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ASYMPTOTIC PROPERTIES OF SELF-NORMALIZED LINEAR PROCESSES WITH LONG MEMORY

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In this paper we study the convergence to fractional Brownian motion for long memory time series having independent innovations with infinite second moment. For the sake of applications we derive the self-normalized version of this theorem. The study is motivated by models arising in economic applications where often the linear processes have long memory, and the innovations have heavy tails.

1. INTRODUCTION AND NOTATION

In this paper we study the asymptotic properties of a causal linear process

$$X_k = \sum_{i\geq 0} a_i \varepsilon_{k-i} \tag{1}$$

when the independent and identically distributed (i.i.d.) innovations $\{\varepsilon, \varepsilon_n; n \in \mathbb{Z}\}$ have infinite variance and $\{a_i; i \geq 0\}$ is a sequence of real constants such that X_k is well defined. More precisely, everywhere in the paper, we assume that the innovations are centered and in the domain of attraction of a normal law. This means that the variables are i.i.d.,

$$\mathrm{E}\varepsilon = 0 \tag{2}$$

and

$$l(x) = \mathrm{E}\varepsilon^2 I(|\varepsilon| \leq x) \quad \text{is a slowly varying function at } \infty. \tag{3}$$

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We say that $h(t)$, defined for $t \geq 0$, is slowly varying if it is positive and measurable on $[A, \infty)$, for some $A > 0$, and if for any $\lambda > 0$, we have $\lim_{x\to\infty} h(\lambda x)/h(x) = 1$ (Seneta, 1976, Def. 1.1).

We define

$$S_n = \sum_{i=1}^{n} X_i.$$

The central limit theorem (CLT) for S_n with i.i.d. innovations and infinite variance when $\sum_{i\geq 0} |a_i| < \infty$ was studied by many authors. We mention among others Knight (1991), Mikosch, Gadrich, Kliipelberg, and Adler (1995), and Wu (2003). For this case the CLT was obtained under a normalization that is regularly varying with exponent $\frac{1}{2}$.

The purpose of this paper is to investigate the CLT in its functional form for the case when

$$a_n = n^{-\alpha} L(n), \quad \text{where } \tfrac{1}{2} < \alpha < 1, \qquad n \geq 1 \tag{4}$$

and $L(n)$ is a slowly varying function at ∞ in the strong sense (i.e., there is a slowly varying function $h(t)$ such that $L(n) = h(n)$). Notice that, by the definition of slowly varying function, the coefficients a_n are positive for n sufficiently large. We shall obtain convergence in distribution under a normalization that is regularly varying with exponent $\frac{3}{2} - \alpha$ which is strictly larger than $\frac{1}{2}$. This is the reason why the time series we consider has long memory.

To give an example of a linear process of this type we mention the fractionally integrated processes because they play an important role in financial time series modeling and they are widely studied. Such processes are defined for $0 < d < \frac{1}{2}$ by

$$X_k = (1 - B)^{-d} \varepsilon_k = \sum_{i\geq 0} a_i \varepsilon_{k-i} \quad \text{with } a_i = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)}, \tag{5}$$

where B is the backward shift operator, $B\varepsilon_k = \varepsilon_{k-1}$. For this example, by the well-known fact that for any real x, $\lim_{n\to\infty} \Gamma(n+x)/n^x \Gamma(n) = 1$, we have $\lim_{n\to\infty} a_n/n^{d-1} = 1/\Gamma(d)$.

The CLT in its functional form was intensively studied for the case of i.i.d. innovations with finite second moment. We refer to Davydov (1970), Taqqu (1975), Phillips and Solo (1992), Wang, Lin, and Gulati (2003), Wu and Min (2005), and Dedecker, Merlevède, and Peligrad (2011), among others. Invariance principles (or functional CLTs) play an important role in econometrics and statistics. For example, to obtain asymptotic distributions of unit root test statistics, researchers have applied invariance principles of various forms; see Phillips (1987) and Wu (2006).

We shall derive here the CLT and its functional form, i.e., convergence to fractional Brownian motion, for the case when the innovations are in the domain of attraction of the normal distribution and the constants satisfy (4). The normalizer in this theorem depends on the slowly varying function $l(x)$ that is in general unknown. To make our results easily applicable we also study the CLT in its self-normalized form.

The self-normalized CLT for sums of i.i.d. random variables was treated in the paper by Giné, Götze, and Mason (1997). The case of self-normalized sums in the domain of attraction of other stable laws was considered by Chistyakov and Götze (2004). A systematic treatment of self-normalized limit theory under the independence assumption is given by de la Peña, Lai, and Shao (2009). The self-normalized version of the CLT for this case was treated in Csörgő, Szyszkowicz, and Wang (2003). Kulik (2006) studied the self-normalized functional CLT when $\sum_{i\geq 0}|a_i| < \infty$. We shall consider the long memory case when coefficients satisfy (4).

Our paper is organized as follows. Section 2 contains the definitions and the results; an application to unit root testing is discussed in Section 3; the proofs are given in Section 4. For convenience, in the Appendix we give some auxiliary results, and we also mention some known facts needed for the proofs.

In this paper we shall use the following notation: a double indexed sequence with indexes n and i will be denoted by a_{ni} when no confusion is possible, and sometimes by $a_{n,i}$. We use the notation $a_n \sim b_n$ instead of $a_n/b_n \to 1$. For positive sequences, the notation $a_n \ll b_n$ replaces Vinogradov symbol O, and it means that a_n/b_n is bounded; $a_n = o(b_n)$ stays for $a_n/b_n \to 0$. The term $[x]$ denotes the integer part of x, the notation $\Rightarrow$ is used for weak convergence, and $\stackrel{P}{\to}$ denotes convergence in probability. By $\mathrm{var}(X)$ we denote the variance of the random variable X and by $\mathrm{cov}(X, Y)$ the covariance of X and Y. The weak convergence to a constant means convergence in probability. We denote by $D[0, 1]$ the space of all functions on $[0, 1]$ that have left-hand limits and are continuous from the right, and $N(0, 1)$ denotes a standard normal random variable.

2. RESULTS

To introduce our results we define a normalizing sequence in the following way. Recall (3) and (4). Let $b = \inf\{x \geq 1 : l(x) > 0\}$, define

$$\eta_j = \inf\left\{s : s \geq b+1, \frac{l(s)}{s^2} \leq \frac{1}{j}\right\}, \qquad j = 1, 2, \ldots, \tag{6}$$

and set

$$B_n^2 := c_\alpha l_n n^{3-2\alpha} L^2(n) \quad \text{with } l_n = l(\eta_n), \tag{7}$$

where

$$c_\alpha = \left\{\int_0^\infty [x^{1-\alpha} - \max(x-1, 0)^{1-\alpha}]^2 dx\right\} \Big/ (1-\alpha)^2. \tag{8}$$

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THEOREM 2.1. *Define* $\{X_n; n \geq 1\}$ *by (1) and the random element* $W_n(t) = S_{[nt]}/B_n$ *on the space* $D[0,1]$. *Assume conditions (2)–(4) are satisfied. Then,* $W_n(t)$ *converges weakly on the space* $D[0,1]$ *endowed with Skorokhod topology to the fractional Brownian motion* W_H *with Hurst index* $H = \frac{3}{2} - \alpha$. *In particular, for* $t = 1$, *we have that* S_n/B_n *converges in distribution to a standard normal variable.*

Remark 2.1. In a recent paper (Peligrad and Sang, 2011) we treat the CLT for the situation when B_n^2 is not necessarily regularly varying. However, for that situation the convergence to the fractional Brownian motion might fail. As a matter of fact, in the context of Theorem 2.1 a necessary condition for the convergence to the fractional Brownian motion W_H with Hurst index $H = \beta$ is the representation $B_n^2 = n^{2\beta} h(x)$ for a function $h(x)$ that is slowly varying at infinity (see Lamperti, 1962).

