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Approximate transmission conditions through a rough thin layer: The case of periodic roughness

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We study the behaviour of the steady-state voltage potentials in a material composed of a two-dimensional object surrounded by a rough thin layer and embedded in an ambient medium. The roughness of the layer is supposed to be ε-periodic, ε being the magnitude of the mean thickness of the layer. For ε tending to zero, we determine approximate transmission conditions in order to replace the rough thin layer by these conditions on the boundary of the interior material. This paper extends the previous works (Poignard, 2009, *Math. Meth. Appl. Sci*., vol. 32, pp. 435–453; Poignard *et al*., 2008, *IEEE Trans. Magnet*., vol. 44, no. 6, pp. 1154–1157) of the third author, which deal with smooth thin layers.

1 Introduction

In this paper, we study the steady-state potential in a dielectric material with a rough thin layer. The roughness of the layer is supposed to be ε-periodic, and the mean thickness of the layer is of order ε, ε being a small positive parameter. The computation of the electric steady-state potential in such domains leads to numerical difficulties inherent in the geometry.

In order to tackle the problem we derive approximate transmission conditions, which generally speaking consist in replacing its influence by added source terms on the limit curve of the layer (when the thickness equals zero). That amounts to removing the rough layer from the original problem (this avoids geometric difficulties), and in order to remedy the error because of this simplification, we add source terms in the transmission conditions: the steady-state potential and the fluxes across the membrane are no longer continuous.

1.1 Motivation

Rough layers appear in many research area. For instance in geophysics the bottom of the oceans and the shores are rough with respect to the large-scale flow. In tribology, many surfaces present a granular aspect at the microscopic scale. The motivation of this paper

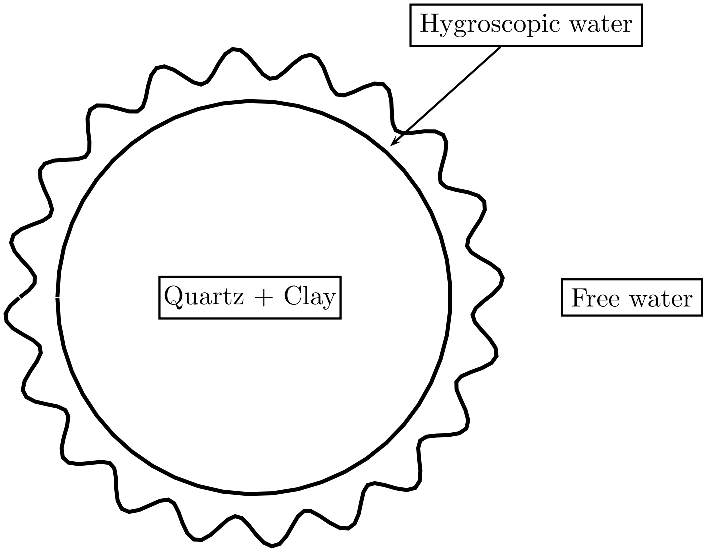

Figure 1. Schematic of the assembly of quartz grain and clay surrounded by a thin layer of the hygroscopic water. The ambient medium is composed by free water.

comes from a collaborative research with Schlumberger on the electrical modelling of silty soils.

In the simplest models, silty soils are composed of water, clay and quartz. The clay leaves are organised on the surface of the quartz grains. Because of their electrical properties, the presence or the lack of clay leaves changes considerably the effective properties of soils. Actually, since they are electrically charged, the clay leaves have the water molecules stuck around them. This changes the electric properties of the so-called bound water or hygroscopic water [14, 15]. This phenomenon occurs on few layers of water molecules; therefore the hygroscopic water is modelled by a rough thin layer. The assembly of the quartz grains and the clay leaves is modelled by an electrically homogeneous domain with non-zero conductivity, while the ambient medium is free water (see Figure 1).

The goal of this paper is to understand the effect of the hygroscopic water on the steady-state electric potential.

1.2 Statement of the problem

For sake of simplicity, we deal with the two-dimensional case; however the three-dimensional case may be studied in the same way up to few appropriate modifications. Let Ω be a smooth bounded domain of $\mathbb{R}^2$ with connected boundary $\partial\Omega$. For $\varepsilon > 0$, we split Ω into three subdomains: $\mathscr{D}^1$, $\mathscr{D}_\varepsilon^m$ and $\mathscr{D}_\varepsilon^0$; $\mathscr{D}^1$ is a smooth domain strictly embedded in Ω (see Figure 2).

We denote by Γ its connected boundary. The domain $\mathscr{D}_\varepsilon^m$ is a thin oscillating layer surrounding $\mathscr{D}^1$. We denote by Γ_ε the oscillating boundary of $\mathscr{D}_\varepsilon^m$:

$$\Gamma_\varepsilon = \partial\mathscr{D}_\varepsilon^m \setminus \Gamma.$$

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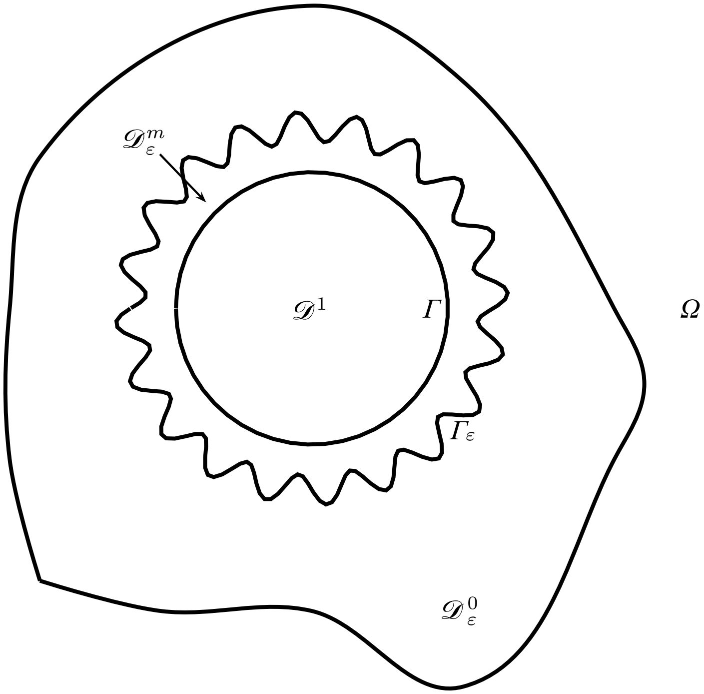

FIGURE 2. Geometry of the problem.

The domain $\mathscr{D}_\varepsilon^0$ is defined by

$$\mathscr{D}_\varepsilon^0 = \Omega \setminus \overline{\left(\mathscr{D}^1 \cup \mathscr{D}_\varepsilon^m\right)}.$$

We also write

$$\mathscr{D}^0 = \Omega \setminus \overline{\mathscr{D}^1}.$$

We define the piecewise-constant function $\sigma : \Omega \to \mathbb{R}$ by

$$\sigma(z) = \begin{cases} \sigma_1, \text{ if } z \in \mathscr{D}^1, \\ \sigma_m, \text{ if } z \in \mathscr{D}_\varepsilon^m, \\ \sigma_0, \text{ if } z \in \mathscr{D}_\varepsilon^0, \end{cases}$$

where σ_1, σ_m and σ_0 are given positive constants[1]. The function σ represents the conductivity of the domain Ω.

Let g belong to $H^s(\partial\Omega)$, for $s \geqslant 1/2$, and denote by u^ε the unique function satisfying

$$\nabla \cdot (\sigma \nabla u^\varepsilon) = 0, \text{ in } \Omega, \tag{1.1a}$$

$$u^\varepsilon|_{\partial\Omega} = g. \tag{1.1b}$$

Observe that for all $\varepsilon > 0$, the domains Ω, $\mathscr{D}^1$, $\mathscr{D}_\varepsilon^m$ and $\mathscr{D}_\varepsilon^0$ are smooth. Hence the above function u^ε belongs to $H^1(\Omega)$, and moreover it belongs to $H^{s+1/2}(\mathscr{D}^1)$, $H^{s+1/2}(\mathscr{D}_\varepsilon^0)$ and

[1] The same following results hold if σ_0, σ_1 and σ_m are given complex functions with imaginary parts (and respective real parts) with the same sign.

$H^{s+1/2}(\mathscr{D}_\varepsilon^m)$. Our aim is to give the first two terms of the asymptotic expansion of u^ε for ε tending to zero.

Several papers are devoted to rough boundaries and derivations of equivalent boundary conditions [1, 2, 3, 8]. In a recent paper Basson and Gérard-Varet [4] derived approximate boundary condition for a boundary with random roughness. The analysis of these previous papers is essentially based on the construction of the so-called wall law, which is a boundary condition imposed on an artificial boundary inside the domain. The wall law only reflects the large-scale effect on the oscillations. Note also that in Chapter 8 of their book, Marchenko and Khrushlov [10] presented equivalent boundary conditions in the very general framework of the elliptic operators with an even degree using homogenisation techniques. We emphasise that all the previous works deal with a problem simpler than ours, since the partial differential equation is studied either in $\Omega \setminus \overline{\mathscr{D}_\varepsilon^0}$ or in $\Omega \setminus \overline{\mathscr{D}_\varepsilon^m}$, and homogeneous Dirichlet boundary conditions are imposed on the rough boundaries.

In this paper, since the rough thin layer is an imperfectly conducting material embedded in an ambient domain, the electric potential satisfies the steady-state potential in the whole domain Ω, including the rough layer $\mathscr{D}_\varepsilon^m$. Hence we are definitely interested in the approximate transmission conditions, and therefore we cannot apply straightforwardly the previous results to our problem. In an earlier paper [11], the third author of the current paper derived the first-order approximate conditions for a thin weakly oscillating layer. This paper is an extension of [11] to the case of an ε-periodic thin layer with thickness of order ε. The main idea of the analysis comes from the paper of Abboud et Ammari [1] and is completely different from the analysis of [11], which was based on an appropriate change of variables. Here, the boundary-layer correctors are obtained by solving an elliptic partial differential equation in an appropriate infinite band with width equal to 1. The well-posedness of the problems and the optimal error estimates are proved to justify our expansion.

The outline of the paper is as follows: In Section 2 we perform a suitable change of variables in order to derive our asymptotics in the simplest way. We also give a preliminary result, which will be useful to prove the formal asymptotics of Section 3. Section 4 is devoted to the justification of our equivalent transmission conditions. We conclude by presenting few numerical simulations performed by Ciuperca, Perrussel and Poignard [6]. We shall first present our main result.

1.3 Main result

Notation 1.1 *We first present the conventions used all along the paper:*

- *All the closed curves are trigonometrically (counterclockwise) oriented.*
- *We generically denote by n the normal to a closed smooth curve of $\mathbb{R}^2$ outwardly directed to the domain enclosed by the curve. Moreover $n^\perp$ is the tangent vector to Γ.*
- *Let $\mathscr{C}$ be a curve of $\mathbb{R}^2$, and let u be a function defined in a tubular neighbourhood of $\mathscr{C}$. We define $u|_{\mathscr{C}\pm}$ by*

$$\forall x \in \mathscr{C}, \quad u|_{\mathscr{C}\pm}(x) = \lim_{t \to 0^+} u(x \pm tn(x));$$

moreover if u is differentiable, we define $\partial_n u|_{\mathscr{C}^\pm}$ *and* $\partial_t u|_{\mathscr{C}^\pm}$ *by*

$$\forall x \in \mathscr{C}, \quad \partial_n u|_{\mathscr{C}^\pm}(x) = \lim_{t\to 0^+} \nabla u(x \pm t n(x)).n(x),$$
$$\partial_t u|_{\mathscr{C}^\pm}(x) = \lim_{t\to 0^+} \nabla u(x \pm t n(x)).n^\perp(x),$$

where '.' denotes the Euclidean scalar product of $\mathbb{R}^2$.

