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Rings With Comparability

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Abstract. The class of rings studied in this paper properly contains the class of right distributive rings which have at least one completely prime ideal in the Jacobson radical. Amongst other results we study prime and semiprime ideals, right noetherian rings with comparability and prove a structure theorem for rings with comparability. Several examples are also given.

Introduction

A right distributive ring is a ring whose lattice of right ideals is distributive. It is well-known that the class of commutative distributive domains coincides with the class of Prüfer domains. Noncommutative right distributive rings were investigated in a paper of Stephenson [10]. Brungs [3] proved that right distributive domains are locally right chain rings (see [2] and the literature quoted therein). Recently several papers showed that some features for right chain rings can be carried over to right distributive rings which have at least one completely prime ideal contained in the Jacobson radical [4], [5], [6], [7], [8].

Elements in a right chain ring R are comparable in the sense that for $a, b \in R$ we have either $aR \subseteq bR$ or $bR \subseteq aR$. Also, if R is a right distributive ring and P is a completely prime ideal contained in the Jacobson radical of R, then we can compare elements of R. In fact, if $a, b \in R$, then one of the following holds: $aR \subseteq bR$, $bR \subseteq aR$ or $(aR)S^{-1} = (bR)S^{-1}$, where $(aR)S^{-1} = \{x \in R : \exists s \in S \text{ with } xs \in aR\}$ and $S = R \setminus P$ ([7, Lemma 3.1]; [5, Section 3]).

A ring R is said to satisfy right P-comparability if for all $a, b \in R$ one of the following conditions holds: $aR \subseteq bR$, $bR \subseteq aR$ or $(aR)S^{-1} = (bR)S^{-1}$, where $S = R \setminus P$. We can prove that several results which are known for right distributive rings are also true for rings having right P-comparability. In this way we can extend results of several papers, mainly [5], [6], [7]. However the main purpose of this paper is to obtain results which are new even for right distributive rings. Finally, examples show that the class of rings with right comparability is bigger than the class of right distributive rings.

In Section 1 we give the basic definitions and results. In Section 2 we give examples. In particular, it follows that rings with right comparability can be obtained as a pullback of a right chain ring T with a maximal ideal M and domains $D \subseteq T/M$ provided that the skew field of fractions of D exists and equals T/M. In Section 3 we prove a converse of the main result of Section 2 for prime rings having both left and right comparability. The corresponding result does not hold if we assume one-sided comparability.

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As we said above most of the results in former papers on distributive rings ([5], [6], [7]) can be extended to the class of rings with comparability. In Section 4 we give an example of this obtaining an extension of Theorem 2.1 of [6].

Finally, in Section 5 we obtain a criterion for a ring with comparability to be right noetherian. As a consequence a prime right noetherian ring R with left and right comparability is a chain ring, provided there exists a completely prime ideal $Q \neq 0$ contained in the Jacobson radical of R.

It should be mentioned here that in [1] the authors studied rings which have some type of comparability, different from our comparability.

Throughout this paper R is always a ring with identity element. By $J(R)$ we denote the Jacobson radical of R, $\beta(R)$ the prime (lower nil) radical of R, and $N_g(R)$ the generalized nil radical of R. The set of units of R is denoted by $U(R)$. The notations $\subset$ and $\supset$ will mean strict inclusions. Ideals are assumed to be two-sided unless otherwise stated.

1 Definitions and Basic Results

Let R be a ring. A right ideal P of R is said to be *completely prime* if $ab \in P$ implies either $a \in P$ or $b \in P$. Thus, if P is completely prime then $R \setminus P$ is multiplicatively closed.

A right ideal $I \neq R, (0)$ of R is said to be a (right) waist if for every right ideal K of R we have either $I \subseteq K$ or $K \subset I$. We point out that I is a waist if for every $a \in R \setminus I$ we have $I \subset aR$. Also, every waist is contained in the Jacobson radical.