To successfully apply this theorem we have to know l_n that depends on the distribution of ε. This can be avoided by constructing a self-normalizer. Denote $\sum_{i=0}^{\infty} a_i^2 = A^2$. Our result is as follows.

THEOREM 2.2. *Under the same conditions as in Theorem 2.1 we have*

$$\frac{1}{nl_n} \sum_{i=1}^{n} X_i^2 \xrightarrow{P} A^2 \tag{9}$$

and therefore

$$\frac{S_{[nt]}}{na_n \sqrt{\sum_{i=1}^{n} X_i^2}} \Rightarrow \frac{\sqrt{c_\alpha}}{A} W_H(t).$$

In particular

$$\frac{S_n}{na_n \sqrt{\sum_{i=1}^{n} X_i^2}} \Rightarrow N\left(0, \frac{c_\alpha}{A^2}\right).$$

3. APPLICATION TO UNIT ROOT TESTING

Invariance principles play an important role in characterizing the limit distribution of various statistics arising from the inference in economic time series.

Let us consider a stochastic process generated according to

$$Y_n = \rho Y_{n-1} + X_n \quad \text{for } n \geq 1,$$

where $Y_0 = 0$, $(X_n)_{n \geq 1}$ is a stationary sequence, and ρ is a constant. Denote the ordinary least squares estimator of ρ by

$$\hat{\rho}_n = \sum_{k=1}^{n} Y_k Y_{k-1} \Big/ \sum_{k=1}^{n} Y_{k-1}^2.$$

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To test $\rho = 1$ against $\rho < 1$, a key step is to derive the limit distribution of the well-known Dickey–Fuller test statistic (Dickey and Fuller, 1979, 1981):

$$\hat{\rho}_n - 1 = \sum_{k=1}^{n} Y_{k-1}(Y_k - Y_{k-1}) \Big/ \sum_{k=1}^{n} Y_{k-1}^2.$$

As shown by Phillips (1987), under the null hypothesis $\rho = 1$, the asymptotic properties of the Dickey–Fuller test statistic rely heavily on the invariance principles. This problem was widely studied under various assumptions on the sequence X_n. Among them Sowell (1990) and Wu (2006) considered the unit root testing problem for long memory processes. By combining our Theorems 2.1 and 2.2 with arguments similar to Phillips (1987), we can formulate the following result obtained for variables that do not necessarily have finite second moment.

PROPOSITION 3.1. *Assume that $(X_n)_{n\geq 1}$ is as in Theorem 2.1. Then the following results hold:*

$$(i) \quad \frac{\sum_{k=1}^{n} Y_{k-1}^2}{n^3 a_n^2 \sum_{i=1}^{n} X_i^2} \Rightarrow \frac{c_\alpha}{A^2}\int_0^1 W_H^2(t)dt,$$

$$(ii) \quad \frac{\sum_{k=1}^{n} Y_{k-1}(Y_k - Y_{k-1})}{n^2 a_n^2 \sum_{i=1}^{n} X_i^2} \Rightarrow \frac{c_\alpha W_H^2(1)}{2A^2},$$

$$(iii) \; n(\hat{\rho}_n - 1) \Rightarrow \frac{W_H^2(1)/2}{\int_0^1 W_H^2(t)dt}.$$

The proof of this proposition requires us only to make obvious changes in the proofs of (A1) and (A2) on page 296 in Phillips (1987), and it is left to the reader.

4. PROOFS

4.1. Proof of Theorem 2.1

To prove the CLT in its functional form, i.e., the weak convergence of $S_{[nt]}/B_n$ on the space $D[0,1]$ to the fractional Brownian motion W_H with Hurst index $H = \frac{3}{2} - \alpha$, we shall first reduce the problem to truncated random variables. For the truncated process we establish tightness on $D[0,1]$ and the convergence of finite-dimensional distributions.

Without loss of generality, in the rest of the paper, we assume for convenience $a_0 = 0$ in definition (1).

We shall divide the proof into several steps.

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Step 1. Existence. To show that X_1 is well defined we use stationarity and Lemma A.2 from the Appendix. First of all we have

$$\sum_{i=1}^{\infty} P(|a_i \varepsilon_{1-i}| > 1) = \sum_{i=1}^{\infty} P(|\varepsilon| > |a_i|^{-1}) = \sum_{i=1}^{\infty} a_i^2 o(l(|a_i|^{-1})).$$

Then, by taking into account that (2) implies $\mathrm{E}\varepsilon I(|\varepsilon| \le |a_i|^{-1}) = -\mathrm{E}\varepsilon I(|\varepsilon| > |a_i|^{-1})$,

$$\sum_{i=1}^{\infty} |\mathrm{E}a_i \varepsilon_{1-i} I(|a_i \varepsilon_{1-i}| \le 1)| \le \sum_{i=1}^{\infty} |a_i| \mathrm{E}|\varepsilon| I(|\varepsilon| > |a_i|^{-1}) = \sum_{i=1}^{\infty} a_i^2 o(l(|a_i|^{-1}))$$

and

$$\sum_{i=1}^{\infty} \mathrm{E}a_i^2 \varepsilon_{1-i}^2 I(|a_i \varepsilon_{1-i}| \le 1) = \sum_{i=1}^{\infty} a_i^2 \mathrm{E}\varepsilon^2 I(|\varepsilon| \le |a_i|^{-1}) = \sum_{i=1}^{\infty} a_i^2 l(|a_i|^{-1}).$$

Notice that

$$\sum_{i=1}^{\infty} a_i^2 l(|a_i|^{-1}) = \sum_{i=1}^{\infty} i^{-2\alpha} L^2(i) l(i^{2\alpha} L^{-2}(i)) < \infty,$$

because $\frac{1}{2} < \alpha < 1$ and $L^2(i) l(i^{2\alpha} L^{-2}(i))$ is a slowly varying function at ∞. The existence in the almost sure sense follows by combining these arguments with the three series theorem.

Step 2. Truncation. For the case when $\mathrm{E}\varepsilon^2 = \infty$, which is relevant to our paper, the truncation is necessary. The challenge is to find a suitable level of truncation. For any integer $1 \le k \le n$ define

$$X'_{nk} = \sum_{i=1}^{\infty} a_i \varepsilon_{k-i} I(|\varepsilon_{k-i}| \le \eta_{n-k+i}) \quad \text{and} \quad S'_n = \sum_{k=1}^{n} X'_{nk}. \tag{10}$$

This definition has the advantage that S'_n can be expressed as a simple sum of a linear process of an array of independent variables. For every $m \ge 1$ we denote

$$b_m = a_1 + \cdots + a_m, \tag{11}$$

and then we introduce the coefficients

$$b_{nj} = b_j = a_1 + \cdots + a_j \quad \text{for } j < n, \tag{12}$$

$$b_{nj} = b_j - b_{j-n} = a_{j-n+1} + \cdots + a_j \quad \text{for } j \ge n.$$

With this notation and recalling definition (6), by changing the order of summation,

$$S'_n = \sum_{i \geq 1} b_{ni} \varepsilon_{n-i} I(|\varepsilon_{n-i}| \leq \eta_i). \tag{13}$$

We shall reduce next the study of limiting distribution of S_n/B_n to the sequence S'_n/B_n. It is enough to show that

$$\frac{1}{B_n} \mathrm{E}|S_n - S'_n| \to 0. \tag{14}$$

To see this we use the fact that by Lemma A.2 stated in the Appendix

$$\mathrm{E}|\varepsilon| I(|\varepsilon| > \eta_i) = o(\eta_i^{-1} l_i).$$

We also know that

$$\eta_n^2 \sim n l_n \tag{15}$$

(see, e.g., relation (13) in Csörgő et al., 2003). Then, by the triangle inequality and relation (A.4) of Lemma A.4 from the Appendix applied with $p = 1$, we obtain

$$\begin{aligned} \mathrm{E}|S_n - S'_n| &\leq \sum_{i \geq 1} |b_{ni}| \mathrm{E}|\varepsilon| I(|\varepsilon| > \eta_i) = \sum_{i \geq 1} |b_{ni}| o(\eta_i^{-1} l_i) \\ &= \sum_{i \geq 1} |b_{ni}| o(i^{-1/2} l_i^{1/2}) = o(n^{3/2-\alpha} l_n^{1/2} L(n)) = o(B_n), \end{aligned} \tag{16}$$

and so (14) is established.

