U_p) = \left(\frac{1}{(m-n+2)(2m-n+3)}, -\frac{1}{2m-n+3}\right). \tag{2.32c}$$

As in § 2.1.2, when the current is propagating toward the origin, the integral curve that connects point O to point A corresponds to the similarity solution of the second kind. Depending on the values of n and m, we can numerically determine the value of δ using the numerical technique outlined after (2.16), where the linearized eigenspace at point O is now

$$\lambda_1 = -\delta, \quad \boldsymbol{e}_1 = (0, \pm 1), \tag{2.33a}$$

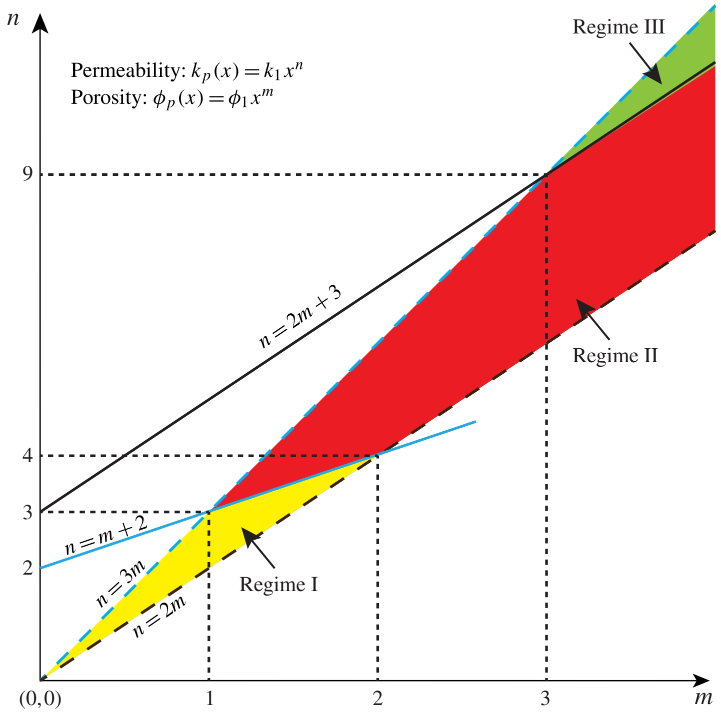

FIGURE 10. (Colour online) Flow regimes for gravity current propagation in heterogeneous porous media with power-law permeability $k_p(x) = k_1 x^n$ and porosity $\phi_p(x) = \phi_1 x^m$ variations in the horizontal direction. When a gravity current is propagating toward the origin, a self-similar solution exists only in regime I, and it is of the second kind. When a gravity current is propagating away from the origin, a first kind self-similar solution is found in both regimes I and II. In regime III, there are neither advancing nor physically meaningful self-similar solutions.

$$\lambda_2 = 0, \quad \boldsymbol{e}_2 = \pm\left(\frac{\delta}{1-(2+m-n)\delta}, 1\right). \tag{2.33b}$$

The types of self-similar solutions that can be expected on the basis of this analysis are summarized in figure 10, in which we colour different regions of the (m, n) plane to illustrate the possible behaviours. As noted earlier, $n/m = 2$ and $n/m = 3$ correspond to two limiting cases of the porous medium's microstructure. Restricting ourselves only to (m, n) such that $2 < n/m < 3$, self-similar solutions are to be expected in the wedge $2m < n < 3m$ as shown by the dashed lines in figure 10. Furthermore, for $n < m + 2$, we see from (2.32*c*) that point B is in the fourth quadrant of the (H_p, U_p) phase plane, thus we expect second-kind self-similar solutions in this region (regime I) of the (m, n) plane. The region in which $m + 2 < n < 2m + 3$ (in addition to $2m < n < 3m$) is denoted as regime II, in which we only expect self-similar solutions of the first kind. The latter are advancing fronts because the exponent of t in (2.22) is positive for (m, n) in this regime. Finally, for $n > 2m + 3$ (regime III), there are no advancing first-kind self-similar solutions and, moreover, the first-kind self-similar solution is unphysical because it predicts $f_p < 0$. This classification is analogous to that in figure 6.

