THE COSMOLOGICAL PRINCIPLE

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Many models of the universe have been proposed, by de Sitter, Milne, Bondi and Gold, Hoyle and others. The observed data being insufficient, the models are usually based on some simple hypothesis. The simplest is the cosmological principle, namely, that apart from local irregularities the universe presents the same general aspect at every point. Milne (5) has used a restricted form of the principle, namely, that the aspect is independent of spatial position but is dependent on the observed time from some fixed epoch in the past. Bondi and Gold (1) have proposed the 'perfect cosmological principle' that the aspect is completely independent of space and time.

It can be shown, almost without mathematical calculation, that the perfect cosmological principle leads to a perfectly precise model for the universe which has some remarkable characteristics. As in Milne's cosmology, it is convenient to introduce both a t-scale and a τ-scale. But whereas in Milne's cosmology, according to the τ-scale there is an infinite past and future, but according to the t-scale only a finite past, in the universe of the perfect cosmological principle there is an infinite past and future in the t-scale, but in the τ-scale only a finite future. There is not an isolated point event, as in the past of Milne's cosmology, but there is a bounding flat future 3-space in a flat space-time such that all processes are speeded up to infinity as they approach it, thus making the apparent future time appear infinite. According to the more conventional t-scale the space-time has a constant curvature. It is in fact the geometry of de Sitter.

It is assumed that the speed of light *in vacuo* is independent of the frequency. This defines a local metric at every point. The geometry of space-time therefore must necessarily be Riemannian subject to an arbitrary conformal transformation.

It may be remarked here that whereas the Riemannian nature of space-time can thus be demonstrated *a priori*, such elaborations as affine connexions, torsion, non-symmetric $g_{\mu\nu}$'s are superfluous conceptions which may have interesting mathematical consequences, but could only be justified in a description of the universe if remarkable coincidences could be demonstrated between mathematical predictions and observed events.

The Riemannian nature of the universe is subject to an arbitrary conformal transformation. This arbitrariness may be eliminated if it is assumed that a certain spectral line of, say, a calcium atom gives a frequency independent of position. The cosmological principle would then imply that all spectral lines have fixed frequencies. The Riemannian geometry is then precise. The metric implied will be referred to as the t-scale.

It is assumed that there is at every point a preferred direction in space-time corresponding to the mean velocity of the stars in the neighbourhood. Taking local co-ordinates (x, y, z, t) with metric

$$x^2+y^2+z^2-t^2,$$

if the preferred time axis is in the direction (x_1, y_1, z_1, t_1), consider the differential equation

$$x_1\,dx+y_1\,dy+z_1\,dz-t_1\,dt = 0.$$

It is a consequence of the cosmological principle that this differential equation should be integrable, since the spatial directions normal to the preferred time axis must be isotropic.

Integration of the differential equation thus leads to a one-parameter system of 3-spaces Σ_t. The suffix t will be a measure of the time interval in proceeding from one Σ_t to another in the direction of the preferred time axis. By the cosmological principle this time interval is independent of position on Σ_t.

The cosmological principle also implies that these 3-spaces Σ_t are isotropic and have a constant curvature which might be positive, zero or negative.

Since the universe appears to be expanding these 3-spaces cannot be geodesically flat in space-time, since a parallel displacement into the future would expand any length.

It is now convenient to make a conformal transformation. Instead of defining our metric by the frequencies of spectral lines, let us define it in relation to the mean distance between galaxies. Call this the τ-scale. The metrics of the t-scale and the τ-scale could coincide for one specific Σ_t, say when $t = 0$, but shifting from one Σ_t to another, the τ-scale metric would change from the t-scale metric so that the distances between galaxies for the τ-scale metric would remain constant.

Suppose that the apparent age of the universe as measured by observed velocities of recession is t_0. Put

$$dt = \theta\,d\tau.$$

Then

$$\frac{d\theta}{\theta} = \frac{dt}{t_0},$$

giving

$$\theta = e^{t/t_0}.$$

Thence

$$d\tau = dt/\theta = e^{-t/t_0}\,dt,$$

$$\tau = K - t_0\,e^{-t/t_0}.$$

Setting $\tau = 0$ when $t = 0$, we have

$$\tau = t_0(1-e^{-t/t_0}).$$

As $t\to\infty$, τ becomes t_0, so that in the τ-scale the future is finite.

Since, in the τ-scale, a parallel displacement of a distance in Σ_t into the future would leave the length invariant, it is clear that Σ_t is geodesically flat in space-time. Its intrinsic curvature could still be positive, zero or negative, however. But since it is geodesically flat, this curvature will not change with time. Let its constant value be $\pm 1/R^2$. Then the intrinsic curvature of Σ_t according to the t-scale is $\pm 1/R^2\theta^2$. But according to the cosmological principle this intrinsic curvature must be independent of time, and $\theta = e^{t/t_0}$ is certainly dependent on the time. The only possibility is that R is infinite, there is no curvature in Σ_t, which is in fact a Euclidean 3-space.