Step 3. Central limit theorem. To make the proof more transparent we shall present first the CLT for S_n/B_n. By step 2 it is enough to find the limiting distribution of S'_n/B_n. We start by noticing that by (16) and the fact that the variables are centered we have

$$|\mathrm{E}S'_n| = |\mathrm{E}(S_n - S'_n)| = o(B_n). \tag{17}$$

One of the consequences of this observation is that S'_n/B_n has the same limiting distribution as $(S'_n - \mathrm{E}S'_n)/B_n$. Furthermore,

$$\operatorname{var}\left(\frac{S'_n}{B_n}\right) = \frac{1}{B_n^2} \sum_{i \geq 1} b_{ni}^2 l_i - \frac{1}{B_n^2} (\mathrm{E}S'_n)^2 \to 1$$

by relation (A.3) in Lemma A.4 and (17).

Moreover, by Lemma A.4(i) for $k_n = n^{4/(2\alpha-1)}$

$$\begin{aligned} \operatorname{var}\left(\sum_{i \geq k_n} b_{ni} \varepsilon_{n-i} I(|\varepsilon_{n-i}| \leq \eta_i)\right) &\ll \sum_{i \geq k_n} n^2 (i-n)^{-2\alpha} L^2(i) l_i \\ &= o(1) \quad \text{as } n \to \infty. \end{aligned} \tag{18}$$

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Then, by Theorem 4.1 in Billingsley (1968), for proving the CLT it is enough to verify Lyapunov's condition for $B_n^{-1}(\bar{S}_n' - \mathrm{E}\bar{S}_n')$ where

$$\bar{S}_n' = \sum_{i=1}^{k_n} b_{ni}\varepsilon_{n-i} I(|\varepsilon_{n-i}| \le \eta_i).$$

Clearly, by (18), $\mathrm{var}(\bar{S}_n'/B_n) \to 1$. In the estimate that follows we use Lemma A.2(iv) along with (15), followed by relation (A.4) of Lemma A.4 applied with $p = 3$ and the fact that $B_n \to \infty$ to get

$$\begin{aligned}\sum_{j=1}^{k_n} |b_{nj}|^3 \mathrm{E}|\varepsilon' - \mathrm{E}\varepsilon'|^3 &\le 8\sum_{j=1}^{k_n} |b_{nj}|^3 \mathrm{E}|\varepsilon|^3 I(|\varepsilon| \le \eta_j) \\ &= \sum_{j=1}^{k_n} |b_{nj}|^3 \eta_j o(l_j) \le \sum_{j=1}^{\infty} |b_{nj}|^3 j^{1/2} o\left(l_j^{3/2}\right) \\ &= o(n^{3(3/2-\alpha)} l_n^{3/2} L^3(n)) = o(B_n^3).\end{aligned} \tag{19}$$

By Lyapunov's CLT and the preceding considerations, S_n/B_n converges to $N(0,1)$ in distribution.

Step 4. Preliminary considerations for the convergence to fractional Brownian motion. For $n \ge 1$ fixed we implement the same level of truncation as before and construct $\{X_{nj}'; 1 \le j \le n\}$ by definition (10). Then we introduce the processes

$$W_n'(t) = \frac{1}{B_n}\sum_{j=1}^{[nt]} X_{nj}' \quad \text{and} \quad W_n''(t) = W_n(t) - W_n'(t).$$

We shall show first that $W_n''(t)$ is negligible for the weak convergence on $D[0,1]$ and then, in the next steps, that $W_n'(t)$ is weakly convergent to the fractional Brownian motion.

To explain this step, it is convenient to express the process in an expanded form. By using notation (11)

$$\begin{aligned}W_n''(t) &= \frac{1}{B_n}\sum_{i=0}^{[nt]-1} b_{[nt]-i}\varepsilon_i I(|\varepsilon_i| > \eta_{n-i}) \\ &\quad + \frac{1}{B_n}\sum_{i\ge 1}(b_{[nt]+i} - b_i)\varepsilon_{-i} I(|\varepsilon_{-i}| > \eta_{n+i}).\end{aligned}$$

We notice that by the triangle inequality,

$$\begin{aligned}\mathrm{E}\left(\sup_{0\le t\le 1}|W_n''(t)|\right) &\le \frac{1}{B_n}\mathrm{E}\left(\sup_{0\le t\le 1}\left|\sum_{i=0}^{[nt]-1} b_{[nt]-i}\varepsilon_i I(|\varepsilon_i| > \eta_{n-i})\right|\right) \\ &\quad + \frac{1}{B_n}\mathrm{E}\left(\sup_{0\le t\le 1}\left|\sum_{i\ge 1}(b_{[nt]+i} - b_i)\varepsilon_{-i} I(|\varepsilon_{-i}| > \eta_{n+i})\right|\right).\end{aligned}$$

Then, by monotonicity and using the notation (12)

$$\mathrm{E}\left(\sup_{0\le t\le 1}|W_n''(t)|\right) \le \frac{1}{B_n}\sum_{i=0}^{n-1}|b_{n-i}|\mathrm{E}|\varepsilon|I(|\varepsilon|>\eta_{n-i}) + \frac{1}{B_n}\sum_{i\ge 1}|b_{n+i}-b_i|\mathrm{E}|\varepsilon|I(|\varepsilon|>\eta_{n+i}) = \frac{1}{B_n}\sum_{i\ge 1}|b_{ni}|\mathrm{E}|\varepsilon|I(|\varepsilon|>\eta_i), \tag{20}$$

which is exactly the quantity shown to converge to 0 in (16). By Theorem 4.1 in Billingsley (1968), it is enough to study the limiting behavior of $W_n'(t)$.

Step 5. Tightness. As before, we reduce the problem to studying the same problem for $W_n'(t)-\mathrm{E}W_n'(t)$. This is easy to see because by the fact the variables are centered and by (20) we clearly obtain

$$\sup_{0\le t\le 1}|\mathrm{E}W_n'(t)| = \sup_{0\le t\le 1}|\mathrm{E}W_n''(t)| \le \mathrm{E}\left(\sup_{0\le t\le 1}|W_n''(t)|\right)\to 0. \tag{21}$$

To show that $W_n'(t)-\mathrm{E}W_n'(t)$ is tight in $D[0,1]$ we shall verify the conditions from Lemma A.5 in the Appendix for the triangular array $B_n^{-1}(X_{nk}'-EX_{nk}')$, $1\le k\le n$. This will be achieved in the following two lemmas.

By the properties of slowly varying functions (see Seneta, 1976; Lemma A.1 in the Appendix) we construct first an integer N_0 and positive constants K_i such that for all $m>N_0$ we have simultaneously

$$\max_{1\le j\le m} b_j^2 \le K_1 m^{2-2\alpha}L^2(m), \tag{22}$$

$$l_{2m} \le K_2 l_m, \tag{23}$$

$$\sup_{k\geq 2m}\frac{(b_k-b_{k-m})^2}{k^{-2\alpha}L^2(k)} \le K_3 m^2, \tag{24}$$

$$\sum_{j\ge m} j^{-2\alpha}L^2(j)l_j \le K_4 m^{1-2\alpha}L^2(m)l_m, \tag{25}$$

and

$$\sum_{j\ge m} j^{-2\alpha}L^2(j) \le K_4 m^{1-2\alpha}L^2(m). \tag{26}$$

This is possible by Lemmas A.4 and A.1.