Experiment	Liquid (% glycerol)	n	b_1 (m^{1-n})	L_{cell} (m)	x_0 (m)	w (kg)	t_c (s)
No. 1	100 %	0.2	0.01589	0.75	0.560	0.1489	86.0
No. 2	100 %	0.2	0.01589	0.75	0.550	0.1569	66.1
No. 3	100 %	0.2	0.01589	0.75	0.550	0.1480	78.7
No. 4	100 %	0.5	0.01732	0.75	0.575	0.1731	89.0
No. 5	100 %	0.5	0.01732	0.75	0.578	0.1602	104.1
No. 6	95 %	0.5	0.01732	0.75	0.578	0.1559	54.9
No. 7	95 %	0.5	0.01732	0.75	0.600	0.1228	76.7
No. 8	90 %	0.5	0.01732	0.75	0.585	0.1556	23.0
No. 9	90 %	0.5	0.01732	0.75	0.581	0.1574	22.3
No. 10	100 %	0.8	0.03776	0.75	0.530	0.3169	81.5
No. 11	100 %	0.8	0.03776	0.75	0.515	0.3494	64.5
No. 12	100 %	0.8	0.03776	0.75	0.525	0.2464	102.5

TABLE 1. Summary of the parameters of the different experiments that we performed of viscous gravity currents propagating toward the origin of a horizontal channel. Glycerol–water solutions were used with various glycerol mass concentrations, as shown in the second column. The gap thickness of the horizontal channel is given by $b(x) = b_1 x^n$, L_{cell} is the total length of the channel, x_0 is the location of the lock gate, w is the total weight of the liquid in the cell and t_c is the time when the moving front reaches the origin in an experiment.

3. Results from experiments and numerical simulation

We conducted a series of constant-volume gravity current experiments in horizontal Hele-Shaw cells whose gap thicknesses vary as power laws of the streamwise coordinate. We also solved numerically the corresponding governing PDE (2.2), which we derived above under the lubrication approximation. The numerical scheme is described in appendix B. In this section, we present a discussion of and a comparison between the experiments, numerical simulations and the theoretical considerations from self-similar intermediate asymptotics for the case of a gravity current propagating toward the origin of a horizontal channel.

We designed three Hele-Shaw cells with different power-law shapes: (*a*) $n = 0.2$, (*b*) $n = 0.5$ and (*c*) $n = 0.8$; recall figure 2. The cells were constructed from scratch-resistant clear cast acrylic sheets (McMaster-Carr, No. 8560K247) using an automatic manufacturing machine. The liquids we used were glycerol–water solutions with various glycerol concentrations, whose physical properties were looked up in published tables. The glycerol is coloured with food dye so the profile shapes of the liquid can be recorded at different times using a USB camera.

We begin each experiment by setting up a lock gate at a certain location $x = x_0$ from the origin, where x_0 was chosen near the end of the channel ($x = L_{cell}$) to allow for a larger distance over which the gravity current can propagate. We then filled the gap between the lock and the end with the glycerol–water solution and suddenly removed the lock gate, allowing the fluid to propagate toward the origin ($x = 0$). We recorded the location of the propagating front $x_f(t)$ at different times.

Details of different experimental designs are summarized in table 1. In these experiments, we can vary different parameters: (*a*) the gap thickness; (*b*) the glycerol concentration, thus the viscosity and density of the fluid; (*c*) the location x_0 of the lock gate; and (*d*) the initial volume of liquid placed behind the lock gate. To justify the lubrication approximation utilized in § 2, we estimate the product of the aspect

