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QUEUING, SOCIAL INTERACTIONS, AND THE MICROSTRUCTURE OF FINANCIAL MARKETS

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We consider an agent-based model of financial markets with asynchronous order arrival in continuous time. Buying and selling orders arrive in accordance with a Poisson dynamics where the order rates depend both on past prices and on the mood of the market. The agents form their demand for an asset on the basis of their forecasts of future prices and their forecasting rules may change over time as a result of the influence of other traders. Among the possible rules are "chartist" or extrapolatory rules. We prove that when chartists are in the market, and with choice of scaling, the dynamics of asset prices can be approximated by an ordinary delay differential equation. The fluctuations around the first-order approximation follow an Ornstein–Uhlenbeck dynamics with delay in a random environment of investor sentiment.

Keywords: Microstructure of Financial Markets, Social Interaction, Queuing Networks

1. INTRODUCTION

In recent years there has been increasing interest in agent-based models of financial markets where the demand for a risky asset comes from many agents with interacting preferences and expectations. These models are capable of reproducing, often through simulations, many "stylized facts" such as the emergence of herding behavior [Lux (1995)], volatility clustering [Lux and Marchesi (2000)], or fat-tailed distributions of stock returns [Cont and Bouchaud (2000)] that are observed in financial data. In contrast to the traditional framework of an economy with a utility-maximizing representative agent, behavioral finance models comprise many *heterogeneous* traders who are *boundedly rational*. The market participants do not necessarily share identical expectations about the future evolution of asset prices or assessments of a stock's fundamental value. Instead, agents are allowed

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to use rule-of-thumb strategies when making their investment decisions and to switch randomly between them as time passes. Following up on the seminal work of Frankel and Froot (1986), one typically distinguishes *fundamentalists*, *noise traders*, and *chartists*. Different types of traders often coexist,[1] with their proportions varying over time as agents are allowed to change their strategies in reaction to the strategies' performance or the choices of other market participants. This may lead to temporary deviations of prices from their benchmark rational expectations value generating bubbles or crashes in periods when technical trading predominates.

An array of agent-based models have been suggested over the past two decades. The underlying mathematical methods and techniques range from central limit theorems for stochastic processes in random media to deterministic dynamical systems. Föllmer and Schweizer (1993) and Horst (2005), for instance, model asset prices as a sequence of temporary equilibrium prices in a random environment of investor sentiment. They show that in a noise trader framework, and after suitable scaling, the asset price process can be approximated by an Ornstein–Uhlenbeck process with random coefficients. Their approach captures some interaction and imitation effects such as word-of-mouth advertising, but the dynamics of the environment lacks a dependence on asset prices. This gap is filled by Föllmer et al. (2005), where the agents are allowed to use technical trading rules. This generates a feedback from past prices into the environment. It turns out that asset prices converge to a unique limiting distribution if the impact of chartists is not too strong. Similar results were obtained in a different setting by Böhm and Wenzelburger (2005); we refer to Bayraktar et al. (in press b) for a more detailed discussion of probabilistic agent-based models.

The approach pioneered by Day and Huang (1990) and Brock and Hommes (1997) analyzes financial markets using deterministic dynamical systems. The idea is to view agent-based models as highly nonlinear deterministic dynamical systems and markets as complex adaptive systems, with the evolution of expectations and trading strategies coupled to market dynamics. Their models display a quite complex dynamics, so only a few analytical characterizations of asset price processes are available. However, when simulated, these models generate realistic time paths of prices explaining many of the stylized facts observed in real financial markets. For further details we refer to recent surveys by Hommes (2006) and LeBaron (2006).

These models differ considerably in their degree of complexity and analytical tractability, but they are all based on the idea that asset prices can be described by a sequence of equilibrium prices. All agents submit their demand schedules to a market maker who matches individual demands in such a way that markets clear in every period. Although such an approach is consistent with dynamic microeconomic theory, a closer examination of the microstructure of securities markets raises the question of whether the standard economic paradigm of a Walrasian auctioneer can actually be applied. In real markets buyers and sellers arrive at different points in time. Moreover, almost all electronic trading systems are based on order books in which all unexecuted limit orders are stored and displayed while awaiting execution.

Analytically tractable models of order book dynamics were of considerable value, but their development has been hindered by the inherent complexity of limit order markets. Rigorous mathematical results have so far only been established under rather restrictive assumptions by, e.g., Mendelson (1982), Luckock (2003), and Kruk (2003). At the same time, there is a considerable (econophysics) literature [Chiarella and Iori (2002), Potters and Bouchaud (2003), Smith et al. (2003), and Farmer et al. (2005), among others] on continuous double auctions with "minimal intelligence agents." Here, interest is not so much in probabilistic models for the resulting price dynamics, but in statistical properties of sample paths. Underlying this approach is the idea that the dynamics of order arrivals follows a Poisson process and that nonexecuted orders are canceled at random points in time. Incoming orders typically follow an i.i.d. dynamics with no dependence on past prices. "Minimal" or "zero intelligence agent" models make many testable predictions for basic properties of markets such as price volatility, and despite their many simplifying assumptions on trader behavior these models have successfully reproduced some of the stylized facts of financial time series.

Microstructure models with asynchronous order arrivals, where incoming orders are executed immediately rather than awaiting the arrival of a matching order, were studied in a series of papers by, e.g., Lux (1995, 1997) and more recently by Bayraktar et al. (in press a, b). These models may be viewed as a first step toward bridging the gap between the econophysics literature, with its many models that generate a rich dynamics and realistic time series, but are not amenable to analytic solutions (beyond statistical properties), and the more traditional temporary equilibrium models, which allow for analytic solutions but do not accurately capture the microstructure of automated trading systems. The idea is that an incoming order changes the stock price by a fixed amount and that agents may switch their investment behavior as a result of the behavior of others and/or the performance of different trading strategies. A convenient mathematical framework is based on the theory of state-dependent queuing networks [Mandelbaum and Pats (1998); Mandelbaum et al. (1998)].

This paper proposes a mathematical framework for analyzing financial market models with asynchronous order arrivals. Our model is flexible enough to capture chartist behavior. As such, it extends earlier work of Lux (1995). He studied a noise trader framework where the joint dynamics of asset prices and opinion indices can be approximated by a system of ordinary differential equations. The ODE approach provides a first approximation to stock prices in a noise trader model, but it does not capture situations where agents base their demands rather than their opinions on price patterns. To capture trend-chasing strategies, we consider a model in which the order rates depend on historic asset prices and opinion indices. We show that when the number of speculators tends to infinity, the joint dynamics of asset prices and trader type distributions can be approximated by a *delay* differential equation. The delay effect reflects the presence of chartists. Our numerical simulations suggest that it has a major effect on stock prices.

More important than the first-order approximation are the random fluctuations around the deterministic trajectory of the delay equation. In our model they can be described by a coupled system of Ornstein–Uhlenbeck processes with delay. Stochastic delay differential equations are a continuous-time analogue of higher-order discrete-time difference equations. Although random difference equations have been widely used as a mathematical basis for modeling stock price dynamics, stochastic delay differential equations have attracted less attention in the finance literature. They have primarily been used in stochastic volatility models [Hobson and Rogers (1998); Kazmerchuk and Wu (2004)] and more recently in the context of insider models by Stoica (2005). In this paper we show that delay equations arise naturally in behavioral finance models when the agents base their investment decisions on the performances of trading strategies and identify the delay effect as a major determinant of financial price fluctuations. For a noise trader model the second-order approximation is given by an Ornstein–Uhlenbeck process, as in Föllmer and Schweizer (1993) and Horst (2005).

The remainder of this paper is organized as follows. In Section 2 we introduce our model and state the main results. Section 3 illustrates the impact of chartists by means of numerical simulation. All proofs appear in Section 4.