LEMMA 4.1. *There are a constant K and an integer N_0 such that for any two integers p and q with $1\le p<q\le n$ with $q-p\ge N_0$ and any $n\ge N_0$*

$$\frac{1}{B_n^2}\mathrm{var}\left(\sum_{i=p+1}^{q}X_{ni}'\right)\le K\left(\frac{q}{n}-\frac{p}{n}\right)^{2-\alpha}. \tag{27}$$

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Proof. We shall use N_0 that was constructed earlier. We start from the decomposition

$$\sum_{i=p+1}^{q} X'_{ni} = \sum_{i=p}^{q-1} b_{q-i}\varepsilon_i I(|\varepsilon_i| \le \eta_{n-i}) + \sum_{i=2p-q}^{p-1} (b_{q-i} - b_{p-i})\varepsilon_i I(|\varepsilon_i| \le \eta_{n-i})$$
$$+ \sum_{i \ge q-2p+1} (b_{q+i} - b_{p+i})\varepsilon_{-i} I(|\varepsilon_{-i}| \le \eta_{n+i}) = I + II + III.$$

We shall estimate the variance of each term separately.

Using the fact that l_n is increasing and (23) we obtain

$$\operatorname{var}(I) \le \sum_{i=p}^{q-1} b_{q-i}^2 l_{n-i} = \sum_{j=1}^{q-p} b_j^2 l_{n-q+j} \le l_n(q-p) \max_{1\le j\le q-p} b_j^2$$
$$\le K_1(q-p)^{3-2\alpha} L^2(q-p) l_n.$$

Then, by taking into account that l_n is increasing and (22) and (23) we have

$$\operatorname{var}(II) \le \sum_{i=2p-q}^{p-1} (b_{q-i} - b_{p-i})^2 l_{n-i} \le l_{2n} 2(q-p) \max_{1\le j\le 2(q-p)} b_i^2$$
$$\le K_1 K_2 (q-p)^{3-2\alpha} L^2(q-p) l_n.$$

To estimate the variance of the last term, we use first (24) to obtain

$$\operatorname{var}(III) = \sum_{i\ge q-2p+1} (b_{i+q} - b_{i+p})^2 l_{n+i} \le \sum_{j\ge 2(q-p)+1} (b_j - b_{j-(q-p)})^2 l_{n+j-q}$$
$$\le K_3 (q-p)^2 \sum_{j\ge 2(q-p)+1} j^{-2\alpha} L^2(j) l_{n+j-q}.$$

Now, by the monotonicity of l_n, because $l_{n+j-q} \le l_{2n}$ for $j \le n$ and $l_{n+j-q} \le l_{2j}$ for $j > n$ by (23), (25), and (26)

$$\sum_{j\ge 2(q-p)+1} j^{-2\alpha} L^2(j) l_{n+j-q} \le K_2 K_5 (q-p)^{-2\alpha+1} L^2(q-p) l_n$$
$$+ K_2 K_4 (q-p)^{-2\alpha+1} L^2(q-p) l_{q-p}.$$

So, for $K_6 = K_2 K_3 (K_4 + K_5)$

$$\operatorname{var}(III) \le K_6 (q-p)^{3-2\alpha} L^2(q-p) l_n.$$

Overall we have so far for a certain constant K_7 that does not depend on p or q,

$$\operatorname{var}\left(\sum_{i=p+1}^{q} X'_{ni}\right) \le K_7 (q-p)^{3-2\alpha} L^2(q-p) l_n. \qquad \textbf{(28)}$$

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By simple algebra, because $1 \leq p < q \leq n$ we derive

$$\mathrm{var}\left(\sum_{i=p+1}^{q} X'_{ni}\right) \leq K_7 (q-p)^{2-\alpha} l_n n^{1-\alpha} L^2(n) \max_{1\leq k\leq n} \frac{k^{1-\alpha}}{n^{1-\alpha}} \frac{L^2(k)}{L^2(n)}.$$

Finally, by Lemma A.1(v),

$$\mathrm{var}\left(\sum_{i=p+1}^{q} X'_{ni}\right) \leq K_8 \left(\frac{q}{n} - \frac{p}{n}\right)^{2-\alpha} l_n n^{3-2\alpha} L^2(n).$$

Therefore, (27) is established by taking into account (7). ■

LEMMA 4.2. *Condition (A.12) is satisfied, namely:*

$$\lim_{n\to\infty} P\left(\max_{1\leq k\leq n} |X'_{nk} - \mathrm{E}X'_{nk}| \geq \varepsilon B_n\right) = 0.$$

Proof. We start from

$$P\left(\max_{1<k\leq n} |X'_{nk} - \mathrm{E}X'_{nk}| \geq \varepsilon B_n\right) \leq \frac{1}{\varepsilon^4 B_n^4} \sum_{k=1}^{n} \mathrm{E}|X'_{nk} - \mathrm{E}X'_{nk}|^4.$$

We now use Rosenthal inequality (de la Peña and Giné, 1999, Thm.1.5.13), which can be easily extended to an infinite sum of independent random variables, by truncating the sum and passing to the limit. So, there is a constant C, such that

$$\mathrm{E}|X'_{nk} - \mathrm{E}X'_{nk}|^4 \leq C \sum_{i=1}^{\infty} a_i^4 \mathrm{E}\varepsilon^4 I(|\varepsilon| \leq \eta_{n-k+i}) + C\left(\sum_{i=1}^{\infty} a_i^2 l_{n-k+i}\right)^2 = I_k + II_k.$$

By Lemma A.2(iv) and (15) it follows that

$$\begin{aligned} a_i^4 \mathrm{E}\varepsilon^4 I(|\varepsilon| \leq \eta_{n-k+i}) &\ll a_i^4 (\eta_{n-k+i}^2) l_{n-k+i} \\ &\ll i^{-4\alpha} L^4(i)(n-k+i) l_{n-k+i}^2. \end{aligned}$$

So

$$\sum_{k=1}^{n} I_k \leq \sum_{i=1}^{\infty} i^{-4\alpha} L^4(i) \sum_{k=i}^{n+i} k l_k^2 \ll n^2 l_n^2.$$

Then, by simple computations involving the partition of sum in two parts, one up to $2n$ and the rest, and then using the properties of regular functions and the fact that $2\alpha > 1$ we obtain

$$\sum_{k=1}^{n} II_k \leq n\left(\sum_{i=1}^{\infty} i^{-2\alpha} L^2(i) l_{n+i}\right)^2 \leq n l_n^2.$$

Finally by (7) we notice that

$$\frac{n^2 l_n^2}{B_n^4} \to 0. \qquad \blacksquare$$

Step 6. Convergence of finite-dimensional distributions. Let $0 \le t_1 < t_2 < \cdots < t_m \le 1$. We shall show next that the vector $(W_n'(t_j); 1 \le j \le m)$ converges in distribution to the finite-dimensional distributions of a fractional Brownian motion with Hurst index $\frac{3}{2} - 2\alpha$, i.e., of a Gaussian process with covariance structure $\frac{1}{2}(t^{3-2\alpha} + s^{3-2\alpha} - (t-s)^{3-2\alpha})$ for $s < t$.