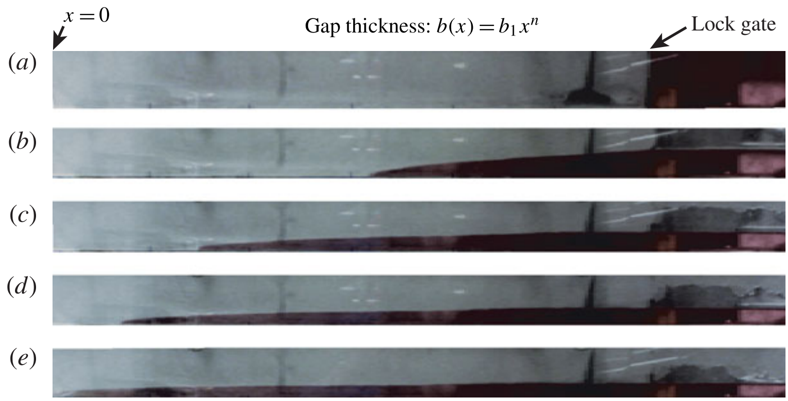

FIGURE 11. (Colour online) Time-lapse images of a viscous gravity current (experiment no. 8 from table 1) propagating toward the origin of a horizontal channel (as in figures 1*d* and 2*b*) at (*a*) $t = 0$ s, (*b*) $t = 1.5$ s, (*c*) $t = 6.5$ s, (*d*) $t = 11.5$ s and (*e*) $t = t_c \approx 22.3$ s, which is when the front is observed to reach the origin ($x_f(t_c) = 0$). The flow was generated by an instantaneous removal of a vertical lock gate. Contrast has been digitally enhanced for visual clarity.

ratio and the Reynolds number: the typical channel width is given by $b_1 x_0^n$, the typical length scale is x_0, the velocity scale is x_0/t_c, where t_c is the time for the front to reach the origin, ρ and μ are the density and viscosity, respectively, of the glycerol solution ($\rho = 1261$ kg m^{-3} and $\mu = 1.412$ Pa s for 100 % glycerol). Thus, for a typical experiment (e.g. experiment no. 1 in table 1), the product of the aspect ratio and Reynolds number is $\rho b_1^2 x_0^{2n}/(\mu t_c) \approx 2 \times 10^{-3} \ll 1$. Also, in a typical experiment, $h/b > 3$ for most of the current (i.e. except at the nose), therefore, drag due to the bottom plate can be neglected. Furthermore, in a typical experiment, the gap thickness b is such that $b(x) > 3$ mm, which is the capillary length for the fluids we use, for 3 cm $< x < L_{cell} = 75$ cm, i.e. for 96 % of the length of the Hele-Shaw cell. Therefore, surface tension can be neglected as well. The fact that drag due to the bottom plate and surface tension can potentially be important near the nose of the current does not affect its global self-similar behaviour, as is evidenced by our experiments and those in the literature.

For Hele-Shaw cells with $n < 1$, we measured a finite critical time t_c at which the front reaches the origin. In figure 12, we plot the rescaled front location $x_f(t)/x_0$ versus the dimensionless time-to-touchdown $\tau/t_c = 1 - t/t_c$ for different experimental conditions. The assumption of self-similarity of the second kind leads us to expect that $x_f(t)/x_0 \propto (1 - t/t_c)^\delta$. Numerical simulation results for $x_f(t)/x_0$ are also plotted on the same figure. As can be seen in figure 12, the experimental results agree very well with the numerical simulation for all of the geometries considered. In particular, as the front approaches the origin, i.e. as $1 - t/t_c \to 0^+$, the initial shape of the current is forgotten, and both the experimental and numerical simulation data for the rescaled location of the nose of the current versus dimensionless time-to-touch-down fall on a straight line in this log–log plot. Therefore, the behaviour is self-similar, as we predicted in our theoretical discussion in § 2.1.2. Furthermore, the values that the slopes of the curves asymptote to agree very well with the predictions from the theory, namely, $\delta \approx 1.14$ for $n = 0.2$, $\delta \approx 1.54$ for $n = 0.5$ and $\delta \approx 2.95$ for $n = 0.8$, where these slopes are predicted via the second-kind phase-plane analysis (recall figure 4 and its discussion).