2. THE MICROECONOMIC MODEL AND THE MAIN RESULTS

It was first argued by Garman (1976) that an exchange market can be characterized by a flow of orders to buy and sell. He also argued that although the orders would arise as the solutions to individual traders' underlying optimization problems, the explicit characterization of such problems is not necessarily important. What matters more is that orders are submitted at different points in time and that imbalances between supply and demand can arise. We shall therefore take a pragmatic approach to modeling financial markets and start right away with the dynamics of order flows. This approach is common in much of the econophysics literature, where interest is not so much in causes of trading, but in phenomenological models and their overall implications. This literature has demonstrated that "zero intelligence" models that drop agent rationality altogether and focus instead on the dynamics of order arrivals are capable of reproducing many statistical properties of financial time series.

2.1. Order Rates and Market Dynamics

We consider a financial market with a large set $\mathbf{A} = \{1, 2, \ldots, N\}$ of economic agents trading a single risky asset. With each agent $a \in \mathbf{A}$ we associate a continuous-time stochastic process $x^a = (x_t^a)$ taking values in some finite set $C = \{c_1, c_2, \ldots, c_m\}$ of investor characteristics. We think of x^a as describing the evolution of the agent's *trader type* or *state*. The agents submit buy and sell orders according to independent Poisson dynamics with the type-dependent rate functions

$$\widetilde{\lambda}_+\big(x_t^a, \cdot\big) \quad \text{and} \quad \widetilde{\lambda}_-\big(x_t^a, \cdot\big).$$

Incoming orders are instantaneously matched by a market maker who sets the price to reflect the degree of market imbalance.

We refer to the empirical distribution ϱ_t^N of trader types at time t as the *mood of the market*,

$$\varrho_t^N := \big\{\varrho_t^N(c)\big\}_{c \in C} \quad \text{with} \quad \varrho_t^N(c) := \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{c\}}\big(x_t^a\big), \tag{1}$$

and allow for a dependence of the order rates on past prices and market moods. To this end, we denote by S_t^N the logarithmic asset price at time t, fix constants $0 < \delta_1 < \delta_2 < \ldots < \delta_l$ along with an $(m+1)$-dimensional continuous function $\tilde{q}$ on $[-\delta_l, 0]$, and put

$$S_{(t)}^N := \big(S_t^N, S_{t-\delta_1}^N, \ldots, S_{t-\delta_l}^N\big) \quad \text{and} \quad \varrho_{(t)}^N := \big(\varrho_t^N, \varrho_{t-\delta_1}^N, \ldots, \varrho_{t-\delta_l}^N\big),$$

where $(S_t^N, \varrho_t^N) = \tilde{q}_t$ on $[-\delta_l, 0]$. In the time interval $[t, t+h]$, an agent $a \in \mathbf{A}$ submits buy and sell orders with probabilities

$$\widetilde{\lambda}_+\big(x_t^a, \varrho_{(t)}^N, S_{(t)}^N\big) * h + o(h) \quad \text{and} \quad \widetilde{\lambda}_-\big(x_t^a, \varrho_{(t)}^N, S_{(t)}^N\big) * h + o(h) \quad \text{as} \quad h \to \infty, \tag{2}$$

respectively. Here $o(h)$ denotes a function that converges faster than linearly to zero when $h \to 0$. In terms of ϱ_t^N the *per capita* order rates take the form

$$\widetilde{\lambda}_\pm\big(\varrho_{(t)}^N, S_{(t)}^N\big) := \int \widetilde{\lambda}_\pm\big(x, \varrho_{(t)}^N, S_{(t)}^N\big)\varrho_t^N(dx).$$

Because the agents act conditionally independent of each other given the histories of past prices and market moods, the probability of some agent submitting a buy/sell order between t and $t+h$ equals

$$N * \widetilde{\lambda}_\pm\big(\varrho_{(t)}^N, S_{(t)}^N, \big) * h + o(h) \quad \text{as} \quad h \to 0.$$

The probabilistic structure of the order arrivals is thus equivalent to assuming that orders arrive according to independent Poisson processes

$$\{\widetilde{\Pi}_+(t)\}_{t \geq 0} \quad \text{and} \quad \{\widetilde{\Pi}_-(t)\}_{t \geq 0}$$

with respective rate functions $N * \widetilde{\lambda}_+$ and $N * \widetilde{\lambda}_-$. The accumulated marketwide net order flow by time t is therefore equal to

$$\widetilde{\Pi}_+\bigg(\int_0^t N * \widetilde{\lambda}_+\big(\varrho_{(u)}^N, S_{(u)}^N\big)\,du\bigg) - \widetilde{\Pi}_-\bigg(\int_0^t N * \widetilde{\lambda}_-\big(\varrho_{(u)}^N, S_{(u)}^N\big)\,du\bigg).$$

Assuming that a buy order increases the logarithmic price by $1/N$ decreases the price by the same amount, we arrive at the following stochastic integral equation for the logarithmic asset prices:

$$S_t^N = S_0^N + \frac{1}{N}\widetilde{\Pi}_+\left(N\int_0^t \widetilde{\lambda}_+\left(\varrho_{(u)}^N, S_{(u)}^N\right)du\right) - \frac{1}{N}\widetilde{\Pi}_-\left(N\int_0^t \widetilde{\lambda}_-\left(\varrho_{(u)}^N, S_{(u)}^N\right)du\right). \tag{3}$$

Remark 2.1. Note that the stock price process is given as a pure jump process in a random environment $\{\varrho_t^N\}$ of investor sentiment. The dynamics of the environment will be endogenous. We allow the agents to switch from one type or forecasting rule to another at random points in time in reaction to historic price patterns, trends, or the performance of competing trading strategies. This generates feedback effects from the price process into the environment. We postpone specific examples to Section 3.

The agents are allowed to switch between different types or trading strategies at random points in time in reaction to a strategy's past performance or the behavior of others. Specifically, we assume that independent of other traders an agent of type $i \in C$ switches to a different state $j \in C$ within the time interval $[t, t+h]$ with probability

$$\overline{\lambda}^{i,j}\left(\varrho_{(t)}^N, S_{(t)}^N\right) * h + o(h) \quad \text{as} \quad h \to 0;$$

the probability that an agent changes her type twice in $[t, t+h]$ is of the order $o(h)$ and hence negligible for small h.

Remark 2.2. Notice that all the other individuals influence one particular trader in the same way. This excludes the existence of a designated "leader" or financial "guru" whose behavior attracts the attention of the majority of market participants.

The average probability that some trader of type i switches to a different state between time t and time $t+h$ equals

$$\widehat{\lambda}_-^i\left(\varrho_{(t)}^N, S_{(t)}^N\right) := \sum_{j \in C} \varrho_t^N(i) * \overline{\lambda}^{i,j}\left(\varrho_{(t)}^N, S_{(t)}^N\right) + o(h), \tag{4}$$

whereas the average probability that an agent switches to state i from a different state $j \neq i$ is given by

$$\widehat{\lambda}_+^i\left(\varrho_{(t)}^N, S_{(t)}^N\right) := \sum_{j \in C} \varrho_t^N(j) * \overline{\lambda}^{j,i}\left(\varrho_{(t)}^N, S_{(t)}^N\right) + o(h). \tag{5}$$

The structure of the agents' migration probabilities allows us to describe the dynamics of the mood of the market in terms of a queuing network with routing, as in Mandelbaum and Pats (1998). There exists a family of Poisson processes $(\widehat{\Pi}_\pm^i)_{i \in C}$ such that the empirical distribution of trader types satisfies the system of stochastic integral equations

$$\varrho_t^N(i) = \varrho_0^N(i) + \frac{1}{N}\widehat{\Pi}_+^i\left(N\int_0^t \widehat{\lambda}_+^i\big(\varrho_{(u)}^N, S_{(u)}^N\big)du\right) - \frac{1}{N}\widehat{\Pi}_-^i\left(N\int_0^t \widehat{\lambda}_-^i\big(\varrho_{(u)}^N, S_{(u)}^N\big)du\right). \tag{6}$$

The Poisson process $\widehat{\Pi}_+^i$ specifies the times at which some agent switches to state i, whereas $\widehat{\Pi}_-^i$ specifies the times when some agent leaves state i. As a result, the processes $(\widehat{\Pi}_\pm^i)_{i \in C}$ are dependent. The next section shows how a strong approximation result for Poisson processes can be applied to represent the joint dynamics of asset prices and empirical distributions in terms of interacting diffusion processes.