Assume that S is a multiplicatively closed subset of R and $a \in R$. We define $(aR)S^{-1}$ by $(aR)S^{-1} = \{x \in R : \exists s \in S \text{ such that } xs \in aR\}$. Recall that S is said to be a *right Ore set* if for every $a \in R$ and $s \in S$ there exist $t \in S$ and $b \in R$ such that $at = sb$. We begin with the following

Lemma 1.1 *Assume that $S \subseteq R$ is a right Ore set. Then $(aR)S^{-1}$ is a right ideal for any $a \in R$.*

Proof If $x, y \in (aR)S^{-1}$ there are $s, t \in S$ such that $xs, yt \in aR$. Since S is a right Ore set there exist $u, v \in S$ with $su = tv$. Then $(x-y)su = xsu - ytv \in aR$ and so $x - y \in (aR)S^{-1}$. Similarly we obtain $xb \in (aR)S^{-1}$ for any $b \in R$. ■

Denote by P a completely prime right ideal of R and let $S = R \setminus P$.

Definition 1.2 We say that R satisfies right comparability with respect to P if for every $a, b \in R$ one of the following conditions holds: $aR \subseteq bR$, $bR \subseteq aR$ or $(aR)S^{-1} = (bR)S^{-1}$.

In the above case we simply say that R has (right) P-comparability and we will omit *right* if there is no possibility of misunderstanding.

Lemma 1.3 *Assume that R has P-comparability, where P is a completely prime right ideal of R. Then P is a waist. In particular, P is a two-sided completely prime ideal contained in $J(R)$.*

Proof Suppose $a \in P$, $b \notin P$. Notice that $bR \subseteq aR$ and $(aR)S^{-1} = (bR)S^{-1}$ imply contradictions. So $aR \subseteq bR$ and thus P is a waist and so $P \subseteq J(R)$. Apply now Lemma 2.5 of [5]. ■

The above lemma shows that P-comparability makes sense only when P is a two-sided completely prime ideal contained in $J(R)$. The following proposition gives several equivalent conditions for R to have P-comparability.

Proposition 1.4 *Let R be a ring, P a (two-sided) completely prime ideal contained in $J(R)$ and $S = R \setminus P$. The following conditions are equivalent:*

(i) R has P-comparability.
(ii) For all $a, b \in R$ we have either $aR \subseteq bR$ or $(bR)S^{-1} \subseteq (aR)S^{-1}$.
(iii) For all $a, b \in R$ we have either $aR \subseteq bR$ or $bR \subseteq (aR)S^{-1}$.
(iv) S is a right Ore set and for all $a, b \in R$ we have either $aR \subseteq bR$ or $b \in (aR)S^{-1}$.
(v) $(aR)S^{-1}$ is a right ideal and a waist, for every $a \in R$.

Proof The proofs of (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are straightforward.

(iii) $\Rightarrow$ (iv). Take $a \in R$, $s \in S$. If $aR \subseteq (sR)S^{-1}$ there exist $t \in S$ and $b \in R$ such that $at = sb$. Otherwise we have $sR \subseteq aR$ and then $s = ar$, for $r \in S$, because $s \notin P$. Therefore, S is a right Ore set. The other part of (iv) is clear.

(iv) $\Rightarrow$ (v). $(aR)S^{-1}$ is a right ideal for every $a \in R$, by Lemma 1.1. Assume $b \notin (aR)S^{-1}$ and suppose there exists $x \in (aR)S^{-1}$ such that $x \notin bR$. Then $b \in (xR)S^{-1}$ and so there exists $s \in S$ with $bs \in xR \subseteq (aR)S^{-1}$. Thus $bst \in aR$ for some $t \in S$ and we obtain $b \in (aR)S^{-1}$, a contradiction. It follows that $(aR)S^{-1} \subset bR$ and so $(aR)S^{-1}$ is a waist.

(v) $\Rightarrow$ (ii). Assume that for $x, y \in R$, $(yR)S^{-1} \nsubseteq (xR)S^{-1}$. Since $(xR)S^{-1}$ is a waist we have $(xR)S^{-1} \subset (yR)S^{-1}$. In this case the assumption $y \in (xR)S^{-1}$ easily gives a contradiction and hence we have $xR \subseteq (xR)S^{-1} \subset yR$. The proof is complete. ■

If $S \subseteq S'$ are multiplicatively closed subsets of R we have $(aR)S'^{-1} \subseteq (aR)S^{-1}$, for any $a \in R$. Using this we can easily see that if $P' \subseteq P$ are completely prime ideals and R has P-comparability, then R also has P'-comparability. The following is an immediate consequence.

