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SEMI-PRIME NOETHERIAN RINGS OF INJECTIVE DIMENSION ONE

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Abstract. Let R be a semi-prime Noetherian ring of injective dimension 1. Let P be a minimal prime ideal of R. In this paper it is shown that R/P need not have injective dimension 1. Necessary and sufficient conditions are given for R/P to have injective dimension 1.

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1. Introduction. Let R be a semi-prime Noetherian ring of injective dimension 1. Because hereditary rings have injective dimension 1, it is reasonable to hope that results in the hereditary case may suggest properties which can be proved about the more general ring R. It was shown by Levy [**3**, Theorem 4.3] that in the hereditary case R is a direct sum of prime rings, but this does not generalise in a straightforward way because the integral group ring $\mathbb{Z}[G]$ of any non-trivial finite group G is a semi-prime Noetherian ring of injective dimension 1 but it is not a direct sum of prime rings. Thus, if P is a minimal prime ideal of R, we cannot expect R/P to be a direct summand of R, but we might at least hope that R/P has injective dimension 1. The two purposes of this paper are firstly to show that R/P need not have injective dimension 1 (Example 2.1) and then to give a necessary and sufficient condition for R/P to have injective dimension 1. The condition is that $(I+P)/P$ is an invertible ideal of R/P, where I is the intersection of all the minimal prime ideals of R except P (Corollary 3.11).

2. The example. It is well known that if G is any finite group then the integral group ring $\mathbb{Z}[G]$ is a semi-prime Noetherian ring of injective dimension 1. Using A_4 as usual to denote the alternating group on four symbols, we shall show that $\mathbb{Z}[A_4]$ has a prime factor ring which does not have injective dimension 1 (we believe that A_4 is the smallest group with this property).

Example 2.1. Set $S = M_3(\mathbb{Z})$. We start by constructing a specific representation of A_4 inside S. Let H consist of all elements of S of the three following types, where in each case a, b, c are any integers with $abc = 1$:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}; \quad \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix}.$$

It is routine to check that the twelve elements of H form a group under multiplication and that $H \cong A_4$.

Let R be the subring of S generated by H. We shall use e_{ij} to denote the 3×3 matrix with 1 in the (i, j)-position and 0's elsewhere. By taking sums of pairs of elements of H it is easy to show that $2e_{ij} \in R$ for all i and j. Thus $2S \subseteq R$. We shall identify $S/2S$ with $M_3(F)$ where $F = \mathbb{Z}/2\mathbb{Z}$. Then $R/2S$ is the group algebra $F[C_3]$ where C_3 is the cyclic group of order 3 generated by the element $e_{12} + e_{23} + e_{31}$ of $M_3(F)$. Thus R is the subring of S such that $2S \subseteq R$ (from which it follows that R is a prime ring) and $R/2S = F[C_3]$, and $R/2S$ is the direct sum of a field with 2 elements and a field with 4 elements. In order to match the notation used in [**1**] we set $T = 2S$. Then T is the intersection of those maximal ideals of R which contain the ideal $2S$ of S, and $\lambda(R/T) = 2$ where λ denotes length as an R/T-module. Set

$$T^* = \{w \in M_3(\mathbb{Q}) : wT \subseteq R\}.$$

Clearly $S \subseteq T^*$ and S/R has 2^6 elements. Also S/R is an R/T-module where R/T is the direct sum of two fields as above, from which it follows that $\lambda(S/R) \geq 3$. Therefore $\lambda(T^*/R) \neq \lambda(R/T)$, so that R does not have injective dimension 1 [**1**, Theorem 5.7]. But $R \cong \mathbb{Z}[A_4]/P$ for some minimal prime ideal P of $\mathbb{Z}[A_4]$. □

3. The condition. Throughout this section R will denote a semi-prime Noetherian ring of injective dimension 1 with quotient ring Q, and I and J will denote ideals of R such that each is the annihilator of the other (note that if U and V are ideals of R then, because R is semi-prime, we have $UV = 0$ if and only if $VU = 0$ if and only if $U \cap V = 0$). We need to establish some general properties of R, I, and J before we can prove the necessary and sufficient condition for R/P to have injective dimension 1 where P is a minimal prime ideal of R.

NOTATION 3.1. For a subset X of Q set

$$X^* = \{q \in Q : qX \subseteq R\}$$

and

$$X_* = \{q \in Q : Xq \subseteq R\}.$$

The following result will be needed; it is an easy consequence of a result in [**2**] and we shall only give it in the form which we need.