By the Cramér–Wold device and taking into account (21) we have to study the limiting distribution of $\sum_{j=2}^{m} \lambda_j (W_n'(t_j) - \mathrm{E}W_n'(t_{j-1}))$, which we express as a weighted sum of independent random variables. By elementary computations involving similar arguments used in the proof of step 3, and taking into account (16) and (19), we notice that Lyapunov's condition is satisfied, and then the limiting distribution is normal with the covariance structure that will be specified next. We compute now the covariance of $W_n'(s)$ and $W_n'(t)$ for $s \le t$. By simple algebra

$$\mathrm{cov}(W_n'(t), W_n'(s)) = \frac{1}{2}(\mathrm{var}(W_n'(t)) + \mathrm{var}(W_n'(s)) - \mathrm{var}(W_n'(t) - W_n'(s))).$$

We analyze now the variance of $W_n'(t)$. For each t fixed, $0 \le t \le 1$

$$\begin{aligned}\mathrm{var}(W_n'(t)) &= \frac{1}{B_n^2} \sum_{i=0}^{[nt]-1} b_{[nt]-i}^2 (\mathrm{E}\varepsilon_0^2 I(|\varepsilon_0| \le \eta_{n-i}) - \mathrm{E}^2 \varepsilon_0 I(|\varepsilon_0| \le \eta_{n-i})) \\ &\quad + \frac{1}{B_n^2} \sum_{i \ge 1} (b_{[nt]+i} - b_i)^2 (\mathrm{E}\varepsilon_0^2 I(|\varepsilon_0| \le \eta_{n+i}) \\ &\quad - \mathrm{E}^2 \varepsilon_0 I(|\varepsilon_0| \le \eta_{n+i})).\end{aligned}$$

Taking into account $\mathrm{E}\varepsilon_0 I(|\varepsilon_0| \le \eta_{n-i}) = -\mathrm{E}\varepsilon_0 I(|\varepsilon_0| > \eta_{n-i})$, by Lemmas A.2 and A.4, after some computations, we obtain

$$\mathrm{var}(W_n'(t)) \sim \frac{1}{B_n^2} \sum_{i=0}^{[nt]-1} b_{[nt]-i}^2 l_{n-i} + \frac{1}{B_n^2} \sum_{i \ge 1} (b_{[nt]+i} - b_i)^2 l_{n+i}.$$

With a similar proof as of relation (A.2) of Lemma A.4, for every $0 \le t \le 1$

$$\mathrm{var}(W_n'(t)) \to t^{3-2\alpha}$$

and for every $0 \le s < t \le 1$

$$\mathrm{var}(W_n'(t) - W_n'(s)) \to (t-s)^{3-2\alpha}. \tag{29}$$

Then

$$\mathrm{cov}(W_n'(t), W_n'(s)) \to \frac{1}{2}\left(t^{3-2\alpha} + s^{3-2\alpha} - (t-s)^{3-2\alpha}\right),$$

which is the desired covariance structure. $\blacksquare$

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4.2. Proof of Theorem 2.2

We notice that it is enough to prove only the convergence in (9). Then (4), (7), and (9) imply

$$B_n^2 \sim c_\alpha n^2 a_n^2 \left(\sum_{j=1}^{n} X_j^2 \right) \Big/ A^2,$$

which we combine with Theorem 2.1, via Slutsky's theorem, to obtain the self-normalized part of the theorem. The proof of (9) will be decomposed in several steps.

Step 1. Truncation. Denote $D_n^2 = A^2 n l_n$. Recall the definition (10) and set $X''_{nj} = X_j - X'_{nj}$. To prove (9) it is enough to establish

$$F_n = \sum_{j=1}^{n} (X''_{nj})^2 \Big/ D_n^2 \xrightarrow{P} 0 \tag{30}$$

and

$$G_n = \sum_{j=1}^{n} (X'_{nj})^2 \Big/ D_n^2 \xrightarrow{P} 1. \tag{31}$$

To see this we square the decomposition $X_j = X'_{nj} + X''_{nj}$; then we sum with j from 1 to n and notice that by the Hölder inequality

$$F_n - 2(G_n F_n)^{1/2} \le \frac{1}{D_n^2} \sum_{j=1}^{n} X_j^2 - G_n \le F_n + 2(G_n F_n)^{1/2}.$$

Step 2. Proof of (30). We start from

$$\begin{aligned}\sum_{k=1}^{n} (X''_{nk})^2 &= \sum_{k=1}^{n} \sum_{i=1}^{\infty} a_i^2 \varepsilon_{k-i}^2 I(|\varepsilon_{k-i}| > \eta_{n-k+i}) \\ &\quad + 2 \sum_{k=1}^{n} \sum_{i<j} a_i a_j \varepsilon_{k-i} I(|\varepsilon_{k-i}| > \eta_{n-k+i}) \varepsilon_{k-j} I(|\varepsilon_{k-j}| > \eta_{n-k+j}) \\ &= I + II\end{aligned}$$

(here and in what follows $\sum_{i<j}$ denotes double summation). By independence, monotonicity, and Lemma A.2(iii), we easily deduce that

$$\begin{aligned}\mathrm{E}|II| &\le 2 \sum_{k=1}^{n} \sum_{i<j} |a_i a_j| \mathrm{E}|\varepsilon_{k-i} I(|\varepsilon_{k-i}| > \eta_{n-k+i}) \varepsilon_{k-j} I(|\varepsilon_{k-j}| > \eta_{n-k+j})| \\ &\le 2n \sum_{i<j} |a_i a_j| \mathrm{E}|\varepsilon| I(|\varepsilon| > \eta_i) \mathrm{E}|\varepsilon| I(|\varepsilon| > \eta_j) \\ &= 2n \sum_{i<j} |a_i a_j| o(\eta_i^{-1} l_i) o(\eta_j^{-1} l_j).\end{aligned}$$

Then, by (15), clearly

$$\mathrm{E}|II| \le n\left(\sum_{i\ge 1}|a_i|i^{-1/2}o\left(l_i^{1/2}\right)\right)^2.$$

Because $\sum_{i\ge 1}|a_i|i^{-1/2} < \infty$, and l_n is increasing, it is easy to see that

$$\mathrm{E}|II| = o(nl_n) = o\left(D_n^2\right).$$

To estimate the contribution of the term I, by changing the order of summation we express this term in the following way:

$$I = \sum_{j=1}^{n}\left(\sum_{i=1}^{j}a_i^2\right)\varepsilon_{n-j}^2 I(|\varepsilon_{n-j}| > \eta_j) + \sum_{j=n+1}^{\infty}\left(\sum_{i=j-n+1}^{j}a_i^2\right)\varepsilon_{n-j}^2 I(|\varepsilon_{n-j}| > \eta_j).$$

We implement now the notation

$$A_{nj}^2 = A_j^2 = \sum_{i=1}^{j}a_i^2 \quad \text{when } j \le n \quad \text{and} \quad A_{nj}^2 = \sum_{i=j-n+1}^{j}a_i^2 \quad \text{when } j > n, \tag{32}$$

and then we express I as

$$I = \sum_{j=1}^{\infty}A_{nj}^2\varepsilon_{n-j}^2 I(|\varepsilon_{n-j}| > \eta_i).$$

Clearly A_{nj}^2 are uniformly bounded by a constant. In addition, by relation (A.5), for $j > 2n$, these coefficients have the following order of magnitude:

$$\begin{aligned} A_{nj}^2 &\ll n^2(j-n)^{-2\alpha-1}\max_{j-n\le k\le j}L^2(k) \le n^2(j-n)^{-2\alpha-1}\max_{j/2\le k\le j}L^2(k) \\ &\ll n^2(j-n)^{-2\alpha-1}\min_{j/2\le k\le j}L^2(k) \le n^2(j-n)^{-2\alpha-1}L^2(j). \end{aligned} \tag{33}$$

Now, we use first Khinchin's inequality (see de la Peña and Giné, 1999, Lem. 1.4.13) followed by the triangle inequality and Lemma A.2, and relation (15) to obtain

$$\begin{aligned} \mathrm{E}\sqrt{I} \ll \mathrm{E}\left|\sum_{j=1}^{\infty}A_{nj}\varepsilon_{n-j}I(|\varepsilon_{n-j}| > \eta_j)\right| &= \sum_{j=1}^{\infty}A_{nj}o\left(\eta_j^{-1}l_j\right) \\ &= \sum_{j=1}^{\infty}A_{nj}j^{-1/2}o\left(l_j^{1/2}\right). \end{aligned}$$

