Proceedings of the Royal Society of Edinburgh, **132A**, 1219–1248, *2002*

From the discrete to the continuous coagulation–fragmentation equations

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(MS received 23 July 2001; accepted 9 November 2001)

The connection between the discrete and the continuous coagulation–fragmentation models is investigated. A weak stability principle relying on *a priori* estimates and weak compactness in L^1 is developed for the continuous model. We approximate the continuous model by a sequence of discrete models and, writing the discrete models as modified continuous ones, we prove the convergence of the latter towards the former with the help of the above-mentioned stability principle. Another application of this stability principle is the convergence of an explicit time and size discretization of the continuous coagulation–fragmentation model.

1. Introduction

Coagulation and fragmentation processes arise in the dynamics of cluster growth and describe the mechanisms by which clusters can coalesce to form larger clusters or break apart into smaller pieces. In the simplest coagulation–fragmentation models, the clusters are usually assumed to be fully identified by their size (or volume or number of particles), which might be either a positive real number (continuous models) or a positive integer (discrete models), depending on the physical context. Though the relationship between the discrete and the continuous models has been considered by some authors (see the survey paper [11, p. 127] and [1,5,6,36]), their analysis is either performed at a formal level [1,11] or their approach is restricted either to a particular fragmentation model (scaling technique [36]) or to the coagulation model (via measure-valued solutions) in [6]. The aim of this paper is to provide a rigorous setting for the formal analysis performed in [1,11] under general assumptions on the coagulation and fragmentation coefficients. A related approach

https://doi.org/10.1017/S0308210500002080 Published online by Cambridge University Press

motivated by the study of a numerical scheme is developed in [5] for the coagulation equation with very restrictive assumptions on the coagulation coefficients.

The coagulation–fragmentation models we consider in this paper describe the time evolution of the cluster size distribution as the system of clusters undergoes binary coagulation and binary fragmentation events. More precisely, denoting by C_z the clusters of size z with $z = y \in \mathbb{R}_+ = (0, +\infty)$ or $z = i \in \mathbb{N} \setminus \{0\}$, the basic reactions taken into account herein are

$$C_z + C_{z'} \xrightarrow{a(z,z')} C_{z+z'} \quad \text{(binary coagulation)}$$

and

$$C_z \xrightarrow{b(z-z',z')} C_{z-z'} + C_{z'} \quad \text{(binary fragmentation)},$$

where a and b denote the coagulation and fragmentation rates, respectively, and are assumed to depend only on the sizes of the clusters involved in these reactions.

In the continuous setting, denoting by $f(t, y)$ the size distribution function at time t, the continuous coagulation–fragmentation equation (hereafter referred to as the CCF equation) reads

$$\frac{\partial f}{\partial t} = Q(f), \qquad (t, y) \in (0, +\infty) \times \mathbb{R}_+, \tag{1.1}$$

$$f(0, y) = f^{\text{in}}(y), \qquad y \in \mathbb{R}_+. \tag{1.2}$$

Here, the coagulation–fragmentation reaction term $Q(f)$ is given by

$$Q(f) = Q_1(f) - Q_2(f) - Q_3(f) + Q_4(f), \tag{1.3}$$

with

$$Q_1(f)(y) = \frac{1}{2} \int_0^y a(y', y - y') f(y') f(y - y') \, \mathrm{d}y',$$

$$Q_2(f)(y) = \frac{1}{2} \int_0^y b(y', y - y') \, \mathrm{d}y' f(y),$$

$$Q_3(f)(y) = L(f)(y) f(y), \quad \text{with } L(f)(y) := \int_0^\infty a(y, y') f(y') \, \mathrm{d}y',$$

$$Q_4(f)(y) = \int_0^\infty b(y, y') f(y + y') \, \mathrm{d}y'.$$

The meaning of the different contributions to the reaction term $Q(f)$ are the following. $Q_1(f)$ accounts for the formation of clusters C_y by coalescence of smaller clusters and $Q_2(f)$ for the breakage of clusters C_y into two smaller pieces. The term $Q_3(f)$ describes the depletion of clusters C_y by coagulation with other clusters, while $Q_4(f)$ represents the gain of clusters C_y as a result of the fragmentation of larger clusters.

In the discrete setting, the size distribution function $c_i(t)$ of clusters of size $i \in \mathbb{N} \setminus \{0\}$ (or i-clusters) at time $t \geqslant 0$ obeys the following system of discrete coagulation–fragmentation equations (hereafter referred to as the DCF equations),

$$\frac{\mathrm{d}c_i}{\mathrm{d}t} = Q_i(c) \quad \text{in } (0, +\infty), \quad c_i(0) = c_i^{\text{in}}, \tag{1.4}$$

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for $i \geqslant 1$, where

$$Q_i(c) = Q_{1,i}(c) - Q_{2,i}(c) - Q_{3,i}(c) + Q_{4,i}(c), \quad i \geqslant 1, \tag{1.5}$$

with

$$Q_{1,i}(c) = \frac{1}{2}\sum_{j=1}^{i-1} a_{i-j,j}c_{i-j}c_j,$$

$$Q_{2,i}(c) = \frac{1}{2}\sum_{j=1}^{i-1} b_{i-j,j}c_i,$$

$$Q_{3,i}(c) = L_i(c)c_i, \quad \text{with } L_i(c) = \sum_{j=1}^{\infty} a_{i,j}c_j,$$

$$Q_{4,i}(c) = \sum_{j=1}^{\infty} b_{i,j}c_{i+j}.$$

The reaction terms $Q_{k,i}(c)$ have a similar meaning to that of $Q_k(f)$, $k \in \{1, \ldots, 4\}$. In fact, the DCF equations (1.4), (1.5) were originally derived by Smoluchowski [30, 31] without the fragmentation term ($b_{i,j} = 0$) to describe the coalescence of colloids moving according to a Brownian movement. It had subsequently been extended to the continuous setting by Müller [26] (see also [11] for a more detailed historical viewpoint). Since then, both the DCF and CCF equations have been used in a wide variety of physical and biological situations, including aerosol physics (rain drops formation, etc.), polymer chemistry, astrophysics (formation of the stars and the planets), hematology or population dynamics (animal grouping). The choice of the size range ($\mathbb{N} \setminus \{0\}$ or $\mathbb{R}_+$) is then peculiar to the scale of the phenomenon to be described, but also depends on the desired level of description (so that both equations may be used to model the same phenomenon but at different scales). It thus seems to be relevant to investigate precisely the connection between the two approaches.

We now present the main idea upon which our approach is built. We actually adapt a method introduced for the Boltzmann equation in [25] (see also [28] and the references therein for further developments) and show that solutions to the DCF equations satisfy a 'modified' CCF equation. More precisely, let us start with a solution $c = (c_i)$ to the DCF equations (1.4), (1.5). For any sequence (φ_i) of real numbers (decaying sufficiently rapidly for large values of $i \in \mathbb{N}$), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty} c_i\varphi_i = \frac{1}{2}\sum_{i,j=1}^{\infty}(a_{ij}c_ic_j - b_{ij}c_{i+j})(\varphi_{i+j} - \varphi_i - \varphi_j). \tag{1.6}$$

Note that (1.6) is a weak formulation of the DCF equations (1.4), (1.5) and may be taken as the definition of a (weak) solution to the DCF equations (1.4), (1.5).