Proposition 1.5 *Assume that R has P-comparability, where $P \subseteq J(R)$. Then the set of all the completely prime right ideals contained in P coincides with the set of all the completely prime two-sided ideals contained in P. Furthermore, this set is linearly ordered and so the generalized nil radical $N_g(R)$ is completely prime and a waist.*

Now we give another definition.

Definition 1.6 A ring R is said to be a *ring with right comparability* if R has right P-comparability for every completely prime right ideal P contained in $J(R)$.

As in [5] we say that R satisfies condition (MP) if there exists a completely prime ideal of R contained in $J(R)$. By the above results, if R is a ring with comparability, then we can find a largest completely prime ideal $Q \subseteq J(R)$ and R has Q-comparability. The following is clear.

Corollary 1.7 *Let R be a ring which satisfies condition* (MP). *Then R is a ring with comparability if and only if the set of completely prime ideals of R contained in $J(R)$ has a largest member Q and R has Q-comparability.*

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2 Examples

We begin with the following natural example which initiated the study of the subject in this paper.

Example 2.1 Assume that R is a (right) distributive ring and P is a completely prime ideal contained in $J(R)$. Then R has P-comparability. In fact, if $a, b \in R$, $aR \nsubseteq bR$ and $bR \nsubseteq aR$, then $aP = bP$ [7, Lemma 3.1]. Also, in this case a and b are non-zero and so we may use Lemma 3.4 of [5] to obtain $(aR)S^{-1} = (bR)S^{-1}$. ■

The next proposition leads to a large class of examples. We use here a construction which is an extension of the one used in [6, Proposition 4.2] to obtain examples of distributive rings.

Let T be a right chain ring with maximal ideal M and let D be a domain contained in the skew field $F = T/M$. Consider the canonical mappings $\pi\colon T \to F$ and $j\colon D \to F$. We denote by R the pullback of D and T. Recall that R is the subring of the ring $D \times T$ consisting of all the elements $(a, x) \in D \times T$ such that $j(a) = \pi(x)$. Under this notation, we have the following

Lemma 2.2 *Let Q be the set of all the elements $(0, x) \in R$, where $x \in M$. Then Q is a completely prime two-sided ideal of R and is a waist as right and left ideal.*

Proof It is clear that Q is an ideal of R. Also, Q is completely prime since D is a domain. Assume that $(a, x) \in R \setminus Q$, *i.e.*, $a \neq 0$. So $x \notin M$ and there exists $x^{-1} \in T$. For any $y \in M$ we have $(0, x^{-1}y) \in R$ and $(0, y) = (a, x)(0, x^{-1}y) \in (a, x)R$. Consequently $Q \subset (a, x)R$. In the same way we obtain $Q \subset R(a, x)$ and so the proof is complete. ■

Remark 2.3 Note that since Q is a waist we have $Q \subseteq J(R)$. Also, the projection of R into T is injective and so allows us to identify R with $\pi^{-1}\big(j(D)\big)$. Under this identification $Q = M$, *i.e.*, we may assume $M \subseteq R$.

We show that under some additional assumption R has Q-comparability. As usual we say that F is a right skew field of fractions of D if every $z \in F$ can be written as $z = j(a)j(b)^{-1}$, for $a, b \in D$, $b \neq 0$. Thus we have the following

Theorem 2.4 *Under the above situation, if F is a right skew field of fractions of D, then R has Q-comparability.*

Proof Let $(a, x) \in R$ and $(b, y) \in S = R \setminus Q$. Then we have $y \in U(T)$. If $a = 0$, then $(a, x) \in (b, y)R$ by Lemma 2.2. Assume $a \neq 0$. Then $x \in U(T)$ and it follows that there exists $u \in U(T)$ with $x = yu$. Thus we can write $\pi(u) = j(c)j(d)^{-1}$ for some $c, d \in D \setminus \{0\}$. Let $z \in T$ such that $\pi(z) = j(d)$. We have $j(a) = \pi(x) = \pi(y)\pi(u) = j(b)j(c)j(d)^{-1}$ and it follows that $j(ad) = j(bc)$, *i.e*, $ad = bc$ since j is a monomorphism. Moreover, we have that $(c, uz) \in R$ and $(a, x)(d, z) = (ad, xz) = (bc, yuz) = (b, y)(c, uz)$, where $(d, z), (c, uz) \in S$. The argument shows that S is a right Ore set, $(a, x) \in (b, y)R$ when $a = 0$, and $\big((a, x)R\big)S^{-1} = \big((b, y)R\big)S^{-1} = R$ when $a \neq 0 \neq b$.