We notice that by (33), Lemma A.1(iv), and the fact that $\alpha > \frac{1}{2}$,

$$\begin{aligned} \mathrm{E}\sqrt{I} &= o\left(\sqrt{nl_n}\right) + n\sum_{j\ge n}^{\infty}j^{-\alpha-\frac{1}{2}}L^2(j+n)j^{-\frac{1}{2}}o\left(l_j^{1/2}\right) \\ &= o\left(\sqrt{nl_n}\right) + O\left(n^{-\alpha}L^2(n)l_j^{1/2}\right) = o\left(\sqrt{nl_n}\right). \end{aligned}$$

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As a consequence, $\sqrt{I/nl_n}$ converges in L_1 to 0, and so I/D_n^2 is convergent to 0 in probability. By gathering all these facts we deduce that (30), holds and the proof is reduced to show that (31) holds. ■

Step 3. Proof of (31). We express the sum of squares as

$$\sum_{k=1}^{n}(X_k')^2 = \sum_{k=1}^{n}\sum_{i=1}^{\infty} a_i^2\varepsilon_{k-i}^2 I(|\varepsilon_{k-i}| \le \eta_{n-k+i}) + 2\sum_{k=1}^{n}\sum_{1\le i<j} a_i a_j \varepsilon_{k-i} I(|\varepsilon_{k-i}| \le \eta_{n-k+i})\varepsilon_{k-j} I(|\varepsilon_{k-j}| \le \eta_{n-k+j}).$$

We shall show that

$$\frac{1}{D_n^2}\sum_{k=1}^{n}\sum_{i=1}^{\infty} a_i^2\varepsilon_{k-i}^2 I(|\varepsilon_{k-i}| \le \eta_{n-k+i}) \xrightarrow{P} 1 \tag{34}$$

and

$$\frac{1}{D_n^2}\sum_{k=1}^{n}\sum_{1\le i<j} a_i a_j \varepsilon_{k-i} I(|\varepsilon_{k-i}| \le \eta_{n-k+i})\varepsilon_{k-j} I(|\varepsilon_{k-j}| \le \eta_{n-k+j}) \xrightarrow{P} 0. \tag{35}$$

We establish first (34).

By using the notation (32), we have

$$\sum_{k=1}^{n}\sum_{i=1}^{\infty} a_i^2\varepsilon_{k-i}^2 I(|\varepsilon_{k-i}| \le \eta_{n-k+i}) = \sum_{i=1}^{\infty} A_{ni}^2\varepsilon_{n-i}^2 I(|\varepsilon_{n-i}| \le \eta_i).$$

By independence, Lemma A.2(iv), and relations (33) and (15), and taking into account that $\alpha > \frac{1}{2}$, we get

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^{\infty} A_{ni}^2\varepsilon_{n-i}^2 I(|\varepsilon_{n-i}| \le \eta_i)\right) &\le \sum_{i=1}^{\infty} A_{ni}^4 \mathrm{E}\varepsilon^4 \mathbf{1}(|\varepsilon| \le \eta_i)\\ &= \sum_{i=1}^{2n}\eta_i^2 o(l_i) + \sum_{i\ge 2n}(i-n)^{-4\alpha-2}L^4(i)n^4\eta_i^2 o(l_i)\\ &= o(n^2 l_n^2) = o(D_n^4).\end{aligned}$$

So (34) is reduced to showing that

$$\frac{1}{D_n^2}\sum_{i=1}^{\infty} A_{ni}^2 \mathrm{E}(\varepsilon^2 I(|\varepsilon| \le \eta_i)) = \frac{1}{D_n^2}\sum_{i=1}^{\infty} A_{ni}^2 l_i \to 1 \quad \text{as } n\to\infty.$$

We divide the sum into three parts, one from 1 to n, one from $n+1$ to $2n$, and the rest of the series. We easily see that by (33),

$$\sum_{j=2n+1}^{\infty} A_{nj}^2 l_j \ll \sum_{j=2n+1}^{\infty}(j-n)^{-2\alpha-1}n^2 L^2(j) l_j = o(D_n^2).$$

Then,

$$\sum_{j=n+1}^{2n} A_{nj}^2 l_j \ll \sum_{j=n+1}^{2n} (j-n)^{1-2\alpha} l_j \max_{1\le i\le n} L(i)$$
$$\ll \left(\sum_{k=1}^{n} k^{1-2\alpha} l_{n+k}\right) \max_{1\le i\le n} L(i) = o(D_n^2).$$

Now, by the proof of relation (A.6) in the Appendix with the only difference that we replace a_i by a_i^2 and so b_n^2 by A_j^2 we obtain

$$\sum_{j=1}^{n} A_j^2 l_j \sim l_n \sum_{j=1}^{n} A_j^2. \qquad \textbf{(36)}$$

Finally, by (36), the definition of D_n^2, and the Toeplitz lemma, (A.8) in the Appendix, it follows that

$$\lim_{n\to\infty} \frac{1}{D_n^2} \sum_{j=1}^{n} A_j^2 l_j = \lim_{n\to\infty} \frac{l_n \sum_{j=1}^n A_j^2}{l_n n A^2} = \lim_{n\to\infty} \frac{A_n^2}{A^2} = 1.$$

This completes the proof of (34).

We move now to prove (35). Let N be a fixed positive integer. For each $1 \le k \le n$ we divide the sum into two parts:

$$\sum_{j=1}^{N} \sum_{i=1}^{j-1} a_i a_j \varepsilon_{k-i} I(|\varepsilon_{k-i}| \le \eta_{n-k+i}) \varepsilon_{k-j} I(|\varepsilon_{k-j}| \le \eta_{n-k+j})$$
$$+ \sum_{j>N} \sum_{i=1}^{j-1} a_i a_j \varepsilon_{k-i} I(|\varepsilon_{k-i}| \le \eta_{n-k+i}) \varepsilon_{k-j} I(|\varepsilon_{k-j}| \le \eta_{n-k+j})$$
$$= I_k + II_k.$$

We estimate the variance of the sum of each term separately.

For estimating $\mathrm{var}(\sum_{k=1}^n II_k)$ we apply the Hölder inequality:

$$\mathrm{var}\left(\sum_{k=1}^{n} II_k\right) \le n \sum_{k=1}^{n} \mathrm{var}\left(\sum_{j>N} \sum_{i=1}^{j-1} a_i a_j \varepsilon_{k-i} I(|\varepsilon_{k-i}| \le \eta_{n-k+i}) \varepsilon_{k-j} \times I(|\varepsilon_{k-j}| \le \eta_{n-k+j})\right).$$

By independence, a term corresponding to the combination of indexes $(k-i_1, k-j_1, k-i_2, k-j_2)$ with $i_1 < j_1$ has a nonnull contribution if and only if $i_1 = i_2$ and $j_1 = j_2$, leading to

$$\mathrm{var}\left(\sum_{k=1}^{n} II_k\right) \le n^2 \sum_{j>N} \sum_{i=1}^{j-1} a_i^2 a_j^2 l_{n+i} l_{n+j} = (n^2 l_n^2) o_N(1),$$

where we used first the monotonicity of l_n and in the last part we used the fact that (by monotonicity, the definition of slowly varying functions, and our notation)

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$l_i \leq l_{2n} \ll l_n$ for $i \leq 2n$ and $l_{i+n} \leq l_{\frac{3i}{2}} \ll l_i$ for $i > 2n$ along with the convergence of the series $\sum_i a_i^2 l_i$.