In order to interpret (1.6) as the weak formulation of a 'modified' CCF equation, we introduce some notations. We fix $\varepsilon \in (0, 1)$ and define

$$f_\varepsilon(t, y) = \sum_{i=1}^{\infty} c_i(t)\chi_i^\varepsilon(y) \tag{1.7}$$

and

$$a_\varepsilon(y,y') = \sum_{i,j=1}^{\infty} \frac{a_{i,j}}{\varepsilon}\chi_i^\varepsilon(y)\chi_j^\varepsilon(y'), \qquad b_\varepsilon(y,y') = \sum_{i,j=1}^{\infty} \frac{b_{i,j}}{\varepsilon}\chi_i^\varepsilon(y)\chi_j^\varepsilon(y'), \tag{1.8}$$

for $(t,y,y') \in \mathbb{R}_+^3$, where we have set

$$\Lambda_i^\varepsilon = [(i-\tfrac{1}{2})\varepsilon, (i+\tfrac{1}{2})\varepsilon), \quad \chi_i^\varepsilon = \mathbf{1}_{\Lambda_i^\varepsilon}, \quad i \geqslant 1. \tag{1.9}$$

Next, for $\varphi \in \mathcal{D}(\mathbb{R}_+)$, we define the sequence (φ_ε) of functions by

$$\varphi_\varepsilon(y) = \sum_{i=1}^{\infty} \varphi_i^\varepsilon \chi_i^\varepsilon(y), \qquad \varphi_i^\varepsilon = \frac{1}{\varepsilon}\int_{\Lambda_i^\varepsilon} \varphi(y)\,\mathrm{d}y. \tag{1.10}$$

Finally, for any ε-step function g, that is, g is a measurable function from $\mathbb{R}_+$ to $\mathbb{R}$ such that

$$g(y) = \sum_{i=1}^{\infty} g_i \chi_i^\varepsilon(y), \quad g_i \in \mathbb{R}, \tag{1.11}$$

we define

$$T_\varepsilon(g)(y,y') := \sum_{i,j=1}^{\infty} g_{i+j}\chi_i^\varepsilon(y)\chi_j^\varepsilon(y'), \quad (y,y') \in \mathbb{R}_+^2. \tag{1.12}$$

Let us emphasize here that $T_\varepsilon(g)(y,y')$ must be seen as an approximation of $g(y+y')$ (see lemma 4.1 below).

With these notations, we are in a position to write an alternative formulation of (1.6) in terms of the new functions f_ε, a_ε, b_ε and φ_ε, which reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty f_\varepsilon\varphi_\varepsilon\,\mathrm{d}y = \frac{1}{2}\int_0^\infty\int_0^\infty (a_\varepsilon f_\varepsilon f_\varepsilon' - b_\varepsilon T_\varepsilon(f_\varepsilon))(T_\varepsilon(\varphi_\varepsilon) - \varphi_\varepsilon - \varphi_\varepsilon')\,\mathrm{d}y\mathrm{d}y'. \tag{1.13}$$

Here and below, we use the following notation:

$$g = g(y), g' = g(y') \quad \text{and} \quad g'' = g(y+y').$$

We first prove that, under some growth conditions on a_{ij} and b_{ij}, (f_ε) lies in a weakly compact subset of $L^1((0,T)\times\mathbb{R}_+)$ for each $T>0$. Consequently, there exist $f \in L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}_+)$ and a subsequence of (f_ε) (not relabelled) such that

$$f_\varepsilon \rightharpoonup f \quad \text{weakly in } L^1((0,T)\times\mathbb{R}_+) \quad \text{for each } T \in \mathbb{R}_+. \tag{1.14}$$

Now, if there are some functions a and b such that

$$a_\varepsilon \to a, \quad b_\varepsilon \to b \quad \text{a.e. and weakly in } L^\infty_{\mathrm{loc}}(\mathbb{R}_+^2), \tag{1.15}$$

we are able to pass to the limit in (1.13) and obtain that f satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty f\varphi\,\mathrm{d}y = \frac{1}{2}\int_0^\infty\int_0^\infty (aff' - bf'')(\varphi'' - \varphi - \varphi')\,\mathrm{d}y\mathrm{d}y' \tag{1.16}$$

in $\mathcal{D}'([0,+\infty))$ for any $\varphi \in \mathcal{D}(\mathbb{R}_+)$. In other words, f is a (weak) solution to the CCF equation (1.1)–(1.3). To be more precise, we actually proceed as follows.

Given coagulation and fragmentation rates a and b, we construct a family of discrete kinetic coefficients $(a^{\varepsilon}_{i,j})$ and $(b^{\varepsilon}_{i,j})$ in such a way that (1.15) holds and thus establish the expected connection between the DCF and CCF equations. In fact, as for the Boltzmann equation, the underlying idea is somehow a stability principle; the weak L^1-compactness of sequences of solutions to the CCF equation is also enjoyed by sequences of solutions to suitable perturbations of the CCF equation. In particular, the DCF equations (1.4), (1.5) are such a perturbation. Furthermore, it turns out that an Euler explicit time discretization of the DCF equations also fits into the framework developed in this paper and the convergence of this scheme towards a solution to the CCF equation is studied below. Let us mention at this point that, although several numerical simulations of the CCF equation have been performed with various deterministic or stochastic numerical methods (see, for example, [3,10,14,18,24,27,29] and the references therein), convergence proofs have only been supplied recently and we refer the reader to, for example, [8,13,14,17,29] for the analysis of stochastic algorithms. As for deterministic numerical schemes, the only result we are aware of concerns a time-explicit Euler scheme for the discrete coagulation equations ($b_{i,j} \equiv 0$), which is shown to converge for bounded coefficients $(a_{i,j})$ [29]. Within our approach, such a restriction is not necessary and the convergence of the time-explicit scheme presented below is valid under fairly general assumptions on the kinetic coefficients a and b.

Let us now briefly outline the contents of the paper. The construction of the sequence of DCF equations approximating the CCF equation is described in the next section, where our convergence results are also stated. Let us emphasize here that, besides the connection between the DCF and the CCF equations, we also establish new existence results for the CCF equation as a byproduct. The *a priori* estimates guaranteeing the weak compactness of (f_ε) in L^1 are gathered in § 3, while the passage to the limit is performed in § 4. As already mentioned, our approach is quite general and we discuss several extensions in the next sections. In § 5 we show how a similar approach allows us to obtain the convergence of a time-explicit Euler scheme. In the remaining sections we outline how one can handle other classes of coagulation and fragmentation rates (§ 6) and how our method may be applied to the coagulation–fragmentation equations with diffusion (§ 7). We finally discuss in the appendix the equivalence between two seemingly different notions of solutions to the CCF equation.

