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Maximal entropy measures for piecewise affine surface homeomorphisms

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Abstract. We study the dynamics of piecewise affine surface homeomorphisms from the point of view of their entropy. Under the assumption of positive topological entropy, we establish the existence of finitely many ergodic and invariant probability measures maximizing entropy and prove a multiplicative lower bound for the number of periodic points. This is intended as a step towards the understanding of surface diffeomorphisms. We proceed by building a jump transformation, using not first returns but carefully selected 'good' returns to dispense with Markov partitions. We control these good returns through some entropy and ergodic arguments.

1. *Introduction*

Introduced by Anosov and Smale in the 1960s [**15**], uniform hyperbolicity is at the core of dynamical system theory. The corresponding systems are well understood since, in particular, the works of Sinai, Bowen and Ruelle (see, e.g., [**23**]) and it is now a central challenge to try to extend our understanding beyond these systems [**2**]. We propose that robust entropy conditions provide a way to do this for new open sets of dynamical systems, by implying a non-uniform but global hyperbolic structure, especially with respect to measures maximizing entropy (see §1.1 for definitions).

Such invariant probability measures can be thought of as describing the 'complexity' of the dynamics. These measures exist as soon as the dynamics is compact and C^∞ (see [**29**]) or somewhat hyperbolic [**14**], although they are known to fail to exist in finite smoothness for interval maps [**5, 32**] and diffeomorphisms of four-dimensional tori [**27**]. Uniqueness problems are usually much more delicate and can be solved only after a global analysis of the dynamics which we propose to do under entropy conditions.

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Entropy expansion is such a condition. It requires the topological entropy (see §1.1) to be strictly larger than the supremum of the topological entropies of all smooth hypersurfaces. It is robust in the sense that it is open in the C^∞ topology. Entropy-expanding C^∞ maps $T : M \to M$ have a finite number of ergodic and invariant probability measures maximizing the entropy. Their periodic points satisfy a *multiplicative lower bound*:

$$\liminf_{n\to\infty,\, p|n} e^{-nh_{\mathrm{top}}(T)} \#\{x \in M \mid T^n x = x\} > 0 \tag{1.1}$$

for some integer $p \geq 1$ (a period) (see [**10, 12**] for precise definitions and statements including other results).

Entropy expansion is satisfied by plane maps of the type $(x, y) \mapsto (1.9 - x^2 + \epsilon y, 1.8 - y^2 + \epsilon x)$ for small ϵ (see [**7**]). On the interval, the condition reduces to non-zero topological entropy. In fact, entropy expansion can be understood as a generalization of some aspects of one-dimensional dynamics. Indeed, the previous results were first proved by Hofbauer [**19, 20**] for piecewise monotone maps on the interval and our approach has built on his techniques [**8**].

The techniques used in the above-mentioned papers do not apply to diffeomorphisms (e.g. a diffeomorphism is never entropy-expanding). However, many properties of interval maps generalize to surface diffeomorphisms so the following are generally expected.

CONJECTURE 1. *Consider a $C^{1+\epsilon}$ diffeomorphism ($\epsilon > 0$) of a compact surface with non-zero topological entropy.*

The collection of ergodic and invariant probability measures with maximal entropy is countable *(possibly finite or empty) and the periodic points satisfy a multiplicative lower bound if there exists at least one measure with maximal entropy.*

CONJECTURE 2. *Consider a C^∞ diffeomorphism of a compact surface with non-zero topological entropy.*

The collection of ergodic and invariant probability measures with maximal entropy is finite *and the periodic points satisfy a multiplicative lower bound.*

By a result of Newhouse [**29**], all C^∞ maps of compact manifolds have at least one measure of maximum entropy. Also a classical theorem of Katok [**22**] states that if T is a $C^{1+\epsilon}$, $\epsilon > 0$, diffeomorphism of a compact surface M, then the number of periodic points satisfies a *logarithmic lower bound*:

$$\limsup_{n\to\infty} \frac{1}{n} \# \log\{x \in M \mid T^n x = x\} \geq h_{\mathrm{top}}(T),$$

i.e. a weak version of (1.1).

This paper presents a proof of the analog of Conjecture 2 in the easier case of piecewise affine homeomorphisms. This replaces distortion of smooth diffeomorphisms by the singularities of the piecewise affine maps. However, this preserves substantial difficulties. Indeed, there exist piecewise affine maps on surfaces without a maximum measure (see Appendix C, although the examples known to the author are discontinuous—or continuous but piecewise quadratic) or with infinitely many maximum measures (see also

Appendix C). Thus this setting, beyond its own interest as a simple and rather natural class of dynamics, is challenging enough to allow the development of new tools which we hope will apply to diffeomorphisms.

1.1. *Definitions and statements.* Let M be a compact two-dimensional manifold possibly with boundary, *affine* in the following sense. There exists a distinguished atlas of charts:

- identifying the neighborhood of any point of M with an open subset of $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0\}$;
- inducing affine changes of coordinates.

These charts are called the *affine charts*. The phenomena we are considering are independent of the global topology, so we could in fact restrict ourselves to the special cases $M = \mathbb{T}^2$ or $M = [0, 1]^2$.

A continuous map $T : M \to M$ is said to be *piecewise affine* if there exists a finite partition P of M such that for every $A \in P$, A and $T(A)$ are contained in affine charts which map them to polygons of $\mathbb{R}^2$ with non-empty interiors and $T : A \to T(A)$ is affine with respect to these affine charts. It is convenient to replace the partition P by the collection $\tilde{P}$ of the interiors of the elements of P. Such a partition $\tilde{P}$ (a partition up to the boundaries of its elements) is called an *admissible partition* with respect to T. We drop the tilde in the sequel.

Let us recall some facts about entropy (we refer to [**16, 35**] for further information). The entropy of a non-necessarily invariant subset $K \subset M$ is a measure of the 'number of orbits' starting from K. Recall that the (ϵ, n)-*ball* at $x \in M$ is $\{y \in M \mid \forall k = 0, 1, \ldots, n-1 \; d(T^k y, T^k x) < \epsilon\}$. The entropy of K is, according to the Bowen–Dinaburg formula [**3**]:

$$h(T, K) := \lim_{\epsilon \to 0} h(T, K, \epsilon) \quad \text{with } h(T, K, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log r(\epsilon, n, K) \tag{1.2}$$

where $r(\epsilon, n, K)$ is the minimal number of (ϵ, n)-balls with union containing K. The *topological entropy* is $h_{\text{top}}(T) := h(T, M)$.

The entropy of an ergodic and invariant probability measure μ can be defined similarly, according to [**22**]

$$h(T, \mu) := \lim_{\epsilon \to 0} h(T, \mu, \epsilon) \quad \text{with } h(T, \mu, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log r(\epsilon, n, \mu)$$

where $r(\epsilon, n, \mu)$ is the minimal number of (ϵ, n)-balls whose union has a μ-measure of at least λ, for a constant $\lambda \in (0, 1)$ ($h(T, \mu)$ is independent of λ).

The *variational principle* states that for $T : M \to M$ (in fact, for any continuous self-map of a compact metric space [**35**]):

$$h_{\text{top}}(T) = \sup_{\mu} h(T, \mu) \tag{1.3}$$

where μ ranges over the T-invariant and ergodic probability measures.

The following combinatorial expression for topological entropy follows from observations of Newhouse and will be the starting point of our investigations.

PROPOSITION 1.1. *Let T be a piecewise affine homeomorphism of a compact surface. The topological entropy of T is given by*

$$h_{\text{top}}(T) = \limsup_{n\to\infty} \frac{1}{n} \log \#\{A_0 \cap f^{-1}A_1 \cap \cdots \cap f^{-n+1}A_{n-1} \neq \emptyset \mid A_i \in P\}. \quad (1.4)$$

Remark 1.2. The above entropy formula was also obtained by Sands and Ishii [**21**] by different methods.

Misiurewicz and Szlenk [**28**] established the same formula for piecewise monotone maps of the intervals.

The proof is given in §2.1.

The variational principle (1.3) brings to the fore the ergodic and invariant probability measures μ such that $h(T, \mu) = \sup_\nu h(T, \nu) = h_{\text{top}}(f)$. We call them *maximum measures*. A corollary of the proof of the previous proposition is the following existence result (compare with the examples in Appendix C).

COROLLARY 1.3. *A piecewise affine homeomorphism of a compact surface has at least one maximum measure.*

Our main result, finally established in §5.1, is the following.

THEOREM 1. *Let $T : M \to M$ be a piecewise affine homeomorphism of a compact affine surface. Assume that $h_{\text{top}}(T) > 0$. Then there are finitely many ergodic, invariant probability measures maximizing the entropy (or maximum measures).*

We also obtain as by-products (see §§5.2 and 5.3) the following results.

PROPOSITION 1.4. *Let $T : M \to M$ be a piecewise affine homeomorphism of a compact affine surface with non-zero topological entropy. The periodic points satisfy a multiplicative lower bound.*

PROPOSITION 1.5. *Let $T : M \to M$ be a piecewise affine homeomorphism of a compact affine surface. Let $\mathcal{S}$ be the singularity locus of M, that is, the set of points x which have no neighborhood on which the restriction of T is affine.*

For any $\epsilon > 0$, there is a compact invariant set $K \subset M \setminus \mathcal{S}$ such that

$$h_{\text{top}}(f|K) > h_{\text{top}}(f) - \epsilon.$$

Moreover, $f : K \to K$ is topologically conjugate to a subshift of finite type (see [**26**]*).*

1.2. *Outline of the proof.* We use an alternative approach to the explicit construction of Markov partitions. We ask less of geometry and use more combinatorics, ergodic theory and entropy estimates to accommodate the resulting non-uniqueness of representation. More precisely, we build small rectangles admitting many returns with 'good properties' which allows the construction of a jump transformation and establish *semi-uniform estimates*, that is, uniform estimates holding on subsets of measures that are lower bounded with respect to any large entropy measure. The finite number of maximum measures for the jump transformation follows from the results of Gurevič on countable state Markov shifts. However, the jump transformation is not a first return map to an *a priori* defined good set. Hence, a careful study of the relation between the jump transformation and the original dynamics is needed to conclude that the maximum measures of both systems can be identified. In fact, we analyze more generally *large entropy measures*, i.e. invariant and ergodic probability measures with entropy close enough to the supremum.

Let us outline the proof. We start in §2 by introducing the natural *symbolic dynamics* of the piecewise affine map using the partition defined by the singularities of T. We first show that this symbolic dynamics has the same entropy as T. This is both a significant result and a fundamental step in our approach. We then establish that the *(local) stable* $W^s(x)$ *and unstable* $W^u(x)$ *manifolds* of points $x \in M$, i.e. the sets of points with the same past or future with respect to the partition P, are line segments outside of an entropy-negligible subset. These line segments can be arbitrarily short and their length may vary discontinuously. However, we prove semi-uniform lower bounds for their lengths and angles, using the conditional entropy with respect to the past or the future. A corollary of these bounds is that the boundary of the partition is negligible with respect to all ergodic invariant probability measures with non-zero entropy.

At this point, one would like to conclude by an argument of the following type. If there was a large number of maximum measures, then one could find two of them, both giving positive measure to a set S of points with local stable and unstable manifolds much larger than the diameter of S. Hence, one could 'jump' back and forth between typical orbits of each of the two measures. However, one could expect such mixing to allow the construction of a measure with greater entropy, a contradiction. However, establishing the increase in entropy seems to require estimates that are too fine (of a multiplicative, rather than logarithmic type). We are thus lead to build a jump transformation with Markov properties which will reduce the problem to loop counting on a graph, for which these fine estimates exist (and, indeed, the uniqueness of the maximum measure has been established in this setting by Gurevič).

Section 3 is devoted to building a Markov structure representing the large entropy dynamics. We first build arrays of *Markov rectangles* which contain a significant proportion of the dynamics. These are approximate rectangles in the sense they contain open subsets of points with local manifolds that do not cross ('holes' in the local product structure). However, we can ensure that the relative measure of such points is very small. Our techniques, however, require T to be replaced by some high power T^L.

We then define hyperbolic strips following the geometric picture of Markov rectangles usual in uniformly hyperbolic dynamics. We provide tools to build many such strips around typical orbits of large entropy measures, using visits to the Markov rectangles while the stable and unstable manifolds are both 'rather large'.

These hyperbolic strips are Markov in the sense that they can be freely concatenated as soon as they end and begin in the same rectangle. However, to obtain a useful Markov representation from this, one needs an invariant way of 'cutting' typical orbits into concatenations of such hyperbolic strips. A fundamental difficulty arises here: there exist incompatible decompositions, i.e. which do not admit a common refinement. There does not seem to be an a priori natural set the visits to which could be used to define invariantly the above 'cutting'†.

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† Note that shadowing lemmas *à la* Katok [**22**] give comparable results for surface diffeomorphisms. The problem is to find invariant decompositions.

We conclude §3 by selecting among hyperbolic strips a subset of *admissible* strips to obtain uniqueness in the decomposition of *forward* orbits (this weak uniqueness will require a more detailed ergodic analysis in §5). We obtain a notion of *good return times* and a Markov structure.

The more technical §4 relates the good return times to geometric and combinatorial properties involving the visits to the Markov rectangles and their holes. It is shown that large entropy measures cannot have too large average good return times.

Finally, §5 proves the main results. We lift large entropy measures of T to the jump transformations as finite extensions. Using that the latter is isomorphic to a countable state Markov shift, a result of Gurevič [**17**] allows us to conclude the proof of the theorem.

Proposition 1.4 rests on a classical estimate of Vere-Jones [**34**] on the number of loops of countable oriented graphs together with a combinatorial argument to transfer this estimate to periodic points of T.

Proposition 1.5, the possibility of approximating in entropy of the whole map by a compact set away from the singularity set, follows from a similar property of countable state Markov shifts: they are approximated in entropy by finite state Markov shifts according to Gurevič.

There are three appendices: Appendix A recalls some facts about measure-theoretic entropy, Appendix B proves a lifting theorem for the tower defined by a return time and Appendix C gives some examples of piecewise smooth maps.

1.3. *Some comments.* The results presented here allow an analysis of the maximal entropy measures of a simple and natural class of dynamics by representing them with countable state Markov shifts. Along the same lines, one can probably make the representation more precise to obtain further results, for instance:

- classification by the topological entropy and periods up to isomorphisms modulo entropy-negligible subsets [**4**];
- precise counting of isolated periodic points, e.g. the existence of a meromorphic extension of the Artin–Mazur zeta function as in [**12**];
- uniqueness of the maximal entropy measure under a transitivity assumption and/or a bound on the number of maximum measures in terms of the cardinality of the partition P.

In a slightly different direction, one would like to understand the nature of the symbolic dynamics of piecewise affine surface homeomorphisms (see §2.1). Is there a tractable class of subshifts containing these symbolic dynamics, i.e. a class that would do for piecewise affine surface homeomorphisms what subshifts of quasi-finite type [**11**] do for piecewise monotone interval maps?

It would also be natural to apply the techniques of this paper to more general dynamics. First, piecewise homographic surface homeomorphisms can be analyzed in the same way. Then one could try to analyze other general classes of piecewise affine maps, especially in higher dimensions (e.g. uniformly expanding maps or entropy-expanding maps or entropy-hyperbolic homeomorphisms [**13**]). Most of these questions are still open despite some partial results (see, e.g., [**9, 25, 33**]) and we should stress that new problems appear immediately. From the point of view of entropy alone:

- there exist piecewise affine continuous *maps* on surfaces and piecewise affine homeomorphisms in *dimension three* for which the right-hand side of (1.4) is strictly larger than the entropy (see Examples 1 and 3 in the Appendix C);
- Example 4 in Appendix C is a piecewise affine *discontinuous map* on a surface with no maximum measure (one can give a continuous, piecewise quadratic version of it, see Example 5); however, the author is not aware of examples of continuous piecewise affine maps without maximum measures.

For diffeomorphisms, the main difficulty with our approach is finding a link between short stable/unstable manifolds and small entropy, e.g. one would need to relate small Lyapunov charts to entropy bounds for a smooth diffeomorphism. An analysis of C^1 diffeomorphisms with non-zero topological entropy with dominated splitting is in preparation.

2. *Pointwise estimates*

2.1. *Symbolic dynamics.* We define a symbolic dynamics for the map T using some admissible partition P, that is, a finite collection of disjoint open polygons with dense union. A key step is showing that ∂P has zero measure with respect to any $\mu \in \mathbb{P}^0_{\text{erg}}(T)$, where $\mathbb{P}^h_{\text{erg}}(T)$ denotes the set of ergodic, invariant probability measures of T with $h(T, \mu) > h$.

Definition 2.1. Let P be an admissible partition. $x \in M$ is *nice* if for every $n \in \mathbb{Z}$, $T^n x$ belongs to an element A_n of the admissible partition P. The sequence $A \in P^{\mathbb{Z}}$ thus defined is the *P-itinerary* of x.

The *symbolic dynamics* of (T, P) is

$$\Sigma := \overline{\{A \in P^{\mathbb{Z}} \mid \exists x \in M\ \forall n \in \mathbb{Z}\ T^n x \in A_n\}}$$

endowed with the shift map $\sigma(A) = (A_{n+1})_{n\in\mathbb{Z}}$.

A standard result (see, e.g., [**35**]) states that since Σ is a subshift, it admits at least one maximal entropy measure. Hence, a 'close enough' relation between the invariant measures of Σ and T will imply the existence of a maximum measure also for T. By the variational principle, we also obtain that T and Σ have the same topological entropy. The Misiurewicz–Szlenk formula for T will then follow. Indeed, for a subshift such as Σ, the topological entropy is computed by counting the *cylinders*, $[A_0 \cdots A_{n-1}] := \{x \in \Sigma \mid x_0 \cdots x_{n-1} = A_0 \cdots A_{n-1}\}$, that is

$$h_{\text{top}}(\Sigma) = \lim_{n\to\infty} \frac{1}{n} \log \#\{[A_0 \cdots A_{n-1}] \neq \emptyset \mid A \in P^n\}.$$

Neither T nor its symbolic dynamics is an extension of the other in general, hence it is convenient to introduce the following common extension:

$$\Sigma \ltimes M := \overline{\{(A, x) \in P^{\mathbb{Z}} \times M \mid \forall n \in \mathbb{Z}\ T^n x \in A_n\}} \quad \text{with } \hat{T}(A, x) = (\sigma A, Tx).$$

The close relation between the measures of T and Σ alluded to above is the following.