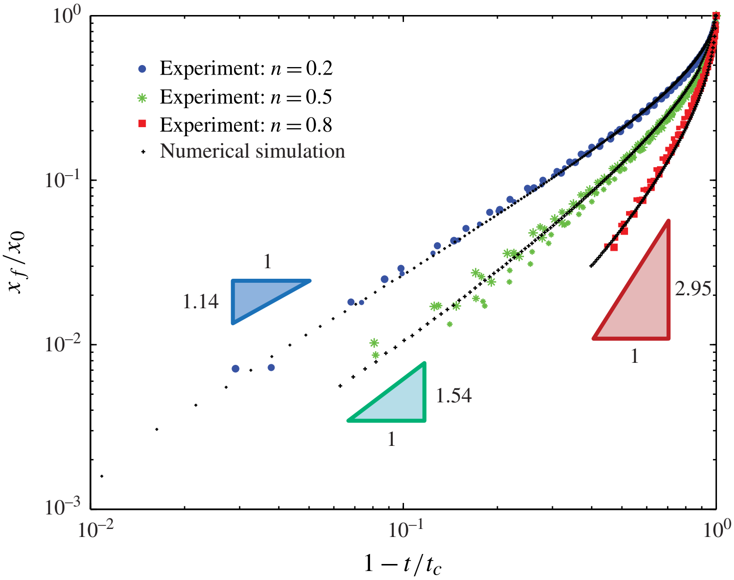

FIGURE 12. (Colour online) Constant-volume gravity currents propagating in Hele-Shaw cells with gap thicknesses of power-law form: $b(x) = b_1 x^n$. Comparison of experiments and numerical simulations for propagation toward the origin. Experiments in three different Hele-Shaw cells were performed: $n = 0.2$ (circles), $n = 0.5$ (stars) and $n = 0.8$ (squares). In both experiments and numerical simulations (crosses), self-similar solutions of the second kind were observed for the three different geometries and the similarity exponents (i.e. the intermediate asymptotic slope on the plot) are in good agreement with the theory (large triangles).

4. Summary and conclusions

In this paper, we investigated the effects of horizontal heterogeneity on the propagation of viscous gravity currents, with an emphasis on second-kind self-similar behaviour. Two geometries were studied in detail as illustrative examples: (*a*) horizontal channels or cracks with a varying gap thickness of power-law form and (*b*) heterogeneous porous media with power-law permeability and porosity variations. In each case, two flow patterns were considered: (*a*) gravity currents propagating away from the origin (defined as the point of vanishing permeability) and (*b*) gravity currents propagating toward the origin.

We employed experimental, theoretical and numerical techniques to study the flow behaviour. Our key findings are: (i) when a viscous gravity current propagates away from the origin in this heterogeneous medium, the behaviour is described by a self-similar solution of the first kind, as is to be expected based on similar problems in the literature; (ii) when the current propagates toward the origin in these models with horizontal heterogeneity, the behaviour is described by self-similarity of the second kind with a non-trivial exponent in the similarity variable. Depending on the form of the heterogeneity, different flow regimes are identified corresponding to the existence of either first-kind or both first- and second-kind self-similarity.

Our study may have possible connections to industrial processes and geological flows in porous media, including recent applications in geological CO_2 storage (see, e.g., Class *et al.* 2009; Zheng *et al.* 2010) and shale gas recovery (see, e.g., Monteiro, Rycroft & Barenblatt 2012). For example, in the presence of horizontal heterogeneity, the propagation regime of a CO_2 plume can change. In the case of shale gas recovery, the propagation of a gas driven by buoyancy in the horizontal channels created by hydraulic fracturing is affected by the shape of the passage. Our study presents and benchmarks a possible theoretical approach to understanding the various flow regimes in a certain class of fissure shapes in a porous medium or crack shapes in a reservoir. Specifically, we show how to obtain the various self-similar behaviours, even for the cases where scaling alone cannot reveal such dynamics.

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Appendix A. Phase-plane formalism for gravity currents propagating toward the origin in a homogeneous porous medium

A.1. *Axisymmetric gravity currents in a homogenous porous medium*

Second-kind self-similar solutions also exist for axisymmetric gravity currents propagating in homogeneous porous media (i.e. constant permeability k and porosity ϕ) as depicted in figure 1(b). We denote the shape of the gravity current by $h_a(r, t)$ and its propagation velocity by $u_a(r, t)$, as before, then Darcy's law and the continuity equation are

$$u_a = -\frac{\Delta\rho g k}{\mu}\frac{\partial h_a}{\partial r}, \tag{A 1a}$$

$$\phi\frac{\partial h_a}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r h_a u_a) = 0, \tag{A 1b}$$

respectively. Following the procedure from the earlier sections, for $t_c < \infty$, we let $\tau = t_c - t$ and

$$u_a(r, t) = \phi\frac{r}{\tau}U_a(r, \tau), \quad h_a(r, t) = \left(\frac{\mu\phi}{\Delta\rho g k}\right)\frac{r^2}{\tau}H_a(r, \tau). \tag{A.2a,b}$$