THEOREM 3.2. (Jans) *Let W be a finitely-generated right R-submodule of Q such that if $q \in Q$ with $qW = 0$ then $q = 0$. Then $W = (W^*)_*$.*

Proof. Because W_R is finitely generated, there is a regular element c of R such that $cW \subseteq R$. Thus W_R embeds in R_R, so that W_R is torsionless. Also it follows from the other assumption made on W that we can identify W^* with $\mathrm{Hom}(W_R, R_R)$ and $(W^*)_*$ with $\mathrm{Hom}({}_R(W^*), {}_R R)$. Because W_R is torsionless and R has injective dimension 1, it follows from [**2**, Corollary, p. 72] that W_R is reflexive and hence $W = (W^*)_*$. □

We shall now study R, I, and J as defined in the first paragraph of this section.

PROPOSITION 3.3. *We have $J^* = J_* = IQ + R = \{q \in Q : qJ \subseteq J\} = \{q \in Q : Jq \subseteq J\}$, and similarly with I and J interchanged.*

Proof. There are central idempotents e and f of Q such that $e + f = 1$, $eQ = IQ = QI$, and $fQ = JQ = QJ$. We shall first show that $(I + J)^* = eR + fR$.

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Set $H = eR + fR$. Clearly $R \subseteq H$. By 3.2 we have $H = (H^*)_*$. Let $w \in H^*$. Then $ew = we \in R$ and $ewJ = 0$, so that $ew \in I$. Similarly $fw \in J$, and $w = ew + fw \in I + J$. Thus $H^* \subseteq I + J$. On the other hand $IH = IeR + IfR$ with $If = 0$. Hence $IH = IeR = Ie = I$, so that $I \subseteq H^*$. Similarly $J \subseteq H^*$, and it follows that $I + J = H^*$. Therefore $H = (H^*)_* = (I + J)_*$. But, because e and f are central, we have $H^* = H_* = \{q \in Q : qe \in R \text{ and } qf \in R\}$, so that by symmetry we also have $H = (H_*)^* = (I + J)^*$.

Next we shall show that $J^* = IQ + R$. We have $IQJ = 0$, so that $IQ + R \subseteq J^*$. Let $x \in J^*$. Then $x = xe + xf$ with $xeJ = 0$. Hence $xe \in J^*$ and so also $xf = x - xe \in J^*$. We have $xfI = 0$ so that $xf \in I^*$. Thus $xf \in I^* \cap J^* = (I + J)^* = H = eR + fR$, from which it follows that $xf \in fR$. Therefore $x = xe + xf \in eQ + fR = eQ + R = IQ + R$, which completes the proof that $J^* = IQ + R$.

By symmetry we also have $J_* = IQ + R$. Also $J^*JI = 0$ with $J^*J \subseteq R$, so that $J^*J \subseteq J$. Therefore $J^* = \{q \in Q : qJ \subseteq J\}$. □

Proposition 3.4. *The injective dimension of I as an R/J-module is* 1.

Proof. By symmetry, it does not matter whether we work on the right or the left. Because Q/R is R-injective it follows that J^*/R (being the annihilator of J in Q/R) is R/J-injective. But by 3.3 we have $J^*/R = (IQ + R)/R \cong IQ/(IQ \cap R) = IQ/I$. Also Q is R-injective so that IQ (being the annihilator of J in Q) is R/J-injective. Thus both IQ and IQ/I are injective as R/J-modules. We note also that IQ is the injective envelope of I as an R/J-module. □

Proposition 3.5. *The ring $R/(I + J)$ is QF.*

Proof. Set $K = I + J$ and $H = eR + fR$ as in the proof of 3.3 (where it was shown that $H = K^*$). Thus H/R is the annihilator of K in Q/R. Because R has injective dimension 1 we know that Q/R is R-injective and hence H/R is R/K-injective. We shall prove that R/K is a self-injective ring by showing that $R/K \cong H/R$ as right R-modules. Define $a : R \to H/R$ by $a(x) = ex + R$ for all $x \in R$. Then a is a right R-module homomorphism, and a is surjective because $eR + R = eR + fR = H$. It is now enough to show that $\mathrm{Ker}(a) = K$. Let $x \in R$. The following statements are equivalent: $ex \in R$; $ex \in R$ and $fx \in R$ (because $fx = x - ex$); $ex \in I$ and $fx \in J$; $x \in I + J$. Therefore $a(x) = 0$ if and only if $x \in K$. □

We next study the quotient ring of R/J, and we find it necessary to be very careful as follows about the notation to be used.