It follows that, according to the τ-scale, space-time is flat. There is a finite future. At a time t_0 hence there is a flat singular 3-space. Frequencies of spectral lines speed up as they approach this singular 3-space according to the formulae

$$\tau = t_0(1-e^{-t/t_0}), \quad t = t_0 \log[1/(t_0-\tau)].$$

Turn now to the more conventional t-scale. The curvature of space-time can be found by formulae given in a previous paper (4) for the change in values of the Riemann–Christoffel tensor due to a conformal transformation. Since space-time is flat according to the τ-scale, the conformal curvature must be identically zero also for the t-scale. It is sufficient therefore to consider the components of the Ricci tensor R_{ij}. The formula is

$$e^{2\phi} R_{ij} = -2(\phi_{ij}-\phi_i\phi_j) - g_{ij}g^{qr}(\phi_{qr}+2\phi_q\phi_r).$$

Take the metric as
$$dx_1^2+dx_2^2+dx_3^2-dx_0^2,$$
so that τ is replaced by x_0. The conformal transformation from the τ-scale to the t-scale is
$$dt = e^{\phi}\,d\tau,$$
where
$$e^{\phi} = e^{t/t_0},$$
so that
$$\phi = t/t_0 = -\log(1-\tau/t_0) = -\log(1-x_0/t_0).$$
Thus
$$\phi_0 = 1/(t_0-x_0), \quad \phi_{00} = 1/(t_0-x_0)^2,$$
and all other first and second derivatives are zero.

Hence
$$\begin{aligned} e^{2\phi} R_{ij} &= 3g_{ij}/(t_0-x_0)^2, \\ R_{ij} &= 3g_{ij}(1-x_0/t_0)^2/(t_0-x_0)^2 \\ &= 3g_{ij}/t_0^2. \end{aligned}$$
Thus
$$R_{11} = R_{22} = R_{33} = -R_{00} = 3/t_0^2.$$

The space-time has a constant curvature which is positive for the spatial and negative for the temporal directions. It should be noted first that space-time is isotropic and the preferred time axis has no geometrical distinction. The metric is that of the de Sitter universe (see (2)).

The fact that the space curvature is positive implies that the spatial universe is finite, a somewhat surprising result, since the 3-spaces Σ_t are obviously infinite. But the Σ_t are not geodesically flat. The spatial universe is generated by the geodesics through a fixed point which are at that point normal to the preferred time axis. The equation to such a geodesic in the (x_0, x_1)-plane will now be found.

The differential equation of a geodesic is given in (4) as

$$\frac{d^2x}{dt^2} = -\left(\frac{\partial\phi}{\partial x} + v\frac{d\phi}{dt}\right)(1-v^2),$$

where $v = dx/dt$. Since the metric in the (x, t)-plane is $x^2 - t^2$, clearly x and t are interchangeable, so that the equation can be written

$$\frac{d^2x_0}{dx_1^2} = -\frac{\partial\phi}{\partial x_0}(1-v^2),$$

where

$$v = \frac{dx_0}{dx_1},$$

$\partial\phi/\partial x_1$ being zero. Thence

$$v\frac{dv}{dx_0} = -\frac{1-v^2}{t_0-x_0}.$$

Integrating, with a suitable constant of integration, we find

$$1-v^2 = 1/(1-x_0/t_0)^2.$$

Further integration, again with a suitable constant of integration, gives

$$x_1^2/t_0^2 = (1-x_0/t_0)^2-1,$$

which is a hyperbola with parametric equations

$$x_1 = t_0 \sinh u, \quad x_0 = t_0(1-\cosh u).$$

The metric in the τ-scale along the geodesic gives

$$ds^2 = t_0^2(\cosh^2 u - \sinh^2 u)\, du^2,$$

so that

$$ds = t_0\, du.$$

In the t-scale the metric is

$$ds' = \frac{t_0\, du}{1-x_0/t_0} = t_0 \operatorname{sech} u\, du.$$

The radius of the universe up to the point $\tau = -\infty$ is thus

$$\int_0^\infty t_0 \operatorname{sech} u\, du = [2t_0 \tan^{-1} e^u]_0^\infty = \tfrac{1}{2}\pi t_0.$$

The circumference of the universe at this radius is

$$\lim_{u\to\infty} 2\pi x_1 \operatorname{sech} u = 2\pi t_0.$$

The volume of the universe is

$$\begin{aligned}\int_0^\infty 4\pi x_1^2 \operatorname{sech}^3 u\, t_0\, du &= 4\pi t_0^3 \int_0^\infty \sinh^2 u \operatorname{sech}^3 u\, du \\ &= 2\pi t_0^3[2\tan^{-1} e^u - \tanh u \operatorname{sech} u]_0^\infty \\ &= \pi^2 t_0^3.\end{aligned}$$

That portion of the universe to which the stars are accessible at the speed of light is confined to the region

$$x_1^2 + x_2^2 + x_3^2 \leqslant t_0^2.$$

this limit corresponds to $\sinh u = 1$, $\cosh u = \sqrt{2}$.