Let $\mathbb{T}$ be the torus $\mathbb{R}/\mathbb{Z}$. Since Γ is a smooth closed curve of $\mathbb{R}^2$ of length 1, it is parameterised by its curvilinear coordinate:

$$\Gamma = \{\Psi(\theta),\ \theta \in \mathbb{T}\}.$$

Let κ be the curvature of Γ, and let f be a smooth 1-periodic and positive function. For sake of simplicity, we suppose that $1/2 \leqslant f \leqslant 3/2$. Choose d_0 such that

$$0 < d_0 < \frac{1}{2\,\|\kappa\|_\infty}. \tag{1.2}$$

For any $\varepsilon \in (0, d_0)$ such that[2] $f(0) = f(1/\varepsilon)$, the rough boundary Γ_ε is taken to be

$$\Gamma_\varepsilon = \{\Psi_\varepsilon(\theta) = \Psi(\theta) + \varepsilon f(\theta/\varepsilon) n(\theta),\ \theta \in \mathbb{T}\}.$$

The closed curves $\mathscr{C}_1$ and $\mathscr{C}_0$ are defined by

$$\mathscr{C}_0 = \{0\} \times \mathbb{T}, \quad \mathscr{C}_1 = \{(f(y), y),\ \forall y \in \mathbb{T}\}.$$

The exterior normal to $\mathscr{C}_1$ is denoted by $n_{\mathscr{C}_1}$ and $n_{\mathscr{C}_0}$ is the exterior normal to $\mathscr{C}_0$:

$$n_{\mathscr{C}_0} = \begin{pmatrix}1\\0\end{pmatrix}, \quad n_{\mathscr{C}_1} = \frac{1}{\sqrt{1+(f'(y))^2}}\begin{pmatrix}1\\-f'(y)\end{pmatrix}. \tag{1.3}$$

Define now the couple (A^0, a^0), where A^0 is a continuous vector field and a^0 is a constant vector, which are given by the unique solution of the following problem:

$$\Delta A^0 = 0, \text{ in } \mathbb{R} \times \mathbb{T} \setminus (\mathscr{C}_0 \cup \mathscr{C}_1), \tag{1.4a}$$
$$\sigma_0 \partial_n A^0|_{\mathscr{C}_1^+} - \sigma_m \partial_n A^0|_{\mathscr{C}_1^-} = (\sigma_m - \sigma_0) n_{\mathscr{C}_1}, \tag{1.4b}$$
$$\sigma_m \partial_n A^0|_{\mathscr{C}_0^+} - \sigma_1 \partial_n A^0|_{\mathscr{C}_0^-} = -(\sigma_m - \sigma_0) n_{\mathscr{C}_0}, \tag{1.4c}$$
$$A^0 \to_{x\to-\infty} 0,\ A^0 \to_{x\to+\infty} a^0. \tag{1.4d}$$

Remark 1.2 *The existence and the uniqueness of the couple* (A^0, a^0) *is discussed in Lemma 2.2. This couple leads to the definition of the boundary-layer corrector (see Theorem 4.1) and enables us to write our approximate transmission conditions.*

[2] For sake of simplicity, we deal with periodic roughness. Since Γ_ε is a closed continuous curve, ε is necessary $1/N$, where N is an integer. All along the paper this assumption holds.

We denote by D_1 and D_2 the two following vectors:

$$D_1 = (\sigma_0 - \sigma_m)\left[\int_0^1 f(y)\,dy\, n_{\mathscr{C}_0} + \int_0^1 A^0(f(y), y)\,dy\right] + (\sigma_m - \sigma_1)\int_0^1 A^0(0, y)\,dy - \sigma_0 a^0$$

and

$$D_2 = (\sigma_m - \sigma_0)\left[\int_0^1 A^0(f(y), y) f'(y)\,dy - \int_0^1 f(y)\,dy \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right].$$

Let u^0 be the unique solution to the following problem:

$$\Delta u^0 = 0, \text{ in } \mathscr{D}^0 \cup \mathscr{D}^1, \tag{1.5a}$$

$$\sigma_0 \partial_n u^0|_{\Gamma^+} = \sigma_1 \partial_n u^0|_{\Gamma^-}, \tag{1.5b}$$

$$u^0|_{\Gamma^+} = u^0|_{\Gamma^-}, \tag{1.5c}$$

$$u^0|_{\partial\Omega} = g. \tag{1.5d}$$

And define u^1 by

$$\Delta u^1 = 0, \text{ in } \mathscr{D}^0 \cup \mathscr{D}^1, \tag{1.5e}$$

$$\sigma_0 \partial_n u^1|_{\Gamma^+} = \sigma_1 \partial_n u^1|_{\Gamma^-} - \kappa D_1 \cdot \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma^+} \end{pmatrix} + D_2 \cdot \partial_t \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma^+} \end{pmatrix}, \tag{1.5f}$$

$$u^1|_{\Gamma^+} = u^1|_{\Gamma^-} + a^0 \cdot \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma^+} \end{pmatrix}, \tag{1.5g}$$

$$u^1|_{\partial\Omega} = 0. \tag{1.5h}$$

In the following remark, Remark 1.3, we discuss the existence and uniqueness of the potentials u^0 and u^1. However, in order to shorten the paper, we leave it to the reader to prove the two following assertions.

Remark 1.3 (Existence and uniqueness of the potentials u^0 and u^1) *We just give here the sketch of the proofs.*

- *With $g \in H^{s+1/2}$ for $s \geqslant 0$, it is well known that the function u^0 exists and is unique in $H^1(\Omega)$. Moreover, u^0 has the following regularity:*

$$u^0|_\omega \in H^{1+s}(\omega), \text{ for } \omega \in \{\mathscr{D}^0, \mathscr{D}^1\}.$$

- *Observe that if u^1 exists, it is obviously unique. Suppose now that $g \in H^{3/2+s}$, with $s \geqslant 0$. Hence $\partial_n u^0|_{\Gamma+}$ and $\partial_t u^0|_{\Gamma^+}$ belong to $H^{1/2+s}(\Gamma)$, and therefore there exists $G \in H^{1+s}(\mathscr{D}^1)$ such that*

$$G|_\Gamma = a^0 \cdot \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma^+} \end{pmatrix}.$$

Define v_G^1 by $v_G^1 = u^1$ in $\mathscr{D}^0$ and $v_G^1 = u^1 + G$ in $\mathscr{D}^1$. Then using the variational formulation of the problem satisfied by v_G^1 and applying straightforwardly the well-known Lax–Milgram theorem we infer the existence and the uniqueness of v_G^1 in $H^1(\Omega)$, and therefore u^1 exists *and is unique. Moreover using the same argument as Li and Vogelius (see the Appendix, p. 147, of [9]) and since* $G \in H^{1+s}(\mathscr{D}^1)$ *we infer*

$$u^1|_\omega \in H^{1+s}(\omega), \text{ for } \omega \in \{\mathscr{D}^0, \mathscr{D}^1\}.$$

Our main result is presented in the next theorem.

Theorem 1.4 *Suppose that* g *belongs to* $H^{7/2}(\partial\Omega)$. *Let* W *be*

$$W = u^\varepsilon - (u^0 + \varepsilon u^1).$$

Then, for any domain ω_0 *and* ω_1 *respectively compactly embedded in* $\Omega \setminus \overline{\mathscr{D}^1}$ *and in* $\mathscr{D}^1$ *and for any* $s \in]1, 2]$ *there exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} \|W\|_{H^1(\omega_0)} &\leqslant C\varepsilon^{1+1/s}, \\ \|W\|_{H^1(\omega_1)} &\leqslant C\varepsilon^{1+1/s}. \end{aligned}$$

Remark 1.5 *It is possible to give the precise behaviour of* u^ε *in a neighbourhood of* Γ *with the help of boundary-layer correctors (see Theorem 4.1).*

Remark 1.6 *Observe that the influence of the curvature appears in the definition of* u^1, *while no curvature was present at the first order for a weakly oscillating thin layer [11]. Actually if no oscillation occurs in the thin layer, i.e. if* f *is constant, then* A^0 *is independent on* y *and equals*

$$A^0 = \begin{pmatrix} A_1^0 \\ A_2^0 \end{pmatrix},$$

where

$$A_1^0(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \dfrac{\sigma_0 - \sigma_m}{\sigma_m} x & \text{if} \quad 0 < x < f, \\ \dfrac{\sigma_0 - \sigma_m}{\sigma_m} f & \text{if} \quad x > f, \end{cases}$$

and $A_2^0 = 0$.

Then we easily obtain

$$D_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 \\ (\sigma_0 - \sigma_m) f \end{pmatrix}.$$

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Hence we obtain the following approximate transmission conditions for u^1*:*

$$\Delta u^1 = 0, \text{ in } \mathscr{D}^0 \cup \mathscr{D}^1, \tag{1.6a}$$

$$\sigma_0 \partial_n u^1|_{\Gamma^+} = \sigma_1 \partial_n u^1|_{\Gamma^-} + (\sigma_0 - \sigma_m) f \partial_t^2 u^0|_{\Gamma^+}, \tag{1.6b}$$

$$u^1|_{\Gamma^+} = u^1|_{\Gamma^-} + \frac{\sigma_0 - \sigma_m}{\sigma_m} f \partial_n u^0|_{\Gamma^+}, \tag{1.6c}$$

$$u^1|_{\partial\Omega} = 0, \text{ on } \partial\Omega, \tag{1.6d}$$

*which are the transmission conditions given in [*11, 12*].*

2 Preliminary analysis

2.1 The equivalent problem in a tubular neighbourhood of Γ

It is convenient to write problem (1.1) in a smooth tubular neighbourhood Ω^{d_0} of Γ, given, for some distance d_0 satisfying (1.2), by

$$\Omega^{d_0} = \{z \in \mathbb{R}^2, \quad dist(z, \Gamma) < d_0\}.$$

Denote by Γ_{-d_0} and Γ_{d_0} the closed curves respectively defined by

$$\Gamma_{-d_0} = \partial\Omega^{d_0} \cap \mathscr{D}^1, \quad \Gamma_{d_0} = \partial\Omega^{d_0} \cap \mathscr{D}^0_\varepsilon.$$

We consider the following Steklov–Poincaré operators:

$$\mathscr{L}_1 : H^{1/2}(\Gamma_{-d_0}) \longrightarrow H^{-1/2}(\Gamma_{-d_0}),$$
$$\mathscr{L}_0 : H^{1/2}(\Gamma_{d_0}) \longrightarrow H^{-1/2}(\Gamma_{d_0}),$$
$$\mathscr{T} : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\Gamma_{d_0}).$$