LEMMA 2.2. *Both maps* $\pi_1 : \Sigma \ltimes M \to \Sigma$ *and* $\pi_2 : \Sigma \ltimes M \to M$ *are entropy preserving: for every invariant probability measure* μ *on* $\Sigma \ltimes M$, $h(\sigma, \pi_1\mu) = h(T, \pi_2\mu) = h(\hat{T}, \mu)$. *Moreover,* π_1 *and* π_2 *induce onto maps between the sets of (ergodic) invariant probability measures.*

In particular, the topological entropies of the three systems are equal by the variational principle recalled in (1.3).

The proof of the above lemma rests on two geometric/combinatorial properties. The first is the following observation by Newhouse, which is very specific to our setting (it is false in higher dimensions or without the invertibility assumption, see Appendix C).

LEMMA 2.3. *The* multiplicity entropy [**6**]

$$h_{\text{mult}}(T) := \limsup_{n\to\infty} \frac{1}{n} \log \max_{x\in M} \text{mult}(P^n, x) \quad \textit{with } \text{mult}(Q, x) := \#\{A \in Q \mid x \in \overline{Q}\}$$

is zero for any piecewise affine homeomorphism of a surface.

The second is a property of linear maps.

LEMMA 2.4. *Let* $d \geq 1$. *For each* $n \geq 0$, *let* $T_n : \mathbb{R}^d \to \mathbb{R}^d$ *be a linear map. Then*

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \max\{\#S \mid \forall 0 \leq k < n \ \text{diam}(T_{k-1} \cdots T_1 T_0 S) \leq 1 \textit{ and }$$
$$\forall x \neq y \in S\ \exists 0 \leq k < n \text{ such that } \|T_{k-1} \cdots T_1 T_0 (x - y)\| > \epsilon\} = 0.$$

We leave the easy proofs of Lemmas 2.3 and 2.4 to the reader.

Proof of Lemma 2.2. Lemma 2.3 (respectively Lemma 2.4) implies that for all $x \in M$ (respectively Σ), for $i = 2$ (respectively $i = 1$),

$$h(\hat{T}, \pi_i^{-1}\{x\}) = 0$$

(this is the entropy of a subset as recalled in (1.2)). Now, $\pi_1 : \Sigma \ltimes M \to \Sigma$ and $\pi_2 : \Sigma \ltimes M \to M$ are both compact topological extensions. Hence, one can apply Bowen's result [**3**]:

$$h(\hat{T}, \hat{\mu}) = h(\sigma, \pi_1\mu) = h(T, \pi_2\mu)$$

for all invariant probability measures $\hat{\mu}$ of $\Sigma \ltimes M$. □

2.2. *Invariant manifolds and Lyapunov exponents.*

For a fixed partition P, we have the following.

Definition 2.5. The *stable manifold* at $A \in \Sigma$ is the following set (convex in the affine charts):

$$W^s(A) := \bigcap_{n\geq 1} \overline{T^{-n} A_n}.$$

The *unstable manifolds* $W^u(A)$ are defined by replacing $n \geq 1$ by $n \leq -1$ in the above equation.

The *Lyapunov exponents* along the stable or unstable direction at $A \in \Sigma$ are

$$\lambda^u(A) := \lim_{n\to\pm\infty} \frac{1}{n} \log \|(T_A^n)'^{\pm 1}\|^{\pm 1} \quad \text{and} \quad \lambda^s(A) := \lim_{n\to\pm\infty} \frac{1}{n} \log \|(T_A^n)'^{\mp 1}\|^{\mp 1}$$

where T_A^n is the affine composition $(T|A_{n-1}) \circ \cdots \circ (T|A_0)$ (if $n \geq 0$) or $[(T|A_{-1}) \circ \cdots \circ (T|A_n)]^{-1}$ (if $n < 0$).

Any nice $x \in M$ defines a unique itinerary A and we write $W^s(x)$ for $W^s(A)$, $\lambda_+^u(x)$ for $\lambda_+^u(A)$ and so on.

The first goal of this section is the following 'non-singularity' result.

PROPOSITION 2.6. *Let $\mu \in \mathbb{P}^0_{\text{erg}}(T)$. The following hold:*

- *$\mu(\partial P) = 0$ (in particular, μ-almost every $x \in M$ is nice);*
- *the Lyapunov exponents exist and satisfy $\lambda^s(x) \leq -h(T, \mu) < 0 < h(T, \mu) \leq \lambda^u(x)$ for μ-almost every $x \in M$;*
- *$W^s(x)$ and $W^u(x)$ are line segments containing x in their relative interiors* $\operatorname{int} W^s(x)$ *and* $\operatorname{int} W^u(x)$ *for μ-almost every $x \in M$.*

Proof. Let $\mu \in \mathbb{P}^0_{\text{erg}}(T)$. As we have not yet proved that almost every $x \in M$ is nice, we have to work in the extension $\Sigma \ltimes M$ to be able to speak of itineraries, invariant manifolds and so on. By compactness, there exists an invariant and ergodic probability measure $\hat{\mu}$ of $\hat{T} : \Sigma \ltimes M \hookleftarrow$ such that $\pi_2\hat{\mu} = \mu$. We have $h(\hat{T}, \hat{\mu}) > 0$ by Lemma 2.2.

We first consider the invariant manifolds.

CLAIM 2.7. *We claim that for $\hat{\mu}$-almost every $(A, x) \in \Sigma \ltimes M$:* (i) *$W^u(A)$ is a line segment;* (ii) *x is not an endpoint of this segment.*

Proof. To begin with, observe that $W^u(\sigma A) \subset T(W^u(A))$ so that $\dim(W^u(\sigma A)) \leq \dim(W^u(A))$. As $\hat{\mu}$ is invariant and ergodic, $\dim(W^u(A)) = d$ $\hat{\mu}$-almost everywhere for some $d \in \{0, 1, 2\}$. Claim (i) above is that $d = 1$.

Let $\hat{P}$ be the natural partition of $\Sigma \ltimes M$ (coming from the canonical partition of Σ). By Lemma 2.2, $(\hat{T}, \hat{\mu})$ has the same entropy as the corresponding symbolic system. Thus, $h(\hat{T}, \hat{\mu}) = h(\hat{T}, \hat{\mu}, \hat{P})$. Using conditional entropy (see Appendix A) we can compute $h(\hat{T}, \hat{\mu}, \hat{P})$ as $H_{\hat{\mu}}(\hat{P}|\hat{P}^-)$ where $\hat{P}^- := \bigvee_{n\geq 1} T^n\hat{P}$. Observe that $A \mapsto W^u(A)$ is $\hat{P}^-$-measurable.

We exclude the cases $d = 0, 2$ by contradiction. Assume first $d = 0$, i.e. $W^u(A)$ is a single point $x \in M$ for $\hat{\mu}$-almost every $(A, x) \in \Sigma \ltimes M$. This implies that

$$h(\hat{T}, \hat{\mu}) = H_{\hat{\mu}}(\hat{P}|\hat{P}^-) = \lim_{n\to\infty} \frac{1}{n} H_{\hat{\mu}}(\hat{P}^n|\hat{P}^-) \leq h_{\text{mult}}(T, P) = 0,$$

a contradiction, excluding the case $d = 0$.

Now assume $d = 2$, so $\hat{\mu}$-almost every $\overline{W^u(A)}$ is the closure of the interior of $W^u(A)$. By construction, two distinct unstable manifolds have disjoint interiors. Therefore, there can only be countably many of them, after discarding a set of zero $\hat{\mu}$-measure. In particular, $W^u(A) = W^u(A^0)$ on a set of positive measure for some A^0. By Poincaré recurrence, there exists an integer $n > 0$ such that $T^n(W^u(A^0)) = W^u(A^0)$. This implies that $\pi_1\hat{\mu}$ is periodic, hence $0 = h(\sigma, \pi_1\hat{\mu}) = h(\hat{T}, \hat{\mu})$. The contradiction proves (i).

We now turn to (ii). If $x \in \partial W^u(A)$, then $T(x) \in \partial W^u(\sigma(A))$. Thus if (ii) is false, then $x \in \partial W^u(A)$ μ-almost everywhere. However, this implies that, for any $\epsilon > 0$ and any large n,

$$\begin{aligned} nh(\hat{T}, \hat{\mu}) = H_{\hat{\mu}}(\hat{P}^n | \hat{P}^-) &\leq \log 2 + \log \max_{x \in M} \#\{A \in P^n \mid \overline{A} \ni x\} \\ &\leq \log 2 + (h_{\text{mult}}(T, P) + \epsilon)n. \end{aligned}$$

As $h_{\text{mult}}(T, P) = 0$, it would follow that $h(\hat{T}, \hat{\mu}) = 0$, a contradiction. □

We now turn to the exponents. First they do exist by the classical Oseledets theorem (see, e.g., [**23**]).

CLAIM 2.8. *For $\pi_1\hat{\mu}$-almost every $A \in \Sigma$, the Lyapunov exponents satisfy $\lambda^s(A) < 0 < \lambda^u(A)$.*

Remark 2.9. The above result is a consequence of the Ruelle–Margulis inequality [**23**, p. 669] *once* we prove that $\mu(\partial P) = 0$.

Proof. We establish the existence of a positive Lyapunov exponent μ-almost everywhere. The existence of a negative exponent will follow by considering T^{-1}. Let $\| \cdot \|_A$ be some measurable family of norms. Consider the family $\| \cdot \|'_A$, $A \in \Sigma$ defined by

$$\|v\|'_A := \|v\|_A / |W^u(A)|_A \quad \text{for } v \parallel W^u(A)$$

where $| \cdot |_A$ is the length with respect to $\| \cdot \|_A$ (using the affine structure). As $T(W^u(A)) \supset W^u(\sigma A)$, we have that $\|T' | E^u(A)\|'_A \geq 1$ (where $E^u(A)$ is the unstable direction at A: the *invariant* family of directions defined by $W^u(A)$) for μ-almost every $A \in \Sigma$. Here $T(W^u(A)) = W^u(\sigma A)$ μ-almost everywhere would imply $h(\hat{T}, \hat{\mu}) = H_{\hat{\mu}}(\hat{P} | \hat{P}^-) = 0$. Hence, $\|T | E^u(A)\|'_A > 1$ on a set of positive measure and

$$\lambda^u(A) = \int \log \|T' | E^u(B)\|'_B \, d\hat{\mu}(B) > 0$$

for μ-almost every $A \in \Sigma$. □

We finish the proof of Proposition 2.6.

Let $\mu \in \mathbb{P}^0_{\text{erg}}(T)$. Let $\hat{\mu}$ be a lift of μ to $\Sigma \ltimes M$. By Lemma 2.2, $h(\hat{T}, \hat{\mu}) = h(T, \mu) > 0$.

Claims 2.7 and 2.8 prove all of the claims of the proposition except $\mu(\partial P) = 0$.

Now, $W^u(A)$ and $W^s(A)$ are line segments μ-almost everywhere by Claim 2.7. Their directions carry distinct Lyapunov exponents by Claim 2.8, hence they must make a non-zero angle μ-almost everywhere. If $x \in \partial P$, then Tx or $T^{-1}x$ would be the end point of at least one of these line segments, a contradiction. Hence, $\mu(\partial P) = 0$. □

That $\mu(\partial P) = 0$ for all ergodic invariant probability measures with non-zero entropy has the following immediate but important consequence.

COROLLARY 2.10. *The partially defined map $\pi : \Sigma' \to M$*

$$\{\pi(A)\} := \bigcap_{n \geq 0} \overline{T^k[A_{-k} \cdots A_k]}$$

with Σ' the subset of Σ where the above intersection is indeed a single point, defines an entropy-preserving bijection between the sets of ergodic, invariant probability measures of T and of Σ with non-zero entropy.

2.3. *Semi-uniform estimates.* We obtain now more quantitative estimates, which we call *semi-uniform* in the sense that they are uniform on a set of uniformly lower-bounded mass for all large entropy measures. To state these results, we need the following 'distortion' estimate. By compactness of M and invertibility of T,

$$d(T) := \sup\left\{\log \frac{\|T'(x).u\|}{\|T'(x).v\|} \;\middle|\; x \in M,\ u, v \in \mathbb{R}^2 \setminus \{0\},\ \|u\| = \|v\|\right\} < \infty.$$

PROPOSITION 2.11. *For any $\mu_0 < h_{\rm top}(T)/d(T)$, there exist $h_0 < h_{\rm top}(T)$, $\theta_0 > 0$ and $\ell_0 > 0$ such that for any $\mu \in \mathbb{P}^{h_0}_{\rm erg}(T)$, the following properties occur jointly on a set of measure at least μ_0:*

$$\rho(x) := \min_{\sigma=s,u} d(x, \partial W^\sigma(x)) \geq \ell_0, \tag{2.1}$$

$$\alpha(x) := \angle(W^s(x), W^u(x)) > \theta_0. \tag{2.2}$$

Here $\angle(W^s(x), W^u(x))$ is the angle between the two lines defined by $W^s(x)$ and $W^u(x)$. We declare $\alpha(x) = \rho(x) = 0$, if $W^s(x)$ or $W^u(x)$ fail to be line segments.

Remark 2.12. In fact we obtain a number $\mu_0 < 1$ arbitrarily close to one satisfying (2.1). However, this is not the case for (2.2). We believe that this cannot be done. Indeed, one can easily build a smooth surface diffeomorphism with non-zero entropy such that for some $\mu_* > 0$ and $h_* > 0$, there are invariant probability measures with entropy at least h_* such that the stable and unstable directions make an arbitrarily small angle on a set of measure at least μ_*. We do not know whether these measures can be taken to have entropy arbitrarily close to the topological entropy or if piecewise affine examples exist.

We first prove the lower bound on angles by comparing the distortion with the entropy.

CLAIM 2.13. *For any $0 < h_1 < h_{\rm top}(T)$, there exists $\theta_1 > 0$ such that the set where $\alpha(x) > \theta_1$ has measure at least $h_1/d(T)$ for all measures $\mu \in \mathbb{P}^{h_1}_{\rm erg}$.*

The Ruelle–Margulis inequality applied to (T, μ) and (T^{-1}, μ) (which is valid as T' is constant on each element of P and $\mu(\partial P) = 0$) yields

$$h_{\rm top}(T) \leq \frac{\lambda^u(\mu) - \lambda^s(\mu)}{2} = \frac{1}{2}\int_M \log \frac{\|T'(x)|E^u(x)\|}{\|T'(x)|E^s(x)\|}\, d\mu(x). \tag{2.3}$$

By continuity there exists $\theta_1 > 0$ such that, for all $u, v \in \mathbb{R}^2 \setminus \{0\}$ with $\angle(u, v) \leq \theta_1$,

$$\text{for all } x \in M \setminus \partial P, \quad \log \frac{\|T'(x) \cdot u\|}{\|T'(x) \cdot v\|} \leq h_1.$$

Therefore, setting $m := \mu(\{x \in M : \alpha(x) > \theta_1\})$:

$$2h(T, \mu) \leq m \cdot d(T) + (1 - m) \cdot h_1$$

so that, assuming $h(T,\mu) > h_1$:

$$m \geq \frac{2h(T,\mu)-h_1}{d(T)-h_1} \geq \frac{2h(T,\mu)-h_1}{d(T)} > \frac{h_1}{d(T)}.$$

This proves Claim 2.13.

CLAIM 2.14. *For any* $\mu_3 < 1$, *there exists* $\ell_0 > 0$ *such that*

$$\text{for all } \mu \in \mathbb{P}^{h_3}_{\text{erg}}(T), \quad \mu(\{x \in M \mid d(x, \partial W^u(x)) > \ell_0\}) > \mu_3 \tag{2.4}$$

for $h_3 = h_{\text{top}}(T)(1-(1-\mu_3)/4)$.

To prove (2.4) let $\epsilon = (1-\mu_3)h_{\text{top}}(T)/2 > 0$, Λ be a Lipschitz constant for T, $n \geq \log 2/\epsilon$ be a large integer and $r = r(\epsilon, n) > 0$ be a small number such that

$$\max_{x\in M} \#\{A \in P^n \mid B(x,r) \cap A \neq \emptyset\} \leq \tfrac{1}{2}e^{(h_{\text{mult}}(T,P)+\epsilon)n} \quad \text{and} \quad \#P^n \leq e^{(h_{\text{top}}(T)+\epsilon)n}.$$

Let $\mu \in \mathbb{P}^{h_3}_{\text{erg}}(T)$, $X_0 := \{x \in M \mid d(x, \partial W^u(x)) \leq \Lambda^{-n} r\}$ and denote by $\mu|X_0$ the normalized restriction of μ to X_0. Using standard facts about entropy (see Appendix A) we obtain

$$\begin{aligned} nh(T,\mu) &= H_\mu(P^n|P^-) \leq H_\mu(P^n \vee \{X_0, M\setminus X_0\}|P^-) \leq H_\mu(\{X_0, M\setminus X_0\}) \\ &\quad + \mu(X_0)H_{\mu|X_0}(P^n|P^-) + (1-\mu(X_0))H_{\mu|M\setminus X_0}(P^n|P^-) \\ &\leq \log 2 + \mu(X_0) \sup_{x\in X_0} \log \#\{A \in P^n \mid A \cap X_0 \cap W^u(x) \neq \emptyset\} \\ &\quad + (1-\mu(X_0)) \log \#P^n \\ &\leq \log 2 + \mu(X_0)(h_{\text{mult}}(T,P)+\epsilon)n + (1-\mu(X_0))(h_{\text{top}}(T)+\epsilon)n. \end{aligned}$$

Hence,

$$\begin{aligned} h(T,\mu) &\leq (1-\mu(X_0))h_{\text{top}}(T) + \mu(X_0)h_{\text{mult}}(T,P) + \epsilon + \frac{1}{n}\log 2 \\ &= h_{\text{top}}(T) + 2\epsilon - \mu(X_0)(h_{\text{top}}(T) - h_{\text{mult}}(T,P)). \end{aligned}$$

implying that

$$\mu(X_0) \leq \frac{h_{\text{top}}(T) - h(T,\mu) + 2\epsilon}{h_{\text{top}}(T) - h_{\text{mult}}(T,P)} \leq \frac{h_{\text{top}}(T) - h(T,\mu) + 2\epsilon}{h_{\text{top}}(T)} < 1 - \mu_3$$

using $h_{\text{mult}}(T,P) = 0$ and $h(T,\mu) > h_3$. The claim is proved.