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The volume of the accessible part of the universe is thus

$$\begin{aligned}&2\pi t_0^3[2\tan^{-1}e^u - \tanh u \operatorname{sech} u]_0^{\log(1+\sqrt{2})}\\ &= 2\pi t_0^3[\tfrac{1}{4}\pi - \tfrac{1}{2}]\\ &= \tfrac{1}{2}\pi t_0^3(\pi - 2).\end{aligned}$$

The accessible portion of the universe is thus confined to the fraction $(\pi - 2)/2\pi$ of the total volume.

Here is the paradox of a finite universe which is continually expanding but yet remaining the same size. The explanation is that in the far inaccessible portion the velocity of recession is sufficiently near to the velocity of light for the Fitzgerald contraction effect to outweigh the effect of expansion.

The question has been raised as to whether the velocity of recession of distant galaxies exceeds the velocity of light or only approaches light speed as a limit. In the 3-space Σ_t the velocity of recession will exceed the velocity of light. This is possible because the 3-space is not geodesic. In the finite geodesic 3-space the recession velocity approaches the speed of light as a limit. Because of the Fitzgerald effect the density of matter increases along this geodesic 3-space, so that the total amount of matter in the universe is infinite.

The fact that the curvature is isotropic is remarkable. One would expect the preferred time axis to be geometrically distinct. The isotropy occurs only because $\phi_{00} - \phi_0^2$ happens to be zero.

On the face of it there would appear to be an irreconcilable conflict with Einstein's general theory. It is fundamental to Einstein's theory that matter is associated with an anisotropic curvature. Even when allowance is made for cosmological terms, the difficulty remains. The only possible means of reconciling the two theories is to postulate a cosmological term which is itself anisotropic and which would exactly cancel out, in the mean, the anisotropic terms due to the presence of matter. There are two objections to such a hypothesis.

The introduction of a cosmological term at all is rather distasteful. That it should be anisotropic is even more objectionable. This objection, however, is not fatal, and the hypothesis could be accepted if it proved to be necessary.

The second objection is the improbability that the two anisotropic terms should exactly cancel in the mean. This objection might be considered fatal if it were valid, but a closer examination shows that in fact it is not valid. It will be shown that the introduction of an anisotropic cosmological term will lead to matter distribution such that the smoothed out curvature will converge to the isotropic de Sitter metric.

Thus, if we consider as above a conformal transformation from a non-expanding space, the component of curvature

$$R_{11} - \tfrac{1}{2}g_{11}R = e^{-2\phi}(\phi'' + 2\phi'^2)$$

will only trivially be affected by the presence of matter, and must be identified with the corresponding component of the anisotropic cosmological term, i.e. it will be constant. With a suitable choice of unit, put

$$e^{-2\phi}(\phi'' + 2\phi'^2) = 3.$$

Changing to $\theta = e^{\phi}$, we obtain $$\theta\theta'' + \theta'^2 = \theta^4,$$

and putting $p = \theta'$, $$\theta p \frac{dp}{d\theta} + p^2 = 3\theta^4.$$

Integration gives $$\theta^2 p^2 = \theta^6 + K.$$

If $K = 0$ this gives very readily the isotropic de Sitter metric. If $K \neq 0$ integration involves elliptic functions, but the behaviour of the integral can be deduced without integration.

Suppose that measurements are taken again after a period in which the universe doubles in size, but new units are chosen so that the new rate of expansion is taken to be the same as in the original measurements. Clearly θ must be replaced by

$$\theta' = \tfrac{1}{2}\theta.$$

In order that the equation $$\theta p \frac{dp}{d\theta} + p^2 = 3\theta^4$$

may be conserved, clearly p must be replaced by

$$p' = \tfrac{1}{4}p.$$

The equation $$\theta^2 p^2 = \theta^6 + K$$

then becomes $$\theta'^2 p'^2 = \theta'^6 + \tfrac{1}{8}K,$$

and the effective value of K is replaced by $K/8$. Clearly the universe converges fairly rapidly to the de Sitter metric.

The only valid objection to the cosmological principle, from this aspect, would therefore seem to be the necessity for postulating an anisotropic cosmological term. The seriousness of such an objection must be a matter for individual judgement.

This modification of Einstein's equations was proposed by Hoyle (3). The only slight difference is that Hoyle proposes a constant rate of creation of matter. With a given constant anisotropic cosmological term the rate of creation of matter will depend on the rate of expansion of space. But this will approach a constant value as the metric approaches the de Sitter metric, so that the distinction converges to zero.

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