If the flow is to be self-similar, then U_a and H_a should depend solely on a similarity variable $\xi \equiv r/\tau^\delta$, hence (A.1) becomes

$$\frac{\mathrm{d}U_a}{\mathrm{d}H_a} = \frac{H_a(2U_a - 2\delta + 1) - U_a(U_a + \delta)}{H_a(2H_a + U_a)}, \tag{A 3a}$$

$$\frac{\mathrm{d}\ln|\xi|}{\mathrm{d}H_a} = -\frac{1}{U_a + 2H_a}. \tag{A 3b}$$

We note that (A3) corresponds to the nonlinear heat conduction problem described in (15) of Gratton & Minotti (1990) with $m = n = 1$ (in their notation).

Equation (A 3a) has three finite critical points:

$$\begin{aligned} \mathrm{O}:(H_a, U_a) &= (0, 0), && \text{(A 4a)}\\ \mathrm{A}:(H_a, U_a) &= (0, -\delta), && \text{(A 4b)}\\ \mathrm{B}:(H_a, U_a) &= \left(\tfrac{1}{8}, -\tfrac{1}{4}\right). && \text{(A 4c)} \end{aligned}$$

A second-kind self-similar solution is found for $\delta = \delta_c$ such that there exist a heteroclinic orbit from point O to point A in the phase plane. Since point B is in the fourth quadrant in this physical situation, and there are no parameters in this model, there always exist both first- and second-kind self-similar solutions.

A.2. *Gravity currents in a homogenous porous medium with converging boundaries*

Second-kind self-similar solutions also exist for a viscous gravity current propagating in a homogeneous porous medium (i.e. constant permeability k and porosity ϕ) with converging boundaries of power-law form $b_c(x) = b_{c1}x^n$ as depicted in figure 1(*d*). We denote the shape of the gravity current by $h_c(x, t)$ and its propagation velocity by $u_c(x, t)$, then Darcy's law and the continuity equation are

$$u_c = -\frac{\Delta\rho g k}{\mu}\frac{\partial h_c}{\partial x}, \qquad \text{(A 5a)}$$

$$\phi\frac{\partial h_c}{\partial t} + \frac{1}{x^n}\frac{\partial}{\partial x}(x^n h_c u_c) = 0, \qquad \text{(A 5b)}$$

respectively. Again, following the same procedure as before, for $t_c < \infty$, we let $\tau = t_c - t$ and

$$u_c(x, t) = \phi\frac{x}{\tau}U_c(x, \tau), \quad h_c(x, t) = \left(\frac{\mu\phi}{\Delta\rho g k}\right)\frac{x^2}{\tau}H_c(x, \tau). \qquad \text{(A6a,b)}$$

If the flow is to be self-similar, H_c and U_c depend only on a similarity variable $\xi \equiv x/\tau^\delta$, hence (A5) becomes

$$\begin{aligned} \frac{\mathrm{d}U_c}{\mathrm{d}H_c} &= \frac{H_c[(n+1)U_c - 2\delta + 1] - U_c(U_c + \delta)}{H_c(2H_c + U_c)}, && \text{(A 7a)}\\ \frac{\mathrm{d}\ln|\xi|}{\mathrm{d}H_c} &= -\frac{1}{U_c + 2H_c}. && \text{(A 7b)} \end{aligned}$$

Equation (A 7a) has three finite critical points:

$$\begin{aligned} \mathrm{O}:(H_c, U_c) &= (0, 0), && \text{(A 8a)}\\ \mathrm{A}:(H_c, U_c) &= (0, -\delta), && \text{(A 8b)}\\ \mathrm{B}:(H_c, U_c) &= \left(\frac{1}{2(n+3)}, -\frac{1}{n+3}\right). && \text{(A 8c)} \end{aligned}$$

A second-kind self-similar solution exists for $\delta = \delta_c$ such that there exist a heteroclinic orbit from point O to point A in the phase plane. We expect that both first- and second-kind self-similar solutions exist for all n, since point B is in the fourth quadrant for all n in this physical situation.