Proof of Proposition 2.11. Claim 2.13 gives $\theta_0 > 0$ such that (2.2) holds on a set of measure at least $h_{\text{top}}(T)/2d(T)$ with respect to all measures in $\mathbb{P}^{h_{\text{top}}(T)/2}_{\text{erg}}(T)$. Claim 2.14 applied to T and T^{-1} with $\mu_3 = 1 - h_{\text{top}}(T)/8d(T)$, shows that for

$$h_0 = h_{\text{top}}(T)\left(1 - \frac{1}{32}\frac{h_{\text{top}}(T)}{d(T)}\right) \geq h_{\text{top}}(T)/2,$$

(2.2) and (2.1) hold jointly on a set of measure at least $h_{\text{top}}(T)/4d(T)$ with respect to all measures in $\mathbb{P}^{h_0}_{\text{erg}}(T)$. □

3. *Construction of the Markov structure*

Roughly speaking, the estimates of the previous section allow us to build a collection of (non-uniform) 'Markov rectangles' which will 'control enough' of the dynamics to analyze all measures of large entropy.

3.1. *Markov rectangles*

Definition 3.1. A (*Markov*) *rectangle* is a closed topological disk R contained in an affine chart and bounded by four line segments, alternatively included in stable and unstable manifolds, making the unstable boundary, $\partial^u R = \partial_1^u R \cup \partial_2^u R$, and the stable boundary, $\partial^s R = \partial_1^s R \cup \partial_2^s R$, respectively. See Figure 2.

A *Markov array* is a finite collection of Markov rectangles with disjoint interiors.

Not every passage of an orbit inside a rectangle is useful. We need the following properties.

Definition 3.2. A point x is *controlled* by a rectangle R if x is nice, belongs to R and if $W^s(x)$ and $W^u(x)$ each intersects ∂R in two points. Note that control depends on the partition P used to define $W^u(x)$ and $W^s(x)$. If necessary we speak of *control with respect* to P.

Here $x \in R$ is 10-*controlled* if, moreover, $\rho(x) > 10 \operatorname{diam} R$ where ρ was defined in (2.1); $x \in R$ is s-controlled if x is nice, $x \in R$ and $W^s(x)$ intersects ∂R in two points.

The set of controlled (respectively 10-controlled, s-controlled) points is denoted by $\kappa(R)$ (respectively $\kappa_{10}(R)$, $\kappa_s(R)$).

A point is controlled by a Markov array $\mathcal{R}$ if it is controlled by one of the rectangles of the array. We define $\kappa(\mathcal{R})$, $\kappa_{10}(\mathcal{R})$, $\kappa_s(\mathcal{R})$ in the obvious way.

Using the previous lower bounds on the lengths and angles of invariant manifolds we first prove the following lemma.

LEMMA 3.3. *There exist numbers $h_0 < h_{\mathrm{top}}(T)$ and $\mu_0 > 0$ and a Markov array $\mathcal{R}$ such that for all $\mu \in \mathbb{P}_{\mathrm{erg}}^{h_0}(T)$,*

$$\mu(\kappa_{10}(\mathcal{R})) > \mu_0.$$

Our analysis requires the following slightly stronger statement (i.e. we only tolerate 'small holes').

LEMMA 3.4. *There is $\mu_0 > 0$ such that for any $\epsilon_0 > 0$, there exist a number $h_0 < h_{\mathrm{top}}(T)$ and a Markov array $\mathcal{R}$ such that for any $\mu \in \mathbb{P}_{\mathrm{erg}}^{h_0}(T)$:*

- $\mu(\kappa_{10}(\mathcal{R})) > \mu_0$;
- $\mu(\mathcal{R} \setminus \kappa_{10}(\mathcal{R})) < \epsilon_0 \mu_0$.

This will be obtained by subdividing the rectangles in the Markov array from Lemma 3.3 into sub-rectangles that are much smaller than most stable/unstable manifolds.

The final twist is that as we replace the partition P by the convex partition $P^{\mathcal{R}}$ generated by P and the Markov array $\mathcal{R}$ (see Figure 1), some invariant manifolds may shrink, say $\tilde{W}^u(x) := \bigcap_{n\geq 1} \overline{T^n P^{\mathcal{R}}(T^{-n}x)} \subsetneq W^u(x)$, diminishing the set of controlled points. Indeed, $\tilde{W}^u(f(x)) \subsetneq W^u(f(x))$ when $W^u(x)$ crosses the boundary of a rectangle from

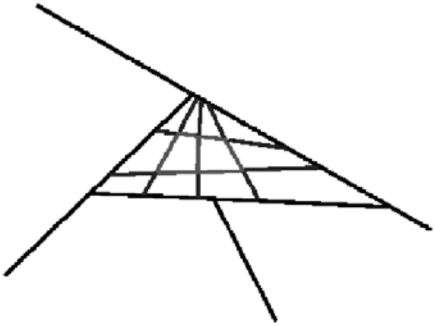

FIGURE 1. The convex partition $\tilde{P}$ refining both P (outside lines) and $\mathcal{R}$ (the two quadrilaterals at the center).

$\mathcal{R}$ before crossing $\partial P(x)$. We shall see, however, that if these intersections are sufficiently separated in time, then $\tilde{W}^u(x) = W^u(x)$ for most points $x \in \mathcal{R}$ with respect to large entropy measures. To guarantee that large separation, we use the following construction.

Definition 3.5. If $\mathcal{R}$ is an array of Markov rectangles contained in an element of P and L is a positive integer, then the $(\mathcal{R}, L)$-*extension* of $(M, T, P, \mathcal{R})$ is $(M_+, T_+, P_+, \mathcal{R}_+)$, defined in the following way:

- $M_+ = M \times \{0, \dots, L-1\}$;
- $T_+(x, k) = (Tx, k+1 \pmod L)$;
- P_+ is the finite partition of M_+ which coincides with a copy of P on each $M \times \{k\}$ for $k \neq 0$ and coincides on $M \times \{0\}$ with a copy of $P^{\mathcal{R}}$;
- $\mathcal{R}_+ = \{R \times \{0\} \mid R \in \mathcal{R}\}$.

The conclusion of this section is the following result.

PROPOSITION 3.6. *Let (M, T, P) be a piecewise affine surface homeomorphism with non-zero entropy. There exist $\mu_0 > 0$, $h_0 < h_{\mathrm{top}}(T)$ such that for any $\epsilon_0 > 0$, there is a Markov array $\mathcal{R}$ and a positive integer L_0 with the following properties. Fix any $L_+ \geq L_0$ and let $(M_+, T_+, P_+, \mathcal{R}_+)$ be the $(\mathcal{R}, L_+)$-extension of $(M, T, P, \mathcal{R})$.*

For each $\mu \in \mathbb{P}^{h_0}_{\mathrm{erg}}(T)$, there exists an ergodic invariant probability measure μ_+ of T_+ with $\pi(\mu_+) = \mu$ (where $\pi(x, k) = x$) such that:

(i) $L_+ \cdot \mu_+(\kappa_{10}(\mathcal{R}_+)) > \mu_0$;

(ii) $L_+ \cdot \mu_+(\mathcal{R}_+ \setminus \kappa_{10}(\mathcal{R}_+)) < \epsilon_0 \cdot \mu_0$.

The above controlled sets are defined with respect to the invariant manifolds relative to P_+ (which contains the Markov array $\mathcal{R}_+$).

Note that it is enough to prove our results (Theorem 1 and Propositions 1.5 and 1.4) for some periodic extension.

We now prove Lemmas 3.3 and 3.4 and Proposition 3.6. We begin by the following lemma.

LEMMA 3.7. *Given $\ell_0 > 0$ and $0 < \theta_0 < 2\pi$, there exists a finite collection of rectangles $R^{(1)}, \dots, R^{(Q)}$ such that:*

(1) *$\mathrm{diam}(R^{(i)}) < \ell_0/10$;*
(2) *any $x \in M$ with $\rho(x) > \ell_0$ and $\angle(W^u(x), W^s(x)) > \theta_0$ belongs to at least one $R^{(i)}$.*

This easily implies Lemma 3.3 using Proposition 2.11 and observing that the finite collection of rectangles above can be subdivided by boundary lines such as those of Fact 3.9 below so that their interiors become disjoint, defining the required Markov array $\mathcal{R}$.

Proof of Lemma 3.7. Let

$$K_* := \{x \in M \mid \rho(x) > \ell_0 \text{ and } \angle(W^u(x), W^s(x)) > \theta_0\}.$$

Let $\{K_j\}_{j=1}^{Q}$ be a finite partition of K_* whose elements have diameter less than $\theta_0\ell_0/100$ and lie within an affine chart of M. We fix j.

Recall that the collection of closed subsets $\mathcal{K}$ of the compact metric space M is a compact space with respect to the Hausdorff metric:

$$d(A, B) = \inf\{\epsilon > 0 \mid A \subset B(B, \epsilon) \text{ and } B \subset B(A, \epsilon)\}$$

where $B(A, \epsilon)$ is the ϵ-neighborhood of the set A, i.e. $\{x \in M \mid d(A, x) < \epsilon\}$.

The easy proofs of the following two facts are left to the reader.

FACT 3.8. *Let $A^n \in \Sigma(T, P)$ converge to A_+. By taking a subsequence, $W^s(A^n)$ also converges in the Hausdorff metric, say to $H \subset M$. Then $H \subset W^s(A_+)$.*

FACT 3.9. *Assume that K_j is as above. Then there exist two points $x_1, x_2 \in \overline{K_j}$, two non-trivial line segments L_1, L_2 and two itineraries $A^1, A^2 \in \Sigma(T, P)$ with the following properties:*

- *L_i is contained in the boundary of $W^s(A^i)$ as a subset of M;*
- *in some affine chart, K_j lies between the two lines supporting L_1 and L_2.*

We call (L_1, L_2) a pair of stable boundary lines of K_j.

We now prove Lemma 3.7. Consider two distinct one-dimensional stable manifolds $W^s(A)$ and $W^s(B)$ which intersect in a single point p. The point p must be the endpoint of at least one of them: otherwise, if $A_n \neq B_n$, then $p \in \partial A_n \cap \partial B_n$ and both $W^s(A)$ and $W^s(B)$ are parallel to the same segment of $\partial A_n \cap \partial B_n$. Thus, their intersection contains a non-trivial line segment.

Observe that if $x, y \in K_j$, $W^u(x)$ and $W^u(y)$, which are line segments, must have disjoint relative interiors or be parallel and overlapping. Thus,

$$\angle(W^u(x), W^u(y)) \leq \theta_0/50.$$

As $\angle(W^u(z), W^s(z)) > \theta_0$ for all $z \in K$, we obtain

$$\angle(W^u(x), W^s(y)) > \theta_0/2.$$

Consider a pair of 'stable boundary lines' (respectively 'unstable boundary lines') given by Fact 3.9 applied to (T, K_j) (respectively applied to (T^{-1}, K_j)). Let $R^{(j)}$ be the rectangle bounded by these four line segments; $R^{(j)}$ is contained in the intersection of two strips with almost parallel sides of width $\leq \mathrm{diam} K_j$ and making an angle at least $\theta_0/2$. Hence,

$$\mathrm{diam}(R^{(j)}) < 5\,\mathrm{diam}(K_j)/\theta_0 < \ell_0/20.$$

On the other hand, $R^{(j)} \supset K_j$, hence $\bigcup_j R^{(j)} \supset K_*$. □

Proof of Lemma 3.4. Apply Lemma 3.3 to obtain $\mathcal{R}$, $\mu_0 > 0$ and $h_0 < h_{\text{top}}(T)$. Recall that $\rho(x)$ is the distance between x and the endpoints of its invariant manifolds (or zero if one of those is not a line segment).

By Claim 2.14 applied with $\mu_3 := 1 - \epsilon_0\mu_0/2$ to T and T^{-1}, there exist $h_1 < h_{\text{top}}(T)$ and $\ell_1 = \ell_1(\epsilon_0\mu_0)$ such that

$$\text{for all } \mu \in \mathbb{P}^{h_1}_{\text{erg}}(T), \quad \mu(\{x \in M \mid \rho(x) < \ell_1\}) < \epsilon_0\mu_0.$$

Let us cut each large rectangle R from $\mathcal{R}$ into sub-rectangles R' with diameter at most $\ell_1/10$, obtaining a new Markov array $\mathcal{R}'$. Using Fact 3.9 again, we can do this by finitely many stable and unstable manifolds (or line segments bounding those). Observe that $\kappa_{10}(\mathcal{R}') \supset \kappa_{10}(\mathcal{R})$ and that the points in $\mathcal{R}' \setminus \kappa_{10}(\mathcal{R}')$ which have line segments as invariant manifolds are ℓ_1-close to an endpoint of their stable/unstable manifold. Hence,

$$\mu(\kappa_{10}(\mathcal{R}')) \geq \mu(\kappa_{10}(\mathcal{R})) \geq \mu_0 \quad \text{and} \quad \mu(\mathcal{R}' \setminus \kappa_{10}(\mathcal{R}')) \leq \epsilon_0\mu_0$$

for all $\mu \in \mathbb{P}^{h_1}_{\text{erg}}(T)$. □

Proof of Proposition 3.6. We apply Lemma 3.4 with $\epsilon_0/2$ obtaining $\mu_0 > 0$ (independent of ϵ_0), $h_0 < h_{\text{top}}(T)$ and a Markov array $\mathcal{R}$. Let $P^{\mathcal{R}}$ be the convex partition defined previously. We go to the $(\mathcal{R}, L_+)$-extension (M_+, T_+, P_+) of (M, T, P) for some large integer L_+ to be specified. As we have observed, there always exists an ergodic, T_+-invariant measure μ_+ extending μ. After replacing it by its image under $(x, i) \mapsto (T^j x, i + j \mod L_+)$ for some constant $j \in \{0, 1, \ldots, L_+ - 1\}$, we may obtain an ergodic extension μ_+ such that

$$L_+ \cdot \mu_+(\kappa_{10}(\mathcal{R}) \times \{0\}) \geq \mu(\kappa_{10}(\mathcal{R})) \geq \mu_0.$$

As the extension is finite-to-one, μ_+ has the same entropy as μ.

Let $x \in M$. If the unstable manifold for T_+,

$$W^u_+(x, 0) := \bigcap_{n \geq 1} \overline{T^n_+ P_+(T^{-n}_+(x, 0))},$$

is strictly shorter than $W^u(x) \times \{0\}$, then it is bounded by $T^{kL_+}_+(y, 0)$ with y an intersection point of $W^u(T^{-kL_+}x)$ for some $k \geq 1$, with one of the new boundary segments, I, of $P^{\mathcal{R}}$. Hence, $P^{kL_+}(T^{-L}x)$ is determined by the past of $T^{-kL_+}x$ and I picked among finitely many choices. Note that this number of choices depends only on P and $\mathcal{R}$ *but not on* L_+.

A standard counting argument shows that if this happened on a subset of $M \times \{0\}$ with μ_+-measure at least $\frac{1}{2}\epsilon_0\mu_0 \cdot L_+^{-1}$, then

$$h(T, \mu) = h(T_+, \mu_+) \leq (1 - \tfrac{1}{2}\epsilon_0\mu_0)h_{\text{top}}(T) + \epsilon(L_+)$$

where $\epsilon(L) \to 0$ as $L \to \infty$. This is strictly less than $h_{\text{top}}(T)$ if L_+ is large enough (which we ensure by taking L_0 large). So it is excluded for large entropy measures. The $(\mathcal{R}, L)$-periodic extension $(M_+, T_+, P_+, \mathcal{R}_+)$ has the required properties for all large integers L_+. □

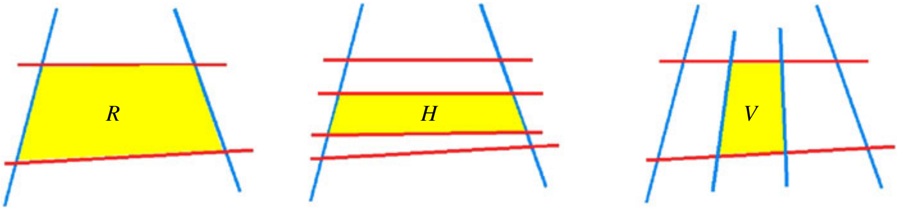

FIGURE 2. From left to right: a rectangle R, an s-rectangle H and a u-rectangle V. The (approximately) red horizontal (respectively blue vertical) line segments are segments of stable (respectively unstable) manifolds.

3.2. *Hyperbolic strips.* The homeomorphism (M, T, P) is some piecewise affine surface homeomorphism with non-zero entropy and $\mathcal{R}$ is some Markov array with $\mathcal{R} \subset P$ (eventually $(M, T, P, \mathcal{R})$ will be the previously built periodic extension $(M_+, T_+, P_+, \mathcal{R}_+)$). We use Figure 2 to define finite itineraries that can be freely concatenated. This is adapted from uniformly hyperbolic dynamics.

Definition 3.10. A quadrilateral Q *u-crosses* a rectangle $R \in \mathcal{R}$ if $Q \subset R$ and its boundary is the union of two subsegments of the stable boundary of R (the stable boundary of Q) and two line segments (the unstable boundary of Q), these four segments being pairwise disjoint, except for their endpoints. An *s-crossing* is defined similarly.