It remains to consider the case $a = b = 0$. In this case $x, y \in M$ and we may assume that there exists $t \in T$ with $x = yt$. If $t \in M$, then $(0, t) \in R$ and we have $(0, x) = (0, y)(0, t)$.

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If $t \notin M$ then t is invertible in T and there exist non-zero elements $c, d \in D$ with $\pi(t) = j(c)j(d)^{-1}$. A similar argument as above shows that $\big((0,x)R\big)S^{-1} = \big((0,y)R\big)S^{-1}$, which completes the proof. ■

Note that in general if $T = F \oplus M$ is a right chain domain with maximal ideal M, then the pullback $R \simeq D \oplus M$ (*cf.* Proposition 4.2 in [6]). We have a particular case of this situation if we take $R = D \oplus tF[[t;\sigma]]$, where D is a right Ore domain, F is the skew field of fractions of D, σ is the extension to F of an automorphism of D and $F[[t;\sigma]]$ is the skew power series ring over F.

Remark 2.5 In former papers ([4], [5], [6], [7]) another comparability condition has been used instead of the one of Definition 1.2. In fact, we say that R satisfies weak (right) P-comparability if for every $a, b \in R$ one of the following conditions holds: $aR \subseteq bR$, $bR \subseteq aR$ or $aP = bP$, where P is as above. It can easily be seen that comparability implies weak comparability. The following example shows that the converse is not true.

Example 2.6 Let $R = \mathbf{Z} \oplus t\mathbf{Q}(X)[[t]]$, where $\mathbf{Z}$ denotes the ring of integer numbers and $\mathbf{Q}(X)$ the field of rational functions over the field of rational numbers $\mathbf{Q}$. Thus R is a commutative domain contained in $T = \mathbf{Q} \oplus t\mathbf{Q}(X)[[t]]$ and $P = t\mathbf{Q}(X)[[t]]$ is a (completely) prime ideal of R since $R/P \simeq \mathbf{Z}$.

Assume that $f = t^n h \in P$, $h = b + tk \in \mathbf{Q}(X)[[t]]$, $b \neq 0$ and $n \geq 1$. Then $h \in U\big(\mathbf{Q}(X)[[t]]\big)$ and it follows that $fP = t^{n+1}\mathbf{Q}(X)[[t]] = t^n P$. From this remark we can easily see that for $f, g \in R$, we have $fR \subseteq gR$, $gR \subseteq fR$ or $fP = gP$.

Finally, R does not have P-comparability. In fact, for $f = tX$ and $g = t(X-1)$ it is not hard to show that $g \notin fR$ and $f \notin (gR)S^{-1}$, $S = R \setminus P$. ■

3 Prime Rings With Comparability

The purpose of this section is to prove a converse of Theorem 2.4 when R is a prime ring. Let us first state the following proposition whose easy proof is left to the reader.

Proposition 3.1 *Assume that R and T are as in Theorem 2.4. Then R is a prime ring if and only if T is a prime ring.*

We will need also the following

Lemma 3.2 *Let R be a prime ring which has right Q-comparability and suppose that Q is a waist as a left ideal. Then the localization R_Q exists and is a prime right chain ring which is an extension of R.*

Proof Assume $s \in S = R \setminus Q$ and $sx = 0$, for $x \in R$. Then $Rsx = 0$ and since $Rs \supset Q$ we have $x = 0$. Thus the localization R_Q does exist since S is a right Ore set (Proposition 1.4(iv)). We easily can see that R_Q is prime and the canonical mapping $R \to R_Q$ is injective. Finally, take $\frac{a}{1}, \frac{b}{1} \in R_Q$. Then $a, b \in R$ and so either $aR \subseteq bR$ or $b \in (aR)S^{-1}$. Consequently we have either $\frac{a}{1}R_Q \subseteq \frac{b}{1}R_Q$ or $\frac{b}{1}R_Q \subseteq \frac{a}{1}R_Q$, and the result follows easily. ■

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Now we are in position to prove the main result of this section.