2.2. Approximation of Poisson Processes and Financial Market Dynamics

The joint dynamics of asset prices and empirical distributions can be described in terms of a higher-dimensional non-Markovian queuing network. To this end, we introduce the vector

$$\widehat{\lambda} = \widehat{\lambda}_+ - \widehat{\lambda}_-, \quad \text{where} \quad \widehat{\lambda}_\pm = (\widehat{\lambda}_\pm^1, \ldots, \widehat{\lambda}_\pm^m)^t,$$

which specifies the agents' instantaneous propensities to adopt new trading strategies and put

$$Q_t^N = \big(\varrho_t^N, S_t^N\big), \quad Q_{(t)}^N = \big(\varrho_{(t)}^N, S_{(t)}^N\big), \quad \text{and} \quad \lambda_\pm\big(Q_{(t)}^N\big) = \begin{pmatrix} \widehat{\lambda}_\pm\big(\varrho_{(t)}^N, S_{(t)}^N\big) \\ \widetilde{\lambda}_\pm\big(\varrho_{(t)}^N, S_{(t)}^N\big) \end{pmatrix}.$$

With suitably defined $(m+1)$-dimensional Poisson processes $\Pi_\pm = \{\Pi_\pm^i\}_{i=1}^{m+1}$, it follows from equations (3) and (6) that the ith component $Q^{N,i}$ of the vector $Q^N = \{Q^{N,i}\}_{i=1}^{m+1}$ satisfies

$$Q_t^{N,i} = Q_0^{N,i} + \frac{1}{N}\Pi_+^i\left(N\int_0^t \lambda_+^i\big(Q_{(u)}^N\big)\,du\right) - \frac{1}{N}\Pi_-^i\left(N\int_0^t \lambda_-^i\big(Q_{(u)}^N\big)\,du\right); \tag{7}$$

here we use the convention that $Q_t^N \equiv \tilde{q}_t$ on $[-\delta_l, 0]$. The first m components of the vector process Q^N describe the dynamics of the distribution of states, whereas the last component describes the evolution of the logarithmic asset price: $\Pi_\pm^{m+1} = \widetilde{\Pi}_\pm$. Our goal is then to prove a limit theorem for the processes Q^N as the number of market participants tends to infinity. To obtain a well-defined price dynamics in the limit of an infinite number of investors, we impose the following conditions on the agents' order rates.

Assumption 2.3.

1. The rate functions $\widetilde{\lambda}_\pm$ and $\widehat{\lambda}_\pm$ are uniformly bounded.
2. For each $x \in C$, the rate functions $\widetilde{\lambda}_\pm(x, \cdot)$ and $\widehat{\lambda}_\pm(x, \cdot)$ are continuously differentiable with bounded first derivative.

The convergence results will be based on a strong approximation result that allows pathwise approximation of a Poisson process by a standard Brownian motion living in the same probability space.

LEMMA 2.4 (Kurtz 1978). *A standard Poisson process* $\{\Pi(t)\}_{t\geq 0}$ *can be realized in the same probability space as a standard Brownian motion* $\{B(t)\}_{t\geq 0}$ *in such a way that the random variable*

$$Y := \sup_{t\geq 0} \frac{|\Pi(t) - t - B(t)|}{log(\max\{2, t\})}$$

has a finite moment-generating function in the neighborhood of the origin and hence a finite mean. In particular, Y is almost surely finite.

By Assumption 2.3 (i), the strong approximation result allows us to realize all the Poisson processes in the same probability space as the $(m+1)$-dimensional Wiener processes

$$\{B_+(t)\}_{t\geq 0} \quad \text{and} \quad \{B_-(t)\}_{t\geq 0}$$

in such as way that we have the alternative representation of the logarithmic asset price process and sequence of empirical distributions of trader types

$$Q_t^{N,i} = Q_0^{N,i} + \frac{1}{N}\left\{N\int_0^t \lambda^i\left(Q_{(u)}^N\right)du + B_+^i\left(N\int_0^t \lambda_+^i\left(Q_{(u)}^N\right)du\right)\right.$$
$$\left. -B_-^i\left(N\int_0^t \lambda_-^i\left(Q_{(u)}^N\right)du\right)\right\} \tag{8}$$

up to a correction term that is of the order $\log N/N$ uniformly on compact time intervals. Here we define

$$\lambda^i\left(Q_{(t)}^N\right) := \lambda_+^i\left(Q_{(t)}^N\right) - \lambda_-^i\left(Q_{(t)}^N\right).$$

Notice that the correction term vanishes almost surely uniformly on compact time intervals when the number of market participants tends to infinity. We shall therefore drop it to simplify our notation. The aim is thus to prove approximation results for the sequence of $(m+1)$-dimensional stochastic processes $\{Q^N\}_{N\in\mathbf{N}}$ defined by (8). The convergence concept we use for the first-order approximation is almost sure convergence on compact time intervals. The convergence concept for the second-order approximation is weak convergence of probability measures on the set $\mathbf{D}_T$ of all real-valued right continuous functions with left limits on $[0, T]$. We write $\mathcal{L}\text{-}\lim_{n\to\infty} X_n = X$ if the $\mathbf{D}_T$-valued random variables X_n converge in distribution to X as n tends to infinity.

2.3. Approximation Results

We are now going to state a first-approximation result for the market dynamics; the proof requires some preparation and will be carried out in Section 4. It turns out that the joint dynamics of logarithmic asset prices and distributions of trader types can almost surely be approximated by the trajectory of an ordinary delay differential equation. The delay effect reflects the presence of chartists.

THEOREM 2.5 (First-Order Approximation). *Under Assumption* 2.3 *the following hold*:

(i) *For a given continuous initial function* $\tilde{q} : [-\delta_l, 0] \to \mathbf{R}^{m+1}$ *and any terminal time* $T > 0$ *there exists a unique process* $q = \{q_t\}_{-\delta_l \le t \le T}$ *that satisfies the delay differential equation*

$$dq_t = \lambda(q_{(t)})dt \quad \textit{with initial condition} \quad q \equiv \tilde{q} \quad \textit{on} \quad [-\delta_l, 0]. \tag{9}$$

(ii) *The sequence of stochastic processes* $\{Q^N\}_{N \in \mathbf{N}}$ *converges almost surely to* q *where the convergence is uniform on compact time intervals*:

$$\lim_{N\to\infty} \sup_{0 \le t \le T} \left| q_t - Q_t^N \right| = 0 \qquad \mathbf{P}\text{-}a.s.$$

In a second step, we study the joint distribution of asset prices and trader types around their first-order approximation. For this we use the self-similarity property of a Wiener process W. It assents that $\{W(t)\}$ and $\{\frac{1}{\sqrt{c}} W(ct)\}$ have the same distribution for any positive constant c. In studying the second-order approximation, we may hence assume that the process Q^N is *defined* by

$$Q_t^{N,i} = \int_0^t \lambda^i \left(Q_{(u)}^N\right) du + \frac{1}{\sqrt{N}} B_+^i \left(\int_0^t \lambda_+^i \left(Q_{(u)}^N\right) du \right) - \frac{1}{\sqrt{N}} B_-^i \left(\int_0^t \lambda_-^i \left(Q_{(u)}^N\right) du \right). \tag{10}$$

It turns out that the fluctuations of Q^N around the first-order approximation can be described by a coupled system of interacting Ornstein–Uhlenbeck processes with delay driven by the Gauss processes

$$X_t^i := B_+^i \left(\int_0^t \lambda_+^i \big(q_{(u)}\big) du \right) - B_-^i \left(\int_0^t \lambda_-^i \big(q_{(u)}\big) du \right). \tag{11}$$

We are now ready to state the main result of this paper. Its proof will be carried out in Section 4.