Using the convention of the direction of the normals (see Notation 1.1), we define the operator $\mathscr{L}_1$ by

$$\forall \phi \in H^{1/2}(\Gamma_{-d_0}),\ \mathscr{L}_1(\phi) = \left.\frac{\partial u}{\partial n}\right|_{\Gamma_{-d_0}},$$

where u is the harmonic function in $\Omega \setminus \overline{(\mathscr{D}^0 \cup \Omega^{d_0})}$ equal to ϕ on Γ_{-d_0}. The operator $\mathscr{L}_0$ is defined by

$$\forall \phi \in H^{1/2}(\Gamma_{d_0}),\ \mathscr{L}_0(\phi) = -\left.\frac{\partial u}{\partial n}\right|_{\Gamma_{d_0}},$$

where u is the harmonic function in $\Omega \setminus \overline{(\mathscr{D}^1 \cup \Omega^{d_0})}$ equal to ϕ on Γ_{d_0} and vanishing on $\partial\Omega$. Similarly $\mathscr{T}$ equals:

$$\forall \chi \in H^{1/2}(\partial\Omega),\ \mathscr{T}(\chi) = -\left.\frac{\partial u}{\partial n}\right|_{\Gamma_{d_0}},$$

where u is the harmonic function in $\Omega \setminus \overline{(\mathscr{D}^1 \cup \Omega^{d_0})}$ equal to χ on $\partial\Omega$ and vanishing on Γ_{d_0}. Moreover the operators $\mathscr{T}$, $\mathscr{L}_0$ and $\mathscr{L}_1$ satisfy the following inequalities, for a

d_0-independent constant C:

$$\forall g \in H^{1/2}(\partial\Omega), \quad |\mathscr{T} g|_{H^{-1/2}(\Gamma_{d_0})} \leqslant C\, |g|_{H^{1/2}(\partial\Omega)}, \tag{2.1a}$$

$$\forall u \in H^{1/2}(\Gamma_{d_0}), \quad |\mathscr{L}_0 u|_{H^{-1/2}(\Gamma_{d_0})} \leqslant C\, |u|_{H^{1/2}(\Gamma_{d_0})}, \tag{2.1b}$$

$$\forall u \in H^{1/2}(\Gamma_{-d_0}), \quad |\mathscr{L}_1 u|_{H^{-1/2}(\Gamma_{-d_0})} \leqslant C\, |u|_{H^{1/2}(\Gamma_{-d_0})}. \tag{2.1c}$$

Furthermore, the following coercivity inequalities hold:

$$\forall u \in H^{1/2}(\Gamma_{d_0}), \quad (\mathscr{L}_0 u, u)_{L^2(\Gamma_{d_0})} \geqslant C\, |u|^2_{H^{1/2}(\Gamma_{d_0})}, \tag{2.1d}$$

$$\forall u \in H^{1/2}(\Gamma_{-d_0}), \quad (\mathscr{L}_1 u, u)_{L^2(\Gamma_{-d_0})} \geqslant C\, |u|^2_{H^{1/2}(\Gamma_{-d_0})}. \tag{2.1e}$$

Problem (1.1) is then equivalent to

$$\nabla \cdot (\sigma \nabla u^\varepsilon) = 0, \text{ in } \Omega^{d_0}, \tag{2.2a}$$

$$\partial_n u^\varepsilon|_{\Gamma_{d_0}} + \mathscr{L}_0 u^\varepsilon|_{\Gamma_{d_0}} = -\mathscr{T} g, \text{ on } \Gamma_{d_0}, \tag{2.2b}$$

$$\partial_n u^\varepsilon|_{\Gamma_{-d_0}} - \mathscr{L}_1 u^\varepsilon|_{\Gamma_{-d_0}} = 0, \text{ on } \Gamma_{-d_0}. \tag{2.2c}$$

2.2 The problem in local coordinates

Denote by Φ the smooth diffeomorphism

$$\forall (\eta, \theta) \in (-d_0, d_0) \times \mathbb{T}, \quad \Phi(\eta, \theta) = \Psi(\theta) + \eta n(\theta).$$

Since $d_0 < 1/\|\kappa\|_\infty$, the open neighbourhood of Γ denoted by Ω^{d_0} can be parameterised as follows:

$$\Omega^{d_0} = \{\Phi(\eta, \theta), (\eta, \theta) \in (-d_0, d_0) \times \mathbb{T}\}.$$

For $\varepsilon > 0$, Ω^{d_0} is split into the three subdomains, namely Ω^1, Ω^m_ε and Ω^0_ε, which are defined with the help of the smooth 1-periodic function f:

$$\Omega^1 = \{\Phi(\eta, \theta), (\eta, \theta) \in (-d_0, 0) \times \mathbb{T}\},$$

$$\Omega^m_\varepsilon = \{\Phi(\eta f(\theta/\varepsilon), \theta), (\eta, \theta) \in (0, \varepsilon) \times \mathbb{T}\},$$

$$\Omega^0_\varepsilon = \Omega^{d_0} \setminus (\overline{\Omega^1 \cup \Omega^m_\varepsilon}).$$

Let $\mathscr{O} = (-d_0, d_0) \times \mathbb{T}$ and denote respectively by $\mathscr{O}^1$, $\mathscr{O}^m_\varepsilon$ and $\mathscr{O}^0_\varepsilon$ the domains:

$$\mathscr{O}^1 = (-d_0, 0) \times \mathbb{T},$$

$$\mathscr{O}^m_\varepsilon = \{(\eta f(\theta/\varepsilon), \theta) \in (0, \varepsilon) \times \mathbb{T}\},$$

$$\mathscr{O}^0_\varepsilon = \mathscr{O} \setminus \overline{\mathscr{O}^1 \cup \mathscr{O}^m_\varepsilon}.$$

We also denote by Ω^0 and $\mathscr{O}^0$ the respective domains $\Omega \setminus \Omega^1$ and $\mathscr{O} \setminus \mathscr{O}^1$. Define the oscillating curve γ_ε by

$$\gamma_\varepsilon = \{(\varepsilon f(\theta/\varepsilon), \theta), \theta \in \mathbb{T}\}.$$

We write $\gamma^s = \{s\} \times \mathbb{T}$ for any $s \in \mathbb{R}$. The Laplacian written in (η, θ)-coordinates equals

$$\Delta_{\eta,\theta} = \frac{1}{1+\eta\kappa(\theta)}\partial_\eta((1+\eta\kappa(\theta))\partial_\eta) + \frac{1}{1+\eta\kappa(\theta)}\partial_\theta\left(\frac{1}{1+\eta\kappa(\theta)}\partial_\theta\right).$$

We also need the normal derivatives on Γ and Γ_ε in (η, θ)-coordinates. In the following, the notation $\nabla_{\eta,\theta}$ denotes the derivative operator:

$$\nabla_{\eta,\theta} = \begin{pmatrix} \partial_\eta \\ \partial_\theta \end{pmatrix}.$$

Let u be defined on Ω, and define v on $(-d_0, d_0) \times \mathbb{T}$ by

$$\forall (\eta, \theta) \in (-d_0, d_0) \times \mathbb{T}, \quad v(\eta, \theta) = u \circ \Phi(\eta, \theta).$$

The following equalities hold, for a sufficiently regular function u:

$$\begin{aligned} \partial_n u|_\Gamma &= \partial_\eta v|_{\eta=0}, \\ \partial_n u|_{\Gamma_{d_0}} &= \partial_\eta v|_{\eta=d_0}, \\ \partial_n u|_{\Gamma_{-d_0}} &= \partial_\eta v|_{\eta=-d_0} \end{aligned}$$

and

$$\partial_n u|_{\Gamma_\varepsilon} = \frac{1}{\sqrt{(1+\varepsilon\kappa f(\theta/\varepsilon))^2 + (f'(\theta/\varepsilon))^2}} \left[(1+\varepsilon\kappa f(\theta/\varepsilon))\partial_\eta v - \frac{f'(\theta/\varepsilon)}{1+\varepsilon\kappa f(\theta/\varepsilon)}\partial_\theta v\right]\Bigg|_{\gamma_\varepsilon}.$$

Moreover, we define the bounded linear operators $\Lambda_0 : H^{1/2}(\gamma^{d_0}) \to H^{-1/2}(\gamma^{d_0})$ and $\Lambda_1 : H^{1/2}(\gamma^{-d_0}) \to H^{-1/2}(\gamma^{-d_0})$ as follows:

$$\begin{aligned} (\Lambda_0 \varphi, \psi) &= (\mathscr{L}_0(\varphi \circ \Phi^{-1}),\ \psi \circ \Phi^{-1}), \quad \forall\, \varphi, \psi \in H^{1/2}(\gamma^{d_0}), \\ (\Lambda_1 \varphi, \psi) &= (\mathscr{L}_1(\varphi \circ \Phi^{-1}),\ \psi \circ \Phi^{-1}), \quad \forall\, \varphi, \psi \in H^{1/2}(\gamma^{-d_0}). \end{aligned}$$

According to (2.1), there exists an ε-independent constant $C > 0$ such that

$$\forall u \in H^{1/2}(\mathbb{T}), \quad (\Lambda_0 u, u)_{L^2(\mathbb{T})} \geqslant C\, |u|^2_{H^{1/2}(\mathbb{T})}, \tag{2.3a}$$

$$\forall u \in H^{1/2}(\mathbb{T}), \quad (\Lambda_1 u, u)_{L^2(\mathbb{T})} \geqslant C\, |u|^2_{H^{1/2}(\mathbb{T})}. \tag{2.3b}$$

It is convenient to denote by $\partial_n^\Phi v|_{\gamma_\varepsilon}$ the following quantity:

$$\partial_n^\Phi v|_{\gamma_\varepsilon} = \frac{1}{\sqrt{(1+\varepsilon\kappa f(\theta/\varepsilon))^2 + (f'(\theta/\varepsilon))^2}} \left[(1+\varepsilon\kappa f(\theta/\varepsilon))\partial_\eta v - \frac{f'(\theta/\varepsilon)}{1+\varepsilon\kappa f(\theta/\varepsilon)}\partial_\theta v\right]\Bigg|_{\gamma_\varepsilon}.$$

With this notation, we can write our initial problem (1.1) in local coordinates. Denoting by v^ε the solution to problem (1.1) in (η, θ)-coordinates,

$$v^\varepsilon = u^\varepsilon \circ \Phi.$$

Then v^ε is continuous and satisfies

$$\Delta_{\eta,\theta} v^\varepsilon = 0, \text{ in } \mathscr{O}^1 \cup \mathscr{O}^m_\varepsilon \cup \mathscr{O}^0_\varepsilon, \tag{2.4a}$$

$$(1 + d_0\kappa)\partial_\eta v^\varepsilon|_{\eta=d_0} + \Lambda_0 v^\varepsilon|_{\eta=d_0} = -(\mathscr{T}g)\circ\Phi, \tag{2.4b}$$

$$(1 - d_0\kappa)\partial_\eta v^\varepsilon|_{\eta=-d_0} - \Lambda_1 v^\varepsilon|_{\eta=-d_0} = 0, \tag{2.4c}$$

with the following transmission conditions:

$$\sigma_0\, \partial_n^\Phi v^\varepsilon\big|_{\gamma_\varepsilon^+} = \sigma_m\, \partial_n^\Phi v^\varepsilon\big|_{\gamma_\varepsilon^-}, \tag{2.4d}$$

$$\sigma_m\, \partial_\eta v^\varepsilon\big|_{\eta=0^+} = \sigma_1\, \partial_\eta v^\varepsilon\big|_{\eta=0^-}. \tag{2.4e}$$