To treat the other term we start from

$$\operatorname{var}\left(\sum_{k=1}^{n} I_k\right) = \operatorname{var}\left(\sum_{j=1}^{N}\sum_{i=1}^{j-1} a_i a_j \sum_{k=1}^{n} \varepsilon_{k-i} I(|\varepsilon_{k-i}| \leq \eta_{n-k+i})\varepsilon_{k-j} \times I(|\varepsilon_{k-j}| \leq \eta_{n-k+j})\right),$$

and then, because we compute the variance of at most N^2 sums and because the coefficients a_i are bounded, clearly,

$$\operatorname{var}\left(\sum_{k=1}^{n} I_k\right) \ll N^4 \max_{1\leq i<j\leq N} \operatorname{var}\left(\sum_{k=1}^{n} \varepsilon_{k-i} I(|\varepsilon_{k-i}| \leq \eta_{n-k+i})\varepsilon_{k-j} I(|\varepsilon_{k-j}| \leq \eta_{n-k+j})\right).$$

We notice now that

$$\operatorname{var}\left(\sum_{k=1}^{n} \varepsilon_{k-i} I(|\varepsilon_{k-i}| \leq \eta_{n-k+i})\varepsilon_{k-j} I(|\varepsilon_{k-j}| \leq \eta_{n-k+j})\right) \leq \sum_{k=1}^{n} \mathrm{E}\varepsilon_{k-i}^2 I(|\varepsilon_{k-i}| \leq \eta_{n-k+i}) \mathrm{E}\varepsilon_{k-j}^2 I(|\varepsilon_{k-j}| \leq \eta_{n-k+j}),$$

because by independence and the fact that $i \neq j$ all the other terms are equal to 0. The result is

$$\operatorname{var}\left(\sum_{k=1}^{n} I_k\right) \ll N^4 \sum_{k=1}^{n} l_{n,n-k+i} l_{n,n-k+j} \ll N^4 (n l_n^2).$$

Overall

$$\frac{1}{D_n^4}\operatorname{var}\left(\sum_{k=1}^{n}\sum_{1\leq i<j} \varepsilon_{k-i} I(|\varepsilon_{k-i}| \leq \eta_{n-k+i})\varepsilon_{k-j} I(|\varepsilon_{k-j}| \leq \eta_{n-k+j})\right) \leq \frac{2}{D_n^4}\operatorname{var}\left(\sum_{k=1}^{n} I_k\right) + \frac{2}{D_n^4}\operatorname{var}\left(\sum_{k=1}^{n} II_k\right) = o_N(1) + O\left(N^4 \frac{1}{n}\right).$$

We conclude that (35) holds by letting first $n \to \infty$ followed by $N \to \infty$. ■

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APPENDIX

We formulate in the first lemma several properties of the slowly varying function. Their proofs can be found in Seneta (1976).

LEMMA A.1. *A slowly varying function $l(x)$ defined on $[A,\infty)$ has the following properties:*

(i) There exists $B \ge A$ such that for all $x \ge B$, $l(x)$ is representable in the form $l(x) = g(x)\exp(\int_B^x a(y)/y\,dy)$, where $g(x) \to c_0 > 0$ and $a(x) \to 0$ as $x \to \infty$. In addition $a(x)$ is continuous.

(ii) *For* $B < c < C < \infty$, $\lim_{x\to\infty} l(tx)/l(x) = 1$ *uniformly in* $c \le t \le C$.

(iii) *For any* $\theta > -1$, $\int_B^x y^\theta l(y)dy \sim x^{\theta+1} l(x)/\theta + 1$ *as* $x \to \infty$.

(iv) *For any* $\theta < -1$, $\int_x^\infty y^\theta l(y)dy \sim x^{\theta+1} l(x)/-\theta - 1$ *as* $x \to \infty$.

(v) *For any* $\eta > 0$, $\sup_{t\ge x}(t^\eta l(t)) \sim x^\eta l(x)$ *as* $x \to \infty$. *Moreover* $\sup_{t\ge x}(t^\eta l(t)) = x^\eta \bar{l}(x)$ *where* $\bar{l}(x)$ *is slowly varying and* $\bar{l}(x) \sim l(x)$.

The following lemma contains some equivalent formulation for variables in the domains of attraction of normal law (3). It is Lemma 1 in Csörgő et al. (2003); see also Feller (1966).

LEMMA A.2. *The following statements are equivalent.*

(i) $l(x) = \mathrm{E}X^2 I(|X| \le x)$ is a slowly varying function at ∞;

(ii) $P(|X| > x) = o(x^{-2} l(x))$;

(iii) $\mathrm{E}|X| I(|X| > x) = o(x^{-1} l(x))$;

(iv) $\mathrm{E}|X|^\alpha I(|X| \le x) = o(x^{\alpha-2} l(x))$ for $\alpha > 2$.

To clarify the behavior of the sequence of normalizer B_n^2 defined by (7) we state the following lemma that follows from relations (3.33) and (3.44) in Kuelbs (1985).

LEMMA A.3. *Assume (3) and define* η_n *by (6). Then,* $l_n = l(\eta_n)$ *is a slowly varying function at* ∞.

The next lemma is useful to study the variance of partial sums for truncated random variables.

LEMMA A.4. *Under the conditions of Theorem 2.1 and with the notation (8) and (12) we have these results.*

(i) The coefficients have the following order of magnitude: There are constants C_1 and C_2 such that for all $n \ge 1$,

$$|b_{ni}| \le C_1 i^{1-\alpha} |L(i)| \quad \text{for } i \le 2n \quad \text{and} \quad |b_{ni}| \le C_2 n (i-n)^{-\alpha} |L(i)| \quad \text{for } i > 2n \tag{A.1}$$

$$\sum_{i=1}^{\infty} b_{ni}^2 \sim c_\alpha n^{3-2\alpha} L^2(n). \tag{A.2}$$

(ii) The asymptotic equivalence for the variance is

$$\sum_{i\ge1} b_{ni}^2 l_i \sim l_n \sum_{i\ge1} b_{ni}^2 \sim B_n^2, \tag{A.3}$$

where B_n^2 is defined by (7).

(iii) For any $p \ge 1$ and any function $h(x)$ slowly varying at ∞,

$$\sum_{i\ge1} |b_{ni}|^p i^{-1+p/2} |h(i)| \ll h(n) n^{p(3/2-\alpha)} L^p(n). \tag{A.4}$$

Proof. The fact that $|b_{ni}| \le C_1 i^{1-\alpha} |L(i)|$ for $i \le 2n$ follows easily by the properties of slowly varying functions listed in Lemma A.1.

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For $i > 2n$, by the properties of strong slowly varying functions, for n sufficiently large,

$$(i-n)^{-\alpha}L(i-n)+\cdots+i^{-\alpha}L(i) \le [(i-n)^{-\alpha}+\cdots+i^{-\alpha}] \max_{i-n\le j\le i} L(j).$$

Then,

$$\max_{i-n\le j\le i} L(j) \le \max_{i/2\le j\le i} L(j) \ll L(i)$$

because

$$\frac{\max_{m\le j\le 2m} L(j)}{\min_{m\le j<2m} L(j)} \to 1. \tag{A.5}$$

The asymptotic equivalence in (A.2) is well known. See, e.g., Theorem 2 in Wu and Min (2005).

We turn now to show (A.3). Let M be a positive integer. We divide the sum into three parts, one from 1 to n, one from $n+1$ to nM, and the third one with all the other terms. The idea of the proof is that for n and M large, the sum from 1 to nM dominates the sum of the rest of the terms.

We treat each of these three sums separately.

By using the definition of $b_{ni} = a_1+\cdots+a_i = b_i$ for $1\le i\le n$ by analogy with Lemma A.1(iii) we show that

$$\sum_{i=1}^{n} b_i^2 l_i \sim l_n \sum_{i=1}^{n} b_i^2. \tag{A.6}$$

To see this, by the first part of Lemma A.1 we have $l_n = g_n h_n$ where $h_n = \exp\left(\int_B^n \frac{a(y)}{y}dy\right)$, $g_n \to c > 0$, $a(x)\to 0$ as $x\to\infty$, and $a(x)$ is continuous. It is easy to show that

$$h_n - h_{n-1} = o(h_n/n) \quad \text{as } n\to\infty \tag{A.7}$$

and also, by part (iii) of the same lemma, we get $\sum_{i=1}^{n-1} b_i^2 \ll n b_n^2$.