THEOREM 2.6 (Second-Order Approximation). *Under Assumption* 2.3 *the following hold*:

(i) *There exists a unique pathwise solution* $Z = (Z^1, \dots, Z^{m+1})$ *to the stochastic delay integral equation*

$$Z_t^i = \int_0^t \left\langle \nabla \lambda^i \left(q_{(u)}\right), Z_{(u)} \right\rangle du + X_t^i \quad \textit{for} \quad i \in \{1, \dots, m+1\} \tag{12}$$

with initial function $Z_t^i = 0$ *on* $[-\delta_l, 0]$. *Here* $\nabla \lambda^i$ *and* $\langle \cdot, \cdot \rangle$ *denote the gradient vector of the function* λ^i *and the standard inner product, respectively.*

(ii) *The fluctuation of the process Q^N around its first-order approximation converges in distribution to $Z = (Z^i)_{i=1}^{m+1}$:*

$$\mathcal{L}\text{-}\lim_{N\to\infty}\left\{\sqrt{N}\left(Q_t^N - q_t\right)\right\}_{0\le t\le T} = \{Z_t\}_{0\le t\le T}.$$

We notice that although the logarithmic asset price process takes values in **R**, the empirical distributions of trader types takes only non-negative values. It would be hence more appropriate to approximate the fluctuations of the process ϱ^N by a reflected diffusion. As this would render our analysis considerably more involved and because our focus is on the impact of trend chasers on the diffusion approximation, we chose a second-order approximation in terms of a "regular" diffusion process.

Remark 2.7. When only fundamentalists and noise traders are active on the market, the first-order approximation reduces to an ordinary differential equation as in Lux (1995) and the second-order approximation is given by an Ornstein–Uhlenbeck process as in Föllmer and Schweizer (1993). Under standard assumptions the first-order approximation converges to some steady state $q^* = (s^*, \varrho^*)$ as time tends to infinity. In this case $\lim_{t\to\infty}\{\lambda_+(q_t) - \lambda_-(q_t)\} = 0$. In the long run markets clear on average and asset prices fluctuate around the equilibrium level in accordance with a standard Wiener process with volatility

$$\sigma^* = \lim_{t\to\infty}\sqrt{\lambda_+(q_t) + \lambda_-(q_t)}.$$

3. EXAMPLES AND NUMERICAL SIMULATIONS

In this section we obtain Lux's noise trader model as a limiting case of our framework. Numerical simulations suggest that although his model displays unstable behavior for very small time lags if the impact of noise traders is too strong, stability may be gained when the time lags exceed some critical level. Our second example can be viewed as a continuous-time version of the model of Föllmer et al. (2005). In this case delay equations arise rather naturally as the agents switch their states in reaction to the past performances of trading strategies. Throughout, we put $x_t^a = 0$ if the agent $a \in \mathbf{A}$ is a fundamentalist at time t while $x_t^a = +1$ and $x_t^a = -1$ indicate (optimistic/pessimistic) noise traders or chartists.

Example 3.1. In our setting the demand function of a fundamentalist in Lux (1995) corresponds to linear order rates of the form

$$\widetilde{\lambda}_+\left(0, \varrho_{(t)}^N, S_{(t)}^N\right) = \begin{cases} \gamma\left(F - S_t^N\right) & \text{if } F - S_t^N > 0 \\ 0 & \text{else} \end{cases}$$

and

$$\widetilde{\lambda}_-\left(0, \varrho_{(t)}^N, S_{(t)}^N\right) = \begin{cases} \gamma\left(S_t^N - F\right) & \text{if } F - S_t^N < 0 \\ 0 & \text{else.} \end{cases}$$

The demand depends on the difference between some fundamental value (F) and the current price; the constant γ measures the trading volume. A noise trader's order rates are price independent. An optimistic noise trader buys the asset whereas a pessimist sells it:

$$\widetilde{\lambda}_{\pm}\left(\pm 1, \varrho^N_{(t)}, S^N_{(t)}\right) \equiv 1 \quad \text{and} \quad \widetilde{\lambda}_{\pm}\left(\mp 1, \varrho^N_{(t)}, S^N_{(t)}\right) \equiv 0.$$

Let us assume that the proportion of fundamentalists is fixed and that chartists switch between optimism and pessimism according to prevailing price trends and denote by x_t^N the average opinion of noise traders. In Lux's model the price dynamics follows the trajectory of an ordinary differential equation $\dot{s} = f(x, s)$ because the agents base their opinion on $\dot{s}$, a purely fictitious benchmark for the current trend. Our market participants, in contrast, react to observed market data. With the performance index

$$U_{t,t-\delta} := \frac{a_1}{\delta}\left(S_t^N - S_{t-\delta}^N\right) + a_2 x_t^N.$$

Lux's transition rates (9) are, in our framework, to be replaced by $\widehat{\lambda}^{0,\pm 1} = 0$ and

$$\widehat{\lambda}^{-1,1}\left(\varrho^N_{(t)}, S^N_{(t)}\right) = e^{U_{t,t-\delta}} \quad \text{and} \quad \widehat{\lambda}^{1,-1}\left(\varrho^N_{(t)}, S^N_{(t)}\right) = e^{-U_{t,t-\delta}}.$$

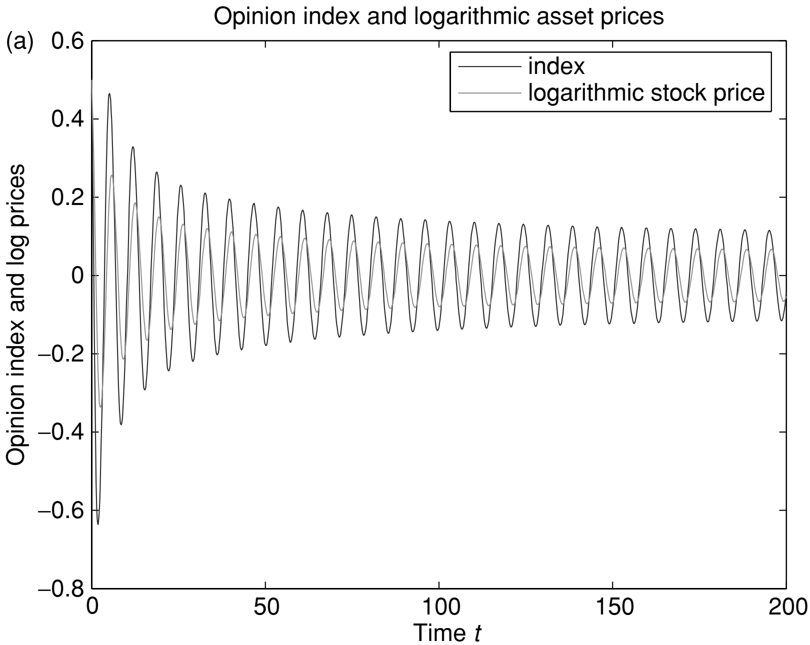
For large N the joint evolution of logarithmic asset prices and opinion indices can then be approximated by the delay differential equation

$$\dot{x}_t = 2\{\tanh(U_{t,t-\delta}) - x_t\}\cosh(U_{t,t-\delta})\,dt \quad \text{and} \quad \dot{s}_t = (x_t + \gamma(F - s_t))\,dt. \tag{13}$$

The equation for the change of stock prices depends only on s_t because the agents' order rates do not depend on past prices.