A *u-rectangle* is a quadrilateral which u-crosses some rectangle $R \in \mathcal{R}$ and whose unstable boundary is made of two segments of unstable manifolds. An *s-rectangle* is defined similarly (see Figure 2).

For $n \geq 1$, a *hyperbolic n-strip* (or just n-strip) is an s-rectangle S such that int $T^k(S)$ is included in some element of P for each $k = 0, \ldots, n-1$ and $T^n(S)$ is a u-rectangle. A *hyperbolic strip* is an n-strip for some $n \geq 1$.

We write $P_a^b(x)$ for $\bigcap_{k=a}^b \overline{T^{-(k-a)} P(T^k x)}$ (we assume implicitly that x is nice—this fails only on an entropy-negligible set by Proposition 2.6).

FACTS 3.11. *The following is immediate.*

(1) *A hyperbolic n-strip is necessarily of the form $P_0^n(x)$ for some $x \in R$.*

(2) *Two hyperbolic strips are either nested or have disjoint interiors.*

We now give some tools to build hyperbolic strips.

LEMMA 3.12. *For $0 < m < n$, if $P_0^m(x)$ and $P_m^n(x)$ are both hyperbolic strips, then so is $P_0^n(x)$.*

This is easy to show using Figure 3. Sufficiently long invariant manifolds allow the construction of hyperbolic strips from scratch.

LEMMA 3.13. *Let $x \in \kappa_{10}(\mathcal{R})$ and $n \geq 1$ such that $T^n x \in \kappa_{10}(\mathcal{R})$. Then $P_0^n(x)$ is a hyperbolic strip.*

Observe that the weaker condition $x \in \kappa(\mathcal{R}) \cap T^{-n}\kappa(\mathcal{R})$ does not imply that $P_0^n(x)$ is a hyperbolic strip.

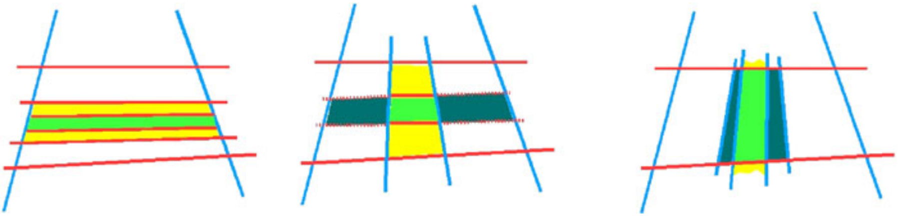

FIGURE 3. Proof of Lemma 3.12: (a) $P_0^n(x) \subset P_0^m(x) \subset R$; (b) the u-rectangle $T^m(P_0^m(x))$ crossing the s-rectangle $P_m^n(x)$ (both inside R'); (c) $T^n P_0^n(x) \subset T^m P_m^n(x) \subset R''$ $(R, R', R'' \in \mathcal{R})$.

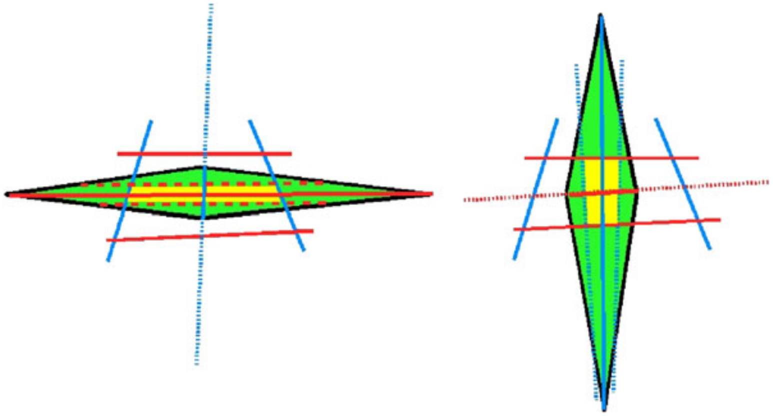

FIGURE 4. Construction of the hyperbolic strip in the proof of Lemma 3.13 (left: time zero around R; right: time n around R'; approximately (vertical) horizontal lines are segments of (un)stable manifolds; dashed lines are (pre)images of regular manifolds. The outer (green) 'diamond' is L. The central (yellow) rectangle inscribed in L is the hyperbolic strip.

Proof. See Figure 4. Let R, R' be the elements of $\mathcal{R}$ containing x and $T^n x$. Consider the diamond (the quadrilateral) L generated by $W^s(x)$ and $T^{-n}W^u(T^n x)$. By convexity, L is contained in $T^{-1}P_1^n(x)$.

Consider one side $[uv]$ of $T^n L$ with $u \notin R'$ and $d(x, u) > 10 \cdot \operatorname{diam}(R')$. Let $\{a\} := [uv] \cap \partial^s R'$. We have $d(u, v) \geq d(u, T^n x) - \operatorname{diam}(R') \geq 9\operatorname{diam}(R')$. Hence, $d(v, a) \leq \operatorname{diam}(R') \leq (1/9)d(u, v)$.

Let $(abcd)$ be the quadrilateral defined by the points of $\partial T^n L \cap \partial R'$. It u-crosses R'. It remains to prove that L s-crosses R so that $R \cap T^{-n}Q$ is the desired hyperbolic strip and must be $P_0^n(x)$.

It is enough to check that $T^{-n}\{a, b, c, d\}$ lies outside of R. The map being piecewise affine, $d(T^{-n}v, T^{-n}a) \leq (1/9)d(T^{-n}u, T^{-n}v)$. Hence, $d(x, T^{-n}v) \geq 10 \cdot \operatorname{diam}(R)$, so $T^{-n}a$ is outside of R. The same holds for the pre-images of b, c, d. □

COROLLARY 3.14. *Let $n \geq 1$ be such that $P_0^n(x)$ is a hyperbolic strip and $T^n x \in \kappa_{10}(\mathcal{R})$. If $m > n$ satisfies $T^m x \in \kappa_{10}(\mathcal{R})$, then $P_0^m(x)$ is also a hyperbolic strip.*

Proof. By Lemma 3.13, $P_n^m(x)$ is a hyperbolic strip. Apply Lemma 3.12 to conclude. □

We need the following technical fact.

LEMMA 3.15. *Let μ be an atomless invariant probability measure. For μ-almost every $x \in \kappa_s(R)$, for some $R \in \mathcal{R}$ and all $n \geq 1$, the intersection of $W^s(x)$ with $\partial^u R$ is disjoint from all of the vertices of P^n, $n \geq 1$. In particular, $\partial^u R \cap \partial P^n(x)$ is the union of two non-trivial segments.*

Proof. We proceed by contradiction assuming that the above fails: on a subset of $\kappa_s(R_i)$ with positive measure at least one of these intersection points coincides with a vertex z of the polygon $\partial P^n(x)$ (so $W^s(z) = W^s(x)$). Reducing this subset, we assume the vertex z to be fixed, say z_+.

By Poincaré recurrence, there must exist infinitely many $n \geq 0$ such that $T^n x \in W^s(z_+)$. Considering two such integers $n_1 < n_2$, we obtain that $T^{n_2 - n_1}(W^s(z_+)) \subset W^s(z_+)$. This implies that all points of $W^s(z_+)$ converge to a periodic orbit. Thus, the ergodic decomposition of μ has an atom, a contradiction. □

We show that if $x \in \kappa_s(\mathcal{R})$, then subsequent visits to $\kappa_{10}(\mathcal{R})$ either give a hyperbolic strip or a shadowing property which will lead to an entropy bound.

LEMMA 3.16. *Let $x \in \kappa_s(\mathcal{R})$ and $0 \leq m < n$ be such that $T^m x$, $T^n x \in \kappa_{10}(\mathcal{R})$. Excluding a set of zero measure of points x, if $P_0^n(x)$ is not a hyperbolic strip then $P_0^m(x)$ determines $P_0^n(x)$ up to a choice of multiplicity four.*

Proof. We know that $W^s(x)$ crosses the rectangle $R \in \mathcal{R}$ containing x. Hence, Lemma 3.15 implies that $\partial P_0^m(x) \cap \partial^u R$ is the union of two unstable, non-trivial segments: $[a, b]$, $[c, d]$. Let $[a', b']$, $[c', d']$ be their images by $T^m | P_0^m(x)$. Let Q' be the quadrilateral generated by them. By convexity $Q' \subset T^m(P_0^m(x))$.

Here $H := P_m^n(x)$ is a hyperbolic strip by Lemma 3.13. We know that Q' and H intersect. If $\operatorname{int} H \cap \{a', b', c', d'\} = \emptyset$, then Q' would go across H, and $P_0^m(x)$ would be a hyperbolic strip, contrary to assumption.

Thus, at least one of the four vertices a, b, c, d determined by $P_0^m(x)$ is contained in $\operatorname{int} H$: this point determines H and therefore $P_0^n(x)$ as claimed. □

3.3. *Admissible strips and good returns.*

In this section, $\mathcal{R}$ is some Markov array with $\mathcal{R} \subset P$. Hyperbolic strips defined above have no uniqueness property: a point $x \in \kappa_s(\mathcal{R})$ sits in an infinite sequence of nested hyperbolic strips. This motivates the following notion.

Definition 3.17. For $n \geq 1$, the *admissible n-strips* are defined by induction on n. A 1-strip is always admissible. For $n > 1$, an admissible n-strip S is an n-strip such that for all $1 \leq m < n$ such that S is included in an admissible m-strip, $T^m(S)$ meets the interior of no hyperbolic strip. An *admissible strip* is an admissible n-strip for some $n \geq 1$.

Figure 5 shows a hyperbolic n-strip S (hatched) which is not admissible.

Definition 3.18. Let (M, T, P) be a piecewise affine surface homeomorphism with a Markov array contained in P.

For a point $x \in M$, the (*good*) *return time* is $\tau = \tau(x)$, the minimal integer $\tau \geq 1$ such that both of the following conditions hold:

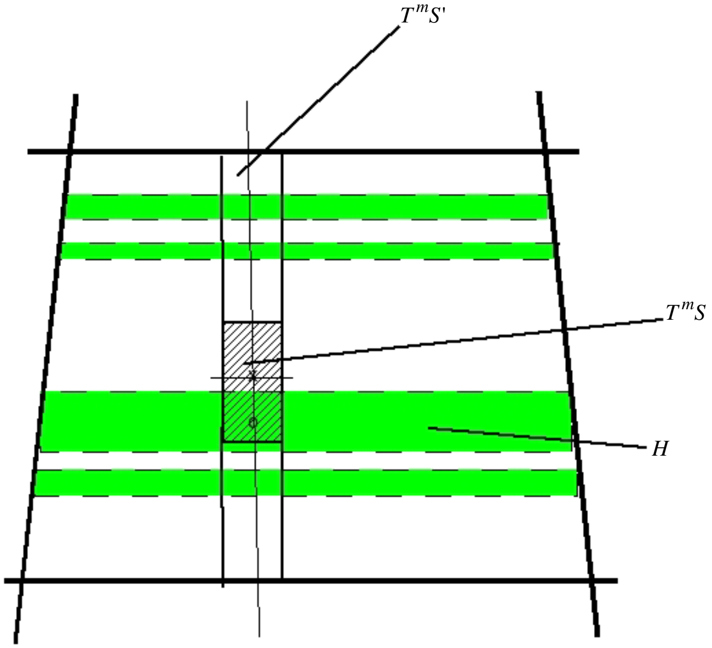

FIGURE 5. An example of a non-admissible n-strip S. For some $0 < m < n$, S is contained in some m-strip S' (hence $T^m S'$ u-crosses) and $T^m S$ meets the interior of some hyperbolic strip H. The above represents $T^m S \subset T^m S'$ and H. Point x is at the crossing of the stable and unstable lines and point o is slightly below, on the same unstable line. The four (green) s-crossing rectangles are the maximum hyperbolic strips. In such a situation, it might be possible to then split the itinerary of S into that of S' followed by that of H, yielding a choice for representing the itinerary of any point like o above, at each of its visit to S. Compare with Lemma 3.22.

- x belongs to an admissible τ-strip;
- $T^\tau(x) \in \kappa_s(\mathcal{R})$.

These conditions are defined *with respect to* some partition containing some Markov array.

If there is no such integer τ, then we set $\tau(x) = \infty$.

Remark 3.19. Note that, at this point, we break the symmetry between the future and the past.

We shall use repeatedly the following obvious observation.

FACT 3.20. *If n is the smallest integer such that $P_0^n(x)$ is a hyperbolic strip (equivalently: $P_0^n(x)$ is an n-strip which is not contained in a k-strip for any $k < n$; $P_0^n(x)$ does not meet a k-strip distinct from $P_0^k(x)$ for any $k < n$; $P_0^n(x)$ is a hyperbolic strip which is maximum with respect to inclusion) then $P_0^n(x)$ is an admissible n-strip.*

Remark 3.21. One could consider the following changes in the definition of admissibility.

(i) Replacing '$T^m(S)$ meets no hyperbolic strip' by '$T^m(S)$ meets no admissible strip' would not change the notion. Indeed, suppose that $T^m(S)$ meets a hyperbolic strip H. Let $k \geq 1$ be the smallest integer such that H is contained in a k-strip, say H_k. The minimality of k implies that H_k is admissible and $H_k \supset H$ so that $T^m(S)$ meets H_k.

(ii) Replacing 'S is included in an admissible m-strip' by 'S is included in a *hyperbolic* m-strip' would exclude some admissible strips and so would cause a problem in the proof of the (key) Claim 4.2 (for the proof that $k = n_i$ in the notation there).

Admissibility gives the following uniqueness property. Denote the one-sided symbolic dynamics by $\Sigma_+(T, P) = \{A_0 A_1 A_2 \cdots \in P^{\mathbb{N}} \mid A \in \Sigma(T, P)\}$.

LEMMA 3.22. *A positive itinerary $A \in \Sigma_+(T, P)$ can be decomposed into at most one way as an infinite concatenation of admissible strips.*

Proof. Consider two distinct decompositions of A into admissible strips, that is, $n_0 = 0 < n_1 < n_2 < \cdots$ and $m_0 = 0 < m_1 < m_2 < \cdots$, such that $A_{n_i} \cdots A_{n_{i+1}}$ and $A_{m_i} \cdots A_{m_{i+1}}$ are admissible strips for all $i \geq 0$. By deleting the identical initial segments, we can assume that the decompositions differ from the beginning, say $n_1 < m_1$. It follows that the admissible m_1-strip $H := \overline{[A_0 \cdots A_{m_1}]}$ is contained in the n_1-admissible strip $\overline{[A_0 \cdots A_{n_1}]}$. Thus, $T^{n_1}(H)$ meets $\overline{[A_{n_1} \cdots A_{n_2}]}$ which is another admissible strip, contradicting admissibility. □

4. *Analysis of large return times*

In this section (M, T, P) is a piecewise affine homeomorphism with positive topological entropy. We first analyze the implications of a long return time $\tau(x)$ from a geometric and then a combinatorial point of view. We then apply this to invariant measures with very large 'average' return times to bound the entropy of these measures.

4.1. *Geometric analysis.*

We analyze geometrically the implications of a large return time.

PROPOSITION 4.1. *Let (M, T, P) be a piecewise affine surface homeomorphism and let $\mathcal{R} \subset P$ be a Markov array. Let $x \in \kappa_s(\mathcal{R})$ and let $1 \leq N \leq \tau(x)$.*

Let $0 \leq N_1 \leq N_2 \leq N_0 < N$ be defined as follows:

- *$0 < N_0 < N$ is the smallest integer such that $T^{N_0}x \in \kappa_{10}(\mathcal{R})$ and $P_0^{N_0}(x)$ is a hyperbolic strip (we set $N_0 := N$ if there is no such integer);*
- *$0 \leq N_1 < N$ is the smallest integer such that $T^{N_1}x \in \kappa_{10}(\mathcal{R})$ (we set $N_1 := N_0 = N$ if there is no such integer);*
- *$0 \leq N_2 < N_0$ is the largest integer such that $T^{N_2}x \in \kappa_{10}(\mathcal{R})$ (we set $N_2 := N_1 = N_0$ if there is no such integer);*
- *$n_1, \ldots, n_r$ ($r \geq 0$) are the admissible times, that is the successive integers in*
$$\{0 \leq k < N \mid P_0^k(x) \text{ is an admissible strip}\}$$
with the convention $n_{r+1} = N$;
- *$m_{i1}, \ldots, m_{is(i)}$ ($s(i) \geq 0$) are the hyperbolic times, that is, for each i, the successive integers*
$$\{n_i < m < n_{i+1} \mid P_0^m(x) \text{ is a } m\text{-strip and } T^m x \in \kappa_s(\mathcal{R})\}$$
with the convention $m_{is(i)} := m_{i1} := n_{i+1}$ and $s(i) = 0$ if the above set is empty.

Then $P^N(x)$ is determined, up to a choice of multiplicity $4 \cdot 2^r$, by:

(1) *the integers* N_1, N_2, r *and* n_i, m_{i1}, $m_{is(i)}$ *for* $1 \leq i \leq r$;

(2) $P(T^k x)$ *for* $k \in [\![0, N_1]\!] \cup [\![N_2, n_1[\![$;

(3) $P(T^k x)$ *for* $k \in \bigcup_{i=1}^r [\![n_i, m_{i1}]\!] \cup [\![m_{is(i)}, n_{i+1}]\!] \setminus [\![0, N_1]\!]$.

(Here $[\![a, b[\![$ *denotes the integer interval with a included and b excluded, etc.)*

Proof. The inequality $0 \leq N_1 \leq N_2 \leq N_0$ is checked easily.

We can assume that $N_1 < N$, as otherwise there is nothing to show.

We first claim that $P_0^{N_1}(x)$ determines $P_0^{N_2}(x)$ up to a choice of multiplicity four.