2. Main results

Throughout the paper we make the following symmetry and growth assumptions:

$$a(y,y') = a(y',y) \quad \text{and} \quad b(y,y') = b(y',y), \quad (y,y') \in \mathbb{R}_+^2; \tag{2.1}$$

$$0 \leqslant a(y,y'), \quad b(y,y') \leqslant A(1+y)(1+y'), \quad (y,y') \in \mathbb{R}_+^2. \tag{2.2}$$

Note that (2.2) is physically natural for the coagulation rates and encompasses unbounded fragmentation rates. In fact, it can be slightly relaxed (see § 6). Under the sole bound (2.2), we do not know if the analysis presented herein is still valid. Furthermore, it is an open problem to prove the existence of a solution to (1.1)–(1.3) under the two assumptions (2.1), (2.2). One actually needs to make some additional structural assumptions on a and b [15, 16, 19], or to impose stronger

https://doi.org/10.1017/S0308210500002080 Published online by Cambridge University Press

growth conditions [12, 33]. Here, we relax the assumptions made in [33] and make the following growth assumption. For each $R \in \mathbb{R}_+$, there holds

$$\lim_{y'\to+\infty} \sup_{y\in(0,R)} \frac{a(y,y')}{y'} = \lim_{y'\to+\infty} \sup_{y\in(0,R)} \frac{b(y,y')}{y'} = 0. \tag{2.3}$$

For example, equation (2.3) holds if the kinetic coefficients satisfy

$$a(y,y'), b(y,y') \leqslant A(1+y^\alpha)(1+(y')^\alpha)$$

for some $A \in \mathbb{R}_+$ and $\alpha \in [0,1)$.

We also assume

$$f^{\mathrm{in}} \in L^1_1(\mathbb{R}_+) := L^1(\mathbb{R}_+;(1+y)\,\mathrm{d}y) \quad \text{and is non-negative a.e.} \tag{2.4}$$

We may now introduce the approximating DCF equations of the CCF equation. We fix $\varepsilon \in (0,1)$ and define the discrete kinetic coefficients $a^\varepsilon_{i,j}$ and $b^\varepsilon_{i,j}$ for $i, j \geqslant 1$ either by

$$a^\varepsilon_{i,j} = \frac{1}{\varepsilon}\int_{\Lambda^\varepsilon_i\times\Lambda^\varepsilon_j} a(y,y')\,\mathrm{d}y'\mathrm{d}y, \qquad b^\varepsilon_{i,j} = \frac{1}{\varepsilon}\int_{\Lambda^\varepsilon_i\times\Lambda^\varepsilon_j} b(y,y')\,\mathrm{d}y'\mathrm{d}y, \tag{2.5}$$

or by

$$a^\varepsilon_{i,j} = \varepsilon a(\varepsilon i, \varepsilon j), \qquad b^\varepsilon_{i,j} = \varepsilon b(\varepsilon i, \varepsilon j) \tag{2.6}$$

if a and b are continuous functions. In both cases, it readily follows from (2.1)–(2.3) that the discrete kinetic coefficients satisfy the symmetry condition

$$a^\varepsilon_{i,j} = a^\varepsilon_{j,i} \quad \text{and} \quad b^\varepsilon_{i,j} = b^\varepsilon_{j,i}, \quad i, j \in \mathbb{N}\setminus\{0\}, \tag{2.7}$$

and the growth conditions

$$0 \leqslant a^\varepsilon_{ij}, \quad b^\varepsilon_{ij} \leqslant A\varepsilon(1+\varepsilon i)(1+\varepsilon j), \quad i, j \geqslant 1, \tag{2.8}$$

and

$$\lim_{j\to+\infty} \frac{a^\varepsilon_{i,j}}{j} = \lim_{j\to+\infty} \frac{b^\varepsilon_{i,j}}{j} = 0, \quad i \geqslant 1. \tag{2.9}$$

We also define the discrete initial data $c^{\mathrm{in},\varepsilon}$ by

$$c^{\mathrm{in},\varepsilon}_i = \frac{1}{\varepsilon}\int_{\Lambda^\varepsilon_i} f^{\mathrm{in}}(y)\,\mathrm{d}y, \quad i \geqslant 1. \tag{2.10}$$

Notice that, by (2.4), we have

$$\varepsilon^2 \sum_{i=1}^\infty i c^{\mathrm{in},\varepsilon}_i \leqslant 2\int_0^\infty f^{\mathrm{in}}(y) y\,\mathrm{d}y \tag{2.11}$$

and

$$\varepsilon \sum_{i=1}^\infty c^{\mathrm{in},\varepsilon}_i \leqslant \int_0^\infty f^{\mathrm{in}}(y)\,\mathrm{d}y. \tag{2.12}$$

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We then consider a solution $c^\varepsilon = (c_i^\varepsilon)_{i\geqslant 1}$ to the DCF equation with kinetic coefficients $(a_{i,j}^\varepsilon)$ and $(b_{i,j}^\varepsilon)$ and initial datum $c_i^{\mathrm{in},\varepsilon}$. As established in [21, 32], such a solution c^ε exists and satisfies

$$\sum_{i=1}^{\infty} i c_i^\varepsilon(t) \leqslant \sum_{i=1}^{\infty} i c_i^{\mathrm{in},\varepsilon}, \quad t \geqslant 0. \tag{2.13}$$

We are now in a position to introduce the continuous formulation of the discrete quantities c^ε, $a_{i,j}^\varepsilon$, $b_{i,j}^\varepsilon$ and define the new functions f_ε by (1.7) (with c_i^ε instead of c_i) and a_ε, b_ε by (1.8) (with $a_{i,j}^\varepsilon$, $b_{i,j}^\varepsilon$ instead of $a_{i,j}$, $b_{i,j}$). Notice that, with these notations, a_ε and b_ε satisfy (2.1) and the growth conditions (2.2) and (2.3) uniformly with respect to $\varepsilon \in (0,1)$. Namely, for any $R \in \mathbb{R}_+$, there exists a bounded function $\omega_R(M)$, which decreases to zero as $M \to +\infty$ and such that

$$\sup_{y'\geqslant M} \sup_{y\in(0,R)} \frac{a^\varepsilon(y,y')}{y'} + \sup_{y'\geqslant M} \sup_{y\in(0,R)} \frac{b^\varepsilon(y,y')}{y'} \leqslant \omega_R(M) \tag{2.14}$$

and

$$a_\varepsilon \to a, \quad b_\varepsilon \to b \quad \text{a.e. and weakly in } L^\infty_{\mathrm{loc}}(\mathbb{R}_+). \tag{2.15}$$

Notice also that f_ε satisfies

$$\int_0^\infty f_\varepsilon(t,y)\,\mathrm{d}y = \varepsilon \sum_{i=1}^{\infty} c_i^\varepsilon(t) \quad \text{and} \quad \int_0^\infty f_\varepsilon(t,y) y\,\mathrm{d}y = \varepsilon^2 \sum_{i=1}^{\infty} i c_i^\varepsilon(t). \tag{2.16}$$

Before stating our result, let us make precise the notion of a solution to (1.1) to be used in the sequel.