2.3 Preliminary result

Denote by $\boldsymbol{C}$ the infinite cylinder $\mathbb{R}\times\mathbb{T}$. Split $\boldsymbol{C}$ into the three subdomains Y_0, Y_m and Y_1 defined by

$$Y_1 = (-\infty, 0)\times\mathbb{T},$$
$$Y_m = \{(\eta f(\theta), \theta),\ \forall(\eta,\theta)\in(0,1)\times\mathbb{T}\},$$
$$Y_0 = \mathbb{R}\times\mathbb{T}\setminus\left(\overline{Y_1\cup Y_m}\right).$$

The closed curves $\mathscr{C}_1$ and $\mathscr{C}_0$ are defined by

$$\mathscr{C}_0 = \{0\}\times\mathbb{T}, \quad \mathscr{C}_1 = \{(f(y), y),\ \forall y\in\mathbb{T}\}.$$

Remember that the normals to the respective curves $\mathscr{C}_0$ and $\mathscr{C}_1$, which are respectively denoted by $n_{\mathscr{C}_0}$ and by $n_{\mathscr{C}_1}$, are defined by (1.3). We also introduce the piecewise constant function $\tilde\sigma : \boldsymbol{C}\to\mathbb{R}$ by

$$\forall(x,y)\in\boldsymbol{C}, \quad \tilde\sigma(x,y) = \begin{cases}\sigma_1 & \text{in } Y_1,\\ \sigma_m & \text{in } Y_m,\\ \sigma_0 & \text{in } Y_0.\end{cases} \tag{2.5}$$

Notation 2.1 *In this paper, we use the following notation:*

- $\mathscr{S}_{per}(\mathbb{R}\times\mathbb{T})$ *is the space of the periodic square-integrable functions in the second variable, which decay exponentially when the first variable goes to infinity.*
- *For any function* $g\in L^2(\mathbb{T})$ *(respectively* $g\in L^2(\mathbb{R}\times\mathbb{T})$*) we denote by* $\{\hat g_k\}_{k\in\mathbb{Z}}$ *(respectively* $\{\hat g_k(x)\}_{k\in\mathbb{Z}}$*) the coefficients of the Fourier expansion of* g *(respectively* $g(x,\cdot)$*), that is*

$$g(y) = \sum_{k\in\mathbb{Z}} \widehat{g}_k e^{2\pi i k y} \quad (\textit{respectively} \quad g(x,y) = \sum_{k\in\mathbb{Z}} \widehat{g}_k(x) e^{2\pi i k y}).$$

Lemma 2.2 *Let F belong to $\mathscr{S}_{per}(\mathbb{R}\times\mathbb{T})$, and suppose that there exist $M > \sup_{y\in\mathbb{T}}|f(y)|$, $C>0$, $\delta_0\in[0,2\pi[$ and a family $\{p_k(x)\}_{k\in\mathbb{Z}}$ of regular functions defined on $|x|>M$ satisfying*

$$|p_k(x)| \leqslant Ce^{\delta_0|k|(x-M)} \quad \textit{for} \quad x>M,$$
$$|p_k(x)| \leqslant Ce^{-\delta_0|k|(x+M)} \quad \textit{for} \quad x<-M, \tag{2.6}$$

such that F has the following Fourier expansion on $|x|>M$:

$$F(x,y) = \sum_{k\in\mathbb{Z}^*} \widehat{F}_k(M)p_k(x)e^{-2\pi|k|(x-M)}e^{2\pi iky}, \quad \forall\, x\geqslant M,$$

$$F(x,y) = \sum_{k\in\mathbb{Z}^*} \widehat{F}_k(-M)p_k(x)e^{2\pi|k|(x+M)}e^{2\pi iky}, \quad \forall\, x\leqslant -M.$$

Let φ and ψ be two smooth functions respectively defined on $\mathscr{C}_0$ and $\mathscr{C}_1$ such that

$$\int_{\mathscr{C}_0}\varphi(\sigma)\,d\sigma + \int_{\mathscr{C}_1}\psi(\sigma)\,d\sigma + \int_{\mathbb{R}\times\mathbb{T}}\tilde{\sigma}(x,y)F(x,y)\,dx\,dy = 0.$$

Then there exists a unique couple (α, a), where α is a continuous function and a is a constant, such that

$$\Delta\alpha = F, \textit{ in } Y_1\cup Y_m\cup Y_0, \tag{2.7a}$$
$$\sigma_0\partial_n\alpha|_{\mathscr{C}_1^+} = \sigma_m\partial_n\alpha|_{\mathscr{C}_1^-} + \psi, \tag{2.7b}$$
$$\sigma_m\partial_n\alpha|_{\mathscr{C}_0^+} = \sigma_1\partial_n\alpha|_{\mathscr{C}_0^-} + \varphi, \tag{2.7c}$$
$$\alpha \to_{x\to-\infty} 0,\ \alpha\to_{x\to+\infty} a. \tag{2.7d}$$

Moreover, α decays exponentially for x tending to $-\infty$, and $\alpha - a$ decays exponentially for x tending to $+\infty$.

Proof To prove this lemma, we rewrite problem (2.7) in the finite strip $[-M,M]\times\mathbb{T}$. The solution α can be written as a Fourier expansion:

$$\alpha(x,y) = \sum_{k\in\mathbb{Z}}\widehat{\alpha}_k(x)e^{2\pi iky}.$$

Replacing α by its expansion in (2.7) and according to the hypothesis on F we infer

$$\widehat{\alpha}_0(x) = \widehat{\alpha}_0(M) \qquad \text{for } x\geqslant M,$$
$$\widehat{\alpha}_k(x) = \left(\widehat{\alpha}_k(M) + \frac{\widehat{F}_k(M)}{4\pi|k|}\int_M^{+\infty} p_k(t)e^{-4\pi|k|(t-M)}\,dt - \frac{\widehat{F}_k(M)}{4\pi|k|}\int_M^x p_k(t)\,dt\right)e^{-2\pi|k|(x-M)}$$
$$-\frac{\widehat{F}_k(M)}{4\pi|k|}\int_x^{+\infty} p_k(t)e^{-4\pi|k|(t-M)}\,dt\cdot e^{2\pi|k|(x-M)} \qquad \text{for } x\geqslant M,\ k\neq 0$$

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and

$$\widehat{\alpha}_0(x) = \widehat{\alpha}_0(-M) \quad \text{for } x \leqslant -M,$$
$$\widehat{\alpha}_k(x) = \left(\widehat{\alpha}_k(-M) + \frac{\widehat{F}_k(-M)}{4\pi|k|}\int_{-\infty}^{-M} p_k(t)e^{4\pi|k|(t+M)}\,dt - \frac{\widehat{F}_k(-M)}{4\pi|k|}\int_{x}^{-M} p_k(t)\,dt\right)e^{2\pi|k|(x+M)}$$
$$-\frac{\widehat{F}_k(-M)}{4\pi|k|}\int_{-\infty}^{x} p_k(t)e^{4\pi|k|(t+M)}\,dt \cdot e^{-2\pi|k|(x+M)} \quad \text{for } x \leqslant -M,\ k \neq 0.$$

Let us now introduce the operators $\mathscr{Q} : H^{1/2}(\mathbb{T}) \to H^{-1/2}(\mathbb{T})$ and $\mathscr{R}_M,\ \mathscr{R}_{-M} : H^{-1}(\mathbb{T}) \to L^2(\mathbb{T})$ by

$$\left.\begin{aligned}
&\forall u \in H^1(\mathbb{T}), \quad \mathscr{Q}u(y) = 2\pi\sum_{k\in\mathbb{Z}^*}|k|\widehat{u}_k e^{iky},\\
&\forall u \in H^{-1}(\mathbb{T}), \quad \mathscr{R}_M u(y) = \sum_{k\in\mathbb{Z}^*}\left(\int_M^{+\infty} p_k(t)e^{-4\pi|k|(t-M)}\,dt\right)\widehat{u}_k e^{2\pi iky},\\
&\forall u \in H^{-1}(\mathbb{T}), \quad \mathscr{R}_{-M} u(y) = \sum_{k\in\mathbb{Z}^*}\left(\int_{-\infty}^{-M} p_k(t)e^{4\pi|k|(t+M)}\,dt\right)\widehat{u}_k e^{2\pi iky}.
\end{aligned}\right\} \tag{2.8}$$

Because of hypothesis (2.6) these operators are well defined. We infer the following mixed boundary condition on $x = M$:

$$\partial_x\alpha|_{x=M} + \mathscr{Q}\alpha|_{x=M} = -\mathscr{R}_M F|_{x=M}. \tag{2.9}$$

Similarly, on $x = -M$ we obtain

$$\partial_x\alpha|_{x=-M} - \mathscr{Q}\alpha|_{x=-M} = \mathscr{R}_{-M} F|_{x=-M}. \tag{2.10}$$

Problem (2.7) is then equivalent to problem (2.11), written in the strip $[-M, M] \times \mathbb{T}$,

$$\Delta\alpha = F, \text{ in } (Y_1 \cup Y_m \cup Y_0) \cap [-M, M] \times \mathbb{T}, \tag{2.11a}$$
$$\sigma_0\partial_n\alpha|_{\mathscr{C}_1^+} = \sigma_m\partial_n\alpha|_{\mathscr{C}_1^-} + \psi, \tag{2.11b}$$
$$\sigma_m\partial_n\alpha|_{\mathscr{C}_0^+} = \sigma_1\partial_n\alpha|_{\mathscr{C}_0^-} + \varphi, \tag{2.11c}$$

with the mixed boundary conditions

$$\partial_x\alpha|_{x=M} + \mathscr{Q}\alpha|_{x=M} = -\mathscr{R}_M F|_{x=M}, \tag{2.11d}$$
$$\partial_x\alpha|_{x=-M} - \mathscr{Q}\alpha|_{x=-M} = \mathscr{R}_{-M} F|_{x=-M}. \tag{2.11e}$$

The variational formulation of (2.11) is then

$$\forall v \in H^1([-M, M] \times \mathbb{T}), \quad \mathscr{A}(\alpha, v) = \mathscr{B}(v), \tag{2.12}$$

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where $\mathscr{A}$ and $\mathscr{B}$ are given by

$$\mathscr{A}(u,v) = \int_{[-M,M]\times\mathbb{T}} \tilde{\sigma}\nabla\alpha.\nabla v\, dxdy + \sigma_0 < \mathscr{Q}\alpha|_{x=M},\ v|_{x=M} >$$
$$+\, \sigma_1 < \mathscr{Q}\alpha|_{x=-M},\ v|_{x=-M} >,$$
$$\mathscr{B}(v) = -\int_{[-M,M]\times\mathbb{T}} \tilde{\sigma} F(x,y)v(x,y)\, dxdy - \int_{\mathscr{C}_0} \phi(\sigma)v(\sigma)d\sigma - \int_{\mathscr{C}_1} \psi(\sigma)v(\sigma)d\sigma$$
$$-\,\sigma_0 \int_{\mathbb{T}} \mathscr{R}_M F(M,y)v(M,y)\, dy - \sigma_1 \int_{\mathbb{T}} \mathscr{R}_{-M} F(-M,y)v(-M,y)\, dy.$$

The continuity of $\mathscr{B}$ on $H^1([-M,M]\times\mathbb{T})$ and of $\mathscr{A}$ on $(H^1([-M,M]\times\mathbb{T}))^2$ easily come from calculations in the Fourier variable. From the Poincaré inequality we also deduce that $\mathscr{A}$ is coercive on $H^1([-M,M]\times\mathbb{T})/\mathbb{R}$.