Next, we just have to use the well-known Toeplitz lemma:

$$\lim_{n\to\infty}\frac{c_n}{d_n} = \lim_{n\to\infty}\frac{c_n - c_{n-1}}{d_n - d_{n-1}}, \tag{A.8}$$

provided $d_n\to\infty$ and the limit on the right-hand side exists. Then it follows that

$$\lim_{n\to\infty}\frac{\sum_{i=1}^n b_i^2 l_i}{l_n\sum_{i=1}^n b_i^2} = \lim_{n\to\infty}\frac{\sum_{i=1}^n b_i^2 l_i}{c h_n\sum_{i=1}^n b_i^2} = \lim_{n\to\infty}\frac{b_n^2 h_n}{h_n\sum_{i=1}^n b_i^2 - h_{n-1}\sum_{i=1}^{n-1} b_i^2}.$$

We shall show that the limit on the right-hand side is equal to 1. We start by writing

$$h_n\sum_{i=1}^n b_i^2 - h_{n-1}\sum_{i=1}^{n-1} b_i^2 = (h_n - h_{n-1})\sum_{i=1}^{n-1} b_i^2 + b_n^2 h_n.$$

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Then, by (A.7),

$$\lim_{n\to\infty} \frac{b_n^2 h_n}{h_n \sum_{i=1}^n b_i^2 - h_{n-1}\sum_{i=1}^{n-1} b_i^2} = \lim_{n\to\infty} \frac{b_n^2 h_n}{o(h_n/n)\sum_{i=1}^n b_i^2 + b_n^2 h_n} = 1,$$

and (A.6) follows.

To treat the second sum, notice that l_n is increasing and then

$$\sum_{i=n+1}^{nM} b_{ni}^2 l_i \sim l_n \sum_{i=n+1}^{nM} b_{ni}^2 \tag{A.9}$$

because

$$l_n \sum_{i=n+1}^{nM} b_{ni}^2 \le \sum_{i=n+1}^{nM} b_{ni}^2 l_i \le l_{nM} \sum_{i=n+1}^{nM} b_{ni}^2,$$

and l_n is a function slowly varying at ∞.

We treat now the last sum. By (A.2) and Lemma A.1,

$$\sum_{i=nM+1}^{\infty} b_{ni}^2 l_i \ll n^2 \sum_{i=nM+1}^{\infty} (i-n)^{-2\alpha} L^2(i) l_i \ll n^2[n(M-1)]^{1-2\alpha} L^2[nM] l_{nM}.$$

We obtain

$$\sum_{i=nM+1}^{\infty} b_{ni}^2 l_i \ll B_n^2 M^{1-2\alpha} \quad \text{as } n\to\infty. \tag{A.10}$$

We combine now the estimates in (A.6) and (A.9). For $\delta > 0$ fixed and n sufficiently large

$$(1-\delta) l_n \sum_{i\ge 1}^{nM} b_{ni}^2 \le \sum_{i\ge 1}^{nM} b_{ni}^2 l_i \le (1+\delta) l_n \sum_{i\ge 1}^{nM} b_{ni}^2.$$

Therefore,

$$(1-\delta) l_n \left(\sum_{i\ge 1}^{\infty} b_{ni}^2 - \sum_{i>nM} b_{ni}^2 \right) \le \sum_{i\ge 1}^{\infty} b_{ni}^2 l_i \le (1+\delta) l_n \sum_{i\ge 1}^{nM} b_{ni}^2 + \sum_{i>nM} b_{ni}^2 l_i. \tag{A.11}$$

Then, by (A.10), for a positive constant C_1 we have

$$\limsup_{n\to\infty} \frac{1}{B_n^2} \sum_{i>nM} b_{ni}^2 l_i \le \frac{C_1}{M^{2\alpha-1}}.$$

We also know that for a certain positive constant C_2,

$$\limsup_{n\to\infty} \frac{1}{B_n^2} \sum_{i>nM} b_{ni}^2 \le \frac{C_2}{M^{2\alpha-1}}.$$

The result follows by dividing (A.11) by B_n^2 and taking first lim sup and also lim inf when $n\to\infty$ followed by $M\to\infty$, and finally we let $\delta\to 0$.

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The proof of (A.4) is similar, and it is sufficient to divide the sum into only two parts, one from 1 to $2n$ and the rest. More exactly, by using (A.1),

$$\sum_{i=1}^{2n} |b_{ni}|^p i^{-1+p/2}|h(i)| \ll \sum_{i=1}^{2n} i^{p(1-\alpha)}|L(i)|^p i^{-1+p/2}|h(i)| \ll h(n)n^{p(3/2-\alpha)}L^p(n)$$

and

$$\sum_{i\geq 2n} |b_{ni}|^p i^{-1+p/2}|h(i)| \ll n^p \sum_{i\geq 2n} i^{-1+(1/2-\alpha)p}|L(i)|^p|h(i)| \ll h(n)n^{p(3/2-\alpha)}L^p(n).$$

The proof is complete. ■

The next lemma is a variant of Theorem 12.3 in Billingsley (1968).

LEMMA A.5. *Assume that $(X_{nk})_{1\leq k\leq n}$ is a triangular array of centered random variables with finite second moment. For $0\leq m\leq n$ let $S_m=\sum_{j=1}^m X_{nj}$ and for $0\leq t\leq 1$, $W_n(t)=S_{[nt]}$. Assume that for every $\varepsilon>0$*

$$P(\max_{1\leq i\leq n}|X_{ni}|>\varepsilon)\to 0 \tag{A.12}$$

and there are a positive constant K and an integer N_0 such that for any $1\leq p<q\leq n$ with $q-p>N_0$ we have

$$\mathrm{E}(S_{nq}-S_{np})^2\leq K\left(\frac{q}{n}-\frac{p}{n}\right)^\gamma \tag{A.13}$$

for some $\gamma>1$. Then $W_n(t)$ is tight in $D[0,1]$, endowed with Skorokhod topology.

Proof. We shall base our proof on a blocking argument. We divide the variables into blocks of size N_0. Let $k=[n/N_0]$. For $1\leq j\leq k$ denote $Y_{nj}=\sum_{i=(j-1)N_0+1}^{jN_0} X_{ni}$ and $Y_{n,k+1}=\sum_{i=kN_0+1}^{n} X_{ni}$. Define $V_n(t)=\sum_{j=1}^{[kt]} Y_{nk}$.

Then we notice that it is enough to show that $V_n(t)$ is tight in $D[0,1]$ because by the fact that $[nt]-[kt]\leq 2N_0$ and by (A.12)

$$P(\sup_t|W_n(t)-V_n(t)|>\varepsilon)\leq P(\max_{1\leq i\leq n}|X_{ni}|>\varepsilon/2N_0)\to 0.$$

By Theorem 8.3 in Billingsley (1968) formulated for random elements of D (see Billingsley, 1968, p. 137) we have to show that for every $0\leq t\leq 1$ and $\varepsilon>0$ fixed,

$$\lim_{\delta\searrow 0}\lim_{n\to\infty}\sup\frac{1}{\delta}P\left(\max_{[kt]\leq j\leq[k(t+\delta)]}\left|\sum_{i=[kt]}^{j}Y_{ni}\right|\geq\varepsilon\right)=0.$$

By Theorem 12.2 in Billingsley (1968), because $\gamma>1$, there is a constant K such that

$$P(\max_{[kt]\leq j\leq[k(t+\delta)]}\left|\sum_{i=[kt]}^{j}Y_{ni}\right|\geq\varepsilon)\leq\frac{K}{\varepsilon^2}\left(\left[\frac{k(t+\delta)N_0}{n}\right]-\left[\frac{ktN_0}{n}\right]\right)^\gamma,$$

and the result follows by multiplying with $1/\delta$ and passing to the limit with $n\to\infty$ and then with $\delta\to 0$. ■

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