Definition 2.1. Assume that a and b satisfy (2.1), (2.2). We say that $f = f(t,y)$ is a weak solution to the CCF equation (1.1)–(1.3), with the initial datum f^{in} satisfying (2.4), if

$$0 \leqslant f \in L^\infty(0,T; L^1_1(\mathbb{R}_+)) \quad \text{for each } T \in \mathbb{R}_+ \tag{2.17}$$

and (1.1) holds in $\mathcal{D}'([0,+\infty)\times\mathbb{R}_+)$, that is,

$$\int_0^\infty \int_0^\infty f \frac{\partial \psi}{\partial t}\,\mathrm{d}y\mathrm{d}t + \int_0^\infty f^{\mathrm{in}} \psi(0,\cdot)\,\mathrm{d}y = \int_0^\infty \int_0^\infty Q(f)\psi\,\mathrm{d}y\mathrm{d}t \tag{2.18}$$

for any $\psi \in \mathcal{D}([0,+\infty)\times\mathbb{R}_+)$.

Note that the boundedness assumptions (2.2) and (2.17) guarantee that the reaction terms $Q_k(f)$ belong to $L^1((0,T)\times(0,R))$ for any $k \in \{1,\dots,4\}$, $T \in \mathbb{R}_+$ and $R \in \mathbb{R}_+$. In particular, the last term in (2.18) makes sense.

Another possible definition of solution is the following.

Definition 2.2. Assume that a and b satisfy (2.1), (2.2). We say that $f = f(t,y)$ is a mild solution to the CCF equation (1.1)–(1.3), with the initial datum f^{in} satisfying (2.4), if

$$0 \leqslant f \in \mathcal{C}([0,+\infty); L^1(\mathbb{R}_+)) \cap L^\infty(0,T; L^1_1(\mathbb{R}_+)) \quad \text{for each } T \in \mathbb{R}_+, \tag{2.19}$$

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with $f(0) = f^{\text{in}}$ and (1.1) holds in the mild sense; for $0 \leqslant t_0 < t_1$, there holds

$$f(t_1,\cdot) - f(t_0,\cdot) = \int_{t_0}^{t_1} Q(f(t,\cdot))\,\mathrm{d}t \quad \text{a.e. in } \mathbb{R}_+. \tag{2.20}$$

Here again, equation (2.20) makes sense thanks to the bounds (2.2) and (2.19). At first glance, it may seem that being a mild solution in the sense of definition 2.2 is a stronger notion of a solution than being a weak solution in the sense of definition 2.1. In fact, we prove in the appendix that these two notions are equivalent.

Our first result makes precise the connection between the DCF and CCF equations.

Theorem 2.3. *Assume that a and b satisfy (2.1)–(2.3) and that f^{in} satisfies (2.4). The family (f_ε) of approximate solutions being defined above, there exists a weak solution f to the CCF equation (1.1)–(1.3), with the initial datum f^{in}, such that, extracting a subsequence if necessary,*

$$f_\varepsilon \to f \quad \textit{weakly in } L^1((0,T)\times\mathbb{R}_+) \quad \textit{for each } T \in \mathbb{R}_+. \tag{2.21}$$

Obviously, if the weak solution to the CCF equation (1.1)–(1.3) is unique, it is the whole family (f_ε) that converges. This is, in particular, the case when

$$a(y,y') \leqslant K(1+y)^{1/2}(1+y')^{1/2},$$
$$\int_0^y (1+y')^{1/2} b(y', y-y')\,\mathrm{d}y' \leqslant K(1+y)^{1/2}$$

for $(y,y') \in \mathbb{R}_+^2$ and some constant $K > 0$ [34]. Another uniqueness result may be found in [12].

Remark 2.4. We have actually the stronger convergence

$$f_\varepsilon \to f \quad \text{in } \mathcal{C}([0,T]; w - L^1(\mathbb{R}_+))$$

for every $T \in \mathbb{R}_+$, where $\mathcal{C}([0,T]; w - L^1(\mathbb{R}_+))$ denotes the space of weakly continuous functions from $[0,T]$ in $L^1(\mathbb{R}_+)$.

The conditions made on a and b may be relaxed in several directions and we refer to § 6 for precise statements. A case of some particular interest is the case of sublinear coagulation coefficients, namely

$$a(y,y') \leqslant A_0(1+y+y'), \quad (y,y') \in \mathbb{R}_+^2, \tag{2.22}$$

for some $A_0 > 0$. In this case, we have the following result.

Theorem 2.5. *Assume that a and b satisfy (2.1), (2.2) and (2.22) and that f^{in} satisfies (2.4). With the kinetic coefficients $(a^\varepsilon_{i,j})$ and $(b^\varepsilon_{i,j})$ being still defined by (2.5) or (2.6), we denote by c^ε a solution to the corresponding DCF equations (1.4), (1.5), with initial datum $c^{\text{in},\varepsilon} = (c_i^{\text{in},\varepsilon})$ given by (2.10) satisfying*

$$\sum_{i=1}^{\infty} i c_i^\varepsilon(t) = \sum_{i=1}^{\infty} i c_i^{\text{in},\varepsilon}, \quad t \in [0,+\infty) \tag{2.23}$$

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(the existence of such a solution follows from [4]). Putting

$$f_\varepsilon(t,y)=\sum_{i=1}^{\infty}c_i^\varepsilon(t)\chi_i^\varepsilon(y),\quad (t,y)\in\mathbb{R}_+^2,$$

as before, there exists a weak solution f to the CCF equation (1.1)–(1.3) with the initial datum f^{in} such that, extracting a subsequence if necessary,

$$f_\varepsilon\to f\quad \textit{weakly in } L^1((0,T)\times\mathbb{R}_+;(1+y)\,\mathrm{d}t\mathrm{d}y)\quad \textit{for each } T\in\mathbb{R}_+ \tag{2.24}$$

and

$$\int_0^\infty f(t,y)y\,\mathrm{d}y=\int_0^\infty f^{\mathrm{in}}(y)y\,\mathrm{d}y\quad \textit{for } t\geqslant 0. \tag{2.25}$$

Here again, the convergence (2.24) may be improved to

$$f_\varepsilon\to f\quad \text{in } \mathcal{C}([0,T];w-L^1(\mathbb{R}_+;(1+y)\,\mathrm{d}y))$$

for every $T\in\mathbb{R}_+$. A byproduct of theorem 2.5 is the existence of solutions to the CCF equation satisfying (2.25) under only assumption (2.4) on the initial data f^{in}. Stronger assumptions on f^{in} are required in the analysis of [12] and theorem 2.5 thus extends the results of [12] for fragmentation coefficients satisfying simultaneously (2.2) and the growth conditions of [12].

3. *A priori* estimates

In this section we consider the family of discrete kinetic coefficients $(a_{i,j}^\varepsilon)$ and $(b_{i,j}^\varepsilon)$ defined by (2.5) or (2.6). We are going to establish several estimates on f_ε defined by (1.7), which are uniform with respect to $\varepsilon>0$ and ultimately imply that (f_ε) lies in a weakly compact set of L^1. Let us emphasize here that the estimates derived in this section are valid under the sole assumptions (2.1) and (2.2) and, in particular, do not rely on (2.3). We put

$$M=\int_0^\infty f^{\mathrm{in}}(y)(1+y)\,\mathrm{d}y.$$

In the following, we denote by C any positive constant depending only on A and M. The dependence of C upon additional parameters will be indicated explicitly.