Theorem 3.3 *Let B be a ring and P a completely prime ideal contained in $J(B)$. Then B is a prime ring having right P-comparability such that P is also a left waist of B if and only if there exists a prime right chain ring T and a right Ore domain $D \subseteq T/J(T)$ such that $T/J(T)$ is the right skew field of fractions of D, and an isomorphism $\varphi\colon B \xrightarrow{\sim} R$ such that $\varphi(P) = Q$, where R and Q are as in Theorem 2.4.*

Proof The "if" part is an immediate consequence of Lemma 2.2, Theorem 2.4 and Proposition 3.1. Assume that B is a prime ring having right P-comparability such that P is a left waist of B. Then B_P is a prime right chain ring, by Lemma 3.2. Denote by $f\colon B \to B_P$ the canonical mapping: $f(x) = \frac{x}{1}$, for all $x \in B$. If $a \in P$, then $f(a) \in PB_P$, where PB_P is the maximal ideal of B_P. Hence f induces a mapping $j\colon B/P \to B_P/PB_P$, where B/P is a domain, and j is injective because P is completely prime. Also, if $xs^{-1} + PB_P \in B_P/PB_P$, then $xs^{-1} + PB_P = j(x + P)j(s + P)^{-1}$. Hence B_P/PB_P is a skew field of fractions of B/P.

Denote by R the pullback defined by $R = \pi^{-1}\big(j(B/P)\big)$, where $\pi\colon B_P \to B_P/PB_P$ is canonical (see Remark 2.3). Take $x \in B$ and consider $\frac{x}{1} \in B_P$. Then $\pi(\frac{x}{1}) = \frac{x}{1} + PB_P = j(x + P)$ and so $\frac{x}{1} \in R$. Put $\varphi(x) = \frac{x}{1}$. We easily see that $\varphi\colon B \to R$ is a monomorphism of rings with $\varphi(P) = Q$, where $Q = PB_P$.

Assume that $ys^{-1} \in R$, $s \notin P$. Then $ys^{-1} + PB_P = j(x + P) = \frac{x}{1} + PB_P$, for some $x \in B$, and so $y - xs \in P$. Since P is a waist as a left ideal of B we have that $y - xs = ps$, for some $p \in P$. Hence $ys^{-1} = \frac{x+p}{1} = \varphi(x + p)$, where $x + p \in B$. Thus φ is a surjective mapping and the proof is complete. ∎

The above theorem applies when B has both left and right Q-comparability. In particular we have

Corollary 3.4 *Assume that R is a prime left and right distributive ring with* (MP) *and let Q be the largest completely prime ideal of R contained in $J(R)$. Then there exists a prime two-sided chain ring T and a two-sided Ore domain D such that the right skew field of fractions of D is $T/J(T)$ and R is isomorphic to the pullback of D and T as in Theorem 2.4.*

Now we give an example to show that the converse of Theorem 2.4 does not hold when we assume one-sided conditions.

Example 3.5 Let K be a field, $B = K(X_2, X_3, \dots)[X_1]$ and $F = K(X_1, X_2, X_3, \dots)$, where $K(X)$ denotes the field of fractions of the integral domain $K[X]$, for $X = (X_2, X_3, \dots)$ (resp. $X = (X_1, X_2, X_3, \dots)$). Then F is the field of fractions of B. Let $\sigma\colon B \to B$ be the K-monomorphism of rings defined by $\sigma(X_i) = X_{i+1}$ $(i \geq 1)$, and denote also by σ its extension to a monomorphism of F. It is clear that $\sigma(F) \subseteq U(B)$. Put $T = F[[t;\sigma]]$ the skew power series ring defined by $at = t\sigma(a)$, $a \in F$, and consider the subring $R = B[[t;\sigma]]$ of T. Then $P = tB[[t;\sigma]]$ is a completely prime ideal of R since $R/P \simeq B$. We show that R has right P-comparability and R is not a pullback.