If $N_1 \geq N_0$, then $N_1 = N_2 = N_0$ and there is nothing to show. Otherwise $N_1 \leq N_2 < N_0$ and $T^{N_1}x, T^{N_2}x \in \kappa_{10}(\mathcal{R})$. As $P_0^{N_2}(x)$ is not hyperbolic this implies, by Lemma 3.16, the above claim.

It remains to prove the following claim. □

CLAIM 4.2. *Except for an entropy-negligible subset of points* $x \in M$, *the following holds. Given some* $1 \leq i \leq r$ *with* $s(i) > 0$, $Q := P_{n_i}^{m_{i1}}(x)$ *and integers* n_i, m_{i1}, m_{is}, *there are only two possibilities for* $P_{n_i}^{m_{is}}(x)$ *(s denotes* $s(i)$*).*

Proof. Let $R, R' \in \mathcal{R}$ be the rectangles containing $T^{n_i}x, T^{m_{i1}}x$ and let ℓ be the line segment through $T^{n_i}x$, directed by $W^s(T^{n_i}x)$ and bounded by ∂R. We first show that $\ell \not\subset Q$.

We have $T^{n_i}x \notin \kappa_s(R)$ as $\tau(x) > n_i$. Thus, $W^s(T^{n_i}x)$ does not s-cross R: $\ell \not\subset W^s(T^{n_i}x)$. There exists $k > n_i$ such that $T^{k-n_i}\ell$ is not contained in the closure of an element of P. Take $k \geq 1$ minimal. If one had $k > m_{i1}$, then $T^{m_{i1}-n_i}\ell \subset W^s(T^{m_{i1}}x)$ (recall that $T^{m_{i1}}x \in \kappa_s(R')$) so that for all $k \geq m_{i1}$, $T^{k-n_i}\ell$ would be contained in an element of P, implying $\ell \subset W^s(T^{n_i}x)$, a contradiction. Thus, $k \leq m_{i1}$ and $\ell \not\subset Q$ as claimed.

Disregarding an entropy-negligible set of points x, we can assume that ℓ divides Q into two subsets with non-empty interiors, say Q_+, Q_-. There cannot exist stable manifolds that s-cross R both in Q_+ and Q_-: by convexity this would imply that $W^s(T^{n_i}x)$ also s-crosses. Thus, there is an s-rectangle in R disjoint from $\kappa_s(R)$ which contains at least one of Q_+, Q_-. Let $W^s(B_+)$ ('above') and $W^s(B_-)$ be the stable manifolds bounding this gap (recall Fact 3.8). Also at least one of $W^s(B_\pm)$ (say $W^s(B_+)$) is not contained in Q so the interior of Q does not meet $W^s(B_+)$. Thus, Q determines $W^s(B_+)$: it is the 'lowermost' stable manifold 'above' Q which crosses R. Likewise $W^s(B^-)$ is the 'uppermost' stable manifold 'below' $W^s(B^+)$ which crosses R.

By definition, $\mathcal{S} := P_0^{m_{is}}(x)$ is hyperbolic. Also $m_{is} \in]\!]n_i, n_{i+1}[\![$ is not admissible, hence there exists $0 < k < m_{is} < n_{i+1}$ such that $\mathcal{S}$ is included in an admissible k-strip and $T^k(\mathcal{S})$ meets an admissible strip. If $k < n_i$, then the same admissible strip would preclude the admissibility of $P_0^{n_i}(x)$ is admissible. The contradiction shows that $k \geq n_i$. As n_{i+1} is the smallest admissible time after n_i, $k = n_i$.

Thus, $T^{n_i}(\mathcal{S})$ meets an admissible strip which must be either 'above' $W^s(B_+)$ or 'below' $W^s(B_-)$. This implies that $P_{n_i}^{m_{is}}(x)$ meets, and therefore contains, $W^s(B_\pm) \cap R$ (for one of the signs ±). It follows that Q determines $P_{n_i}^{m_{is}}(x)$, up to a binary choice. □

4.2. *Combinatorial estimates*

Remark 4.3. In the remainder of this section, the controlled sets and return times are understood to be with respect to the partition P_+ and the Markov array $\mathcal{R}_+ \subset P_+$ of some periodic extension as defined in Definition 3.5.

We extract from Proposition 4.1 the following complexity bound.

PROPOSITION 4.4. *Let (M, T, P) be a piecewise affine surface homeomorphism and let $\mathcal{R}$ be a Markov array. Let $\epsilon_* > 0$ and let $C_* = C_*(\epsilon_*) < \infty$ be such that*

$$\text{for all } n \geq 0 \;\; \#(P^{\mathcal{R}})^n \leq C_* e^{(h_{\text{top}}(T)+\epsilon_*)n}.$$

For each positive integer L, let (M_+, T_+, P_+) be the $(L, \mathcal{R})$-extension of (M, T, P).

Let $L, N \geq 1$, $M, R, S \geq 0$ be some integers. Consider $\mathcal{I} = \mathcal{I}(N, L, M, R, S)$ the set of cylinders $(P_+)^N(x)$ for $x \in \kappa_s(\mathcal{R}_+)$ such that, in the notation of Proposition 4.1 applied to the periodic $(L, \mathcal{R})$-extension:

- $\tau(x) \geq N$;
- $r = R$ *and* $\#\{m_{ij} \geq N_0 \mid 1 \leq i \leq r, 1 \leq j \leq s(i)\} = S$;
- $N_2 - N_1 = M$.

Let $\rho > R/N$. Then

$$\log \#\mathcal{I} \leq (h_{\text{top}}(T) + \epsilon_*)(N - L(S - R) - M + S) + K_*(\rho, N)N + (\rho + 3/N)N \log C_*$$

where $K_*(\cdot)$ and $K_*(\cdot, \cdot)$ are universal† *functions satisfying $K_*(\rho, N) \downarrow K_*(\rho)$ when $N \to \infty$ and $K_*(\rho) \downarrow 0$ when $\rho \to 0$.*

The proof of the above result uses the following lemma.

LEMMA 4.5. *In the notation of Proposition 4.1:*

(1) *n_1 is the smallest integer such that $(P_+)_0^{n_1}(x)$ is hyperbolic and $n_1 \leq N_0$;*

(2) $\{n_i \mid 1 \leq i \leq r\} \subset \{0 \leq k < N \mid T_+^k x \in \mathcal{R}_+ \setminus \kappa(\mathcal{R}_+)\}$;

(3) $\{N_0 \leq k < N \mid T_+^k x \in \kappa_{10}(\mathcal{R}_+)\} \subset \{m_{ij} \mid 1 \leq i \leq r$ *and* $1 \leq j \leq s(i)\}$.

Proof. By definition n_1 is the smallest integer such that $(P_+)_0^{n_1}(x)$ is an admissible strip so (1) is just Fact 3.20.

With $(P_+)_0^{n_i}(x)$ being an admissible strip, $T_+^{n_i} x \in \mathcal{R}_+$. We have $n_i < N$ so $T_+^{n_i} x \notin \kappa(\mathcal{R}_+)$, proving (2).

The m_{ij} are the times $m \in]\!]n_1, N[\![$ (or, equivalently, $m \in [\![0, N[\![$ by property (1)) such that $T_+^m x \in \kappa_s(\mathcal{R}_+)$ and $(P_+)_0^m(x)$ is a hyperbolic, but not admissible, strip. As $(P_+)_0^{N_0}(x)$ is a hyperbolic strip and $T_+^{N_0} x \in \kappa_{10}(\mathcal{R}_+)$, Corollary 3.14 gives that $N_0 \leq k < N$ and $T_+^k x \in \kappa_{10}(\mathcal{R}_+)$ implies that $(P_+)_0^k(x)$ is a hyperbolic strip. This strip cannot be admissible as $T_+^k x \in \kappa_s(\mathcal{R}_+)$ and $k < N$, hence such k is some m_{ij}, proving (3). □

Proof of Proposition 4.4. According to Proposition 4.1, given N, M, R and S, to determine an element of $\mathcal{I}(N, L, M, R, S)$ we need to specify:

---

† Here K_* does not depend on any of the data $T : M \to M, N, M, R, S, L, C_*, \epsilon_*$.

(1) the integers $N_1, N_2, n_1, \ldots, n_R$ and $m_{i1}, m_{is(i)}$ for $i = 1, \ldots, R$;
(2) the itineraries $(P_+)^{N_1}(x)$, $(P_+)^{n_1-1}_{N_2}(x)$ (if $n_1 > N_2$) and $(P_+)^{N-1}_{n_r+1}(x)$;
(3) $(P_+)^{m_{i1}}_{n_i}(x)$, $(P_+)^{n_{i+1}}_{m_{is(i)}}(x)$ for each $i = 1, \ldots, r$;
(4) a choice among $4 \cdot 2^R$.

Observe that $N_1 \le N_2 \le N_0$. Using property (3) of Lemma 4.5 (in particular, that N_0 is some m_{ij}), it follows that

$$\#\left(\bigcup_{i=1}^{r}]\!]m_{i1}, m_{is(i)}[\![\setminus]\!]N_1, N_2[\![\right) \ge \#\left(\bigcup_{i=1}^{r}]\!]m_{i1}, m_{is(i)}[\![\setminus]\!]0, N_0[\![\right)$$
$$\ge \# \bigcup_{\substack{i=1,\ldots,r\\ 1\le j<s(i)\\ m_{ij}\ge N_0}}]\!]m_{ij}, m_{ij}+L[\![\ \ge (S-R)(L-1) \ge (S-R)L - S$$

recalling the definitions of L and S. Hence, the number of choices for those items is bounded by:

(1) $\binom{N}{2}\binom{N}{R}^3$ where $\binom{a}{b} = a!/b!(a-b)!$ is the binomial coefficient;
(2–3) $C_*^{R+2} \exp((h_{\mathrm{top}}(T) + \epsilon_*)(N - (S-R)L - M + S))$;
(4) $4 \cdot 2^R$.

Recalling that† $\binom{n}{\alpha n} \sim (1/\sqrt{2\pi\alpha(1-\alpha)})n^{-1/2}e^{H(\alpha)n}$ as $n \to \infty$, i.e. $\log \binom{n}{\alpha n} \le H(\alpha)n + C(\alpha)$ and that $k \in [\![0, (n-1)/2]\!] \mapsto \binom{n}{k}$ is increasing, the stated bound follows with

$$K_*(\rho, N) = 3H(\rho) + \rho \log 2 + 3N^{-1} \log N + N^{-1} \log 4C(\rho). \qquad \square$$

4.3. *Large average return times and entropy.* We are going to apply the previous estimates linking long return times either to visits to the holes $(\mathcal{R} \setminus \kappa(\mathcal{R}))$ or to low entropy. We show that for a suitable choice of the parameters of our constructions, large entropy measures have finite average return time.

Recall the good return time $\tau : \kappa_s(R) \to \bar{\mathbb{N}}$ (possibly infinite) of Definition 3.18. We define $\tau_n(x)$, $n \ge 1$, inductively by $\tau_1(x) = \tau(x)$ and $\tau_{n+1}(x) = \tau(T^{\tau_n(x)}(x))$ ($\tau_{n+1}(x) = \infty$ if $\tau_n(x) = \infty$).

The essential supremum of a function f over a subset X with respect to a measure μ is

$$\mu\text{-sup}_{x\in X} f(x) := \inf_{X'=X[\mu]} \sup_{x\in X'} f(x)$$

where X' ranges over the measurable subsets of X such that $\mu(X \setminus X') = 0$ (X and f are assumed to be measurable). Our key estimate is as follows.

PROPOSITION 4.6. *There exist $h_2 < h_{\mathrm{top}}(T)$ and $L_2 < \infty$ with the following property. Consider the Markov array $\mathcal{R}$ defined by Proposition 3.6. For any integer $L_+ \ge L_2$, let $(M_+, T_+, P_+, \mathcal{R}_+)$ be the L_+-periodic extension of Definition 3.5. Then, for each $\mu \in \mathbb{P}^{h_2}_{\mathrm{erg}}(T_+)$, the good return time with respect to P_+ and $\mathcal{R}_+$, satisfies*

$$\tau_*(\mu) = \mu\text{-sup}_{x\in\mathcal{R}_+} \tau_*(x) < \infty \quad \textit{where } \tau_*(x) := \limsup_{n\to\infty} \frac{1}{n}\tau_n(x).$$

---

† We have $f(t) \sim g(t)$ if and only if $\lim_{t\to\infty} f(t)/g(t) = 1$ and $H(t) = -t \log t - (1-t) \log(1-t)$.

Remark 4.7. We make the following remarks.

(1) That $\tau_*(x) < \infty$ almost everywhere already ensures that almost every point in $\kappa(\mathcal{R})$ has a good return. However, we need more.

(2) The proof below does *not* provide a semi-uniform bound on τ_* as our estimates below depend on the speed of convergence of some ergodic averages (see Remark 4.8).

Proof of Proposition 4.6. The first step of the proof fixes a Markov array $\mathcal{R}$ and a periodic extension of T and finds a candidate upper bound for $\tau_*(\mu)$. The second step defines a language (a collection of words of increasing lengths) with small entropy. The final step shows that large average return times imply that this language is enough to describe the measure. A large average can therefore happen only for low entropy measures.

Step 1: The Markov array. We apply Proposition 3.6 and obtain first numbers $\mu_0 > 0$ and $h_0 < h_{\rm top}(T)$. We let $0 < \epsilon_0 < \min(h_{\rm top}(T), 1)/200$ be small enough so that in the notation of Proposition 4.4:

$$K_*(\epsilon_0\mu_0) < \frac{\mu_0}{100} h_{\rm top}(T). \tag{4.1}$$

We pick L_* to be so large that, for all $\ell \geq L_*$

$$K_*(\epsilon_0\mu_0, \ell) < \frac{\mu_0}{100} h_{\rm top}(T). \tag{4.2}$$

Proposition 3.6 now gives an integer $L_2(T, \epsilon_0, L_*) \geq L_*$ and a Markov array $\mathcal{R}$ such that the following holds.

For each $L_+ \geq L_2$, any $\mu \in \mathbb{P}^{h_0}_{\rm erg}(T)$ can be lifted to an ergodic invariant probability measure μ_+ on the periodic extension $(M_+, T_+, P_+, \mathcal{R}_+)$ satisfying

$$L_+\mu_+(\kappa_{10}(\mathcal{R}_+)) > \mu_0 \quad \text{and} \quad L_+\mu_+(\mathcal{R}_+ \setminus \kappa(\mathcal{R}_+)) < \epsilon_0\mu_0.$$

Recall that $P^{\mathcal{R}}$ denotes the convex partition of M generated by P and $\mathcal{R}$.

We fix $\epsilon_* := (\mu_0/100)h_{\rm top}(T)$ and define $C_* = C_*(P^{\mathcal{R}}, \epsilon_*) < \infty$ as in Proposition 4.4. Note that C_* does not depend on L_+. Hence, possibly after increasing L_+, we may assume that

$$L_+ > \frac{\log C_*}{(\mu_0/100)h_{\rm top}(T)}.$$

Fix $\ell_* \geq L_*$ so large that $3 \log C_*/\ell_* < (\mu_0/100)h_{\rm top}(T)$.

We omit the sharp subscript in the sequel so that M, T, P, μ, μ_0 in fact denote $M_+, T_+, P_+, \mu_+, \mu_{0+}$ (in particular, $L_+\mu_0$ is the original μ_0). To refer to the original μ or μ_0, we write $\mu_\flat$ or $\mu_{0\flat}$. It will be a convenient exception to continue to write $P^{\mathcal{R}}$ for $P_\flat^{\mathcal{R}_\flat}$.

We let

$$h_2 := \max(h_0, h_{\rm top}(T)(1 - 0.9L_+\mu_0)) < h_{\rm top}(T)$$

and fix some $\mu \in \mathbb{P}^{h_2}_{\rm erg}(T)$ together with μ_+ as above. According to the Birkhoff ergodic theorem, one can find $K_1 \subset M$ and $L_1 < \infty$ such that

$$\mu(M \setminus K_1) < \epsilon_0\mu_0^2/(10^6 \log \#P^{\mathcal{R}}) \tag{4.3}$$

and, for all $x \in K_1$,

$$\text{for all } n \geq L_0 := \frac{\epsilon_0\mu_0}{1000}L_1, \quad \left|\frac{1}{n}\#\{0 \leq k < n \mid T^k x \in \kappa_{10}(\mathcal{R})\} - \mu(\kappa_{10}(\mathcal{R}))\right| < \left(\frac{\mu_0}{1000}\right)^2.$$

$$\text{for all } n \geq L_1, \qquad \frac{1}{n}\#\{0 \leq k < n \mid T^k x \in \mathcal{R} \setminus \kappa(\mathcal{R})\} < \epsilon_0\mu_0. \tag{4.4}$$

Remark 4.8. The above L_1 is the only estimate in the proof of this proposition which does not seem semi-uniform.

Increasing ℓ_* if necessary, we assume

$$\ell_* \geq (1000/\mu_0)L_1$$

and $\ell\binom{\ell}{\ell/\ell_*} \leq e^{(L_+\mu_0/100)h_{\text{top}}(T)\cdot\ell}$ for all $\ell \geq \ell_*$.

We set

$$\tau_{\max} := \frac{1000 \log \#P^{\mathcal{R}}}{\mu_0}\ell_*. \tag{4.5}$$

To prove that $\tau_*(\mu) \leq \tau_{\max}$, we assume by contradiction that

$$M_1 := \left\{x \in \kappa(\mathcal{R}) \;\middle|\; \limsup_{n\to\infty} \frac{1}{n}\tau_n(x) > \tau_{\max}\right\} \text{ has positive } \mu\text{-measure.} \tag{4.6}$$

Step 2: Low entropy language. For each integer $\ell \geq 1$ we define a set $C(\ell)$ of $P^{\mathcal{R}}$-words of length ℓ as

$$C(\ell) := \bigcup_{\substack{\ell_1+\cdots+\ell_k=\ell \\ k\leq\ell/\tau_{\max}}} C(\ell_1, \ldots, \ell_k).$$

Here $C(\ell_1, \ldots, \ell_k)$ is the set of all concatenations $\gamma_1 \cdots \gamma_k$ where each γ_m $(1 \leq m \leq k)$ is a word (i.e. a finite sequence) of length $|\gamma_m| = \ell_m$ satisfying the following requirements.

