ON TOPOLOGICAL INVARIANTS OF THE PRODUCT OF GRAPHS

M. Behzad* and S.E. Mahmoodian

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1. Introduction. We consider ordinary graphs, that is, finite, undirected graphs with no loops or multiple lines. The product (also called cartesian product [4]) $G_1 \times G_2$ of two graphs G_1 and G_2 with point sets V_1 and V_2, respectively, has the cartesian product $V_1 \times V_2$ as its set of points. Two points (u_1, u_2) and (v_1, v_2) are adjacent if $u_1 = v_1$ and u_2 is adjacent with v_2 or $u_2 = v_2$ and u_1 is adjacent with v_1. In this note we investigate the chromatic numbers, planarity and traversability (often referred to as topological invariants) of $G_1 \times G_2$.

2. Notations and Definitions. Let $V_1 = \{v_1^1, v_1^2, \ldots, v_1^{p_1}\}$, $V_2 = \{v_2^1, v_2^2, \ldots, v_2^{p_2}\}$, and let q_i denote the number of lines of G_i, $i = 1, 2$. The graph $G_1 \times G_2$ has $p_1 p_2$ points and $p_1 q_2 + q_2 p_1$ lines. This graph which is isomorphic with $G_2 \times G_1$ contains p_2 disjoint "horizontal" copies $G_1^1, G_1^2, \ldots, G_1^{p_2}$ (ordered from top to bottom) of G_1 and p_1 "vertical" copies $G_2^1, G_2^2, \ldots, G_2^{p_1}$ (ordered from left to right) of G_2. A horizontal copy G_1^i and a vertical copy G_2^j have only one point (v_1^j, v_2^i) in common.

The (point-) chromatic number $\chi(G)$ of a graph G is the minimum number of colors required to color points of G in such a way that no two adjacent points have the same color. The line-chromatic number $\chi'(G)$ is defined similarly. The total-chromatic number $\chi''(G)$ of G [1] is

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the minimum number of colors required to color the elements (points and lines) of G in such a way that no two adjacent elements (two points or two lines) and no two incident elements (a point and a line) have the same color.

A graph is planar if it can be drawn in the plane with no lines crossing.

A graph G is called hamiltonian if it contains a cycle passing through all points of G. A connected graph G is called eulerian if the degree of (that is, the number of lines incident with) every point of G is even.

3. Chromatic Numbers. In this section the point, line and total-chromatic numbers of $G_1 \times G_2$ are investigated. By a proper coloring of, for example, points of G is meant an assignment of colors to points of G in such a way that adjacent points receive different colors. The color of an element e of G will be denoted by $c(e)$. The notation $c(u, v)$ will be used for the color of the point (u, v).

THEOREM 3.1. $\chi(G_1 \times G_2) = \max\{\chi(G_1), \chi(G_2)\}$.

Proof. Assume, without loss of generality, that $\chi(G_1) \geq \chi(G_2)$. Color the points of G_1^1 with colors $1, 2, \ldots, \chi(G_1)$ properly and suppose $c(v_1^1, v_2^1) = 1$. Then color the points of G_2^1 with colors $1, 2, \ldots, \chi(G_2)$ properly in such a way that $c(v_1^1, v_2^1) = 1$. Now color the point (v_1^j, v_2^i), $i, j > 1$, with color $m + n - 1 \mod (\chi(G_1))$, where $m = c(v_1^1, v_2^i)$ and $n = c(v_1^j, v_2^1)$. In order to show that this coloring is a proper coloring of points of $G_1 \times G_2$ it suffices to consider two points with the same first and with the same second entries.

Let, for example (v_1^ℓ, v_2^s) and (v_1^ℓ, v_2^t) be two adjacent points of $G_1 \times G_2$. We have $c(v_1^\ell, v_2^s) = r + w - 1 \mod (\chi(G_1))$ and and $c(v_1^\ell, v_2^t) = v + w - 1 \mod (\chi(G_1))$, where $c(v_1^1, v_2^s) = r$, $c(v_1^\ell, v_2^1) = w$, and $c(v_1^1, v_2^t) = v$. Clearly (v_1^1, v_2^s) and (v_1^1, v_2^t) are adjacent in the subgraph G_2^1 of $G_1 \times G_2$. Hence $r \neq v$.

This implies $c(v_1^\ell, v_2^s) \neq c(v_1^\ell, v_2^t)$.