Appendix B. Numerical scheme for nonlinear diffusion equations with spatially varying coefficients

B.1. *Preliminaries*

In this section, we develop a new finite-difference numerical method capable of handling the nonlinear parabolic PDEs that arise in this work. To this end, let us use the following notation for the purposes of this section:

$$\frac{\partial h}{\partial t} = \frac{A}{x^p}\frac{\partial}{\partial x}\left(x^q h^s \frac{\partial h}{\partial x}\right), \quad (x, t) \in (0, L) \times (0, t_f). \tag{B 1}$$

Here, A, p, q and s are positive real numbers; for example, $p = n$, $q = 3n$ and $s = 1$ gives (2.2), while $h = h_p$, $A = A_p$, $p = m$, $q = n$ and $s = 1$ gives (2.19). In general, p and q are not arbitrary, but determined by the physics at hand. Here L is the size of the computational domain and t_f is the final time in the simulation.

Rather than solving a moving boundary-value problem on $(0, x_f(t))$, we prefer to solve the problem numerically on a fixed domain $(0, L)$ and impose appropriate boundary conditions at $x = 0$ and $x = L$ to automatically enforce the global mass conservation constraint from, e.g. (2.3). To establish the appropriate boundary conditions, we begin by multiplying (B 1) by x^p and integrating from $x = 0$ to $x = L$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L x^p h(x, t)\,\mathrm{d}x = A\left[x^q h^s \frac{\partial h}{\partial x}\right]_0^L. \tag{B 2}$$

Now, we enforce the constraint on the total mass by noting that $h(x, t) = 0$ for $x \geqslant x_f(t)$, hence $\int_0^{x_f} x^p h\,\mathrm{d}x = \int_0^L x^p h\,\mathrm{d}x = Bt^\gamma \Rightarrow \mathrm{d}/\mathrm{d}t \int_0^L x^p h\,\mathrm{d}x = \gamma B t^{\gamma-1}$ $(\gamma \neq 0)$. Now, we require that

$$A\left[x^q h^s \frac{\partial h}{\partial x}\right]_0^L \equiv AL^q(h^s h_x)\Big|_{x=L} - A(x^q h^s h_x)\Big|_{x\to 0} = \begin{cases}\gamma B t^{\gamma-1}, & \gamma \neq 0,\\ 0, & \gamma = 0.\end{cases} \tag{B 3}$$

Assuming injection from a single boundary, we can derive two sets of boundary conditions for the PDE from (B 3):

$$(x^q h^s h_x)|_{x\to 0} = \begin{cases}-\dfrac{\gamma B}{A}t^{\gamma-1}, & \gamma \neq 0,\\ 0, & \gamma = 0,\end{cases} \tag{B 4a}$$

$$(h^s h_x)|_{x=L} = 0 \Rightarrow h_x|_{x=L} = 0, \tag{B 4b}$$

and

$$(x^q h^s h_x)|_{x\to 0} = 0 \Rightarrow h_x|_{x=0} = 0, \tag{B 5a}$$

$$(h^s h_x)|_{x=L} = \begin{cases}\dfrac{\gamma B}{AL^q}t^{\gamma-1}, & \gamma \neq 0,\\ 0, & \gamma = 0.\end{cases} \tag{B 5b}$$

Thus, by imposing either set of boundary conditions above, we automatically satisfy the global mass conservation constraint. Note that for injection $(\gamma \neq 0)$ at the origin $(x = 0)$ we need $(x^q h^s h_x)|_{x\to 0}$ to be finite, which is only the case if $h^s h_x = O(1/x^q)$ as $x \to 0$, i.e. the profile and/or its slope blows up at $x = 0$. Clearly, in this case, the lubrication approximation of one-dimensional flow near the origin is violated locally, however, this does not have a significant effect on the gravity current away from this localized region (Huppert 1982; Golding, Huppert & Neufeld 2013). Alternatively, we could solve the PDE on a domain (ℓ, L), where $\ell \approx 0$, enforcing the flux condition $(x^q h^s h_x)|_{x=\ell} = -(\gamma B/A)t^{\gamma-1} \Rightarrow (h^s h_x)|_{x=\ell} = -(\gamma B/A\ell^q)t^{\gamma-1}$.