- *Type 1 requirement:* γ_m is an itinerary from $\mathcal{I}(\ell_m, L_+, M, R, S)$ (in the notation of Proposition 4.4) with
$$\ell_m \geq \ell_*, \quad L_+(S-R) + M - S \geq \frac{98}{100}L_+\mu_0\ell_m \quad \text{and} \\ R < \frac{\min(h_{\text{top}}(T), 1)}{200}L_+\mu_0\ell_m; \tag{4.7}$$
recall that $L_+\mu_0$ is the original μ_0, independent on L_+.
- *Type 2 requirement:* the sum of the lengths of these segments is less than $(L_+\mu_0/500 \log \#P)\ell$.

Observe that the union defining $C(\ell)$ has at most $\ell\binom{\ell}{[\ell/\tau_{\max}]} \leq e^{(L_+\mu_0/100)h_{\text{top}}(T)\ell}$ terms. It remains to bound $\#C(\ell_1, \ldots, \ell_n)$.

By Proposition 4.4, the logarithm of $\#\mathcal{I}(\ell_m, L_+, M, R, S)$ under condition (4.7) is bounded by

$$\begin{aligned}
&(h_{\text{top}}(T) + \epsilon_*)(\ell_k - (L_+(S-R) + M - S)) + K(\epsilon_0\mu_0, \ell_k)\ell_k + (2R+3)\log C_* \\
&\leq h_{\text{top}}(T)\left(1 + \frac{L_+\mu_0}{100}\right)\left(1 - \frac{98}{100}L_+\mu_0\right)\ell_k + \frac{2L_+\mu_0}{100}h_{\text{top}}(T)\ell_k + 3\log C_* \\
&\leq h_{\text{top}}(T)\left(1 - \frac{94}{100}L_+\mu_0\right)\ell_k + 3\log C_* \\
&\leq h_{\text{top}}(T)(1 - 0.93L_+\mu_0)\ell_k,
\end{aligned} \tag{4.8}$$

Hence,

$$\#C(\ell) \le \exp((L_+\mu_0/100)h_{\text{top}}(T)\ell) \times \exp(h_{\text{top}}(T)(1-0.93L_+\mu_0)\ell) \\ \times(\#\mathcal{P}^{\mathcal{R}})^{(L_+\mu_0/500\log\#\mathcal{P}^{\mathcal{R}})\ell} \le e^{h_2\ell}. \tag{4.9}$$

Step 3: Consequence of large return times. We are going to show that, for all $x \in \kappa(\mathcal{R})$, all large enough integers n:

$$\tau_n(x) > \tau_{\max}\cdot n \implies P^{\tau_n(x)}(x) \in C(\tau_n(x)). \tag{4.10}$$

Observe that this will imply that any ergodic and invariant measure μ such that $\tau_*(\mu) > \tau_{\max}$ has entropy at most h_2 using Proposition A.2 with (4.9) and:

- $M_0 := \{x \in M \mid \{n \ge 0 \mid T^{-n}x \in M_1\}$ is infinite$\}$ (recall (4.6));
- $a_i(x) := \min\{j \ge i \mid T^{-j}x \in M_1\}$ for all $i \ge 1$;
- $b_i(x) := \tau_n(T^{-a_i(x)}x)$ with n a positive integer such that $\tau_{n_i}(x) \ge \max(a_i(x), \tau_{\max}\cdot n_i)$;

concluding the proof of Proposition 4.6. We now prove (4.10).

Let $x \in \kappa(\mathcal{R})$. We consider a large integer n such that $\tau_n(x) > \tau_{\max}\cdot n$. Equation (4.3) and the ergodic theorem give

$$\frac{1}{\tau_n(x)}\#\{0 \le k < \tau_n(x) \mid T^k x \notin K_1\} < \epsilon_0\mu_0^2/(10^6\log\#P^{\mathcal{R}}).$$

Let $N := \tau_n(x)$ and, for $k = 0, \dots, n-1$, let I_k be the integer interval $[\![\tau_k(x), \tau_{k+1}(x)[\![$ and $\ell_k := \#I_k$.

Let $B_1 \subset [\![0, n[\![$ be the set of those integers $0 \le k < n$ such that

$$\#\{m \in I_k \mid T^m x \notin K_1\} \ge \frac{\epsilon_0\mu_0}{1000}\ell_k.$$

The union of those segments I_k occupies only a small proportion of $[\![0, N[\![$:

$$\sum_{k\in B_1}\ell_k \le \frac{1000}{\epsilon_0\mu_0}\times\frac{\epsilon_0\mu_0^2}{10^6\log\#P^{\mathcal{R}}}N \le \frac{\mu_0}{1000\log\#P^{\mathcal{R}}}\cdot N.$$

Let $B_2 \subset [\![0, n[\![$ be the set of k such that $\ell_k \le \ell_*$. They also occupy a small proportion

$$\sum_{k\in B_2}\ell_k \le \ell_* n \le \ell_*\frac{N}{\tau_{\max}} \le \frac{\mu_0}{1000\log\#P^{\mathcal{R}}}\cdot N,$$

by the choice of $\tau_{\max}$.

Therefore, the segments I_k for $k \in B_1 \cup B_2$ satisfy the type 2 requirement in the definition of $C(\ell)$. It is enough to prove that the remaining I_ks satisfy the type 1 requirement.

For such segments I_k, $p_1 := \min\{p \ge 0 \mid T^{p+\tau_k(x)}x \in K_1\}$ satisfies

$$p_1 \le \frac{\epsilon_0\mu_0}{1000}\ell_k \tag{4.11}$$

by the definition of B_1. By the definition of B_2,

$$\ell_k \ge \ell_* \ge \frac{1000}{\mu_0}L_1 = \frac{10^6}{\epsilon_0\mu_0^2}L_0. \tag{4.12}$$

This fulfills the first requirement of (4.7).

Hence, in the notation of Proposition 4.4,

$$N_1 := \min\{j \geq 0 \mid T^{\tau_k(x)+j}x \in \kappa_{10}(\mathcal{R})\} \leq p_1 + L_0 \leq \frac{\epsilon_0\mu_0}{500}\ell_k. \tag{4.13}$$

Also, by (4.4) and $\ell_k - p_1 \geq L_1$:

$$\#\{j \in [\![\tau_k(x) + p_1, \tau_{k+1}(x)[\![\mid T^j x \in R \setminus \kappa(\mathcal{R})\} < \epsilon_0\mu_0(\ell_k - p_1).$$

Hence, using point (2) of Lemma 4.5:

$$R \leq r' := \#\{j \in I_k \mid T^j x \in \mathcal{R} \setminus \kappa(\mathcal{R})\} < 2\epsilon_0\mu_0\ell_k. \tag{4.14}$$

Note that this implies $R \leq \ell_k \cdot \mu_0 \min(h_{\text{top}}(T), 1)/200$, part of the type 1 requirement. It remains to show the lower bound on $L_+(S - R) + M$.

First, similarly to (4.14):

$$|\#\{j \in I_k \mid T^j x \in \kappa_{10}(\mathcal{R})\} - \mu(\kappa_{10}(\mathcal{R}))\ell_k| < \frac{\mu_0}{500}\ell_k.$$

Setting, again as in Proposition 4.4,

$$N_0 := \min\{j \geq 0 \mid T^{\tau_k(x)+j}(x) \in \kappa_{10}(\mathcal{R}) \text{ and } P^j(T^{\tau_k(x)}x) \text{ is hyperbolic}\}$$

(observe that N_0 might be large) and $S := \#\{m_{ij} > N_0 \mid i, j\}$ we obtain, using point (3) of Lemma 4.5,

$$S \geq s' := \#\{j \in]\!]\tau_k(x) + N_0, \tau_{k+1}(x)[\![\mid T^j x \in \kappa_{10}(\mathcal{R})\}.$$

Also, $M := N_2 - N_1 = [N_0]_\kappa - N_1$ where

$$[t]_\kappa := \max\{n \in [\![0, t[\![\mid T^n x \in \kappa_{10}(R)\}$$

(we define $[t]_\kappa := t$ if there is no such integer). To complete our estimate, we consider two cases.

First case: $N_0 < p_1 + L_1$. We use the trivial bound $M \geq 0$, (4.11), (4.12) and (4.4) to obtain

$$\begin{aligned} s' &\geq \#\{j \in [\![\tau_k(x) + p_1 + L_1, \tau_{k+1}(x)[\![\mid T^j x \in \kappa_{10}(R)\} \\ &\geq \frac{499}{500}\mu_0(\ell_k - p_1 - L_1) \geq \frac{498}{500}\mu_0\ell_k. \end{aligned}$$

Hence,

$$L_+S + M \geq \frac{498}{500}L_+\mu_0\ell_k \geq \frac{99}{100}L_+\mu_0\ell_k. \tag{4.15}$$

Second case: $N_0 \geq p_1 + L_1$. Using the definition of p_1, K_1 and L_0:

$$s' \geq \mu(\kappa_{10}(R))((1 - 10^{-3})(\ell_k - p_1) - (1 + 10^{-3})(N_0 - p_1)).$$

From (4.11),

$$s' \geq \frac{998}{1000}\mu(\kappa_{10}(\mathcal{R}))\ell_k - \frac{1001}{1000}\mu(\kappa_{10}(\mathcal{R}))N_0.$$

Hence, using $(1001/1000)L_+\mu(\kappa_{10}(\mathcal{R})) \le 1$ and $M = [N_0]_\kappa - N_1$:

$$\begin{aligned} L_+S + M &\ge \frac{998}{1000}L_+\mu(\kappa_{10}(\mathcal{R}))\ell_k - \frac{1001}{1000}L_+\mu(\kappa_{10}(\mathcal{R}))N_0 + [N_0]_\kappa - N_1 \\ &\ge \frac{998}{1000}L_+\mu_0\ell_k - (N_0 - [N_0]_\kappa) - N_1. \end{aligned}$$

In light of (4.13), to prove that (4.15) also holds in this second case it is enough to show the following.

CLAIM 4.9. *For any* $0 \le t \le \ell_k$, $t - [t]_\kappa \le (\mu_0/250)\ell_k$.

Proof. We distinguish two cases. First assume that $[t]_\kappa < p_1 + L_0$. Then t, the first visit to $\kappa_{10}(R)$ after $[t]_\kappa$, is bounded by the first visit after $p_1 + L_0$, i.e. using (4.13) and $\ell_k \ge (1000/\mu_0^2)$ which follows from (4.12):

$$t - [t]_\kappa \le t \le p_1 + L_0 + \frac{1000}{999}\mu_0^{-1} \le \frac{4}{1000}\mu_0\ell_k$$

proving the claim in this case. Second we assume that $[t]_\kappa > p_1 + L_0$. Then

$$\begin{aligned} &(1 - \mu_0/1000)\mu(\kappa_{10}(\mathcal{R}))(t - p_1) \\ &\quad\le \#\{j \in [\![\tau_k(x) + p_1, \tau_k(x) + t[\![\mid T^j x \in \kappa_{10}(\mathcal{R})\} \\ &\quad= \#\{j \in [\![\tau_k(x) + p_1, \tau_k(x) + [t]_\kappa]\!] \mid T^j x \in \kappa_{10}(\mathcal{R})\} \\ &\quad\le (1 + \mu_0/1000)\mu(\kappa_{10}(\mathcal{R}))([t]_\kappa + 1 - p_1). \end{aligned} \tag{4.16}$$

So $t - p_1 \le (1 + 3\mu_0/1000)([t]_\kappa + 1 - p_1)$. Hence,

$$t - [t]_\kappa \le \frac{3}{1000}\mu_0[t]_\kappa + 2 \le \frac{4}{1000}\mu_0\ell_k,$$

proving the claim. □

In both cases, (4.15) together with (4.14) implies

$$L_+(S - R) + M - S \ge \frac{98}{100}L_+\mu_0\ell_k. \tag{4.17}$$

This establishes the remaining part of the type 1 requirement on $P_{\tau_k(x)}^{\tau_{k+1}(x)}(x)$ for all $k \in [\![0, n[\![\setminus (B_1 \cup B_k)$. Hence, $P^{\tau_n(x)}(x)$ belongs to $C(\tau_n(x))$, concluding the proof of (4.10) and of Proposition 4.6. □

From the above proof the following result also holds.

COROLLARY 4.10. *We have* $\tau : \kappa_s(\mathcal{R}) \to \bar{\mathbb{N}}^*$ *has eventually bounded gaps in the sense of Appendix B with respect to any large entropy measure.*

Proof. Given μ, a large entropy measure, we fix $\tau_{\max}$ as in (4.5) and we proceed by contradiction assuming that for each large $t > \tau_{\max}$, there is a set of positive μ-measure S with the following property. For each $x \in S$, there exist sequences of integers $n_k \in \mathbb{Z}$ and $m_k \in \mathbb{N}^*$ such that

$$\tau_{m_k}(T_k^n x) > t \cdot m_k \quad \text{and} \quad \sup_k \inf\{|i| \mid i \in [\![n_k, n_k + \tau_{m_k}(T_k^n x)[\![\} < \infty.$$

(the case of improper orbits is similar and easier and left to the reader). It is now enough to apply Proposition A.2 using (4.9)–(4.10) to obtain that $h(f, \mu) \le h_2$, a contradiction. □

5. *Proof of the main results*

We finally prove the main results by building a Markov system from the arbitrary concatenations of admissible strips and relating it to the dynamics of the piecewise affine homeomorphism. This is done in the analysis of large entropy measures and then used for the other claims.

5.1. *Maximal entropy measures.*

We prove Theorem 1 about the finite number of maximum measures.

Step 1: Tower. Fix a Markov array $\mathcal{R}$ as in Proposition 3.6, defining a $(\mathcal{R}, L_+)$-periodic extension $(M_+, T_+, P_+, \mathcal{R}_+)$ of $(M, T, P, \mathcal{R})$. This may only increase the number of maximum measures as it is a finite topological extension. Let $\Sigma := \Sigma(T_+, P_+)$ be its symbolic dynamics (see Definition 2.1). Corollary 2.10 shows that it is enough to prove the results for Σ.

We now build an invertible tower (see Appendix B) $\hat{\Sigma}$ over Σ. This is done by defining a return time $\tau : \Sigma_\tau \to \mathbb{N}^*$ for some $\Sigma_\tau \subset \Sigma$.

Definition 5.1. An *extended admissible* P_+*-word* is a word $w_0 \cdots w_n$ over P_+ such that $[w_0 \cdots w_n]$ is an admissible strip; $w_0 \cdots w_{n-1}$ is the associated *admissible* P_+*-word*.

For a sequence $A \in \Sigma$, we define inductively

$$t_1(A) := \sup\{n \geq 1 \mid A_0 \cdots A_n \text{ is an extended admissible word}\} \in \mathbb{N}^* \cup \{-\infty, \infty\}$$

and $t_{n+1}(A) = t_1(\sigma^{t_n(A)}(A))$ (or $t_n(A)$ if it was infinite). Let

$$\Sigma_\tau := \{A \in \Sigma \mid \forall n \geq 1,\ t_n(A) \in \mathbb{N}^*\}.$$

We tacitly exclude entropy-negligible subsets of points of M_+ and of Σ (these correspond by Lemma 2.10).

CLAIM 5.2. *We claim that Σ_τ coincides with the set of P_+-itineraries of the points $x \in \kappa_s(\mathcal{R}_+)$.*

Proof. We know that x has a finite good return time $m := \tau(x)$ by Proposition 4.6. Let $A \in \Sigma$ be its itinerary. Observe that $S^0 := A_0 \cdots A_{m-1}$ is admissible and $T_+^m x \in \kappa_s(\mathcal{R}_+)$. By induction, A splits into a concatenation of admissible words $S^0 S^1 S^2 \cdots$. The cylinders defined by the finite concatenations $S^0 S^1 \cdots S^k A_{\tau_{k+1}(x)}$, $k \geq 0$, are hyperbolic strips.

Clearly $t_1(A) \geq \tau(x)$. Now, if $H := (P_+)_0^n(x)$ is hyperbolic with $n > m$, then H is contained in S^0 (an m-admissible strip) and $T^m H$ meets the hyperbolic strip S^1. Hence, H cannot be admissible, proving that $t_1(A) = \tau(x) < \infty$. That $t_n(A) < \infty$ for all $n \geq 1$ follows from invariance. Hence, $A \in \Sigma_\tau$.

For the converse, let $A \in \Sigma_\tau$ and denote by x the point with itinerary A. Here $[A_0 \cdots A_m]$ is an admissible strip for $m = t_1(A)$ so $A_0 \in \mathcal{R}$. If $k = t_n(A)$, then $[A_0 \cdots A_k]$ is a concatenation of admissible strips, hence a k-strip. As k is arbitrarily large, it follows that $W^s(x)$, the intersection of the previous strips, must cross A_0, proving $x \in \kappa_s(\mathcal{R})$. □

By construction, for all $A \in \Sigma_\tau$, $\sigma^{t_1(A)}(A) \in \Sigma_\tau$. Hence, t_1 is a return time and defines an invertible tower $\hat{T} : \hat{\Sigma} \to \hat{\Sigma}$ in the sense of Appendix B. Moreover, Corollary 4.10 shows that any large entropy measure μ on $\Sigma(T_+, P_+)$ has eventually bounded gaps in the sense of Definition B.2. By Proposition B.3, any such measure can be lifted to $\hat{\Sigma}$ and any invariant probability measure of $\hat{\Sigma}$ is a finite extension of one in $\Sigma(T_+, P_+)$ (in particular, both measures have the same entropy).

It follows also that $h(\hat{T}) := \sup_\mu h(\sigma | hS, \mu) = h_{\text{top}}(\Sigma)$ so that maximum measures of Σ lift to maximum measures of $\hat{T}$.