Lemma 3.1. *For any $t\geqslant 0$, there holds*

$$\int_0^\infty f_\varepsilon(t,y)y\,\mathrm{d}y\leqslant 2M. \tag{3.1}$$

Proof. By (2.11), (2.13) and (2.16), we have

$$\int_0^\infty f_\varepsilon(t,y)y\,\mathrm{d}y=\varepsilon^2\sum_{i=1}^{\infty}ic_i^\varepsilon(t)\leqslant\varepsilon^2\sum_{i=1}^{\infty}ic_i^{\mathrm{in},\varepsilon}\leqslant 2\int_0^\infty f^{\mathrm{in}}(y)y\,\mathrm{d}y,$$

whence (3.1). □

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LEMMA 3.2. *For $T \in \mathbb{R}_+$, $R \in \mathbb{R}_+$ and $t \in [0,T]$, we have*

$$\int_0^R f_\varepsilon(t,y)\,\mathrm{d}y \leqslant C(T,R). \tag{3.2}$$

Proof. Let m be the integer such that $m\varepsilon < R \leqslant (m+1)\varepsilon$. We infer from (1.6) and the non-negativity of $b^\varepsilon_{i,j}$ and c^ε that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{m+1}\varepsilon c_i^\varepsilon &\leqslant \frac{\varepsilon}{2}\sum_{i,j=1}^{\infty} a^\varepsilon_{i,j}c_i^\varepsilon c_j^\varepsilon(\mathbf{1}_{i+j\leqslant m+1}-\mathbf{1}_{i\leqslant m+1}-\mathbf{1}_{j\leqslant m+1}) + \varepsilon\sum_{i=1}^{m+1}\sum_{j=1}^{\infty} b^\varepsilon_{i,j}c^\varepsilon_{i+j}\\ &\leqslant \sum_{k=1}^{\infty}\varepsilon c_k^\varepsilon \sum_{i=1}^{m+1} b^\varepsilon_{i,k-i}.\end{aligned}$$

It follows from (2.8) that, for any $k \geqslant 2$,

$$\sum_{i=1}^{m+1} b^\varepsilon_{i,k-i} \leqslant A\sum_{i=1}^{m+1}\varepsilon(1+\varepsilon i)(1+\varepsilon k) \leqslant C_R(1+\varepsilon k).$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{m+1}\varepsilon c_i^\varepsilon \leqslant C_R\sum_{k=1}^{\infty}\varepsilon c_k^\varepsilon(1+\varepsilon k) \leqslant C_R\biggl(\sum_{i=1}^{m+1}\varepsilon c_i^\varepsilon + 2M\biggr).$$

Using the Gronwall lemma yields

$$\sum_{i=1}^{m+1}\varepsilon c_i^\varepsilon(t) \leqslant C(R,T)\sum_{i=1}^{m+1}\varepsilon c_i^{\mathrm{in},\varepsilon} \tag{3.3}$$

for any $t \in [0,T]$ and we conclude, thanks to (2.12) and (2.16). □

LEMMA 3.3. *Let $\Phi \in \mathcal{C}^1([0,+\infty))$ be a non-negative convex function such that $\Phi(0)=0$, $\Phi'(0)=1$ and Φ' is concave. If*

$$L_\Phi := \int_0^\infty \Phi(f^{\mathrm{in}})(y)\,\mathrm{d}y < \infty, \tag{3.4}$$

there holds

$$\int_0^R \Phi(f_\varepsilon(t,y))\,\mathrm{d}y \leqslant C(T,R)(L_\Phi + \Phi(1)) \tag{3.5}$$

for each $t \in [0,T]$, $T \in \mathbb{R}_+$ and $R \in \mathbb{R}_+$.

Proof. We first recall that the properties of Φ imply that

$$u\Phi'(v) \leqslant \Phi(u) + \Phi(v), \quad u,v \geqslant 0. \tag{3.6}$$

Indeed, owing to the convexity of Φ and the concavity of Φ', we have $v\Phi'(v) \leqslant 2\Phi(v)$ by [20, lemma A.1], and the convexity of Φ further entails that

$$u\Phi'(v) \leqslant \Phi(u) - \Phi(v) + v\Phi'(v) \leqslant \Phi(u) + \Phi(v).$$

We denote again by m the integer such that $m\varepsilon < R \leqslant (m+1)\varepsilon$. We infer from (1.4), (1.5), (3.6) and the non-negativity of $a^{\varepsilon}_{i,j}$, $b^{\varepsilon}_{i,j}$ and c^{ε} that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{m+1}\varepsilon\Phi(c^{\varepsilon}_i) &\leqslant \varepsilon\sum_{i=1}^{m+1}(Q^{\varepsilon}_{1,i}(c^{\varepsilon})+Q^{\varepsilon}_{4,i}(c^{\varepsilon}))\Phi'(c^{\varepsilon}_i)\\ &\leqslant \tfrac{1}{2}\varepsilon\sum_{i=1}^{m+1}\sum_{j=1}^{i-1}a^{\varepsilon}_{i-j,j}c^{\varepsilon}_{i-j}c^{\varepsilon}_j\Phi'(c^{\varepsilon}_i)+\varepsilon\sum_{i=1}^{m+1}\sum_{j=1}^{\infty}b^{\varepsilon}_{i,j}c^{\varepsilon}_{i+j}\Phi'(c^{\varepsilon}_i)\\ &\leqslant \tfrac{1}{2}\varepsilon\sum_{i=1}^{m+1}\sum_{j=1}^{i-1}a^{\varepsilon}_{i-j,j}(\Phi(c^{\varepsilon}_{i-j})+\Phi(c^{\varepsilon}_i))c^{\varepsilon}_j\\ &\qquad\qquad +\varepsilon\sum_{k=2}^{\infty}c^{\varepsilon}_k\sum_{i=1}^{m+1}b^{\varepsilon}_{i,k-i}(\Phi(1)+\Phi(c^{\varepsilon}_i)).\end{aligned}$$

Since we have

$$\sup_{i,j\leqslant m+1}a^{\varepsilon}_{i,j}\leqslant C(R)\varepsilon \quad\text{and}\quad \sup_{i\leqslant m+1}b^{\varepsilon}_{i,k-i}\leqslant C(R)\varepsilon(1+\varepsilon k),$$

by (2.8), for any $k\geqslant 2$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{m+1}\varepsilon\Phi(c^{\varepsilon}_i) &\leqslant C(R)\biggl(\sum_{i=1}^{m+1}\varepsilon\Phi(c^{\varepsilon}_i)\biggr)\biggl(\sum_{j=1}^{m+1}\varepsilon c^{\varepsilon}_j\biggr)\\ &\qquad + C(R)\biggl(\sum_{k=1}^{\infty}\varepsilon(1+\varepsilon k)c^{\varepsilon}_k\biggr)\biggl(\sum_{i=1}^{m+1}\varepsilon(\Phi(1)+\Phi(c^{\varepsilon}_i))\biggr).\end{aligned}$$