B.2. *Construction of a second-order-accurate scheme with internal iterations*

Now, we introduce the grid function $h_i^n \approx h(x_i, t^n)$ on the staggered grid $x_i = (i - 1/2)\Delta x$, where $\Delta x = L/N$. This means that the point $i=0$ is a half-spacing to the left of $x=0$, and the point $i=N+1$ is a half-spacing to the right of $x=L$; thus, $N+2$ is the total number of spatial grid points. Furthermore, we use the notation

$$\mathscr{L} h \equiv \frac{A}{x^p}\frac{\partial}{\partial x}\left(x^q h^s \frac{\partial h}{\partial x}\right), \tag{B 6}$$

and $\mathscr{L}_d$ denotes the discretized counterpart to $\mathscr{L}$, as defined below. We would like to construct a Crank–Nicolson scheme (Crank & Nicolson 1947; Strikwerda 2004) for (B 1) via the time discretization

$$\frac{h_i^{n+1} - h_i^n}{\Delta t} = \frac{1}{2}\left(\mathscr{L}_d h_i^{n+1} + \mathscr{L}_d h_i^n\right), \tag{B 7}$$

which is second-order accurate in time. Now, a second-order central difference approximation to $\mathscr{L}$ can be constructed by treating the 'non-Cartesian part' $(p, q \neq 0)$ via a weighted difference as in Christov & Homsy (2009):

$$\mathscr{L} h \approx \mathscr{L}_d h_i^n = \frac{A}{x_i^p}\left[\frac{x_{i+1/2}^q \psi_{i+1/2}^{n+1/2}\dfrac{(h_{i+1}^n - h_i^n)}{\Delta x} - x_{i-1/2}^q \psi_{i-1/2}^{n+1/2}\dfrac{(h_i^n - h_{i-1}^n)}{\Delta x}}{\Delta x}\right], \tag{B 8}$$

where we introduced an intermediate variable ψ to denote the grid function of the nonlinear term h^s. To have a nonlinear conservative extension of the Crank–Nicolson scheme for (B 1), this term has to be evaluated at the half-time step and on a staggered mesh with respect to x_i (Christov & Deng 2002), i.e.

$$\psi_{i+1/2}^{n+1/2} \equiv \tfrac{1}{2}\left\{\tfrac{1}{2}\left[(h_{i+1}^{n+1})^s + (h_i^{n+1})^s\right] + \tfrac{1}{2}\left[(h_{i+1}^n)^s + (h_i^n)^s\right]\right\}. \tag{B 9}$$

Note that if $p=q=s=0$, then (B 8) reduces to the standard three-point second-order central difference formula, namely $\partial^2 h/\partial x^2 \approx (h_{i+1}^n - 2h_i^n + h_{i-1}^n)/(\Delta x)^2$.

With (B 8) and (B 9) in mind, we shall iteratively find the grid function h_i^{n+1} at the new time stage by replacing it in (B 7) with $h_i^{n,k+1}$, where $h_i^{n,0} \equiv h_i^n$ (Yanenko 1971). Thus, the scheme becomes

$$\begin{aligned}\frac{h_i^{n,k+1} - h_i^n}{\Delta t} &= \frac{A}{2(\Delta x)^2}\left[\frac{x_{i+1/2}^q}{x_i^p}\psi_{i+1/2}^{n,k+1/2}(h_{i+1}^{n,k+1} - h_i^{n,k+1}) - \frac{x_{i-1/2}^q}{x_i^p}\psi_{i-1/2}^{n,k+1/2}(h_i^{n,k+1} - h_{i-1}^{n,k+1})\right] \\ &\quad + \frac{A}{2(\Delta x)^2}\left[\frac{x_{i+1/2}^q}{x_i^p}\psi_{i+1/2}^{n,k+1/2}(h_{i+1}^n - h_i^n) - \frac{x_{i-1/2}^q}{x_i^p}\psi_{i-1/2}^{n,k+1/2}(h_i^n - h_{i-1}^n)\right], \end{aligned} \tag{B 10}$$