On the other hand, from the hypothesis on F, we infer

$$\int_{[M,+\infty[\times\mathbb{T}} F(x,y)\, dy\, dx = \int_M^{\infty} \left(\int_{\mathbb{T}} F(x,y)\, dy \right) dx = 0,$$

and similarly

$$\int_{]-\infty,-M]\times\mathbb{T}} F(x,y)\, dy\, dx = 0.$$

We also have

$$\int_{\mathbb{T}} \mathscr{R}_M F(M,y)\, dy = \int_{\mathbb{T}} \mathscr{R}_{-M} F(-M,y)\, dy = 0,$$

which gives

$$\mathscr{B}(1) = 0.$$

So we have a unique solution of (2.12) up to an additive constant, and we can choose this constant such that

$$\int_{\mathbb{T}} \alpha(-M,y)\, dy = \widehat{\alpha}_0(-M) = 0.$$

We then have

$$a = \widehat{\alpha}_0(M) = \int_{\mathbb{T}} \alpha(M,y)\, dy.$$

The exponential decay of $\alpha - a$ at $+\infty$ and of α at $-\infty$ come from the expressions of $\widehat{\alpha}_k(x)$ for $k \neq 0$. □

Remark 2.3 *Suppose that for $|x|$ large enough F equals $F(x,y) = F_1(x,y)F_2(x)$, where F_1 is a harmonic function and F_2 is such that*

$$\exists\ C > 0 \ \textit{and} \ \ \delta_0 \in [0, 2\pi[\ \ \textit{such that} \quad |F_2(x)| \leqslant Ce^{\delta_0|x|}, \ \ \forall\ x$$

(for example if F_2 is polynomial). Then F satisfies obviously the hypothesis of Lemma 2.2.

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3 Formal asymptotics

3.1 Zeroth-order approximation

Let v^0 be the continuous 'background' solution defined by

$$\begin{aligned}
&\Delta_{\eta,\theta} v^0 = 0, \text{ in } \mathcal{O}^1 \cup \mathcal{O}^0,\\
&(1 + d_0\kappa)\partial_\eta v^0|_{\eta=d_0} + \Lambda_0 v^0|_{\eta=d_0} = -(\mathcal{T} g)\circ \Phi,\\
&(1 - d_0\kappa)\partial_\eta v^0|_{\eta=-d_0} - \Lambda_1 v^0|_{\eta=-d_0} = 0,\\
&\sigma_0 \partial_\eta v^0|_{\eta=0^+} = \sigma_1 \partial_\eta v^0|_{\eta=0^-}.
\end{aligned}$$

A classical regularity result implies that for $g \in H^s(\partial\Omega)$, the potential v^0 belongs to $H^1(\Omega)$, and moreover it has $H^{s+1/2}$-regularity in $\mathcal{O}^1$ and in $\mathcal{O}^0$. In this section, we suppose that g is as regular as necessary so that the involved quantities are well defined. Rigorous proof of the formal asymptotic expansion is performed in the next section.

Denote by w^0 the error $v^\varepsilon - v^0$. This continuous function is the unique solution to

$$\begin{aligned}
&\Delta_{\eta,\theta} w^0 = 0, \text{ in } \mathcal{O}^1 \cup \mathcal{O}^0,\\
&(1 + d_0\kappa)\partial_\eta w^0|_{\eta=d_0} + \Lambda_0 w^0|_{\eta=d_0} = 0,\\
&(1 - d_0\kappa)\partial_\eta w^0|_{\eta=-d_0} - \Lambda_1 w^0|_{\eta=-d_0} = 0,\\
&\sigma_0 \partial_n^\Phi w^0|_{\gamma_\varepsilon^+} = \sigma_m \partial_n^\Phi w^0|_{\gamma_\varepsilon^-} + (\sigma_m - \sigma_0)\partial_n^\Phi v^0|_{\gamma_\varepsilon},\\
&\sigma_m \partial_\eta w^0|_{\eta=0^+} = \sigma_1 \partial_\eta w^0|_{\eta=0^-} + (\sigma_0 - \sigma_m)\partial_\eta v^0|_{\eta=0^+},\\
&w^0|_{\partial\Omega} = 0, \text{ on } \partial\Omega.
\end{aligned}$$

Since we are interested in the derivation of terms up to order 1, we throw away all the terms which are *a priori* of order smaller than ε^2. This approximation will be rigorously justified in the next section.

Using the explicit expression for the normal derivative on γ_ε, we infer

$$\begin{aligned}
\partial_n^\Phi v^0|_{\gamma_\varepsilon} = {} & \frac{1 - \varepsilon\kappa f(\theta/\varepsilon)/(1 + (f'(\theta/\varepsilon))^2) + O(\varepsilon^2)}{\sqrt{1 + [f'(\theta/\varepsilon)]^2}} [(1 + \varepsilon\kappa f(\theta/\varepsilon))\partial_\eta v^0|_{\gamma_\varepsilon}\\
& - f'(\theta/\varepsilon)(1 - \varepsilon\kappa f(\theta/\varepsilon) + O(\varepsilon^2))\partial_\theta v^0|_{\gamma_\varepsilon}]. \qquad (3.1)
\end{aligned}$$

A Taylor expansion in the η-variable implies

$$\partial_\eta v^0|_{\gamma_\varepsilon} = \partial_\eta v^0|_{\eta=0^+} + \varepsilon f(\theta/\varepsilon)\partial_\eta^2 v^0|_{\eta=0^+} + O(\varepsilon^2),$$

$$\partial_\theta v^0|_{\gamma_\varepsilon} = \partial_\theta v^0|_{\eta=0^+} + \varepsilon f(\theta/\varepsilon)\partial_{\eta,\theta}^2 v^0|_{\eta=0^+} + O(\varepsilon^2).$$

From the above equalities, we infer

$$\begin{aligned}\partial_n^{\Phi} v^0|_{\gamma_\varepsilon} = &\frac{1}{\sqrt{1+(f')^2}}\partial_\eta v^0|_{\eta=0^+} - \frac{f'}{\sqrt{1+(f')^2}}\partial_\theta v^0|_{\eta=0^+} \\ &+\varepsilon\left\{\frac{\kappa f}{\sqrt{1+(f')^2}}\left[\left(1-\frac{1}{1+(f')^2}\right)\partial_\eta v^0|_{\eta=0^+}\right.\right. \\ &+\left.\left(1+\frac{1}{1+(f')^2}\right)\partial_\theta v^0|_{\eta=0^+}\right] + \frac{f}{\sqrt{1+(f')^2}}\partial_\eta^2 v^0|_{\eta=0^+} \\ &\left.-\frac{f'f}{\sqrt{1+(f')^2}}\partial_{\eta\theta}^2 v^0|_{\eta=0^+}\right\} + O(\varepsilon^2), \end{aligned} \tag{3.2}$$

where f and f' are regarded as functions of $\frac{\theta}{\varepsilon}$. Since

$$\partial_\eta^2 v^0|_{\eta=0^+} = -\kappa\partial_\eta v^0|_{\eta=0^+} - \partial_\theta^2 v^0|_{\eta=0^+}$$

and on writing

$$n_{\mathscr{C}_1}^{\perp} = \frac{1}{\sqrt{1+(f')^2}}\begin{pmatrix} f' \\ 1\end{pmatrix},$$

we infer

$$\begin{aligned}\left[\sigma\partial_n^{\Phi} v^0\right]_{\gamma_\varepsilon} = &(\sigma_0-\sigma_m)n_{\mathscr{C}_1}(\theta/\varepsilon).\nabla_{\eta,\theta}v^0|_{\eta=0^+} \\ &+\varepsilon(\sigma_0-\sigma_m)\left\{\kappa f(\theta/\varepsilon)\left(\frac{n_{\mathscr{C}_0}(\theta/\varepsilon)}{\sqrt{1+(f'(\theta/\varepsilon))^2}}\right.\right. \\ &\left.-\frac{2+(f'(\theta/\varepsilon))^2}{1+(f'(\theta/\varepsilon))^2}n_{\mathscr{C}_1}(\theta/\varepsilon)\right)\cdot\nabla_{\eta,\theta}v^0|_{\eta=0^+} \\ &\left.-f(\theta/\varepsilon)n_{\mathscr{C}_1}^{\perp}(\theta/\varepsilon).\partial_\theta\nabla_{\eta,\theta}v^0|_{\eta=0^+}\right\} + O(\varepsilon^2). \end{aligned} \tag{3.3}$$

Observe also that

$$\left[\sigma\partial_n v^0\right]_{\eta=0} = (\sigma_0-\sigma_m)n_{\mathscr{C}_0}(\theta/\varepsilon).\nabla_{\eta,\theta}v^0|_{\eta=0^+}.$$

We now define the vector field A^0 from which we will obtain the boundary-layer corrector of order 0. From

$$\int_{\mathscr{C}_1} n_{\mathscr{C}_1}(\sigma)\,d\sigma - \int_{\mathscr{C}_0} n_{\mathscr{C}_0}(\sigma)\,d\sigma = \begin{pmatrix}0\\0\end{pmatrix}$$

and Lemma 2.2, there exists a unique couple (A^0, a^0) (where A^0 is a continuous vector field and a^0 a constant vector) such that

$$\begin{aligned}&\Delta A^0 = 0, \text{ in } \mathbb{R}\times\mathbb{T}\setminus(\mathscr{C}_0\cup\mathscr{C}_1), \\ &\sigma_0\partial_n A^0|_{\mathscr{C}_1^+} - \sigma_m\partial_n A^0|_{\mathscr{C}_1^-} = (\sigma_m-\sigma_0)n_{\mathscr{C}_1}, \\ &\sigma_m\partial_n A^0|_{\mathscr{C}_0^+} - \sigma_1\partial_n A^0|_{\mathscr{C}_0^-} = -(\sigma_m-\sigma_0)n_{\mathscr{C}_0}, \\ &A^0 \to_{x\to-\infty} 0,\ A^0 \to_{x\to+\infty} a^0.\end{aligned}$$

We define now the boundary-layer corrector of order 0, v^0_{BL} on $\mathcal{O}$ by

$$v^0_{BL}(\eta,\theta) = \begin{cases} \varepsilon\left(A^0(\eta/\varepsilon,\theta/\varepsilon) - a^0\right)\cdot\nabla_{\eta,\theta}v^0|_{\Gamma^+}, \text{ if } \eta > 0,\\ \varepsilon A^0(\eta/\varepsilon,\theta/\varepsilon)\cdot\nabla_{\eta,\theta}v^0|_{\Gamma^+}, \text{ if } \eta < 0.\end{cases}$$