It turns out that the quantitative behavior of the system (13) depends on δ. When $a_1 = 1$, $a_2 = 0.75$, and $\gamma = \frac{3}{2}$, Lux's stability condition is violated, so the fundamental equilibrium $x_t \equiv 0$ and $s_t \equiv F$ is unstable for small δ. This is shown in Figure 1a, which displays the first-order approximation for $\delta = 0.01$. When δ is increased to 0.5, asset prices initially display large fluctuations but eventually settle down to the equilibrium level, as displayed in Figure 1b.

The previous example suggests that the time lag δ is a major determinant of stock price fluctuations in a noise trader framework. It also suggests that it is appropriate to reduce the first-order approximation to an ordinary differential equation by replacing the performance index $U_{t,t+\delta}$ with $a_1\dot{s}_t + a_2 x_t$ when δ is sufficiently small. Although such reduction is possible in a noise trader framework, it does not always carry over to models of trend chasing, where the agents base their demand rather than their opinions on price patterns. As an illustration, consider a



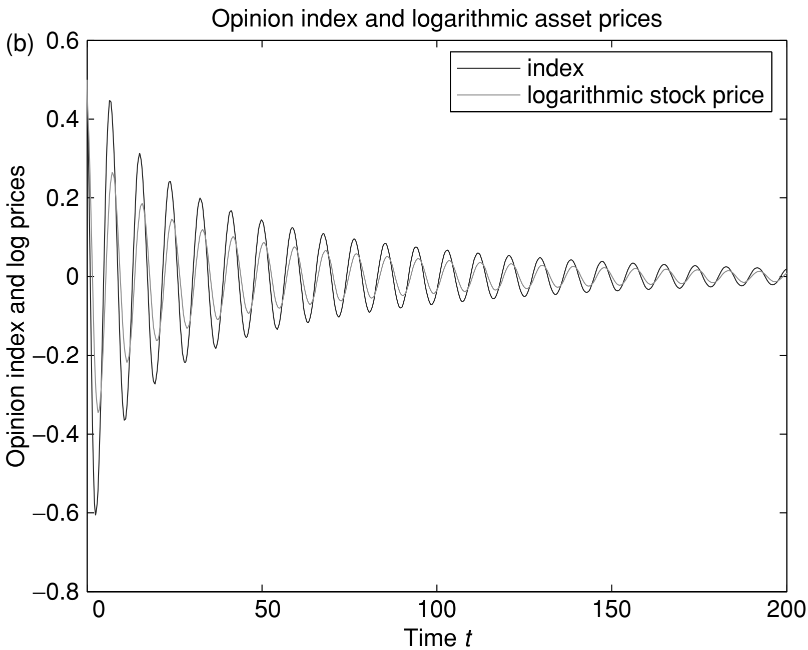


FIGURE 1. Dependence of asset prices and opinion indices of the time lag: (a) $\delta = 0.01$: convergence to a stable limit cycle; (b) $\delta = 0.05$: convergence to equilibrium.

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situation where chartists submit orders in reaction to the actual price trend,

$$\widetilde{\lambda}_{\pm}\big(\pm 1, \varrho^N_{(t)}, S^N_{(t)}\big) = f\left(\frac{S^N_t - S^N_{t-\delta}}{\delta}\right) \quad \text{and} \quad \widetilde{\lambda}_{\pm}\big(\mp 1, \varrho^N_{(t)}, S^N_{(t)}\big) \equiv 0,$$

for some transformation f. For large N and small δ one is tempted to replace $\frac{S^N_t - S^N_{t-\delta}}{\delta}$ by $\dot{s}_t$ and hence the delay equation (13) by

$$\begin{aligned} \dot{x}_t &= 2\{\tanh(a_1\dot{s}_t + a_2x_t) - x_t\}\cosh(a_1\dot{s}_t + a_2x_t) \quad \text{and} \\ \dot{s}_t &= (x_t f(\dot{s}_t) + \gamma(F - s_t)). \end{aligned}$$

However, when f is nonlinear, there is no reason to expect this implicit dynamics to be well defined. Beyond the simple benchmark of a noise trader framework, continuous-time agent-based models thus call for an extension of Lux's approach beyond an ODE approximation. The following example further illustrates this effect.

Example 3.2. Consider a model with a fundamentalist and chartists. A fundamentalist's order rates are as in the previous example and the chartists' rates are given by

$$\widetilde{\lambda}_{+}\big(1, \varrho^N_{(t)}, S^N_{(t)}\big) = \begin{cases} \gamma_C\big(S^N_t - S^N_{t-\delta}\big) & \text{if } S^N_t - S^N_{t-\delta} > 0 \\ 0 & \text{else} \end{cases}$$

and

$$\widetilde{\lambda}_{-}\big(1, \varrho^N_{(t)}, S^N_{(t)}\big) = \begin{cases} -\gamma_C\big(S^N_t - S^N_{t-\delta}\big) & \text{if } S^N_t - S^N_{t-\delta} < 0 \\ 0 & \text{else,} \end{cases}$$

respectively. Let us assume that the agents choose their trading strategies in reaction to a utility index that reflects the strategies' past performances. More precisely, let $P^0_{t-\delta_i}$ and $P^{+1}_{t-\delta_i}$ be the profits over the time periods $(t-\delta_i, t-\delta_{i+1})$ associated with the fundamentalists' and chartists' trading strategy, respectively. The profits are obtained by multiplying the price increment between $t-\delta_{i+1}$ and $t-\delta_i$ with the average demand. For a fundamentalist this quantity is given by

$$P^0_{t-\delta_i} = \gamma(e^{S_{t-\delta_i}} - e^{S_{t-\delta_{i+1}}})\big(F - S_{t-\delta_{i+1}}\big),$$

whereas a chartist's profit function takes the form

$$P^1_{t-\delta_i} = \gamma_C(e^{S_{t-\delta_i}} - e^{S_{t-\delta_{i+1}}})\big(S_{t-\delta_{i+1}} - S_{t-\delta_{i+2}}\big).$$

Following Föllmer et al. (2005), we define the performance index associated with a trading strategy as a weighted average of the profits a trader would have generated in the past if she had implemented this strategy,

$$U^0_t = \sum_{i=1}^{l} \alpha^{i-1} P^0_{t-\delta_i} \quad \text{and} \quad U^1_t = \sum_{i=1}^{l} \alpha^{i-1} P^{+1}_{t-\delta_i}, \tag{14}$$

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for some discount factor $\alpha < 1$. Let us now put $U_t = U_t^0 - U_t^1$ and denote by x_t the proportion of fundamentalists minus the proportion of chartists at time t. A blend of the models of Föllmer et al. (2005) and Lux (1995) is captured by the flip rates:

$$\widehat{\lambda}^{0,1}\big(\varrho^N_{(t)}, S^N_{(t)}\big) = \frac{e^{-\beta_1 U_t - \beta_2 x_t}}{e^{\beta_1 U_t + \beta_2 x_t} + e^{-\beta_1 U_t - \beta_2 x_t}}$$

and

$$\widehat{\lambda}^{1,0}\big(\varrho^N_{(t)}, S^N_{(t)}\big) = \frac{e^{\beta_1 U_t + \beta_2 x_t}}{e^{\beta_1 U_t + \beta_2 x_t} + e^{-\beta_1 U_t - \beta_2 x_t}}.$$

The first-order approximation is then given by the system of delay differential equations

$$\begin{aligned} \dot{x}_t &= \{\tanh(\beta_1 u_t + \beta_2 x_t) - x_t\}\, dt \\ \dot{s}_t &= \left\{\gamma_C \frac{1 - x_t}{2}(s_t - s_{t-\delta_1}) + \gamma \frac{1 + x_t}{2}(F - s_t)\right\} dt, \end{aligned} \tag{15}$$

where u_t is the fundamentalist's excess performance as defined by (14) with the observed prices $S_t^N, S_{t-\delta_1}^N, \ldots, S_{t-\delta_l}^N$ replaced by their respective approximations $s_t, s_{t-\delta_1}, \ldots, s_{t-\delta_l}$.