which can be rewritten as

$$\begin{aligned}&\left[-\frac{A\Delta t}{2(\Delta x)^2}\frac{x^q_{i-1/2}}{x^p_i}\psi^{n,k+1/2}_{i-1/2}\right]h^{n,k+1}_{i-1}\\&\quad+\left[1+\frac{A\Delta t}{2(\Delta x)^2}\left(\frac{x^q_{i+1/2}}{x^p_i}\psi^{n,k+1/2}_{i+1/2}+\frac{x^q_{i-1/2}}{x^p_i}\psi^{n,k+1/2}_{i-1/2}\right)\right]h^{n,k+1}_{i}\\&\quad+\left[-\frac{A\Delta t}{2(\Delta x)^2}\frac{x^q_{i+1/2}}{x^p_i}\psi^{n,k+1/2}_{i+1/2}\right]h^{n,k+1}_{i+1}\\&=h^n_i+\frac{A\Delta t}{2(\Delta x)^2}\left[\frac{x^q_{i+1/2}}{x^p_i}\psi^{n,k+1/2}_{i+1/2}(h^n_{i+1}-h^n_i)-\frac{x^q_{i-1/2}}{x^p_i}\psi^{n,k+1/2}_{i-1/2}(h^n_i-h^n_{i-1})\right].\end{aligned}\tag{B 11}$$

It should be clear that each internal iteration involves the inversion of a tridiagonal matrix, and the scheme has truncation error $O\left[(\Delta x)^2+(\Delta t)^2\right]$. This is in contrast to the scheme of Diez *et al.* (1992), which has truncation error $O\left[(\Delta x)^2+(\Delta t)\right]$, is valid only for $p=q$ and requires a 'precursor' film ahead of the gravity current.

We construct the boundary conditions semi-implicitly, i.e. we do not take the half-time step values of the nonlinear h^s terms but rather the values at the previous time step, so that we can explicitly solve for the h values at the upcoming time step. Thus, the boundary conditions in (B.4) become

$$\tfrac{1}{2}\left[(h^{n,k}_0)^s+(h^{n,k}_1)^s\right]\frac{1}{\Delta x}\left(h^{n,k+1}_1-h^{n,k+1}_0\right)=\begin{cases}-\dfrac{\gamma B}{A}t^{\gamma-1}, & \gamma\neq 0,\\ 0, & \gamma=0,\end{cases}\tag{B 12a}$$

$$\frac{1}{\Delta x}\left(h^{n,k+1}_{N+1}-h^{n,k+1}_N\right)=0,\tag{B 12b}$$

which can be rewritten as

$$h^{n,k+1}_0-h^{n,k+1}_1=\begin{cases}-\dfrac{2\gamma Bt^{\gamma-1}\Delta x}{A[(h^{n,k}_0)^s+(h^{n,k}_1)^s]}, & \gamma\neq 0,\\ 0, & \gamma=0,\end{cases}\tag{B 13a}$$

$$-h^{n,k+1}_N+h^{n,k+1}_{N+1}=0.\tag{B 13b}$$

To perform the internal iterations over k at each time step n, we initialize with $h^{n,0}=h^n$ and continue until K such that $\max_i\left|h^{n,K}_i-h^{n,K-1}_i\right|<10^{-8}\max_i\left|h^{n,K-1}_i\right|$. For the simulations in the present work, we observe that only a few internal iterations are required to meet the convergence criterion. Then, we set $h^{n+1}=h^{n,K}$ to complete an iteration of the nonlinear conservative Crank–Nicolson scheme.

In a typical simulation used for generating the results in § 3, where $s=1$, we used the initial condition

$$h(x,0)=\begin{cases}0, & 0\leqslant x\leqslant 0.8L,\\ 0.1(x-0.8L), & 0.8L<x\leqslant L.\end{cases}\tag{B 14}$$

This corresponds to $x_f(0)/L=x_0/L=0.8$, which is comparable with the experiments, and $h(L,0)/L=0.02\ll 1$ as required by the lubrication approximation. The value of L is set by the value of A. It should be noted that the shape of the initial condition is 'forgotten' in self-similar regime.

B.3. *Convergence of the scheme and conservation properties*

We have verified the second order of accuracy of the scheme constructed above by comparing the numerical solution to the Barenblatt–Pattle exact point-source solution (Barenblatt 1952; Pattle 1959) on the domain $x \in (-L, L)$ for the axisymmetric case in N dimensions, i.e. $p = q = N - 1$, and s a positive integer. By halving Δt and Δx simultaneously, we observe second-order accuracy of the numerical solution in the L^1 and L^2 norms. Similarly, for both the Barenblatt–Pattle solution on $x \in (-L, L)$ and the problems on $x \in (0, L)$ with the nonlinear boundary conditions above, we have verified that $\int_0^L x^p h \, \mathrm{d}x = Bt^\gamma$ up to numerical precision for all t in each simulation.

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