Let $f = a_0 + ta_1 + t^2a_2 + \cdots \in R$ where $a_0 \neq 0$. Thus $f \in U(T)$ and it is easy to show that if $q = tp \in P$, where $p \in B$, we have $q = ft\sigma(f^{-1})p$, where $t\sigma(f^{-1})p \in R$. It follows that P is a waist as a right ideal and so $P \subseteq J(R)$. Also, it is easy to check that $(gR)S^{-1} = R$, for all $g \in R \setminus P$, and consequently $S = R \setminus P$ is a right Ore set.

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Consider $f = t^n h$, $g = t^m l$, where $h = h_0 + th_1 + \cdots \in B$, $l = l_0 + tl_1 + \cdots \in B$, $h_0 \neq 0$ and $l_0 \neq 0$. If $m > n$ we have $g \in fR$. If $m = n$, then $f \in (gR)S^{-1}$ as it is easy to see. Hence R has right P-comparability.

Finally, note that R is not a pullback of the type of Theorem 2.4 since P is not a left waist of R. In fact, $tX_2 \in P$ and $tX_2 \notin RX_1$. ∎

4 Prime and Semiprime Ideals

Following [5], a subset T of R is said to be a *right multiplicative ideal* if for every $a \in T$ and $r \in R$ we have $ar \in T$. A right multiplicative ideal T of R is said to be *prime* (resp. semiprime) if for $a, b \in R$ we have that for $aRb \subseteq T$ (resp. $aRa \subseteq T$) implies either $a \in T$ or $b \in T$ (resp. $a \in T$).

The purpose of this section is to prove an extension of Theorem 2.1 of [6]. Note that the proof here is easier than the one given in that paper. We will use the following remark: if P is a completely prime ideal of R, then P is a waist as a right ideal if and only if $aP = P$, for every $a \notin P$.

Theorem 4.1 *Let R be a ring which has P-comparability, where P is a completely prime ideal of R contained in $J(R)$. Then any semiprime right multiplicative ideal of R contained in P is a prime right ideal and a waist.*

Proof Let L be a semiprime right multiplicative ideal contained in P. Suppose $a, b \in L$. If $a = br$, for some $r \in R$, it follows easily that $a + b \in L$. Assume $b \in (aR)S^{-1}$, $S = R \setminus P$. Thus $bs = at$, for some $s \in S, t \in R$. Hence $(a + b)P = (a + b)sP = a(s + t)P \subseteq L$ and so $(a + b)R(a + b) \subseteq (a + b)P \subseteq L$. Therefore $a + b \in L$ and consequently L is a right ideal.

Now, assume $a \in L$ and $b \notin L$. If $b \in (aR)S^{-1}$ there exists $s \in S$ with $bs \in aR \subseteq L \subseteq P$. Since $s \notin P$ we get $b \in P$, hence $bRb \subseteq bP = bsP \subseteq L$, a contradiction. Thus we have $a \in bR$ and L is a waist as a right ideal.

Finally, suppose $aRb \subseteq L$ and $a \notin L$. If $a \in (bR)S^{-1}$, there exist $s \in S$, $r \in R$ such that $as = br$. Thus $asRas = asRbr \subseteq aRbr \subseteq L$, and so $as \in L$. Hence $a \in P$ since $s \notin P$, and it follows that $aRa \subseteq aP = asP \subseteq L$, a contradiction. Consequently, there exists $t \in R$ such that $b = at$ and so $atRat \subseteq aRb \subseteq L$. Therefore $b = at \in L$. The proof is complete. ∎

The following is an immediate consequence.

Corollary 4.2 *Let R be a ring which has P-comparability.*

(i) If $L \subseteq P$ is a prime right multiplicative ideal, then L is a right ideal and a waist.
(ii) The prime radical $\beta(R)$ is a prime ideal and a waist.

5 Noetherian Rings With Comparability

Recall that a ring R is said to satisfy accw (ascending chain condition on waists) if every family of waists of R has a maximal member. Right distributive rings which satisfy accw were studied in [6, Section 3].