Notation 3.6. Let $'$ denote image in the semi-simple ring $Q' = Q/fQ$. Then $R' = (R + fQ)/fQ \cong R/(R \cap fQ) = R/J$. Thus we can identify R/J with R', and the quotient ring of R' is Q'. Also $I' = (I + fQ)/fQ \cong I/(I \cap fQ) \cong I$ because $I \cap fQ = 0$. Similarly $I \cong (I + J)/J$. Therefore it follows from 3.4 that I' has injective dimension 1 as an R'-module, and from 3.5 that R'/I' is a QF ring.

Proposition 3.7. *With the notation of* 3.6 *set*

$$W = \{w \in Q' : wI' \subseteq I'\}.$$

Then $W = R'$.

Proof. Clearly $R' \subseteq W$. Let $w \in W$. We have $w = q'$ for some $q \in Q$. Because $wI' \subseteq I'$ we have $(qI)' \subseteq I'$, i.e. $qI \subseteq I + fQ$. But $I \subseteq eQ$ where e is central and $ef = 0$. Therefore $qI \subseteq I$. Hence by 3.3 we have $q \in JQ + R = R + fQ$, so that $w = q' \in R'$. □

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COROLLARY 3.8. *Let $x \in Q'$. Then $xI' \subseteq R'$ if and only if $I'x \subseteq R'$.*

Proof. Suppose that $xI' \subseteq R'$. Then $I'xI' \subseteq I'$, so that $I'x \subseteq R'$ by 3.7. □

PROPOSITION 3.9. *The R'-module Q'/I' is injective.*

Proof. This is because Q' is the injective envelope of I', and I' has injective dimension 1 as explained in 3.6. □

We can now prove our main result.

THEOREM 3.10. *The ring R/J has injective dimension 1 if and only if $(I+J)/J$ is an invertible ideal of R/J.*

Proof. With the notation of 3.6 we need to show that R' has injective dimension 1 if and only if I' is an invertible ideal of R'. Suppose that R' has injective dimension 1. By 3.5 we know that R'/I' is a QF ring. Set $W = \{w \in Q' : wI' \subseteq R'\}$ and note that by 3.8 we also have $W = \{w \in Q' : I'w \subseteq R'\}$. Then WI' is an ideal of R' with $I' \subseteq WI'$. Let $x \in R'$ with $xWI' \subseteq I'$. Then $xW \subseteq R'$ by 3.7. Adapting the notation of 3.1 to R' and Q' rather than R and Q, we have $W = (I')^* = (I')_*$ and $x \in W^*$. But R' has injective dimension 1, so that $W^* = ((I')_*)^* = I'$ by 3.2. Therefore $x \in I'$. Thus the ideal WI'/I' of the QF ring R'/I' has zero left annihilator, so that $WI'/I' = R'/I'$, i.e. $WI' = R'$. By symmetry we also have $I'W = R'$, so that I' is an invertible ideal of R'.

Conversely suppose that I' is an invertible ideal of R'. By 3.9 we know that Q'/I' is an injective R'-module. It follows from 3.12 (below) that Q'/R' is injective, and so R' has injective dimension 1. □

COROLLARY 3.11. *Let R be a semi-prime Noetherian ring of injective dimension 1, and let J be a minimal prime ideal of R. Let I be the intersection of all the minimal prime ideals of R other than J (with $I = R$ if there are none). Then the ring R/J has injective dimension 1 if and only if $(I+J)/J$ is an invertible ideal of R/J.*

LEMMA 3.12. *Let R be any ring with an ideal I which is invertible in some over-ring Q of R. Then Q/R is a direct summand of the direct sum of a finite number of copies of Q/I (as right R-modules).*

Proof. We can fix $x_1, \ldots, x_n \in I$ and $w_1, \ldots, w_n \in Q$ such that $w_1x_1 + \ldots + w_nx_n = 1$ and $w_iI \subseteq R$ for all i. Set $X = (Q/I)^{(n)}$ and define $a : Q/R \to X$ by

$$a(q+R) = (x_1q + I, \ldots, x_nq + I)$$

for all $q \in Q$. Then a is a right R-module homomorphism. Define $b : X \to Q/R$ by

$$b((q_1 + I, \ldots, q_n + I)) = w_1q_1 + \cdots + w_nq_n + R$$

for all $q_i \in Q$. It is easy to check that $ba(q+R) = q + R$ for all $q \in Q$, so that a splits and Q/R is isomorphic to a direct summand of X. □

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