Remark 3.1 *Using the equations satisfied by A^0, we observe that A^0 decays exponentially for $x \to -\infty$, and similarly $A^0 - a^0$ decays exponentially for $x \to \infty$.*

From equality (3.1) with v^0 replaced by v^0_{BL}, we deduce

$$\begin{aligned}\partial^{\Phi}_n v^0_{BL}|_{\gamma_\varepsilon} &= \left(\frac{1}{\sqrt{1+(f')^2}}\frac{\partial A^0}{\partial x} - \frac{f'}{\sqrt{1+(f')^2}}\frac{\partial A^0}{\partial y}\right)\cdot\nabla_{\eta,\theta}v^0|_{\eta=0^+}\\ &\quad -\varepsilon\frac{f'}{\sqrt{1+(f')^2}}(A^0-a^0)\cdot\partial_\theta\nabla_{\eta,\theta}v^0|_{\eta=0^+}\\ &\quad +\varepsilon\kappa f\Bigg([(1+(f')^2)^{-1/2} - (1+(f')^2)^{-3/2}]\frac{\partial A^0}{\partial x}\\ &\quad +[(1+(f')^2)^{-1/2} + (1+(f')^2)^{-3/2}]f'\frac{\partial A^0}{\partial y}\Bigg)\cdot\nabla_{\eta,\theta}v^0|_{\eta=0^+} + O(\varepsilon^2).\end{aligned}$$

Taking into account the fact that

$$\frac{1}{\sqrt{1+(f')^2}}\left[\sigma\frac{\partial A^0}{\partial x}\right]_{\mathscr{C}_1} - \frac{f'}{\sqrt{1+(f')^2}}\left[\sigma\frac{\partial A^0}{\partial y}\right]_{\mathscr{C}_1} = \left[\sigma\frac{\partial A^0}{\partial n}\right]_{\mathscr{C}_1},$$

we obtain

$$\begin{aligned}\left[\sigma\partial^{\Phi}_n v^0_{BL}\right]_{\gamma_\varepsilon} &= (\sigma_m-\sigma_0)n_{C_1}\cdot\nabla_{\eta,\theta}v^0|_{\eta=0^+} + \varepsilon(\sigma_m-\sigma_0)\\ &\quad\times\frac{f'}{\sqrt{1+(f')^2}}(A^0-a^0)\cdot\partial_\theta\nabla_{\eta,\theta}v^0|_{\eta=0^+} + \varepsilon\kappa f\Bigg(\frac{2}{\sqrt{1+(f')^2}}\left[\sigma\frac{\partial A^0}{\partial x}\right]_{\mathscr{C}_1}\\ &\quad -(\sigma_m-\sigma_0)\frac{2+(f')^2}{1+(f')^2}n_{\mathscr{C}_1}\Bigg)\cdot\nabla_{\eta,\theta}v^0|_{\eta=0^+} + O(\varepsilon^2).\end{aligned}\tag{3.4}$$

Therefore, using (3.3) and (3.4) we infer

$$\begin{aligned}\left[\sigma\partial^{\Phi}_n(v^0+v^0_{BL})\right]_{\gamma_\varepsilon} &= (\sigma_0-\sigma_m)\varepsilon\Bigg\{\kappa\frac{f(\theta/\varepsilon)}{\sqrt{1+(f'(\theta/\varepsilon))^2}}\left(n_{\mathscr{C}_0} + \frac{2}{\sigma_0-\sigma_m}\left[\sigma\frac{\partial A^0}{\partial x}\right]_{\mathscr{C}_1}\right)\cdot\nabla_{\eta,\theta}v^0|_{\eta=0^+}\\ &\quad -\left(fn^{\perp}_{\mathscr{C}_1} + \frac{f'}{\sqrt{1+(f')^2}}(A^0-a^0)\right)\cdot\partial_\theta\nabla_{\eta,\theta}v^0\Bigg\} + O(\varepsilon^2).\end{aligned}\tag{3.5}$$

On the other hand, for any v regular enough, we have

$$\Delta_{\eta,\theta}v = \partial^2_\eta v + \partial^2_\theta v + ((1+\eta\kappa)^{-2}-1)\partial^2_\theta v + \frac{\kappa}{1+\eta\kappa}\partial_\eta v - \frac{\eta\kappa'}{(1+\eta\kappa)^3}\partial_\theta v.$$

Now from the equation satisfied by A^0 we obtain

$$\begin{aligned}\Delta_{\eta,\theta} v^0_{BL} &= 2\frac{\partial A^0}{\partial y}\cdot \partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+} - \frac{2\eta\kappa+\eta^2\kappa^2}{(1+\eta\kappa)^2}\left(\frac{1}{\varepsilon}\frac{\partial^2 A^0}{\partial y^2}\cdot \nabla_{\eta,\theta} v^0|_{\eta=0^+}\right.\\ &\quad \left. + 2\frac{\partial A^0}{\partial y}\cdot \partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+}\right) + \frac{\kappa}{1+\eta\kappa}\frac{\partial A^0}{\partial x}\cdot \nabla_{\eta,\theta} v^0|_{\eta=0^+}\\ &\quad - \frac{\eta\kappa'}{(1+\eta\kappa)^3}\frac{\partial A^0}{\partial y}\cdot \nabla_{\eta,\theta} v^0|_{\eta=0^+} + O(\varepsilon),\end{aligned}$$

which gives, when applied with $\eta = x\varepsilon$,

$$\Delta_{\eta,\theta} v^0_{BL} = \kappa\left(\frac{\partial A^0}{\partial x} - 2x\frac{\partial^2 A^0}{\partial y^2}\right)\cdot \nabla_{\eta,\theta} v^0|_{\eta=0^+} + 2\frac{\partial A^0}{\partial y}\cdot \partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+} + \tilde{O}(\varepsilon), \tag{3.6}$$

where $\tilde{O}(\varepsilon)$ denotes any function of order ε when $|\eta|/\varepsilon$ is bounded and decays exponentially for $|\eta|/\varepsilon$ tending to infinity.

3.2 First-order approximation

For a function u defined in $\mathcal{O}^0 \cup \mathcal{O}^1$, we denote by $[u]|_{\gamma_\varepsilon}$ the jump of u across γ_ε:

$$[u]|_{\gamma_\varepsilon} = u|_{\gamma_\varepsilon^+} - u|_{\gamma_\varepsilon^-}.$$

Using equalities (3.3), (3.4) and (3.6) we deduce that the function W_0 defined by $W_0 = v^\varepsilon - (v^0 + v^0_{BL})$ satisfies

$$\begin{aligned}&\Delta_{\eta,\theta} W^0 = \kappa(\theta) G_1\left(\frac{\eta}{\varepsilon},\frac{\theta}{\varepsilon}\right)\cdot \nabla_{\eta,\theta} v^0|_{\eta=0^+} + G_2\left(\frac{\eta}{\varepsilon},\frac{\theta}{\varepsilon}\right)\cdot \partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+} + \tilde{O}(\varepsilon),\\ &(1+d_0\kappa)\partial_\eta W^0|_{\eta=d_0} + \Lambda_0 W^0|_{\eta=d_0} = g^\varepsilon_{0+},\\ &(1-d_0\kappa)\partial_\eta W^0|_{\eta=-d_0} - \Lambda_1 W^0|_{\eta=-d_0} = g^\varepsilon_{0-},\end{aligned}$$

with the following transmission conditions:

$$\begin{aligned}&\sigma_0 \partial_n^\Phi W^0|_{\gamma_\varepsilon^+} - \sigma_m \partial_n^\Phi W^0|_{\gamma_\varepsilon^-} = \varepsilon\left[\kappa(\theta) B_1(\theta/\varepsilon)\cdot \nabla_{\eta,\theta} v^0|_{\eta=0^+} + B_2(\theta/\varepsilon)\cdot \partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+}\right],\\ &\sigma_m \partial_\eta W^0|_{\eta=0^+} - \sigma_1 \partial_\eta W^0|_{\eta=0^-} = 0\end{aligned}$$

and

$$[W^0]|_{\gamma_\varepsilon} = 0, \quad [W^0]|_{\eta=0} = \varepsilon a^0 \cdot \nabla_{\eta,\theta} v^0|_{\eta=0^+},$$

where G_1, G_2, B_1 and B_2 are given with the help of (3.5) and (3.6) by

$$G_1 = -\left(\frac{\partial A^0}{\partial x} - 2x\frac{\partial^2 A^0}{\partial y^2}\right),$$
$$G_2 = -2\frac{\partial A^0}{\partial y},$$
$$B_1 = -\frac{f}{\sqrt{1+(f')^2}}\left((\sigma_0-\sigma_m)n_{\mathscr{C}_0} + 2\left[\sigma\frac{\partial A^0}{\partial x}\right]_{\mathscr{C}_1}\right),$$
$$B_2 = (\sigma_0-\sigma_m)\left(f n_{\mathscr{C}_1}^{\perp} + \frac{f'}{\sqrt{1+(f')^2}}(A^0-a^0)\right)$$

and

$$g^{\varepsilon}_{0+} = -(1+d_0\kappa)\partial_\eta v^0_{BL}(\eta_0,\theta) - \Lambda_0(v^0_{BL}|_{\eta=\eta_0}),$$
$$g^{\varepsilon}_{0-} = -(1-d_0\kappa)\partial_\eta v^0_{BL}(-\eta_0,\theta) + \Lambda_1(v^0_{BL}|_{\eta=-\eta_0}).$$

We now define the following vectors, for $k=1,2$:

$$D_k = \int_{\mathbb{R}\times\mathbb{T}} \tilde{\sigma} G_k\,dxdy + \int_{\mathscr{C}_1} B_k\,d\sigma, \tag{3.7}$$

with $\tilde{\sigma}$ defined in (2.5). Simple calculations give

$$D_1 = (\sigma_0-\sigma_m)\left[\int_0^1 f(y)\,dy\,n_{\mathscr{C}_0} + \int_0^1 A^0(f(y),y)\,dy\right] + (\sigma_m-\sigma_1)\int_0^1 A^0(0,y)\,dy - \sigma^0 a^0$$

and

$$D_2 = (\sigma_m-\sigma_0)\left[\int_0^1 A^0(f(y),y)f'(y)\,dy - \int_0^1 f(y)\,dy\begin{pmatrix}0\\1\end{pmatrix}\right].$$

Define v^1 by

$$\Delta_{\eta,\theta}v^1 = 0, \text{ in } \mathcal{O}^1\cup\mathcal{O}^0,$$
$$(1+d_0\kappa)\partial_\eta v^1|_{\eta=d_0} + \Lambda_0 v^1|_{\eta=d_0} = 0,$$
$$(1-d_0\kappa)\partial_\eta v^1|_{\eta=-d_0} - \Lambda_1 v^1|_{\eta=-d_0} = 0,$$

with the following transmission conditions:

$$\sigma_0\partial_\eta v^1|_{\eta=0^+} - \sigma_1\partial_\eta v^1|_{\eta=0^-} = \kappa D_1\cdot\nabla_{\eta,\theta}v|^0_{\eta=0^+} + D_2\cdot\partial_\theta\nabla_{\eta,\theta}v|^0_{\eta=0^+},$$
$$v^1|_{\eta=0^+} - v^1|_{\eta=0^-} = a^0\cdot\left(\nabla_{\eta,\theta}v^0\right)|_{\eta=0^+}.$$