Let max deg G denote the maximum degree among the degree of points of G. Concerning $\chi'(G)$, Vizing [5] has shown that max deg G $\leq \chi'(G) \leq$ max deg G + 1. Since max deg $G_1 \times G_2$ = max deg G_1 + max deg G_2 we have

THEOREM 3.2. max deg G_1 + max deg $G_2 \leq \chi'(G_1 \times G_2) \leq$ max deg G_1 + max deg G_2 + 1.

If the line-chromatic number of G_i, $i = 1, 2$, equals its maximal degree, we shall show $\chi'(G_1 \times G_2)$ equals the maximal degree of $G_1 \times G_2$.

THEOREM 3.3. *Suppose* $\chi'(G_i)$ = max deg G_i, $i = 1, 2$. *Then* $\chi'(G_1 \times G_2)$ = max deg G_1 + max deg G_2.

Proof. Clearly $\chi'(G_1) + \chi'(G_2) \leq \chi'(G_1 \times G_2)$. The converse is true for every pair of graphs G_1 and G_2. To see this color the lines of each horizontal copy properly with colors $1, 2, \ldots, \chi'(G_1)$ and each vertical copy properly with colors $\chi'(G_1) + 1$, $\chi'(G_1) + 2, \ldots,$ $\chi'(G_1) + \chi'(G_2)$.

Assuming $\chi'(G_i)$ = max deg G_i + 1, $i = 1, 2$, one might think $\chi'(G_1 \times G_2)$ = max deg G_1 + max deg G_2 + 1. In Fig. 1 G_1 and G_2 are taken to be $K_5 - x$, where K_n is the complete graph of order n and $K_n - x$ denotes K_n minus one line. $\chi'(G_1) = \chi'(G_2)$ = max deg G_1 + 1. But $\chi'(G_1 \times G_2)$ is shown to be max deg G_1 + max deg G_2 = 8. The graph $(K_5 - x) \times (K_5 - x)$ is the smallest graph with the above property.

Given two graphs G_1 and G_2 we have $\chi(G_1) \leq \chi''(G_2)$ or $\chi(G_2) \leq \chi''(G_1)$. Suppose $\chi(G_1) > \chi''(G_2)$. Then $\chi''(G_1) \geq \chi(G_1) > \chi''(G_2) \geq \chi(G_2)$ imply $\chi(G_2) < \chi''(G_1)$.

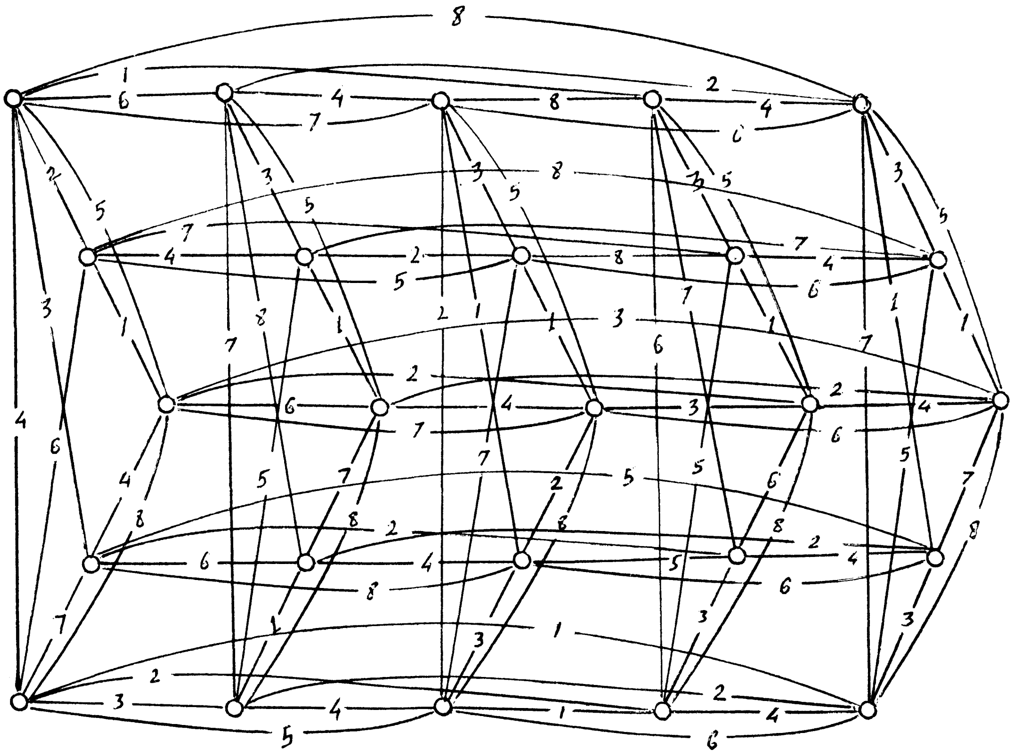

Fig. 1

THEOREM 3. 4. If $\chi(G_1) \leq \chi''(G_2)$, then max deg G_1 + max deg $G_2 + 1 \leq \chi''(G_1 \times G_2) \leq \chi''(G_2) + \chi'(G_1)$.