Our simulation suggest that past asset prices may have a significant impact on stock market dynamics. Figure 2 displays the first-order approximation of the model of Example 3.2 for $\gamma = 1$, $\gamma_C = 3$, $\alpha = 0.9$, $F = 0$, $\beta_1 = 2$, $\beta_2 = 0.5$, and $l = 3$ if $x_t \equiv s_t \equiv 0.4$ for $t < 0$. For these parameter values the delay equation (15) has a steady state at $s = 0$ and $x = 0$. For the small time lags $\delta_1 = 3/10$, $\delta_2 = 5/10$, and $\delta_3 = 7/10$, the first-order approximation converges rapidly to an equilibrium, as shown in Figure 2a. For larger lags $\delta_1 = 1$, $\delta_2 = 2$, and $\delta_3 = 3$ the system displays erratic though regular and persistent fluctuations; see Figure 2b. Such a history dependence of asset prices and market moods is not and cannot be captured if the dynamics is reduced to a simple ODE. It turns out that the strength of social interactions as measured by β_2 also has an important impact on the magnitude of the fluctuations. A stronger social interaction decreases the relative importance of the past performances of trading strategies and seems to dampen price fluctuations. This effect is illustrated by Figure 3, which shows the first-order approximation for $\delta_1 = 1$, $\delta_2 = 2$, and $\delta_3 = 3$ and for $\beta_2 = 0.5$ and $\beta_2 = 1.14$, respectively.

4. PROOF OF THE MAIN THEOREMS

In this section we prove our main results: the pathwise approximation of the processes Q^N by the trajectory of a delay differential equation and the approximation in distribution of the fluctuations around the first-order approximation by a stochastic delay equation.

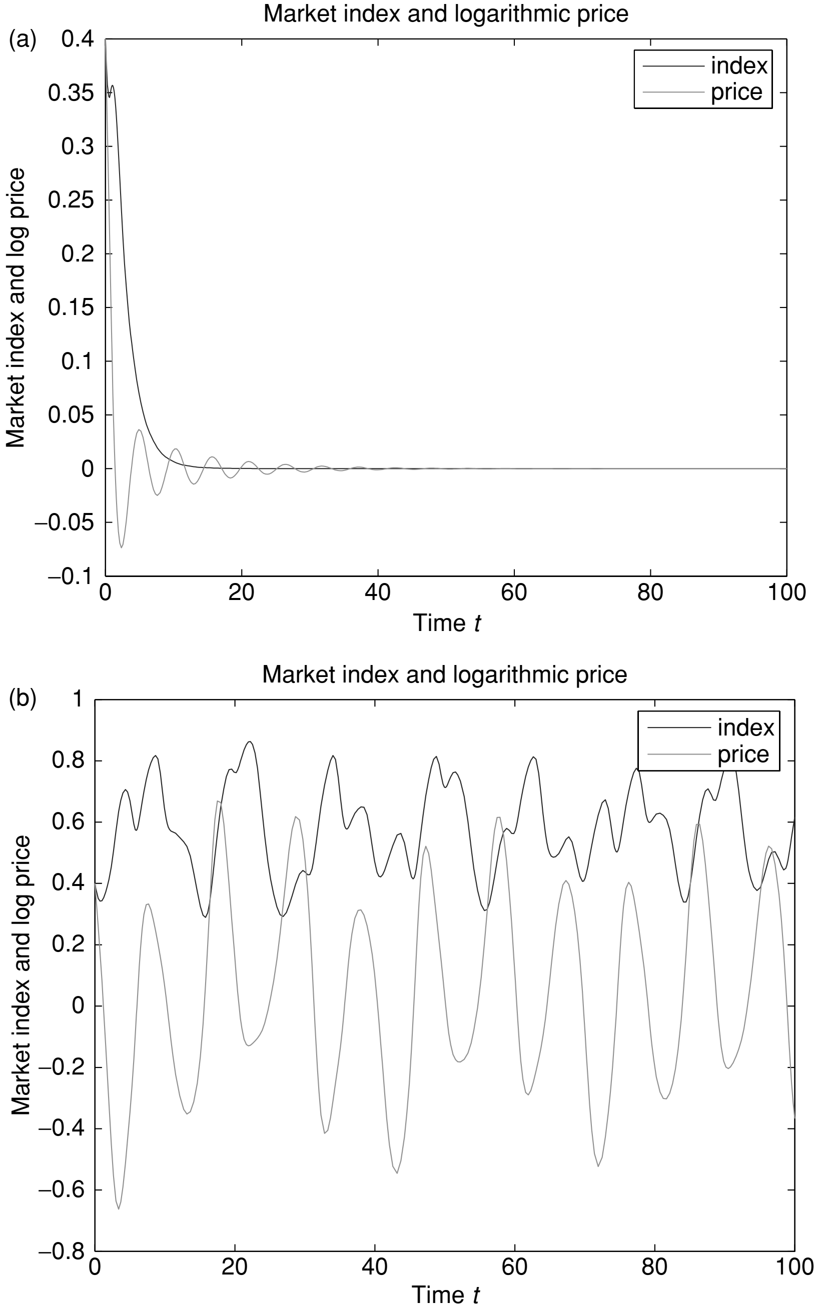

FIGURE 2. Dependence of the market dynamics on time lags: (a) Small lags: rapid convergence to equilibrium; (b) large lags: erratic fluctuations.

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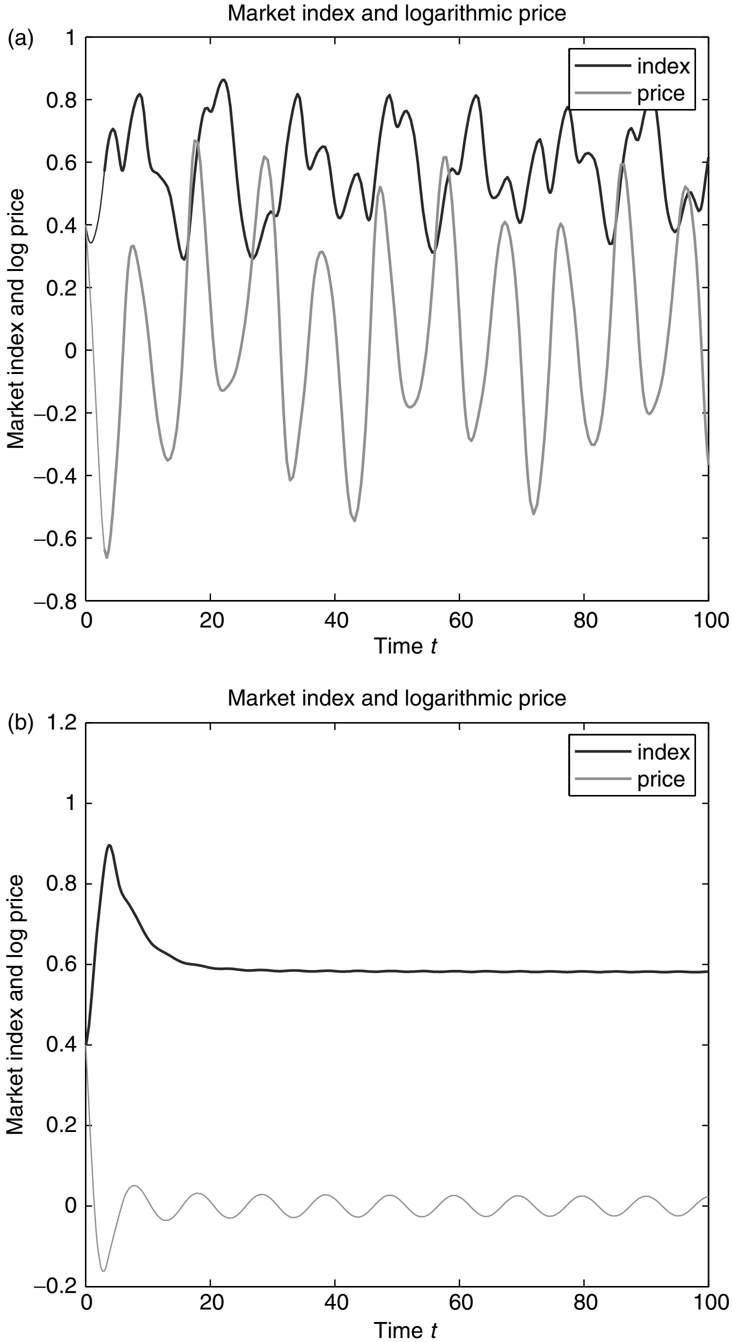