To prove the theorem it is therefore enough to show that the tower $\hat{\Sigma}$ has finitely many ergodic measures of maximal entropy.

Step 2: Markov structure. Recall the following definition (see [**18**] for background).

Definition 5.3. A *Markov shift* is a space of sequences

$$\Sigma(\mathcal{G}) := \{x \in V^{\mathbb{Z}} \mid \forall n \in \mathbb{Z}\ x_n \to x_{n+1} \text{ in } \mathcal{G}\}$$

where $\mathcal{G}$ is an oriented graph with a countable set of vertices V together with the left shift σ.

We also recall that a graph $\mathcal{G}$ as above is *(strongly) irreducible* if for every $(A, B) \in V^2$, there is a path from A to B on $\mathcal{G}$. A (*strongly*) *irreducible component* is a subgraph with this property maximum with respect to inclusion.

It will be convenient to say that w' is a *follower* of w if the two are admissible words w, w' in the sense of Definition 5.1, and s being the first symbol of w', the concatenation ws is an extended admissible word.

CLAIM 5.4. *Let $\mathcal{G}$ be the oriented graph with vertices (w, i) where w is any admissible word and $0 \leq i < |w|$ ($|\cdot|$ is the length of the word) and arrows*

$$(w, i) \to (w, i+1) \quad \text{if } i+1 < |w|, \qquad (w, |w|-1) \to (w', 0) \quad \text{if } w' \text{ is a follower of } w.$$

The tower $\hat{\Sigma}$ is measurably conjugate† *to the Markov shift $\Sigma(G)$.*

Proof. Define $p : \Sigma(\mathcal{G}) \to \hat{\Sigma}$ by $p(\alpha) = (A, \omega) \in \Sigma \times \{0, 1\}^{\mathbb{Z}}$ where, if $\alpha_n =: (w_0 \cdots w_{\ell-1}, k)$, then $A_n = w_k$ and $\omega_n = 1$ if and only if $k = 0$. Observe that the sequence A thus obtained is a concatenation of admissible words so $A \in \Sigma$. Also, whenever m and n are two successive integers with $\omega_m = \omega_n = 1$, $A_n A_{n+1} \cdots A_m$ is an admissible word. Finally $\omega_m = 1$ for infinitely many positive and negative integers m. The proof of Claim 5.2 shows that $\sigma^m A \in \Sigma_\tau$ for all such m, so that $(A, \omega) \in \hat{\Sigma}$. Thus p is well-defined and is clearly a measurable conjugacy. □

Step 3: Conclusion

Proof of Theorem 1. We know that $\mathcal{G}$ has at most one irreducible component for each vertex of the type $(w_0, 0)$. These correspond to the finitely many rectangles in the Markov array $\mathcal{R}$. Hence, by a classical result of Gurevič [**17**], $\Sigma(\mathcal{G})$ with $h(\Sigma(\mathcal{G})) < \infty$ has finitely many maximum measures, proving the main theorem. □

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† That is, there is a bimeasurable bijection $\psi : \hat{\Sigma} \to \Sigma(\mathcal{G})$ such that $\psi \circ \hat{T} = \sigma \circ \psi$.

5.2. *Number of periodic points.* We prove Proposition 1.4 about the number of periodic points. We use the construction of the proof of Theorem 1. To prove the lower bound (1.1), it is enough to prove it for T_+, as T_+ is a finite extension of T. We assume by contradiction that the number of points fixed under T_+^n is such that for any integer $p \geq 1$, there is a sequence $n_k \to \infty$ of multiples of p such that

$$\lim_{n_k\to\infty} \frac{N_{T_+}(n_k)}{e^{n_k h_{\mathrm{top}}(T)}} = 0. \tag{5.1}$$

In the following we denote $(M_+, T_+, P_+, \mathcal{R}_+)$ by $(M, T, P, \mathcal{R})$.

The starting point is the following estimate for $\Sigma(\mathcal{G})$. Consider a maximum measure. It is carried on one irreducible component. For simplicity, we replace $\mathcal{G}$ by that irreducible component. By Gurevič [**17**], the existence of a maximum measure for $\Sigma(\mathcal{G})$ implies that $\mathcal{G}$ is *positive recurrent* with parameter $R = e^{-h_{\mathrm{top}}(T)}$. By Vere-Jones [**34**], this implies that the number $N_{\mathcal{G}}(n)$ of loops of length n based at a given vertex satisfies, for some positive integer p,

$$\lim_{n\to\infty,\ p|n} N_{\mathcal{G}}(n)e^{-h_{\mathrm{top}}(T)n} = p. \tag{5.2}$$

Each n-periodic sequence A in the symbolic dynamics Σ is associated to a closed, convex set

$$\bigcap_{j\geq 0} \overline{T_+^j[A_{-j}\cdots A_j]}$$

invariant under T_+^n. This set contains at least one point fixed by T_+^n which we denote by $\pi(A)$. It remains to show that π does not identify too many points.

Consider π from the set $\Sigma(n)$ of n-periodic sequences to the set $M_+(n)$ of n-periodic T_+-orbits. Our assumption (5.1) implies that, for some sequence $m_n \to \infty$ (where $p|n$) this map is at least m_n-to-one on a subset $\Sigma'(n)$ of $\Sigma(n)$ with cardinality at least $e^{nh_{\mathrm{top}}(T)}/3$. We use the following observation.

LEMMA 5.5. *Let* $A^1, \ldots, A^m \in \Sigma$ *be such that* $\pi(A^1) = \cdots = \pi(A^m) =: x$. *If the finite words* $A_0^i \cdots A_{n-1}^i$, $i = 1, \ldots, m$, *are pairwise distinct, then either:*

(1) $T^k x$ *is a vertex of* P_+ *for some* $0 \leq k < n$; *or*

(2) *there exist* $r \geq (m-1)/2$ *distinct integers* $0 \leq n_1 < \cdots < n_r < n$ *such that for all* (k, l) *with* $1 \leq k < \ell < n$, $T^{n_k}x$ *lies on the interior of an edge of* P_+; *moreover, if* v_k *is the direction of the open edge containing* $T^{n_k}x$, *then the image by* $T^{n_\ell - n_k}$ *of* v_k *is transverse to* v_ℓ.

Proof. Let $A_0^i \cdots A_{n-1}^i$, $i = 1, \ldots, m$ be finite words as in the above statement. We show that the failure of (1) implies (2).

Each word $A_0^i \cdots A_{n-1}^i$ defines an element of P_+^n containing x in its closure so $\mathrm{mult}(x, P_+^n) \geq m$. We assume that (1) fails. Observe that:

(i) $\mathrm{mult}(x, P_+^{k+1}) = \mathrm{mult}(x, P_+^k)$ if $T^k x$ is in the interior of an element of P_+ or if, for all $A \in P_+^k$,

$$P_+^k T_+^k(A \cap B(x, \epsilon)) \subset B \quad \text{for some } B \in P_+ \text{ and } \epsilon > 0;$$

(ii) $\mathrm{mult}(x, P_+^{k+1}) \leq \mathrm{mult}(x, P_+^k) + 2$.

Observation (ii) uses that T_+ is a piecewise affine surface homeomorphism so the preimage of an edge may locally divide into at most two subsets of at most two of the elements of P_+^k touching x. This implies that $\operatorname{mult}(x, P_+^n) \leq 1 + 2\#\{0 \leq k < n \mid T_+^k x \text{ is on an edge of } P_+\}$. The lemma follows. □

Proof of Proposition 1.4. Only const $\cdot\, n$ points of T_+ can satisfy assertion (1) in the above lemma. As $h_{\text{mult}}(T_+, P_+) = 0$, their preimages in $\Sigma'(n)$ are in subexponential number. The remainder of $\Sigma'(n)$ corresponds to points x whose orbit stays off the vertices of P_+ and which admit distinct sequences $A^i \in \Sigma'(n)$, $i = 1, \ldots, m_n$ with $\pi(A^i) = x$.

Given such an x, fix $0 < n_1 < \cdots < n_r < n$ as in point (2) of the lemma. Pick j such that $0 \leq n_{j+1} - n_j \leq 2n/(m_n - 1)$. Thus, $T_+^{n_j} x$ is a vertex of $P^{n_{j+1} - n_j + 1}$. The number of such vertices, for given n_j, is bounded by const $\cdot\, \# P_+^{2n/(m_n - 1)}$. Taking into account the choice of n_j, the number of such xs is bounded by $e^{(2/m_n)(h_{\text{top}}(T) + \epsilon)n}$ for all large n. Thus, for large multiples n of p,

$$\#\Sigma(n) \leq 3\#\Sigma'(n) \leq 3e^{(2/m_n)(h_{\text{top}}(T)+\epsilon)n} + 3e^{\epsilon n}.$$

As $m_n \to \infty$, this contradicts the Vere-Jones estimate (5.2), proving the lower bound (1.1) and Proposition 1.4. □

5.3. *Entropy away from the singularities.*

Proposition 1.5 is a corollary of the proof of Theorem 1 using the following result.

PROPOSITION 5.6. (Gurevič [**17**]) *Let G be a countable, oriented graph. Let $G_0 \subset G_1 \subset \cdots$ be a non-decreasing sequence of finite subgraphs exhausting G: any vertex and any arrow of G belong to G_n with n large enough.*

Then $\lim_{n\to\infty} h_{\text{top}}(G_n) = h_{\text{top}}(G)$.

Proof of Proposition 1.5 In the proof of Theorem 1, one has shown that there is a countable oriented graph $\mathcal{G}$ such that the corresponding countable state Markov shift whose maximum measures have entropy $h_{\text{top}}(T)$.

Let $\mathcal{G}_n$ be the finite subgraph defined by keeping only the vertices and arrows of $\mathcal{G}$ which are on a loop of length at most n and based at one of the finitely many symbols representing an element of the Markov array. The sequence $\mathcal{G}_n$ exhausts $\mathcal{G}$. The above proposition therefore ensures that, for any $\epsilon > 0$, there is some $\mathcal{G}_N$ with topological entropy at least $h_{\text{top}}(T) - \epsilon$.

The projection of this subshift is a compact invariant subset K of M which contains only points with infinitely many visits to the controlled set $\kappa(\mathcal{R})$ in the future and in the past. If K met the singularity lines of T, there would be such a point x with the additional property that $Tx \in \partial W^u(Tx)$ or $T^{-1}x \in W^s(T^{-1}x)$. However, this would prevent any future or past visit to $\kappa(\mathcal{R})$, a contradiction.

Finally, the above construction makes the isomorphism with $\Sigma(\mathcal{G}_n)$ obvious (it is indeed one-to-one as K does not meet the boundary of P), but the latter is a subshift of finite type as $\mathcal{G}_n$ is finite. □

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A. *Appendix. Bounds on metric entropy*

We recall (see, e.g., [**1**]) some standard notation and facts about entropy and a few consequences: $(X, \mathcal{B}, \mu)$ is a probability space; $H_\mu(P) := -\sum_{A\in P} \mu(A) \log \mu(A)$ denotes the mean entropy of a partition (we leave implicit all measurability assumptions). For $Y \subset X$, $(\mu|Y)(\cdot) := (\mu(Y))^{-1}\mu(\cdot \cap Y)$ (zero if $\mu(Y) = 0$). For a sub-σ-algebra $\mathcal{A}$ of $\mathcal{B}$, the *conditional entropy* is

$$H_\mu(P|\mathcal{A}) := \int_X - \sum_{A\in P} 1_A \log E(1_A|\mathcal{A})\, d\mu,$$

where $E(\cdot|\mathcal{A})$ is the conditional expectation with respect to μ.

First, if P is a partition, $Y \subset X$ and $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{A}$, then

$$\begin{aligned} &H_\mu(P \vee \{Y, X \setminus X\}|\mathcal{A}) \\ &\quad \le H_\mu(\{Y, X \setminus X\}|\mathcal{A}) + \mu(Y) H_{\mu|Y}(P|\mathcal{A}) + \mu(X \setminus Y) H_{\mu|(X\setminus Y)}(P|\mathcal{A}). \end{aligned} \tag{A.1}$$

Second, the entropy of a measure can be computed as the average of the entropies given the past. More precisely, we have the following statement.

LEMMA A.1. *Let μ be an invariant probability measure for some bimeasurable bijection $T : X \to X$. Let P be a finite, measurable partition. Then*

$$h(T, \mu, P) = \int_X - \sum_{A\in P} 1_A \log E(1_A|P^-)\, \mu(dx) \tag{A.2}$$

where with P^- is the past partition *generated by $T^n P$, $n \ge 1$.*

In particular, if $N(n, x, P) = \#\{P_0^{n-1}(y) \mid y \in X$ and $P_{-\infty}^{-1}(y) = P_{-\infty}^{-1}(x)\}$ where $P_a^b(x) := (A_n)_{a\le n\le b}$ with $T^n x \in A_n$, then

$$h(T, \mu, P) \le \int_X \frac{1}{n} \log N(n, x, P)\, \mu(dx). \tag{A.3}$$

Proof. For (A.2), see, e.g., [**30**, Ex. 4(b), p. 243] for entropy as an average of conditional information.

Observe that

$$E(1_A \log E(1_A|P^-)|P^-)(x) = E(1_A|P^-)(x) \log E(1_A|P^-)(x).$$

Hence, the integrand in (A.2) can be replaced by the above right-hand side. Equation (A.3) now follows from the standard bound:

$$- \sum_{A\in P^N} E(1_A|P^-)(x) \log E(1_A|P^-)(x) \le \log \#\{A \in P^N \mid A \cap W^u(x) \ne \emptyset\}. \qquad \square$$

In combination with Rudolph's backward Vitali lemma [**31**, Theorem 3.9, p. 33], this yields the following convenient estimate.

PROPOSITION A.2. *Let μ be an ergodic, σ-invariant probability measure on $\mathcal{A}^{\mathbb{Z}}$ with finite alphabet $\mathcal{A}$. Assume that there exist a measurable family of subsets $W(A^-, \ell) \subset P^\ell$ (for $A^- \in \mathcal{A}^{\mathbb{Z}_-}$, $\ell \geq 1$) with cardinality bounded by $Ce^{H\ell}$ and a subset $\Sigma_0 \subset \mathcal{A}^{\mathbb{Z}}$ of positive measure such that, for all $A \in \Sigma_0$, there are sequences of integers $a_i = a_i(A)$, $b_i = b_i(A)$, $i \geq 1$ (depending measurably on A) satisfying*

(1) $\lim_{i\to\infty} b_i - a_i = \infty$;

(2) $\sup_i \inf\{|k| \mid k \in [\![a_i, b_i]\!]\} < \infty$;

(3) $A_{a_i} A_{a_i+1} \cdots A_{b_i-1} \in W(\cdots A_{a_i-2}A_{a_i-1}, b_i - a_i)$.

Then

$$h(\sigma, \mu) \leq H.$$

Condition (2) above means that the intervals $[\![a_i, b_i]\!]$ do not escape to infinity: they all intersect some $[-R, R]$ for some R large.

Proof. To apply Rudolph's backward Vitali lemma, we need

$$a_i(A) \leq 0 \leq b_i(A) \tag{A.4}$$

for all large enough i, for all $A \in \Sigma_0$. By passing to subsequences, depending on A, we can assume the existence of the (possibly infinite) limits $\lim_{i\to\infty} a_i(A)$, $\lim_{i\to\infty} b_i(A)$ for all $A \in \Sigma_0$. Assume, for instance, that $\lim_i a_i(A) = -\infty$ and $\lim_{i\to\infty} b_i(A) < 0$ for almost every $A \in \Sigma_0$, the other cases being similar or trivial. By assumption, $\inf_i b_i(A) > -\infty$ for all $A \in \Sigma_0$. Restricting Σ_0 we can assume that this infimum is some fixed number $b \in \mathbb{Z}$. Replacing Σ_0 by $\sigma^{\min(b,0)}\Sigma_0$ ensures that (A.4) holds.

The Rudolph lemma implies that for any $\epsilon > 0$, for μ-almost every A, for all large enough integers n, one can find a disjoint cover of a fraction at least $1-\epsilon$ of $[\![0, n[\![$ by at most ϵn intervals $[\![a_i, b_i]\!]$ such that $A_{a_i} \cdots A_{b_i} \in W(\cdots A_{a_i-2}A_{a_i-1}, b_i - a_i)$ Applying (A.3) with

$$N(A, n) \leq \binom{n}{2\epsilon n} e^{Hn} \times \#\mathcal{A}^{\epsilon n}.$$

gives that $h(\sigma, \mu) \leq H + 3\epsilon \log \epsilon + \epsilon \log \#\mathcal{A}$. We conclude by letting $\epsilon \to 0^+$. □

B. *Appendix. Tower lifts*

We study towers from a point of view closely related to that of Zweimuller [**36**]. Let T be an ergodic invertible transformation of a probability space (X, μ) and let B be a measurable subset of X. A *return time* is a function $\tau : B \to \bar{\mathbb{N}}^* := \{1, 2, \ldots, \infty\}$ which is measurable and such that $T^{\tau(x)}(x) \in B$ for all $x \in B$ with $\tau(x) < \infty$ (but τ is not necessarily the first return time).

We are interested in lifting T-invariant measures to the following *invertible tower*:

$$\begin{aligned}\hat{X} := \{(x, \omega) \in X \times \{0, 1\}^{\mathbb{Z}} \mid \omega_n = 1 \implies T^n x \in B \text{ and}\\ \tau(T^n x) = \min\{k \geq 1 \mid \omega_{n+k} = 1\}\} \setminus \hat{X}_*\end{aligned} \tag{B.1}$$

with $\hat{T}(x, \omega) = (Tx, \sigma(x))$ and $\hat{X}_*$ is the set of (x, ω) with only finitely many ones either in the future or in the past of ω.

Observe that

$$(x,\omega),\,(x,\omega')\in\hat X \text{ and } \omega_n=\omega'_n=1 \text{ for some } n \implies \text{for all } k\ge n,\ \omega_k=\omega'_k. \quad \text{(B.2)}$$

Here $(\hat X,\hat T)$ is an extension of a subset of (X,T) through $\hat\pi:\hat X\to X$ defined by $\hat\pi(x,\omega)=x$.