Proof. The first inequality is obvious. Color the elements of G_2^1 and the lines of each horizontal copy properly with colors $1, 2, \ldots, \chi(G_1), \ldots, \chi''(G_2)$ and colors $\chi''(G_2) + 1$, $\chi''(G_2) + 2, \ldots, \chi''(G_2) + \chi'(G_1)$, respectively. Suppose $c(v_1^1, v_2^1) = 1$. Then color the points of G_1^1 with colors $1, 2, \ldots, \chi(G_1)$ properly in such a way that the point (v_1^1, v_2^1) receives color 1. Next, consider G_2^j, $j = 2, \ldots, p_1$ and let e be

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an element of G_2^j. To e corresponds an element e' of G_2^1. Let $c(e) = c(v_1^j, v_2^1) + c(e') - 1 \mod (\chi''(G_2))$. Now it is an easy matter to check that this coloring is a proper coloring of the elements of $G_1 \times G_2$; completing the proof.

Remarks. (i) The bounds given in Theorem 3.4 cannot, in general, be improved. That is, for two positive integers m and n there exist two graphs G_1 and G_2 with $\chi'(G_1) = m$, $\chi''(G_2) = n$, and $\chi''(G_1 \times G_2) = \chi'(G_1) + \chi''(G_2)$. In fact, let $G_1 = K_{1,m}$ and $G_2 = K_{1,n-1}$, where $K_{m,n}$ denotes the complete bigraph of order $m+n$. Incidently, for these graphs max deg G_1 + max deg $G_2 + 1$ equals $\chi''(G_1 \times G_2)$, too.

(ii) The second inequality in the theorem cannot be changed to an equality as can be seen by considering $C_4 \times C_4$, where C_n, $n \geq 3$, denotes the cycle of length n.

(iii) It was conjectured by one of the authors [1] that for any graph G max deg $G + 1 \leq \chi''(G) \leq$ max deg $G + 2$. This conjecture has been proved to be true for many special classes of graphs [1, 3]. However, Theorem 3.4 together with the theorem of Vizing stated earlier imply that if the conjecture is true for prime graphs, then for a composite graph G the number max deg $G + 3$ is an upper bound for $\chi''(G)$.

(iv) If $\chi(G_1) \leq \chi''(G_2)$ and $\chi(G_2) \leq \chi''(G_1)$, then $\chi''(G_1 \times G_2) \leq \min\{\chi''(G_2) + \chi'(G_1), \chi''(G_1) + \chi'(G_2)\}$.

4. Planarity. Without loss of generality, in this section, we consider only connected graphs. For $G_1, G_2 \notin \{K_1, K_2\}$ we have

THEOREM 4.1. *If $G_1, G_2 \notin \{K_1, K_2\}$, then $G_1 \times G_2$ is planar if and only if both are paths or one is a path and the other is a cycle.*

Proof. If G_1 and G_2 are both paths or one is a path and the other is a cycle, then it is clear that $G_1 \times G_2$ is planar. In order to prove the converse we consider two cases.

(i) G_1, for example, has a point of degree three. Then $K_{1,3}$

is a subgraph of G_1 and $K_{1,2}$ is a subgraph of G_2. The graph $K_{1,3} \times K_{1,2}$ which is homeomorphic with $K_{3,3}$ is a subgraph of $G_1 \times G_2$. Hence by a well-known theorem of Kuratowski – a graph G is planar if and only if it has no subgraph homeomorphic with K_5 or $K_{3,3}$ – $G_1 \times G_2$ is not planar.

(ii) Both G_1 and G_2 are cycles. It is not difficult to show that $C_m \times C_n$ contains a subgraph homeomorphic with $K_{3,3}$ in this case, too. Hence, in order for $G_1 \times G_2$ to be planar neither factors can have a point of degree three nor both can be cycles; implying the theorem.

In Theorem 4.1 we assumed that $G_1, G_2 \notin \{K_1, K_2\}$. If $G_2 = K_1$, then clearly $G_1 \times K_1$ is planar if and only if G_1 is planar. Before we study the case $G_2 = K_2$, we consider outer-planar graphs.