Therefore, using lemma 3.2 (and, more precisely, equation (3.3)) and equations (2.16) and (3.2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{m+1}\varepsilon\Phi(c^{\varepsilon}_i)\leqslant C(T,R)\biggl(\sum_{i=1}^{m+1}\varepsilon\Phi(c^{\varepsilon}_i)+(R+1)\Phi(1)\biggr)$$

for any $t\in[0,T]$, from which we deduce that

$$\sum_{i=1}^{m+1}\varepsilon\Phi(c^{\varepsilon}_i(t))\leqslant C(T,R)\biggl(\Phi(1)+\sum_{i=1}^{m+1}\varepsilon\Phi(c^{\mathrm{in},\varepsilon}_i)\biggr)$$

by the Gronwall lemma. Finally, the Jensen inequality ensures that

$$\sum_{i=1}^{m+1}\varepsilon\Phi(c^{\mathrm{in},\varepsilon}_i)\leqslant\sum_{i=1}^{m+1}\int_{\Lambda^{\varepsilon}_i}\Phi(f^{\mathrm{in}}(y))\,\mathrm{d}y\leqslant L_{\Phi},$$

and thus

$$\int_0^R\Phi(f_{\varepsilon}(t,y))\,\mathrm{d}y\leqslant\varepsilon\sum_{i=1}^{m+1}\Phi(c^{\varepsilon}_i(t))\leqslant C(T,R)(\Phi(1)+L_{\Phi}),$$

which completes the proof of (3.5). □

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Lemma 3.4. *There exist $f \in L^\infty_{\mathrm{loc}}([0,+\infty); L^1(\mathbb{R}_+))$ and a subsequence of f_ε (not relabelled) such that, for any $T \in \mathbb{R}_+$,*

$$f_\varepsilon \rightharpoonup f \quad \textit{weakly in } L^1((0,T)\times\mathbb{R}_+). \tag{3.7}$$

Proof. Since $f^{\mathrm{in}} \in L^1(\mathbb{R}_+)$, it follows from a refined version of the de la Vallée–Poussin theorem [9, 23] that there exists a function Φ fulfilling the assumptions of lemma 3.3 and such that $\Phi(r)/r \to +\infty$ as $r \to +\infty$ and

$$\int_{\mathbb{R}_+} \Phi(f^{\mathrm{in}})\,\mathrm{d}y < \infty.$$

Gathering lemmas 3.1, 3.2 and 3.3, we conclude that

$$\sup_{\varepsilon>0}\sup_{[0,T]}\biggl\{\int_0^\infty (1+y) f_\varepsilon\,\mathrm{d}y + \int_0^R \Phi(f_\varepsilon)\,\mathrm{d}y\biggr\} < \infty \tag{3.8}$$

for any $R \in \mathbb{R}_+$ and $T \in \mathbb{R}_+$, which implies that (f_ε) lies in a weakly compact subset of $L^1((0,T)\times(0,R))$ by the Dunford–Pettis theorem. We deduce that there exists $f \geqslant 0$ such that (3.7) holds true. Moreover, thanks to (3.8), $f \in L^\infty(0,T; L^1_1(\mathbb{R}_+))$. □

We now state a fourth *a priori* bound which will be useful to pass to the limit in the (quadratic) coagulation terms.

Lemma 3.5. *For any $T \in \mathbb{R}_+$, $R \in \mathbb{R}_+$ and $\psi \in L^\infty(0,R)$,*

$$\int_0^R f_\varepsilon(.,y)\psi(y)\,\mathrm{d}y \quad \textit{is bounded in } W^{1,\infty}(0,T). \tag{3.9}$$

Proof. We define the sequence ψ_i^ε from ψ by (1.10) and denote by m the integer such that $m\varepsilon \leqslant R < (m+1)\varepsilon$. We infer from (2.8) that

$$\begin{aligned}
\biggl|\frac{\mathrm{d}}{\mathrm{d}t}\int_0^R f_\varepsilon(t,y)\psi(y)\,\mathrm{d}y\biggr| &\\
&= \varepsilon\biggl|\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{m+1} c_i^\varepsilon\psi_i^\varepsilon\biggr| \\
&\leqslant \varepsilon\sum_{i=1}^{m+1}|Q_i(c^\varepsilon)\psi_i^\varepsilon| \\
&\leqslant \tfrac{3}{2}\varepsilon\|\psi\|_{L^\infty}\biggl(\sum_{i=1}^{m+1}\sum_{j=1}^{\infty} a_{i,j}^\varepsilon c_i^\varepsilon c_j^\varepsilon + \sum_{k=1}^{\infty} c_k^\varepsilon \sum_{\ell=1}^{m+1} b_{k-\ell,\ell}^\varepsilon\biggr) \\
&\leqslant \tfrac{3}{2}A\|\psi\|_{L^\infty}\biggl\{\biggl(\sum_{i=1}^{\infty}\varepsilon(1+\varepsilon i)c_i^\varepsilon\biggr)^2 + \sum_{k=1}^{\infty}\varepsilon(1+\varepsilon k)c_k^\varepsilon\biggl(\sum_{\ell=1}^{m+1}(1+\varepsilon\ell)\varepsilon\biggr)\biggr\}
\end{aligned}$$

and the right-hand side of the above inequality is bounded in $L^\infty(0,T)$ by lemmas 3.1, 3.2 and equation (2.16). □

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Combining lemma 3.5 and (3.8), we obtain the convergence of (f^ε) towards f in a stronger topology. Notice, however, that this stronger convergence will not be used in the proof of theorem 2.3.

Corollary 3.6. *The convergence (3.7) may be improved to*

$$f_\varepsilon \to f \quad \text{in } \mathcal{C}([0,T]; w - L^1(\mathbb{R}_+)). \tag{3.10}$$

Proof. Fix $R \in \mathbb{R}_+$. On the one hand, $(f_\varepsilon(t))$ lies in a weakly compact set of $L^1(0,R)$ by (3.8). On the other hand, lemma 3.5 ensures that (f_ε) is weakly equicontinuous in $L^1(0,R)$ at every $t \in [0,T]$ (in the sense of [35, definition 1.3.1]). Consequently, according to a variant of the Arzelà–Ascoli theorem (see, for example, [35, theorem 1.3.2]), the sequence (f_ε) is relatively compact in $\mathcal{C}([0,T]; w - L^1(0,R))$. This last fact and (3.8) then entail (3.10). □

4. Passing to the limit

The proof of theorem 2.3 is split into two steps. We consider a function $\varphi \in \mathcal{D}([0,+\infty) \times \mathbb{R}_+)$ and define the ε-step function $\varphi_\varepsilon(t)$ by (1.10) for each $t \geqslant 0$. We first infer from (1.4) and (1.5) that f_ε satisfies the following equation:

$$\begin{aligned}\int_0^\infty \int_0^\infty f_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t}\,\mathrm{d}y\mathrm{d}t + \int_0^\infty f_\varepsilon(0)\varphi_\varepsilon(0)\,\mathrm{d}y \\ = -\frac{1}{2}\int_0^\infty \int_0^\infty \int_0^\infty (a_\varepsilon f_\varepsilon f_\varepsilon' - b_\varepsilon T_\varepsilon(f_\varepsilon))(T_\varepsilon(\varphi_\varepsilon) - \varphi_\varepsilon - \varphi_\varepsilon')\,\mathrm{d}y\mathrm{d}y'\mathrm{d}t.\end{aligned} \tag{4.1}$$

Step 1 (convergence results). We prove several convergence results, which will be needed to pass to the limit in (4.1). We begin with two elementary convergence results for test functions.