Remark B.1. The jump transformation $T^\tau:\{x\in B\mid \tau(x)<\infty\}\to B$ is defined by $T^\tau(x):=T^{\tau(x)}(x)$. It is closely related to $\hat T$. Indeed, T^τ is isomorphic to the first return map of $\hat T$ on $[1]:=\{(x,\omega)\in\hat X\mid\omega_0=1\}$ so any $\hat T$-invariant probability measure gives by restriction and normalization a T^τ-invariant probability measure (see [**36**]).

Such lifting requires that τ be 'not too large' (see [**36**] where the classical integrability condition is studied). Our condition is in terms of the following 'iterates' of τ: the functions $\tau_m:B\to\bar{\mathbb N}^*$, $m\ge 1$, are defined, as before, by $\tau_1:=\tau$ and $\tau_{m+1}(x):=\tau(T^{\tau_m(x)}(x))$ if $\tau_m(x)<\infty$, $\tau_{m+1}(x):=\infty$ otherwise.

Definition B.2. We say that $x\in X$ has an *improper orbit* if

$$n(x):=\{n\in\mathbb N\mid T^{-n}x\in B \text{ and } \forall m\ge 1,\ \tau_m(T^{-n}x)<\infty\} \text{ is finite.} \quad \text{(B.3)}$$

We say that $x\in X$ has *t-gaps* for some $0<t<\infty$ if x has an improper orbit or if there exist two integer sequences $(n_k)_{k\in\mathbb N}$ and $(m_k)_{k\in\mathbb N}$, $m_k>0$ for all $k\ge 0$, such that

$$\text{for all } k\ge 0,\ \tau_{m_k}(T^{n_k}x)\ge\max(t\cdot m_k,k) \quad\text{and}$$
$$\sup_{k\ge 1}\min\{|i|\mid i\in[n_k,n_k+\tau_{m_k}(T^{n_k}x)]\}<\infty.$$

A measure has *eventually bounded gaps*, if for some $t<\infty$, the set of points in X with t-gaps has zero measure.

Note that $\tau(T^nx)=\infty$, for a single n, implies that x has t-gaps for any $t<\infty$.

PROPOSITION B.3. *Let $T:X\to X$ be a self-map with a return time $\tau:B\to\mathbb N^*$. Then:*

- *every T-invariant ergodic probability measure μ with eventually bounded gaps can be lifted to a $\hat T$-invariant ergodic probability measure on $\hat X$;*
- *any $\hat T$-invariant, ergodic probability measure $\hat\mu$ is a finite extension of the T-invariant measure $\hat\pi(\hat\mu)$.*

Proof of Proposition B.3. We first prove the existence of a lift for μ like above. We follow the strategy of [**36**] and [**24**] (which was inspired by constructions of Hofbauer) and define the following *non-invertible tower* to obtain a convenient topology:

$$\tilde X:=\{(x,k,\tau)\in X\times\mathbb N\times\mathbb N\mid \exists y\in B\ \tau(y)=\tau,\ k<\tau \text{ and } x=T^ky\},$$
$$\tilde T(x,k,\tau):=(T(x),k+1,\tau) \text{ if } k+1<\ell,\ (T(x),0,\tau(T(x))) \text{ otherwise.}$$

For any integer K, we write $\tilde X_K:=\{(x,k,\tau)\in\tilde X\mid k=K\}$, $\tilde X_{\le K}:=\bigcup_{k\le K}\tilde X_k$ and define $\tilde\pi(x,k,\tau)=x$. Observe that $\tilde\pi\circ\tilde T=T\circ\tilde\pi$ and that $(\hat X,\hat T)$ is a natural extension of $(\tilde X,\tilde T)$ through $(x,\omega)\mapsto(x,k,\ell)$ with $k\ge 0$ minimal such that $\omega_{-k}=1$ and $\ell=\tau(T^{-k}x)$. Hence, it is enough to lift μ to $\tilde X$.

Fix $t < \infty$ such that the set of points of X with t-gaps has zero μ-measure. Let $\tilde{\mu}_0$ be the probability measure defined by

$$\tilde{\mu}_0(\{(x, 0, \tau(x)) \mid x \in A\}) = \mu(A) \text{ for all Borel sets } A$$

(sets disjoint from those above have zero $\tilde{\mu}_0$-measure). We have $\tilde{\pi}(\tilde{\mu}_0) = \mu$ but, except in trivial cases, $\tilde{T}_*\tilde{\mu}_0 \neq \tilde{\mu}_0$ so we consider

$$\tilde{\mu}_n := \frac{1}{n}\sum_{k=0}^{n-1} \tilde{T}^k \tilde{\mu}_0$$

and try to take some accumulation point $\tilde{\mu}$. We identify $\tilde{\mu}_n$ with its density with respect to $\tilde{\mu}_\infty$, the σ-finite measure defined by

$$\tilde{\mu}_\infty(\{(x, k, \tau(T^{-k}x)) \mid x \in A \text{ and } k < \tau(T^{-k}x)\}) = \mu(A \cap \{\tau > k\})$$

for all Borel sets $A \subset B$ and all $k \geq 0$. As $\tilde{\pi}(\tilde{\mu}_n) = \mu$, we must have

$$\frac{d\tilde{\mu}_n}{d\tilde{\mu}_\infty} \leq 1.$$

Using the Banach–Alaoglu theorem, i.e. the weak star compactness of the unit ball of $L^\infty(\tilde{\mu}_\infty)$ as the dual of $L^1(\tilde{\mu}_\infty)$, we obtain an accumulation point of the $\tilde{\mu}_n$, i.e. a measure $\tilde{\mu}$ on $\tilde{X}$ with $d\tilde{\mu}/d\tilde{\mu}_\infty \leq 1$ such that, for some subsequence $n_k \to \infty$,

$$\text{for all } f \in L^1(\tilde{\mu}_\infty), \quad \lim_{k\to\infty} \int f \, d\tilde{\mu}_{n_k} = \int f \, d\tilde{\mu}. \tag{B.4}$$

Observe that $\tilde{\mu}$ is $\tilde{T}$-invariant: indeed, (B.4) together with the T-invariance of μ implies that $d\tilde{\mu} \circ \tilde{T}^{-1}/d\tilde{\mu} \leq 1$ whereas $\tilde{\mu} \circ \tilde{T}^{-1}(\tilde{X}) = \tilde{\mu}(\tilde{X})$ so the previous inequality must be an equality $\tilde{\mu}$-almost everywhere.

This invariance and the ergodicity of μ implies that $\tilde{\pi}\tilde{\mu} = \alpha\mu$ for some $0 \leq \alpha \leq 1$. It remains to prove that $\tilde{\mu} \neq 0$ so that it can be renormalized into the announced lift of μ. Assume by contradiction that $\tilde{\mu} = 0$. Hence, for any $L < \infty$,

$$\int 1_{\tilde{X}_{\leq L}} \, d\tilde{\mu}_{n_k} = \int \frac{1}{n_k} \#\{0 \leq k < n_k \mid \tilde{T}^k(x, 0, \tau(x)) \in \tilde{X}_{\leq L}\} \, d\mu \to 0.$$

So, possibly for a further subsequence,

$$\frac{1}{n_k}\#\{0 \leq k < n_k \mid \tilde{T}^k(x, 0, \tau(x)) \in \tilde{X}_{\leq L}\} \to 0 \quad \mu\text{-almost everywhere.} \tag{B.5}$$

Now,

$$\#\{0 \leq k < n \mid \tilde{T}^k(x, 0, \tau(x)) \in \tilde{X}_0\} < \epsilon n \implies \tau_{\epsilon n}(x) \geq n.$$

Hence, (B.5) implies that x (in fact, any of its preimages in the natural extension) has t-gaps for all $t > 0$, contradicting the assumption on μ.

We now show that any $\hat{T}$-invariant, ergodic probability measure $\hat{\mu}$ is a finite extension of $\mu := \hat{\pi}(\hat{\mu})$. By definition of $\hat{X}$, $\hat{\mu}([1]) > 0$ where $[1] = \{(x, \omega) \in \hat{X} : \omega_0 = 1\}$. Assume that there is some positive measure subset $S \subset \hat{X}$, and some number K of measurable functions:

$$\omega^1, \ldots, \omega^K : S \to \{0, 1\}^{\mathbb{Z}}$$

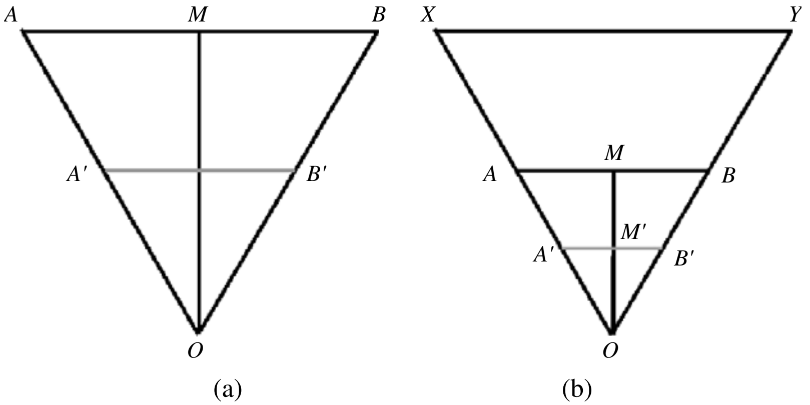

FIGURE 6. Geometry of: (a) a continuous piecewise affine map with $h_{\text{mult}}(T, P) > h_{\text{top}}(T) = 0$; (b) a discontinuous piecewise affine map with no maximum measure.

such that, for all $x \in S$, $(x, \omega^i(x)) \in \hat{X}$, $\omega^i(x) \neq \omega^j(x)$ for $i \neq j$ and, for all $j = 1, \ldots, K$:

$$\lim_{n\to\infty} \frac{1}{n}\#\{0 \le k < n \mid \omega^j_{-k}(x) = 1\} = \hat{\mu}([1]).$$

If $K \cdot \hat{\mu}([1]) > 1$, then, for almost every $x \in S$, there exist two distinct indices j, $j' \in \{1, \ldots, K\}$ and arbitrarily large integers $n_k \to \infty$ such that $\omega^j_{-n_k}(x) = \omega^{j'}_{-n_k}(x)$. However, this implies $\omega^j(x) = \omega^{j'}(x)$ by (B.2). The contradiction proves $K \le \hat{\mu}([1])^{-1} < \infty$: $\hat{\mu}$ is a finite extension of μ. □

C. *Appendix. Examples*

C.1. *Positive multiplicity entropy*

Example 1. (Buzzi [**6**]) There exists a continuous, piecewise affine surface map (M, T, P) with $h_{\text{mult}}(T, P) > 0$ and $h_{\text{top}}(T) = 0$.

Consider some triangle ABO in $\mathbb{R}^2$ with non-empty interior and let M, A', B' be the middle points of $[AB]$, $[AO]$, $[BO]$ (see Figure 6(a)). Let T be affine in each of the triangles $\tau_0 := AMO$ and $\tau_1 := BMO$ with $T(O) = O$, $T(A) = T(B) = A'$, $T(M) = B'$ so that $T : ABO \to ABO$ is conjugate to $(\theta, r) \mapsto (1 - 2|\theta|, r/2)$ on $(-1, 1) \times (0, 1)$. Take $P = \{\tau_0, \tau_1\}$ as the admissible partition.

We have $h_{\text{mult}}(T, P) = \log 2$ (because all words on $\{\tau_0, \tau_1\}$ appear in the symbolic dynamics and the corresponding cylinders contain O in their closure). On the other hand, the only invariant probability measure is the Dirac supported by O, hence $h_{\text{top}}(T) = 0$ as claimed.

Example 2. (Kruglikov and Rypdal [**25**]) There exists a piecewise affine homeomorphism (M, T, P) with $\dim M = 3$ and $h_{\text{mult}}(T, P) = h_{\text{top}}(T) > 0$.

Let $([0, 1]^2, T_2, P_2)$ be a piecewise affine homeomorphism with non-zero topological entropy. Consider the pyramid $M := \widehat{[0, 1]^2}$ where $\widehat{A}$ denotes the convex subset of $\mathbb{R}^3$

generated by $O := (0, 0, 0)$ and $A \times \{1\}$. Define $T : M \to M$ as the piecewise affine map with partition $P := \{\widehat{A} \setminus \{O\} \mid A \in P_2\}$ such that $T(O) = O$, $T(x, y, 1) = (T_2(x, y), 1)$ for each vertex (x, y) of P_2. Observe that $h_{\mathrm{top}}(T) = h_{\mathrm{top}}(T_2)$ and that T has an obvious measure of maximal entropy carried by the invariant set $[0, 1]^2 \times \{1\}$. Finally, considering T^n around $(0, 0, 0)$ it is easy to see that $h_{\mathrm{mult}}(T, P) = h_{\mathrm{top}}(T_2) = h_{\mathrm{top}}(T)$.

Example 3. There exists a piecewise affine homeomorphism (M, S, P) with $\dim M = 3$, $h_{\mathrm{mult}}(S, P) > 0$ and $h_{\mathrm{top}}(S) = 0$.

Define S from the previous example T by $S(x, y, z) := T(x, y, z)/2$ on the pyramid M so that 0 is a sink. To make S onto, add a symmetric pyramid M^- whose summit is a source.

C.2. *No maximal entropy measure*

Example 4. There exists a piecewise affine surface (M, T, P) *discontinuous map* T such that there is no maximum measure. More precisely, there exists a sequence of invariant probability measures μ_n with

$$\lim_{n\to\infty} h(T, \mu_n) = h_{\mathrm{top}}(T) > 0$$

but μ_n converges weakly to an invariant Dirac measure.

Remark. The above formulation excludes trivial examples like $T : [0, 1] \to [0, 1]$ with $T(x) = 1/4 + x/2$ for $x > 1/2$ and $T(x) = x + 1/2$ for $x \leq 1/2$ which has no invariant probability measure.

Let T be a piecewise affine map defined on the triangle XYO with $O = (0, 0)$, $X = (-2, 2)$ and $Y = (2, 2)$. Let $A = (-1, 1)$, $B = (1, 1)$ and $M = (0, 1)$, and $A' = A/2$, $B' = B/2$ and $M' = M/2$ (see Figure 6(b)). We require that:
(1) $T|\overline{XYBA}$ is the identity;
(2) $T : AMO \to A'B'O$ is affine with $A \mapsto A'$, $M \mapsto B'$, $O \mapsto O$;
(3) $T : MBO \to YXO$ is affine with $M \mapsto Y$, $B \mapsto X$, $O \mapsto O$.
It is easy to see that $h_{\mathrm{top}}(T) = \log 2$. We claim that $\sup_\mu h(T, \mu) = \log 2$. Clearly the supremum is bounded by $h_{\mathrm{top}}(f)$. Conversely, for any $h < \log 2$, one can find an invariant measure on the full shift $(\sigma, \{0, 1\}^{\mathbb{N}})$ such that $\mu([1^K]) = 0$ for some $K = K(h) < \infty$ with $h(\sigma, \mu) > h$. It is then easy to construct an isomorphic T-invariant measure (with support included in $y \leq y_0$ for any given $0 < y_0 < 1$), proving that $\sup_\mu h(T, \mu) \geq \log 2$. The equality follows from $h_{\mathrm{top}}(T) = \log 2$. The same observations allow the construction of the sequence μ_n with the claimed properties.

On the other hand, assume that μ is an invariant and ergodic probability measure with $h(T, \mu) = 2$. Here μ must be supported on $y < 1$. Hence, the map π that sends a point of $\mathbb{R}^2$ to the ray from the origin that contains it maps (T, μ) to $(f, \pi_*\mu)$ where $f : \theta \mapsto 1 - 2|\theta|$ on $[-1, 1]$. The fibers of π are contained in line segments originating from O on which T is linear, hence they have zero entropy and π is entropy-preserving [**3**]. This implies that $\pi_*\mu$ is the $(1/2, 1/2)$-Bernoulli measure. Using, say, the central limit theorem, we obtain that, for μ-almost $(x, y) \in ABO$, there exists a positive integer n such that

$$\#\{0 \leq k < n \mid f^k(\pi(x, y)) < 1/2\} < \frac{n}{2} - \frac{|\log y|}{2\log 2}$$

so that $T^m(x, y) \in XYBA$ for some $m \leq n$, contradicting the invariance of the measure: there is no maximum measure.

Example 5. There exists a continuous, piecewise *quadratic* surface map T such that for any invariant probability measure μ:

$$h(T, \mu) < \sup_\nu h(T, \nu).$$

On the rectangle $[1, 2] \times [-1, 1]$, consider $T(x, y) := (x, T_x(y))$ with

$$T_x(y) = \begin{cases} \dfrac{x(2-x)}{2} - x|y| & \text{if } |y| < 2 - x, \\ -\dfrac{x(2-x)}{2} & \text{otherwise.} \end{cases}$$

For each $1 \leq x < 2$, $[-1, 1]$ is mapped into the T_x-forward invariant segment $[-x(2-x)/2, x(2-x)/2]$ on which T_x has constant slope x. Hence, $h_{\text{top}}(T_x) = \log x$ for $x \neq 2$. Clearly, $T(2, y) = (2, 0)$ so $h_{\text{top}}(T_2) = 0$.

C.3. *Infinitely many maximal entropy measures*

Example 6. There is a piecewise affine continuous map (respectively homeomorphism) of $[0, 1]^2$ (respectively $[0, 1]^3$) with uncountably many ergodic invariant probability measures with non-zero, maximal entropy.

Indeed, such examples are trivially obtained from piecewise affine maps on $[0, 1]$ or homeomorphisms on $[0, 1]^2$ with non-zero topological entropy by taking a direct product with the identity on the unit interval. It is the *low dimension* (one for maps, two for homeomorphisms) that prevents the existence of such indifferent factors and ensures the finite number of maximum measures under the simple condition of non-zero topological entropy.

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