A graph G is said to be <u>homeomorphic from</u> a graph H if G is obtained from H by inserting points (of degree 2) on some lines of H. An <u>outer-planar</u> graph is a graph G which can be embedded in the plane so that every point of G lies on the exterior region. Chartrand and Harary [2] have characterized outer-planar graphs as those graphs which do not contain subgraphs homeomorphic from K_4 or $K_{2,3}$.

LEMMA. *If G is an outer-planar graph, then $G \times K_2$ is planar.*

Proof. From the definition of outer-planar graphs it can be assumed that every point of an outer-planar graph lies on a cycle. This and the fact that $C_n \times K_2$ is planar imply the lemma.

THEOREM 4.2. *$G \times K_2$ is planar if and only if G does not contain a subgraph homeomorphic from K_4 or $K_{2,3}$.*

Proof. Since $K_4 \times K_2$ $(K_{2,3} \times K_2)$ has a subgraph homeomorphic with K_5 $(K_{3,3})$ the product of a graph homeomorphic from K_4 $(K_{2,3})$ and K_2 will have a subgraph homeomorphic with K_5, $(K_{3,3}$, respectively) too. Hence by Kuratowski's theorem if G has a subgraph homeomorphic from K_4 or $K_{2,3}$, then $G \times K_2$ is not planar. The preceding lemma implies the converse.

5. Traversability. An important notion in graph theory is that of hamiltonian. No one has found yet a criterion for graphs having a hamilton cycle. Before we give conditions under which $G_1 \times G_2$ is hamiltonian, it might be of value to mention that $G_1 \times G_2$ is eulerian if and only if points of G_1 and G_2 are of the same parity and both are connected.

THEOREM 5.1. *Let G_1 and G_2 be two graphs having spanning paths. Then $G_1 \times G_2$ is not hamiltonian if and only if both have an odd number of points and none has an odd cycle.*

Proof. Assume that $2m + 1$ and $2n + 1$, m and n positive integers, are the orders of G_1 and G_2, and that

$P_1 = \{v_1^1, \ldots, v_1^{2m+1}\}$ and $P_2 = \{v_2^1, \ldots, v_2^{2n+1}\}$ are their spanning paths, respectively. Suppose neither G_1 nor G_2 has an odd cycle. Moreover, assume that $G_1 \times G_2$ is hamiltonian. The length of a hamilton cycle C of $G_1 \times G_2$ is odd. Draw $G_1 \times G_2$ in the plane in such a way that the lines of spanning paths $P_1^i (P_2^j)$ in all copies $G_1^i (G_2^j)$ of $G_1 \times G_2$ are horizontal (vertical) and that neither a line of $G_1^i (G_2^j)$ crosses a line of $G_1^k (G_2^\ell)$ for $i \neq k$ ($j \neq \ell$, respectively) nor a line of G_1^i crosses a line of G_2^j more than once, for $i, k = 1, 2, \ldots, 2m + 1$; $j, \ell = 1, 2, \ldots, 2n + 1$. Draw $2n$ horizontal ($2m$ vertical) lines "between" G_1^i and G_1^{i+1}, $i = 1, 2, \ldots, 2n$ (G_2^j and G_2^{j+1}, $j = 1, 2, \ldots, 2m$). The number of times the lines of C cross each of these horizontal (vertical) lines is even. Since there are $2m + 2n$ horizontal and vertical lines, the length of the cycle C must be even, a contradiction to our assumption.

For the converse we need to consider the following cases.

(i) Suppose the order of G_1 or G_2 is even. It is easy to see that the product of two paths is hamiltonian if at least one has odd length. Hence the assertion is true in this case.

(ii) Suppose the order of G_1 and G_2 is odd, and one, say G_1, has an odd cycle. To complete the proof of the theorem, it suffices to show that $G_1 \times G_2$ is hamiltonian if G_1 is a path of order

$2m + 1$ with $2m + 2$ lines having a cycle C of odd length and G_2 is a path of order $2n + 1$.