Denote by w^1 the following quantity:

$$w^1 = W^0 - \varepsilon v^1.$$

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Since we have

$$\partial_n^{\Phi} v^1|_{\gamma_\varepsilon} = n_{\mathscr{C}_1} \cdot \nabla_{\eta,\theta} v^1|_{\eta=0^+} + O(\varepsilon),$$

and since $\partial_\eta v^1|_{\eta=0^+} = n_{\mathscr{C}_0}.\nabla_{\eta,\theta} v^1|_{\eta=0^+}$, w^1 satisfies

$$\begin{aligned}
&\Delta_{\eta,\theta} w^1 = \kappa(\theta) G_1 \cdot \nabla_{\eta,\theta} v^0|_{\eta=0^+} + G_2 \cdot \partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+} + \tilde{O}(\varepsilon),\\
&(1 + d_0\kappa)\partial_\eta w^1|_{\eta=d_0} + \Lambda_0 w^1|_{\eta=d_0} = g^\varepsilon_{0+},\\
&(1 - d_0\kappa)\partial_\eta w^1|_{\eta=-d_0} - \Lambda_1 w^1|_{\eta=-d_0} = g^\varepsilon_{0-},
\end{aligned}$$

with the following transmission conditions:

$$\begin{aligned}
\left[\sigma \partial_n^{\Phi} w^1\right]_{\gamma_\varepsilon} &= \varepsilon\left(\kappa(\theta) B_1(\theta/\varepsilon).\nabla_{\eta,\theta} v^0|_{\eta=0^+} + B_2(\theta/\varepsilon).\partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+}\right)\\
&\quad - \varepsilon(\sigma_0 - \sigma_m) n_{\mathscr{C}_1}.\nabla_{\eta,\theta} v^1|_{\eta=0^+} + O(\varepsilon^2),\\
\left[\sigma \partial_\eta w^1\right]_{\eta=0} &= -\,\varepsilon\left(\kappa(\theta) D_1.\nabla_{\eta,\theta} v^0|_{\eta=0^+} + D_2.\partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+}\right)\\
&\quad + \varepsilon(\sigma_0 - \sigma_m) n_{\mathscr{C}_0}.\nabla_{\eta,\theta} v^1|_{\eta=0^+}
\end{aligned}$$

and

$$[w^1]|_{\gamma_\varepsilon} = 0, \quad [w^1]|_{\eta=0} = 0.$$

We now introduce the two following problems defined in $\mathbb{R} \times \mathbb{T}$. For $j = 1, 2$, let $(A^{1,j}, a^{1,j})$ with $A^{1,j}$ continuous satisfy

$$\begin{aligned}
&\Delta A^{1,j} = G_j, \text{ in } \mathbb{R} \times \mathbb{T} \setminus (\mathscr{C}_0 \cup \mathscr{C}_1),\\
&\sigma_0 \partial_n A^{1,j}|_{\mathscr{C}_1^+} - \sigma_m \partial_n A^{1,j}|_{\mathscr{C}_1^-} = B_j,\\
&\sigma_m \partial_n A^{1,j}|_{\mathscr{C}_0^+} - \sigma_1 \partial_n A^{1,j}|_{\mathscr{C}_0^-} = -D_j,\\
&Aj^1 \to_{x\to-\infty} 0,\ A^{1,j} \to_{x\to+\infty} a^{1,j}.
\end{aligned}$$

According to Lemma 2.2 and Remark 2.3 and using (3.7) it is clear that the above problems are well posed. Define now the boundary-layer corrector of order 1 on $\mathscr{O}$ by

$$\begin{aligned}
&\forall \eta > 0,\\
&v^1_{BL}(\eta,\theta) = \varepsilon^2(\kappa(\theta)[A^{1,1}(\eta/\varepsilon, \theta/\varepsilon) - a^{1,1}] \cdot \nabla_{\eta,\theta} v^0|_{\eta=0^+}\\
&\qquad + [A^{1,2}(\eta/\varepsilon, \theta/\varepsilon) - a^{1,2}] \cdot \partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+}\\
&\qquad + [A^0(\eta/\varepsilon, \theta/\varepsilon) - a^0] \cdot \nabla_{\eta,\theta} v^1|_{\eta=0^+}),\\
&\forall \eta < 0,\\
&v^1_{BL}(\eta,\theta) = \varepsilon^2(\kappa(\theta) A^{1,1}(\eta/\varepsilon, \theta/\varepsilon) \cdot \nabla_{\eta,\theta} v^0|_{\eta=0^+}\\
&\qquad + A^{1,2}(\eta/\varepsilon, \theta/\varepsilon) \cdot \partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+} + A^0(\eta/\varepsilon, \theta/\varepsilon) \cdot \nabla_{\eta,\theta} v^1|_{\eta=0^+}).
\end{aligned}$$

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4 Justification of the expansion

Suppose now that g belongs to $H^{7/2}(\partial\Omega)$. Then v^0 belongs to $H^4(\mathcal{O}^0)$ and $H^4(\mathcal{O}^1)$. Hence we can differentiate the function $W^1 = W^0 - \varepsilon v^1 - v^1_{BL}$, which therefore satisfies

$$\begin{aligned}
&\Delta_{\eta,\theta} W^1 = F_\varepsilon(\eta,\theta), \text{ in } \mathcal{O}^1 \cup \mathcal{O}^m_\varepsilon \cup \mathcal{O}^0_\varepsilon,\\
&(1 + d_0\kappa)\partial_\eta W^1|_{\eta=d_0} + \Lambda_0 W^1|_{\eta=d_0} = g^\varepsilon_{1+},\\
&(1 - d_0\kappa)\partial_\eta W^1|_{\eta=-d_0} - \Lambda_1 W^1|_{\eta=-d_0} = g^\varepsilon_{1-},
\end{aligned}$$

with the following transmission conditions:

$$\begin{aligned}
&\sigma_0 \partial^\Phi_n W^1|_{\gamma^+_\varepsilon} = \sigma_m \partial^\Phi_n W^1|_{\gamma^-_\varepsilon} + \varepsilon^2 R^\varepsilon_1,\\
&\sigma_m \partial_\eta W^1|_{\eta=0^+} = \sigma_1 \partial_\eta W^1|_{\eta=0^-} + \varepsilon^2 R^\varepsilon_2
\end{aligned}$$

and

$$\begin{aligned}
&W^1|_{\gamma^+_\varepsilon} = W^1|_{\gamma^-_\varepsilon},\\
&W^1|_{\eta=0^+} = W^1|_{\eta=0^-} + \varepsilon^2\big(\kappa a^{1,1}.\nabla_{\eta,\theta} v^0|_{\eta=0^+} + a^{1,2}.\partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+} + a^0.\nabla_{\eta,\theta} v^1|_{\eta=0^+}\big).
\end{aligned}$$

The function F_ε is smooth with respect to the η-variable, while it is square-integrable in the θ-variable. Moreover it satisfies

$$|F_\varepsilon(\eta,.)|_{L^2(\mathbb{T})} \leqslant \begin{cases} C\varepsilon, \text{ if } \eta \leqslant c\varepsilon, \\ \text{decays exponentially for } \eta/\varepsilon \text{ tending to infinity,} \end{cases}$$

which implies that for all $s \in [1,2]$ there exists $c_s > 0$ such that

$$\|F_\varepsilon\|_{L^s(\mathcal{O})} \leqslant c_s \varepsilon^{1+1/s}. \tag{4.1}$$

Simple calculations lead to the following estimates:

$$|R^\varepsilon_1|_{L^2(\gamma_\varepsilon)} \leqslant C, \tag{4.2a}$$
$$|R^\varepsilon_2|_{L^2(\gamma^0)} \leqslant C, \tag{4.2b}$$
$$\left|g^\varepsilon_{1\pm}\right|_{L^2(\gamma^{\pm d_0})} \leqslant C_1 e^{-C_2/\varepsilon}. \tag{4.2c}$$

Theorem 1.4 is a straightforward consequence of the following theorem.

Theorem 4.1 *For any $s \in]1,2]$ there exists $c_s > 0$ independent of ε such that*

$$\left\|W^1\right\|_{H^1(\mathcal{D}^1)} + \left\|W^1\right\|_{H^1(\mathcal{D}^0)} \leqslant c_s \varepsilon^{1+1/s}.$$

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Proof To prove the estimate, we write the problem satisfied by W^1 in local coordinates. We consider a function $p \in H^2(\mathcal{O}^1)$ such that

$$
\begin{aligned}
& p = \kappa a^{1,1}.\nabla_{\eta,\theta} v^0|_{\eta=0^+} + a^{1,2}.\partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+} + a^0.\nabla_{\eta,\theta} v^1|_{\eta=0^+} \quad \text{on} \quad \eta = 0, \\
& \partial_\eta p = 0 \quad \text{on} \quad \eta = 0, \\
& p = \partial_\eta p = 0 \quad \text{on} \quad \eta = -d_0,
\end{aligned}
$$

and we set

$$\hat{W}^1 = W^1 + \varepsilon^2 p 1_{\mathcal{O}^1},$$

where $1_{\mathcal{O}^1}$ denotes the characteristic function of $\mathcal{O}^1$. Then $\hat{W}^1$ is continuous and satisfies

$$
\begin{aligned}
& \Delta_{\eta,\theta} \hat{W}^1 = F_\varepsilon(\eta, \theta), \ \text{ in } \mathcal{O}_\varepsilon^m \cup \mathcal{O}_\varepsilon^0, \\
& \Delta_{\eta,\theta} \hat{W}^1 = F_\varepsilon(\eta, \theta) + \varepsilon^2 \Delta_{\eta,\theta} p, \ \text{ in } \mathcal{O}^1, \\
& (1 + d_0\kappa)\partial_\eta \hat{W}^1|_{\eta=d_0} + \Lambda_0 \hat{W}^1|_{\eta=d_0} = g_{1+}^\varepsilon, \\
& (1 - d_0\kappa)\partial_\eta \hat{W}^1|_{\eta=-d_0} - \Lambda_1 \hat{W}^1|_{\eta=-d_0} = g_{1-}^\varepsilon, \\
& \sigma_0 \partial_n^\Phi \hat{W}^1|_{\gamma_\varepsilon^+} = \sigma_m \partial_n^\Phi \hat{W}^1|_{\gamma_\varepsilon^-} + \varepsilon^2 R_1^\varepsilon, \\
& \sigma_m \partial_\eta \hat{W}^0|_{\eta=0^+} = \sigma_1 \partial_\eta \hat{W}^0|_{\eta=0^-} + \varepsilon^2 R_2^\varepsilon.
\end{aligned}
$$