FIGURE 3. Dependence of the market dynamics on the strength of social interactions: (a) Large lags; weak social interaction; (b) large lags: stronger social intraction.

4.1. Proof of the First-Order Approximation

To establish the strong approximation, we first state a result on the existence and uniqueness of solutions of delay differential equations. Its proof follows from standard arguments given in, e.g., Driver (1977).

LEMMA 4.1. *Under the assumptions of Theorem* 2.5, *for any continuous initial function* $(q_s)_{-\delta_l \le s \le 0}$, *there exists a unique global solution to the delay equation* (9).

We are now ready to establish the approximation of the processes Q^N by the solution to the delay differential equation (9).

Proof of the first-order approximation. Because the rate functions are uniformly bounded, the law of iterated logarithms for Brownian motion yields

$$\lim_{N\to\infty} \sup_{u\le t} \frac{1}{N} B^i_{\pm}\left(N \int_0^u \lambda^i_{\pm}\left(Q^N_{(v)}\right) dv\right) = 0 \qquad \mathbf{P}\text{-a.s.}$$

Thus for every $\epsilon > 0$ there exists $N^* \in \mathbf{N}$ such that

$$\left|Q^N_t - q_t\right| \le \int_0^t \left|\lambda\left(Q^N_{(u)}\right) - \lambda\left(q_{(u)}\right)\right| du + \epsilon \qquad \mathbf{P}\text{-a.s.}$$

for all $N \ge N^*$. Because the rate functions are differentiable with uniformly bounded first derivatives, there exists a constant $L < \infty$ that satisfies

$$\left|Q^N_t - q_t\right| \le L \int_0^t \sup_{-\delta_l \le v \le u} \left|Q^N_v - q_v\right| du + \epsilon \qquad \mathbf{P}\text{-a.s.}$$

By convention, $Q^N_v = q_v$ for $v < 0$, so $\sup_{-\delta_l \le v \le u} |Q^N_v - q_v| = \sup_{0 \le v \le u} |Q^N_v - q_v|$ and

$$\sup_{0\le v\le t} \left|Q^N_v - q_v\right| \le L \int_0^t \sup_{0 \le v \le u} \left|Q^N_v - q_v\right| du + \epsilon \qquad \mathbf{P}\text{-a.s.}$$

As a result, an application of Gronwall's lemma yields

$$\sup_{0\le v\le t} \left|Q^N_v - q_v\right| \le \epsilon e^{Lt} \qquad \mathbf{P}\text{-a.s.}$$

This proves the assertion, as ϵ is arbitrary. ∎

4.2. Proof of the Second-Order Approximation

To keep the paper self-contained we first prove pathwise uniqueness of the solution to the stochastic integral equation (12).

PROPOSITION 4.2. *Under the assumptions of Theorem* 2.6 *there exists an almost surely unique pathwise solution to* (12).

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Proof. To prove the existence of a global solution we first establish the existence and uniqueness of a local solution, that is, of a solution on a time interval $[0, \delta]$ for a sufficiently small $\delta > 0$. In a second step we apply a standard argument to show how the local solution can be extended to a solution on $[0, T]$.

Let $\mathcal{C}_T$, equipped with the standard sup-norm $\|\cdot\|_\infty$, be the Banach space of all continuous $(m+1)$-dimensional functions on $[-\delta_l, T]$. For the continuous initial function $q : [-\delta_l, 0] \to \mathbf{R}^{m+1}$ and a given trajectory $(X_t(\omega))_{t \geq 0}$ we define mappings $\varphi \in \mathcal{C}_T$ and $F : [-\delta_l, T] \times \mathcal{C}_T \to \mathbf{R}^{m+1}$ by

$$\varphi(t) = \begin{cases} q(t) & \text{for } t \in [-\delta_l, 0] \\ q(0) + X_t(\omega) & \text{for } t \in [0, T] \end{cases} \quad \text{and} \quad F^i(t, x) = \left\langle \nabla \lambda^i \left(q_{(t)}\right), x_{(t)} \right\rangle,$$

respectively. By Assumption 2.3, the map $t \to F(\cdot, \varphi)$ is almost surely continuous and hence it is almost surely bounded,

$$\|F(\cdot, \varphi)\|_\infty \leq B,$$

where the random bound B depends on the trajectory of the process X. Let us now fix a positive constant b. For a given $\delta > 0$ we introduce a closed subset of $\mathcal{C}_T$ by

$$\mathcal{E}_\delta = \{\psi \in \mathcal{C}_\delta : \|\psi - \varphi\|_\infty \leq b \text{ and } \psi \equiv q \text{ on } [-\delta_l, 0]\}.$$

Because the rate functions have a uniformly bounded first derivative, we have for all $x \in \mathcal{E}_\delta$ that

$$|F(t, x)| \leq |F(t, x) - F(t, \varphi)| + |F(t, \varphi)| \leq L\|x - \varphi\|_\infty + B \leq Lb + B$$

for some $L < \infty$. Because the constants B and b do not depend on δ, the operator defined by

$$H(x)(t) = \begin{cases} q(t) & \text{for } t \in [-\delta_l, 0] \\ q(0) + \int_0^t F(u, x)du + X_t(\omega) & \text{for } t \in [0, \delta] \end{cases}$$

maps the closed set $\mathcal{E}_\delta$ into itself when δ is sufficiently small. Observe now that

$$|H(x)(t) - H(y)(t)| \leq \int_0^t |F(u, x) - F(u, y)|\, du \leq L\delta \max_{-l \leq s \leq \delta} |x(s) - y(s)|.$$

Hence

$$\max_{-l \leq s \leq \delta} |H(x)(t) - H(y)(t)| \leq L\delta \max_{-l \leq s \leq \delta} |x(s) - y(s)|.$$

This shows that for almost every trajectory of the process X there exists a sufficiently small $\delta > 0$ such that the operator $H : \mathcal{E}_\delta \to \mathcal{E}_\delta$ is a contraction. By Banach's theorem it has a unique fixed point. As a result, the stochastic integral equation (12) has a unique solution on sufficiently small time intervals. By a standard argument, the solution can be extended to a solution on the whole interval $[0, T]$. ∎

As a second step toward the proof of the second-order approximation, we introduce the processes

$$U_t^N = \sqrt{N}\left(Q_t^N - q_t\right)$$

and

$$X_t^N = B_+\left(\int_0^t \lambda_+\left(Q_{(u)}^N\right)du\right) - B_-\left(\int_0^t \lambda_-\left(Q_{(u)}^N\right)du\right).$$

The following lemma shows that the sequence $\{U^N\}$ is bounded in probability.