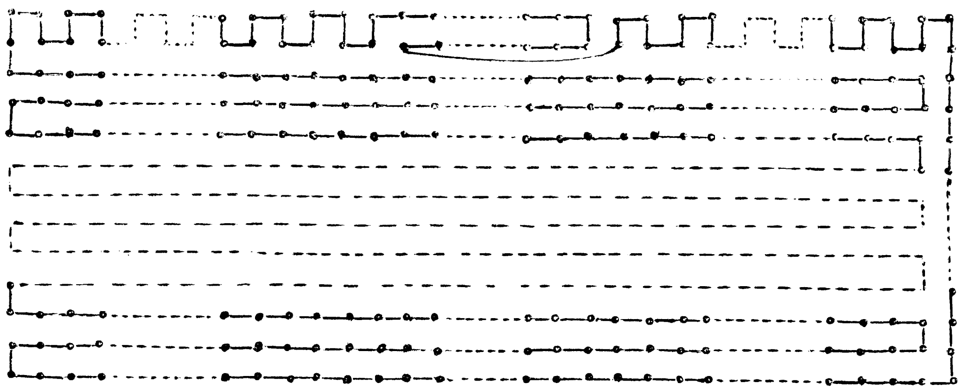
Remove the lines of C from G_1 to obtain two paths P_1 and P_2 and a set of isolated points. (In general P_1 or P_2 might be an isolated point, too; in which case it will be considered as a path of length zero.) The length of these paths are of the same parity. According to their length two cases must be studied.

(i) The length of P_1 and P_2 is odd.

(ii) The length of P_1 and P_2 is even.

Instead of writing a tedious proof for our assertion we show a method of finding a hamilton cycle for each case in the following figures.

COROLLARY. If G_1 and G_2 are hamiltonian, then $G_1 \times G_2$ is hamiltonian also.



Case (i)

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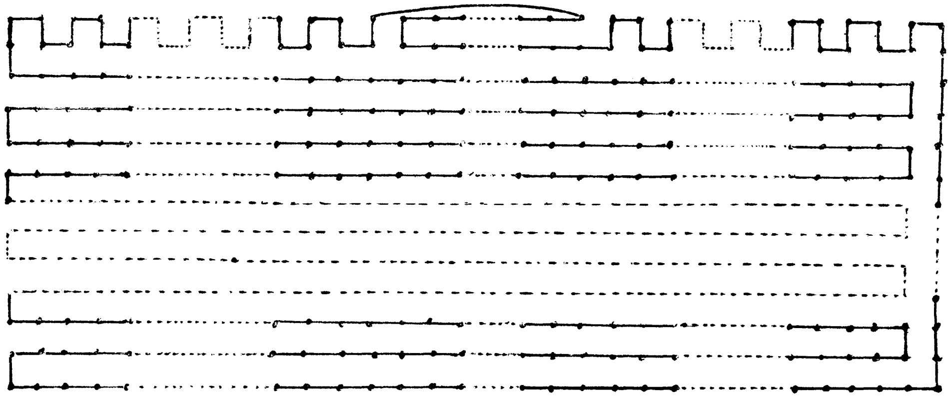

Case (ii)

The next two theorems give sufficient conditions under which $G_1 \times G_2$ is not hamiltonian,

THEOREM 5.2. _If in_ G_1 _three points of degree one are adjacent to a point and if_ G_2 _contains a point of degree one, then_ $G_1 \times G_2$ _is not hamiltonian_.

Proof. Let v_1^1 be adjacent to points v_1^2, v_1^3 and v_1^4 each of degree one in G_1 and let v_2^1 be a point of degree one in G_2. Then the points (v_1^2, v_2^1), (v_1^3, v_2^1), and (v_1^4, v_2^1) in $G_1 \times G_2$ all have degree two and are adjacent with (v_1^1, v_2^1). Since no cycle can contain three lines adjacent with one point, $G_1 \times G_2$ cannot be hamiltonian.

THEOREM 5.3. _Let_ G_i _have two points of degree one adjacent with a point of_ G_i, $i = 1, 2$. _Then_ $G_1 \times G_2$ _is not hamiltonian._

Proof. Let v_i^1 be adjacent with two points v_i^2 and v_i^3 of degree one in G_i, $i = 1, 2$. Suppose $G_1 \times G_2$ is hamiltonian. Then the lines (v_1^2, v_2^2) (v_1^1, v_2^2), (v_1^1, v_2^2) (v_1^3, v_2^2),

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(v_1^3, v_2^2) (v_1^3, v_2^1), (v_1^3, v_2^1) (v_1^3, v_2^3), (v_1^3, v_2^3) (v_1^1, v_2^3),

(v_1^1, v_2^3) (v_1^2, v_2^3), (v_1^2, v_2^3) (v_1^2, v_2^1) and (v_1^2, v_2^1) (v_1^2, v_2^2)

must be in a hamilton cycle of $G_1 \times G_2$. These themselves form a cycle not containing the point (v_1^1, v_2^1), a contradiction.

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The National University of Iran

Western Michigan University

Pahlavi University, Iran