Now multiplying the main equations by $\sigma(1+\eta\kappa)\varphi$ and integrating by parts, we infer the following variational formulation for $\hat{W}^1 \in H^1(\mathcal{O})$:

$$
\begin{aligned}
& \int_{\mathcal{O}} \sigma D \nabla_{\eta,\theta} \hat{W}^1 \cdot \nabla_{\eta,\theta} \varphi + \sigma_0 < \Lambda_0 \hat{W}^1, \varphi > + \sigma_1 < \Lambda_1 \hat{W}^1, \varphi > \\
& \quad = - \int_{\mathcal{O}} \sigma(1+\eta\kappa) F_\varepsilon \varphi - \varepsilon^2 \sigma^1 \int_{\mathcal{O}} (1+\eta\kappa) \Delta_{\eta,\theta} p \varphi + \sigma^0 \int_{\gamma^{d_0}} g_{1+}^\varepsilon \varphi - \sigma^1 \int_{\gamma^{-d_0}} g_{1-}^\varepsilon \varphi \\
& \quad - \varepsilon^2 \int_{\gamma_\varepsilon} \frac{\sqrt{(1+\varepsilon\kappa f)^2 + (f')^2}}{\sqrt{1+(f')^2}} R_1^\varepsilon \varphi - \varepsilon^2 \int_{\gamma^0} R_2^\varepsilon \varphi, \qquad \forall\, \varphi \in H^1(\mathcal{O}),
\end{aligned}
$$

where D is the diagonal matrix with elements $1+\eta\kappa$ and $1/(1+\eta\kappa)$.
Taking $\varphi = \hat{W}^1$, using inequalities (4.1) and (4.2) and using also a result of Bonder *et al.* [5], since the amplitude and the period of the oscillations of Γ_ε^r have the same order, we infer the result. □

Remark 4.2 *Since the boundary-layer corrector v_{BL}^1 is of order ε^2 in $L^2(\mathcal{O})$ we infer that for any $s \in]1,2]$*

$$\|v^\varepsilon - v^0 - v_{BL}^0 - \varepsilon v^1\|_{L^2(\mathcal{O})} \leqslant C_s \varepsilon^{1+1/s}.$$

5 Conclusion

In this paper, we have proved the efficiency of our transmission conditions to tackle the numerical difficulties inherent in the geometry of a rough thin layer. Even though we have considered here the periodic case, the same analysis with much more tedious calculations can treat the quasi-periodic case.

The main feature of our result is the following: unlike the case of the weakly oscillating thin membrane (see [11]), if the period of the oscillations of the rough layer is similar to its thickness, then the layer influence on the steady-state potential cannot be approximated by only considering the mean effect of the rough layer.

Actually, if we were to consider the mean effect of the roughness, the approximate transmission conditions would be these presented in (1.6), by replacing f by its average $\tilde{f}$. Observe that our transmission conditions (1.5) are very different, since they involve the curvature of Γ and the normal and tangent derivatives of u^0. More precisely, denote by $\tilde{u}^1$ the first-order approximation of the mean effect of the layer. Then according to (1.6) $\tilde{u}^1$ satisfies

$$\begin{aligned}
&\Delta \tilde{u}^1 = 0, \text{ in } \mathscr{D}^0 \cup \mathscr{D}^1, \quad u^1|_{\partial\Omega} = 0, \text{ on } \partial\Omega,\\
&\sigma_0 \partial_n \tilde{u}^1|_{\Gamma^+} = \sigma_1 \partial_n \tilde{u}^1|_{\Gamma^-} + (\sigma_0 - \sigma_m)\tilde{f}\partial_t^2 u^0|_{\Gamma^+},\\
&\tilde{u}^1|_{\Gamma^+} = \tilde{u}^1|_{\Gamma^-} + \frac{\sigma_0 - \sigma_m}{\sigma_m}\tilde{f}\partial_n u^0|_{\Gamma^+},
\end{aligned}$$

while we remember that the 'real' first-order u^1 satisfies

$$\begin{aligned}
&\Delta u^1 = 0, \text{ in } \mathscr{D}^0 \cup \mathscr{D}^1, \quad u^1|_{\partial\Omega} = 0,\\
&\sigma_0 \partial_n u^1|_{\Gamma^+} = \sigma_1 \partial_n u^1|_{\Gamma^-} - \kappa D_1 \cdot \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma^+} \end{pmatrix} + D_2 \cdot \partial_t \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma^+} \end{pmatrix},\\
&u^1|_{\Gamma^+} = u^1|_{\Gamma^-} + a^0 \cdot \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma^+} \end{pmatrix}.
\end{aligned}$$

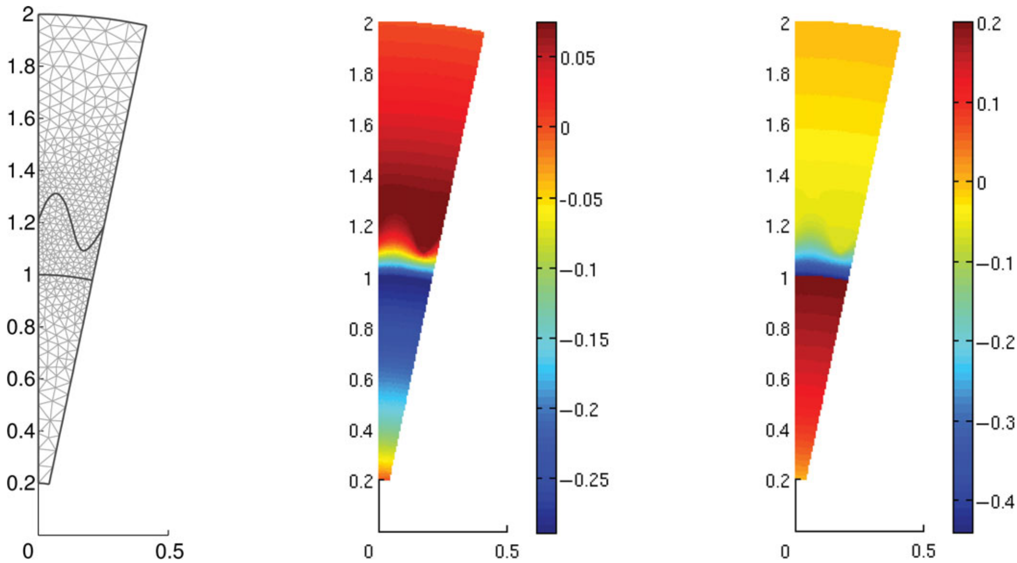
The motivation of this work comes from a collaboration with Schlumberger to model silty soils. However we are confident of the importance of such results in other research areas, for instance in the mathematical study of radiation patterns created by periodic structures of phased-arrays antennas. These antennas are formed by the superposition of a great number of identical electromagnetic horns, and very slight changes in the above analysis would lead to the appropriate transmission conditions. We conclude this paper by illustrating our theoretical results with the numerical simulations obtained by Ciuperca, Perrussel and Poignard and presented in [6]. The mesh generator *Gmsh* [7] and the finite-element library *Getfem++* [13] have been used to perform simulations.

The computational domain Ω is delimited by the circles of radius 2 and of radius 0.2 centred on 0, while $\mathscr{D}^1$ is the intersection of Ω with the concentric disc of radius 1. The rough layer is then described by $f(y) = 1 + \frac{1}{2}\sin(y)$. One period of the domain is shown Figure 3(a). The Dirichlet boundary data are identically 1 on the outer circle and 0 on the inner circle.

The conductivities σ_0, σ_1 and σ_m equal 3, 1 and 0.1 respectively. The computed coefficients[3] issued from problem (1.4) are given in Table 1.

The numerical convergence rates for the H^1-norm in $\mathscr{D}^1$ of the three errors $u^\varepsilon - u^0$, $u^\varepsilon - u^0 - \varepsilon u^1$ and $u^\varepsilon - u^0 - \varepsilon\widetilde{u}^1$ as ε goes to zero are given Figure 4. As predicted by the

[3] The convergences at the infinity in problem (1.4) are exponential, and hence it is sufficient to compute problem (1.4) for $|x| \leqslant M$, with M large enough to obtain a^0 with a good accuracy.



(a) One period. (b) Error order 0. (c) Error order 1.

FIGURE 3. (Colour online) Representation of one period of the domain and the corresponding errors with approximate solutions u^0 and $u^0 + \varepsilon u^1$ for $\varepsilon = 2\pi/30$. Please note that the error in the rough layer should not be consisdered because a proper reconstruction of the solution in it is not currently implemented.

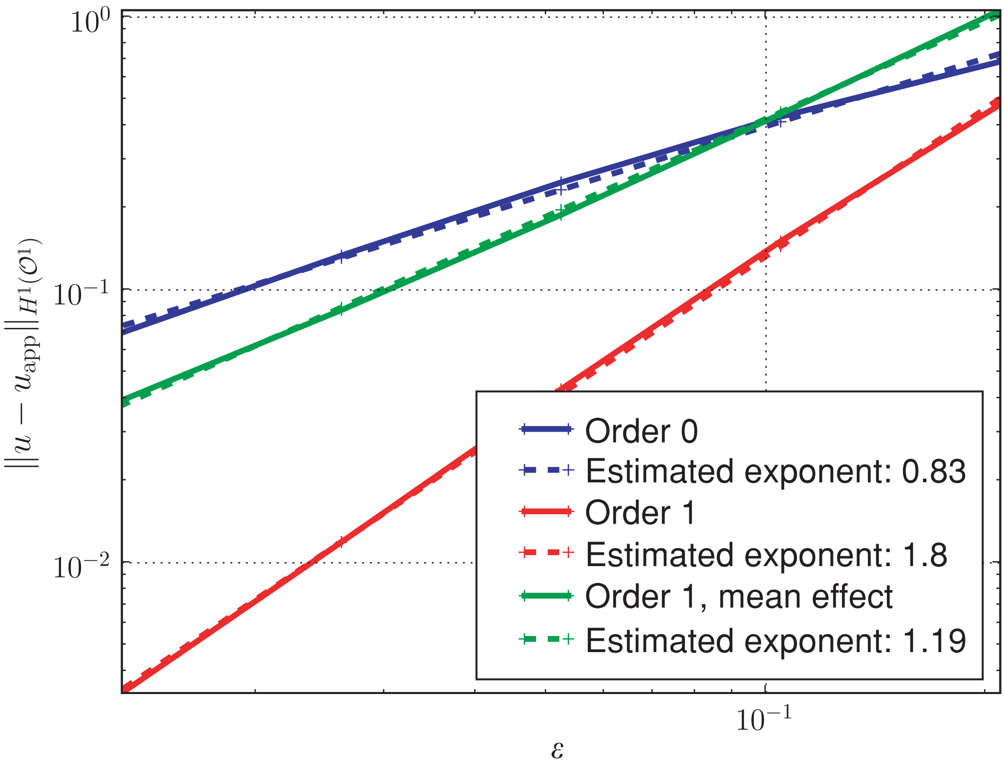


FIGURE 4. (Colour online) The H^1-error in the cytoplasm versus ε for three approximate solutions.

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Table 1. *Coefficients issued from problem* (1.4). *Three significant digits are kept.*

a_1^0	a_2^0	D_1^1	D_2^1	D_1^2	D_2^2
19.3	0	0	0	−0.0499	−3.87

theory, the rates are close to 1 for order 0 and for order 1 with the mean effect, whereas the rate is close to 2 for the 'real' order 1 equal to $u^\varepsilon - u^0 - \varepsilon u^1$.

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