LEMMA 4.3. *For any $\epsilon > 0$, there exist $N^* \in \mathbf{N}$ and $K < \infty$ such that*

$$\mathbf{P}^*\left[\sup_{0\le t\le T}\left|U_t^N\right| > K\right] < \epsilon \quad \textit{for all} \quad N \ge N^*. \tag{16}$$

Proof. The strong approximation for Brownian motion yields the representation

$$U_t^N = \sqrt{N}\int_0^t \left\{\lambda\left(Q_{(u)}^N\right) - \lambda\left(q_{(u)}\right)\right\}du + X_t^N. \tag{17}$$

Because the rate functions are bounded, the sequence $\{X^N\}_{N\in\mathbf{N}}$ is tight, and hence it is bounded in probability. As a result, Lipschitz continuity of the rate functions yields

$$\sup_{0\le t\le T}\left|U_t^N\right| \le L\int_0^T \sup_{0\le t\le u}\left|U_u^N\right|du + \sup_{0\le t\le T}\left|X_t^N\right|$$

for some $L > 0$. Hence, by Gronwall's inequality, we have almost surely that

$$\sup_{0\le t\le T}\left|U_t^N\right| \le e^{3LT}\sup_{0\le t\le T}\left|X_t^N\right|. \qquad \blacksquare$$

The second-order approximation uses the following continuity property of a standard Wiener process W: for any $\alpha \in (0, \frac{1}{2})$ and $T > 0$, there exists an integrable and hence almost surely finite random variable M such that

$$|W(t_1) - W(t_2)| \le M|t_1 - t_2|^\alpha$$

almost surely for all $t_1, t_2 \le T$; see, for instance, Remark 2.12 in Karatzas and Shreve (1991). Thus, the first-order approximation shows that the sequence of stochastic processes $\{X^N\}_{N\in\mathbf{N}}$ converges almost surely to X uniformly on compact time intervals. With this we are now ready to establish the second-order approximation. The proof uses a perturbation of an argument given in Bayraktar et al. (in press a).

Proof of the second-order approximation. For a function $f \in \mathcal{C}_T$ and the continuous initial function $\tilde{q} : [-\delta_l, 0] \to \mathbf{R}$, let $H(f) = (H^1(f), \ldots, H^{m+1}(f))$ be

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the unique function that satisfies the integral equation

$$H_t^i(f) = \begin{cases} q^i(t) & \text{for } t \in [-l, 0] \\ \int_0^t \langle \nabla\lambda^i\big(\tilde{q}_{(u)}\big), H_{(u)}(f)\rangle du + f_t^i & \text{for } t \in [0, T]. \end{cases}$$

Hence $H(X) = Z$, where Z is defined in (12). Because the rate functions have a uniformly bounded derivative, an application of Gronwall's lemma shows that H is a continuous operator. As a result,

$$\lim_{N\to\infty} \|H(X^N) - Z\|_\infty = 0,$$

because the sequence $\{X^N\}_{N\in\mathbf{N}}$ converges almost surely and hence in probability to X. With $E_t^N := U_t^N - H_t(X^N) = (E_t^{N,1}, \dots, E_t^{N,m+1})^t$, it is then enough to prove that

$$\lim_{N\to\infty} \sup_{0\le t\le 1} \left|E_t^N\right| = 0 \tag{18}$$

in probability, because the limit in probability of the sum of two random variables is equal to the sum of the limits in probability. The representation (17) of U_t^N yields

$$\begin{aligned} E_t^{N,i} &= \sqrt{N}\int_0^t \left\{\lambda^i\big(Q_{(u)}^N\big) - \lambda^i\big(q_{(u)}\big)\right\} du - \int_0^t \big\langle\nabla\lambda^i\big(q_{(u)}\big), H_{(u)}(X^N)\big\rangle du \\ &= \sqrt{N}\int_0^t \left\{\lambda^i\big(Q_{(u)}^N\big) - \lambda^i\big(q_{(u)}\big)\right\} du - \int_0^t \big\langle\nabla\lambda^i\big(q_{(u)}\big), U_{(u)}^N\big\rangle du \\ &+ \int_0^t \big\langle\nabla\lambda^i\big(q_{(u)}\big), E_{(u)}^N\big\rangle du. \end{aligned}$$

By the mean value theorem for vector-valued functions, there exists a vector ξ_u^N that lies between $Q_{(u)}^N$ and $q_{(u)}$, such that

$$\lambda^i\big(Q_{(u)}^N\big) - \lambda^i\big(q_{(u)}\big) = \frac{1}{\sqrt{N}}\big\langle\nabla\lambda^i\,\big(\xi_u^N\big)\,,\,U_{(u)}^N\big\rangle.$$

Hence

$$E_t^{N,i} = \int_0^t \big\langle\nabla\lambda^i\,\big(\xi_u^N\big) - \nabla\lambda^i(q_{(u)}), U_{(u)}^N\big\rangle du - \int_0^t \big\langle\nabla\lambda^i\big(q_{(u)}\big), E_{(u)}^N\big\rangle du.$$

In view of the first-order approximation,

$$\lim_{N\to\infty} \sup_{0\le u\le T} \left|\nabla\lambda^i\big(\xi_u^N\big) - \nabla\lambda^i\big(q_{(u)}\big)\right| = 0$$

almost surely. Because the processes U^N are bounded in probability, it now follows from Lemma 3.15 (ii) in Bayraktar et al. (in press a) that the processes

$$\left\{ \int_0^t \left\langle \nabla\lambda^i\left(\xi_u^N\right) - \nabla\lambda^i\left(q_{(u)}\right), U_{(u)}^N \right\rangle du \right\}_{0 \le t \le T}$$

converge to 0 in probability when $N \to \infty$. Now, an application of Gronwall's lemma shows that the processes E^N converge to 0 in probability uniformly on compact time intervals. ∎

5. CONCLUSION

This paper has introduced a mathematical framework for analyzing financial price fluctuations in continuous-time behavioral finance models. When buy and sell orders arrive at random points in time in accordance with a Poisson dynamics and some agents employ technical trading rules, we showed that the joint dynamics of asset prices and trader opinions can be approximated by the trajectory of a delay differential equation. The fluctuations around this first-order approximation follow an Ornstein–Uhlenbeck process with delay. In a benchmark model of noise trading, our first- and second-order approximations resemble the dynamics of Lux (1995) and Föllmer and Schweizer (1993), respectively. Mathematically, our limit results were based on methods and techniques from the theory of state-dependent queuing networks.

The driving feature of the price process is the switching of agents from one forecasting rule to the other. This switching can be attributed to the relative success of the rules. The switching process has the characteristic that agents can, at any point in time, herd on one rule. When this happens, agents' forecasts are self-reinforcing. There is freedom in specifying the order rates, which eventually map past profits from the different forecasting rules into the probability of choosing those rules. This makes our specification rather general and extends previous results on noise trader models.

Several avenues are open for future research. For instance, as pointed out in the Introduction, stochastic delay equations have been used as a mathematical basis for studying complete market models with stochastic volatility. With our choice of scaling, the volatility is deterministic. Under a different limit-taking scheme, it seems possible to obtain a continuous-time version of the popular GARCH models of stochastic volatility. It would also be useful to study the stability properties of the first-order approximation in a more rigorous manner and to identify the key parameters affecting the dynamics of asset prices and market moods.

NOTE

1. The question of when boundedly rational agents will survive in the long run has been studied by, e.g., Blume and Easley (in press) and Horst and Wenzelburger (in press). The closely related issue of evolutionary stability of portfolio rules has been addressed by, e.g., Evstigneev et al. (2006).

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