Lemma 4.1. *The sequence (φ_ε) satisfies*

$$\varphi_\varepsilon \to \varphi \qquad \text{strongly in } L^\infty(\mathbb{R}_+^2), \tag{4.2}$$

$$T_\varepsilon(\varphi_\varepsilon) \to \{(t,y,y') \mapsto \varphi(t,y+y')\} \quad \text{strongly in } L^\infty([0,+\infty) \times \mathbb{R}_+^2). \tag{4.3}$$

Proof. For any fixed $t, y, y' \in \mathbb{R}_+$, we have

$$\begin{aligned}|T_\varepsilon(\varphi_\varepsilon)(t,y,y') - \varphi(t,y+y')| \\ &= \left|\sum_{i,j=1}^\infty \chi_i^\varepsilon(y)\chi_j^\varepsilon(y')\frac{1}{\varepsilon}\int_{\Lambda_{i+j}^\varepsilon} (\varphi(t,z) - \varphi(t,y+y'))\,\mathrm{d}z\right| \\ &\leqslant \sup_{|y+y'-z| \leqslant 2\varepsilon} |\varphi(t,z) - \varphi(t,y+y')| \\ &\leqslant 2\varepsilon \sup_{\mathbb{R}_+^2}\left|\frac{\partial \varphi}{\partial z}(t,z)\right|\end{aligned}$$

and (4.3) follows. Assertion (4.2) may be proved in the same way. □

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LEMMA 4.2. *For any* $\psi \in C_c^1(\mathbb{R}_+^3)$, *we define two functions* $S_\varepsilon(\psi)$ *and* $S(\psi)$ *in* $L^\infty(\mathbb{R}_+^2)$ *by*

$$S_\varepsilon(\psi)(t,z) := \sum_{k=2}^{\infty} \chi_k^\varepsilon(z) \sum_{j=1}^{k-1} \frac{1}{\varepsilon} \int\int_{\Lambda_{k-j}^\varepsilon \times \Lambda_j^\varepsilon} \psi(t,y,y')\,\mathrm{d}y\mathrm{d}y', \tag{4.4}$$

$$S(\psi)(t,z) := \int_0^z \psi(t,z-y',y')\,\mathrm{d}y'. \tag{4.5}$$

There holds

$$S_\varepsilon(\psi) \xrightarrow{\varepsilon\to 0} S(\psi) \quad \textit{strongly in } L^\infty(\mathbb{R}_+^2). \tag{4.6}$$

Proof. Fix $(t,z) \in \mathbb{R}_+^2$, $\varepsilon > 0$ and let k be the integer such that

$$(k - \tfrac{1}{2})\varepsilon \leqslant z \leqslant (k + \tfrac{1}{2})\varepsilon.$$

Either $k = 0$ or $k = 1$ and

$$|(S_\varepsilon(\psi) - S(\psi))(t,z)| = |S(\psi)(t,z)| \leqslant 2\varepsilon\|\psi\|_{L^\infty}.$$

Or $k \geqslant 2$, and we easily compute

$$\begin{aligned}(S(\psi) - S_\varepsilon(\psi))(t,z) &= \int_0^z \psi(t,z-y,y)\,\mathrm{d}y - \sum_{j=1}^{k-1} \frac{1}{\varepsilon} \int\int_{\Lambda_{k-j}^\varepsilon \times \Lambda_j^\varepsilon} \psi(t,y,y')\,\mathrm{d}y\mathrm{d}y' \\ &= \int_0^{\varepsilon/2} \psi(t,z-y,y)\,\mathrm{d}y + \int_{(k-1/2)\varepsilon}^z \psi(t,z-y,y)\,\mathrm{d}y \\ &\quad + \sum_{j=1}^{k-1} \int_{\Lambda_j^\varepsilon} (\psi(t,z-y,y) - \psi(t,(k-j)\varepsilon,\varepsilon j))\,\mathrm{d}y \\ &\quad + \sum_{j=1}^{k-1} \frac{1}{\varepsilon} \int\int_{\Lambda_{k-j}^\varepsilon \times \Lambda_j^\varepsilon} (\psi(t,(k-j)\varepsilon,\varepsilon j) - \psi(t,y,y'))\,\mathrm{d}y\mathrm{d}y'.\end{aligned}$$

This implies that

$$\sup_{(t,z)\in\mathbb{R}_+^2} |(S(\psi) - S_\varepsilon(\psi))(t,z)| \leqslant (2\|\psi\|_{L^\infty} + 2L\|\nabla\psi\|_{L^\infty})\varepsilon,$$

where $L \geqslant 1$ is such that $\operatorname{supp}\psi \subset [0, L-1]^3$. Then (4.6) readily follows. □

Let us recall the following lemma, which is a classical consequence of the Egorov and Dunford–Pettis theorems (see, for example, [22, lemma A.2] for a proof).

LEMMA 4.3. *Let* U *be an open subset of* $\mathbb{R}^m$, $m \geqslant 1$, *and consider two sequences* (v_n) *in* $L^1(U)$ *and* (w_n) *in* $L^\infty(U)$ *and two functions* $v \in L^1(U)$ *and* $w \in L^\infty(U)$ *such that*

$$v_n \rightharpoonup v \quad \textit{weakly in } L^1(U),$$
$$|w_n| \leqslant C \quad \textit{and} \quad w_n \to w \quad \textit{a.e.}$$

for some $C \geqslant 0$. Then

$$\lim_{n\to\infty} \|v_n(w_n - w)\|_{L^1} = 0 \quad \textit{and} \quad v_n w_n \rightharpoonup vw \quad \textit{weakly in } L^1(U).$$

We are now in a position to prove convergence results for the sequence (f_ε).

LEMMA 4.4. *The sequence (f_ε) satisfies the following conditions. For any $T \in \mathbb{R}_+$ and $R \geqslant 1$, we have*

$$f_\varepsilon(t,y)f_\varepsilon(t,y') \rightharpoonup f(t,y)f(t,y') \quad \textit{weakly in } L^1((0,T)\times(0,R)^2), \tag{4.7}$$

$$T_\varepsilon(f_\varepsilon)(t,y,y') \rightharpoonup f(t,y+y') \quad \textit{weakly in } L^1((0,T)\times(0,R)^2) \tag{4.8}$$

and

$$\sup_{[0,T]} \int_0^R \left(\int_0^\infty y' T_\varepsilon(f_\varepsilon)\,\mathrm{d}y' \right) \mathrm{d}y \leqslant C(T,R). \tag{4.9}$$

Proof. We split the proof into three steps.