We begin this section with the following lemma whose proof is straightforward.

Lemma 5.1 *Let R be any ring. If every right ideal of R which is a waist is finitely generated, then R satisfies accw.*

We will see soon that the converse of Lemma 5.1 is not true.

The purpose of this section is to give a criterion for a ring which satisfies P-comparability to be right noetherian. We will need the following

Lemma 5.2 *Let R be a ring with P-comparability. If $I \subseteq P$ is a right ideal and a waist, then aI is also a waist, for every $a \in R$.*

Proof Assume that $x \notin aI$. If $x = ar$, $r \in R \setminus I$, we have $I \subset rR$ and so $aI \subseteq arR = xR$. Otherwise there exists $s \in S$ with $as \in xR$. Thus $aI \subseteq aP = asP \subseteq xP \subset xR$, and we are done. ■

Theorem 5.3 *Assume that R is a ring with P-comparability, $P \subseteq J(R)$. Then the following conditions are equivalent:*

(i) R is right noetherian.
(ii) R/P is right noetherian and every waist of R which is contained in P is finitely generated as a right ideal.

Proof It is enough to prove (ii) $\Rightarrow$ (i). Assume that $I_1 \subseteq I_2 \subseteq \cdots$ is a sequence of right ideals of R. If there exists $n \geq 1$ such that $P \subseteq I_n$, then the sequence must stabilize since R/P is right noetherian. So we may assume $I_j \subset P$, for all j.

For every $i \geq 1$ there exists a smallest waist L_i of R with $I_i \subseteq L_i$, and we have $L_1 \subseteq L_2 \subseteq \cdots$. Hence by Lemma 5.1 there exists $n \geq 1$ such that $L_n = L_{n+1} = \cdots$. Thus it is enough to show, changing notation, that if $I_1 \subseteq I_2 \subseteq \cdots \subseteq L$, where L is the smallest waist of R containing I_1, then the sequence stabilizes.

By assumption $L = a_1R + \cdots + a_nR$. For $i \geq 1$ and $1 \leq l \leq n$ we put

$$H_{il} = \{r \in R : \exists r_1, \ldots, r_{l-1} \in R \text{ with } a_1r_1 + \cdots + a_{l-1}r_{l-1} + a_lr \in I_i\}.$$

Then H_{il} is a right ideal of R. Also, by Lemma 5.2 a_iP is a waist of R and clearly $a_iP \subset L$. Hence $a_iP \subseteq I_1$ and then we have $P \subseteq H_{il}$, for $i \geq 1$ and $1 \leq l \leq n$. Furthermore $H_{1l} \subseteq H_{2l} \subseteq \cdots$ and so there exists $m \geq 1$ such that $H_{ml} = H_{m+jl}$, for all $j \geq 1$ and $1 \leq l \leq n$, because R/P is right noetherian. We show that $I_{m+j} = I_m$, for $j \geq 1$.

If $x = a_1r_1 \in I_{m+j}$, then $r_1 \in H_{m+j1} = H_{m1}$ and so $x \in I_m$. Assume that if $x = a_1r_1 + \cdots + a_{s-1}r_{s-1} \in I_{m+j}$ then $x \in I_m$ and take $y = a_1t_1 + \cdots + a_{s-1}t_{s-1} + a_st_s \in I_{m+j}$, $t_j \in R$. Thus $t_s \in H_{m+js} = H_{ms}$ and so there exists $z = a_1t'_1 + \cdots + a_{s-1}t'_{s-1} + a_st_s \in I_m$. Therefore $z - y = a_1(t'_1 - t_1) + \cdots + a_{s-1}(t'_{s-1} - t_{s-1}) \in I_{m+j}$ and hence $z - y \in I_m$. Thus $y \in I_m$ and the proof is complete. ■

Corollary 5.4 *Let R be a ring with comparability and let Q be the largest completely prime ideal of R contained in $J(R)$. Then R is a right noetherian ring if and only if R/Q is a right noetherian ring and every waist of R is finitely generated.*

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It is natural to ask whether it is enough to assume that R satisfies accw instead of that every waist of R is finitely generated, for R to be noetherian. The following example shows that this is not the case. It also shows that the converse of Lemma 5.1 does not hold.

Example 5.5 The commutative domain $R = \mathbf{Z} \oplus t\mathbf{Q}[[t]]$ is a ring with P-comparability and $R/P \simeq \mathbf{Z}$ is noetherian, where $P = t\mathbf{Q}[[t]]$. As in [6, Example 4.1], we can easily see that every ideal of R is of the type $H = t^n H_0 \oplus t^{n+1}\mathbf{Q}[[t]]$, where H_0 is a $\mathbf{Z}$-submodule of $\mathbf{Q}$ for $n \geq 1$ and of $\mathbf{Z}$ for $n = 0$. It follows that every waist of R is of the type $t^n\mathbf{Q}[[t]]$ since the lattice of $\mathbf{Z}$-submodules of $\mathbf{Q}$ does not have waists. Consequently R has accw. Finally, note that P is not finitely generated over R (since $\mathbf{Q}$ is not finitely generated over $\mathbf{Z}$). Hence R is not a noetherian ring. ∎

Remark 5.6 Assume that R is a right distributive domain. Since accw is not enough for having finitely generated waists it is a natural question to ask under what conditions we have a converse of Lemma 5.1. We can see that every waist of R is finitely generated if and only if R satisfies accw and every prime ideal of R contained in $J(R)$ is finitely generated as a right ideal. In fact, if R satisfies accw and I is a waist of R, then $I = aP$, for some $a \in R$ and $P \subseteq J(R)$ a completely prime ideal [6, Theorem 3.1]. So the result holds. We are able to show that the same is true for domains with comparability, since the results of [6] can be easily extended to our case.

Now we show that under some additional assumption a ring with comparability is not a noetherian ring unless in trivial cases.

Assume that R is a prime ring with left and right comparability and denote by Q the largest completely prime ideal of R in $J(R)$. We also assume that R has at least two prime ideals contained in $J(R)$, *i.e.*, $Q \neq 0$. We have

Proposition 5.7 *Under the above conditions, if R is a right noetherian ring, then R is a chain domain. In particular, a left and right distributive domain having at least two prime ideals in $J(R)$ is noetherian if and only if it is a noetherian chain domain.*

Proof By Theorem 3.3, R is a pullback of a right Ore domain $D = R/Q$ and a prime chain ring T with maximal ideal Q, where the skew field of fractions of D is $F = T/Q$.

Since R is right noetherian and aQ is a waist, for every $a \in R$, there exists $b \in Q$ such that $0 \neq Q^2 = bQ$. Consider the R-homomorphism $\varphi\colon T \to Q$ given by $\varphi(x) = bx$, for all $x \in T$. Then φ induces a homomorphism $\psi\colon T/Q \to Q/Q^2$ which is injective. In fact, if $x \in T \setminus Q$ and $bx \in Q^2 = bQ$ we have $bx = bq$, for $q \in Q$. Then $b(x - q) = 0$ and $x - q \notin Q$. Since $x - q$ is invertible in T we obtain $b = 0$ and so $Q^2 = 0$, a contradiction.

Now Q is a noetherian right R-module. Hence Q/Q^2 is a noetherian D-module as well. Thus $F = T/Q$ is also a noetherian D-module and so F is finitely generated as a right D-module. Since F is the *left* skew field of fractions of D this is impossible if $D \neq F$. It follows that $F = D$ and so $R = \pi^{-1}\big(j(F)\big) = T$ is a chain ring. Finally, since R is a prime noetherian ring it is a domain. ∎

The result of the above proposition is not true if R does not have both left and right comparability.

Example 5.8 Let B be the ring given in Example 3.5. Thus $R = B[[t;\sigma]]$ has right P-comparability but not left P-comparability, where $P = tB[[t;\sigma]]$. We claim that R is right noetherian but it is not a right chain ring. In fact, the same arguments of [6, Example 4.1] show that every right ideal of R is of type $H = t^n H_0 \oplus t^{n+1}B[[t;\sigma]]$, where H_0 is a right ideal of B. Thus it is clear that the waists of R are the ideals of the type $t^n R$, since the ring B has not any waist. So R is right noetherian by Theorem 5.3. ■

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