STEP 1 (we prove (4.8)). On the one hand, let Φ be the function introduced in the proof of lemma 3.4. Denoting by m the integer such that $R \in \Lambda_m^\varepsilon$, we have, for any $t \in [0,T]$,

$$\begin{aligned}
\int_0^R \int_0^R \Phi(T_\varepsilon(f_\varepsilon))\,\mathrm{d}y'\mathrm{d}y &= \sum_{1\leqslant i,j\leqslant m} \Phi(c_{i+j}^\varepsilon)\varepsilon^2 \\
&\leqslant \sum_{k=2}^{2m}\sum_{j=1}^{m} \Phi(c_k^\varepsilon)\varepsilon^2 \\
&\leqslant (R+1)\sum_{k=2}^{2m} \Phi(c_k^\varepsilon)\varepsilon \\
&\leqslant C(R)\int_0^{2(R+1)} \Phi(f_\varepsilon)\,\mathrm{d}y \\
&\leqslant C(R,T)\int_0^{2(R+1)} \Phi(f^{\mathrm{in}})\,\mathrm{d}y.
\end{aligned}$$

Since $\Phi(r)/r \to +\infty$ as $r \to +\infty$, the above estimate implies that $(T_\varepsilon(f_\varepsilon))$ lies in a weakly compact subset of $L^1((0,T)\times(0,R)^2)$.

On the other hand, fix $\psi \in \mathcal{D}(\mathbb{R}_+^3)$ and compute

$$\begin{aligned}
\int_0^\infty\int_0^\infty\int_0^\infty T_\varepsilon(f_\varepsilon)\psi\,\mathrm{d}y'\mathrm{d}y\mathrm{d}t &= \int_0^\infty \sum_{i,j=1}^\infty c_{i+j}^\varepsilon \int\int_{\Lambda_i^\varepsilon\times\Lambda_j^\varepsilon} \psi(t,y,y')\,\mathrm{d}y'\mathrm{d}y\mathrm{d}t \\
&= \int_0^\infty \sum_{k=2}^\infty \varepsilon c_k^\varepsilon \sum_{j=1}^{k-1}\frac{1}{\varepsilon}\int\int_{\Lambda_{k-j}^\varepsilon\times\Lambda_j^\varepsilon} \psi(t,y,y')\,\mathrm{d}y'\mathrm{d}y\mathrm{d}t \\
&= \int_0^\infty\int_0^\infty f_\varepsilon S_\varepsilon(\psi)\,\mathrm{d}z\mathrm{d}t,
\end{aligned} \tag{4.10}$$

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where $S_\varepsilon(\psi)$ is defined by (4.4). From lemmas 3.4, 4.2 and 4.3, we deduce that

$$\int_0^\infty \int_0^\infty f_\varepsilon S_\varepsilon(\psi)\,\mathrm{d}z\mathrm{d}t \xrightarrow{\varepsilon\to 0} \int_0^\infty \int_0^\infty f S(\psi)\,\mathrm{d}z\mathrm{d}t, \tag{4.11}$$

where $S(\psi)$ is defined by (4.5). Finally, combining (4.10), (4.11) and the change of variables $(y, y') \to (y, z = y + y')$, we get

$$\int_0^\infty \int_0^\infty \int_0^\infty T_\varepsilon(f_\varepsilon)\psi\,\mathrm{d}y'\mathrm{d}y\mathrm{d}t \xrightarrow{\varepsilon\to 0} \int_0^\infty \int_0^\infty \int_0^\infty f(t, y + y')\psi(t, y, y')\,\mathrm{d}y'\mathrm{d}y\mathrm{d}t.$$

This exactly means that $T_\varepsilon(f_\varepsilon) \rightharpoonup f(y + y')$ in the sense of distributions and therefore (4.8) follows from the weak L^1 compactness previously established.

STEP 2 (we prove (4.9)). Let m be the integer such that $R \in \Lambda^\varepsilon_{m+1}$. We just have to compute

$$\begin{aligned}
\int_0^R \left(\int_0^\infty y' T_\varepsilon(f_\varepsilon)\,\mathrm{d}y'\right)\mathrm{d}y &\leqslant \sum_{i=1}^{m+1}\sum_{j=1}^{\infty} \int\int_{\Lambda_i^\varepsilon\times\Lambda_j^\varepsilon} y' T_\varepsilon(f_\varepsilon)(y, y')\,\mathrm{d}y\mathrm{d}y' \\
&\leqslant \sum_{i=1}^{m+1}\sum_{j=1}^{\infty} \varepsilon(j + i)c^\varepsilon_{i+j}\varepsilon^2 \\
&\leqslant \sum_{k=1}^{\infty} \varepsilon^2 k c^\varepsilon_k \sum_{i=1}^{m+1} \varepsilon \\
&\leqslant (R + 1)\int_0^\infty y f_\varepsilon\,\mathrm{d}y \\
&\leqslant C(T, R)
\end{aligned}$$

for any $t \in [0, T]$ by lemma 3.1.

STEP 3 (we prove (4.7)). We first claim that, for any $\psi \in L^\infty((0, T) \times (0, R)^2)$, there holds

$$\int_0^R f'_\varepsilon\psi'\,\mathrm{d}y' \xrightarrow{\varepsilon\to 0} \int_0^R f'\psi'\,\mathrm{d}y' \quad \text{strongly in } L^1((0, T) \times (0, R)). \tag{4.12}$$

Indeed, from lemma 3.5, we know that (4.12) holds true for any function ψ of the form

$$\psi(t, y, y') = \sum_{n=1}^{N} u_n(y')v_n(t, y). \tag{4.13}$$

Now, if ψ is an arbitrary function in $L^\infty((0, T) \times (0, R)^2)$, there exists a sequence (ψ_α) of functions of the form (4.13) such that $\psi_\alpha \to \psi$ a.e. and weakly in $L^\infty((0, T)\times (0, R)^2)$. Therefore,

$$\begin{aligned}
&\left\|\int_0^R (f'_\varepsilon - f')\psi'\,\mathrm{d}y'\right\|_{L^1} \\
&\qquad \leqslant \left\|\int_0^R (f'_\varepsilon - f')\psi'_\alpha\,\mathrm{d}y'\right\|_{L^1} + \sup_\varepsilon \int_0^T\int_0^R\int_0^R |f'_\varepsilon - f'||\psi' - \psi'_\alpha|\,\mathrm{d}y'\mathrm{d}y\mathrm{d}t,
\end{aligned}$$

https://doi.org/10.1017/S0308210500002080 Published online by Cambridge University Press

whence

$$\limsup_{\varepsilon\to 0}\left\|\int_0^R (f'_\varepsilon - f')\psi'\,\mathrm{d}y'\right\|_{L^1} \leqslant \sup_\varepsilon \int_0^T\int_0^R\int_0^R |f'_\varepsilon - f'||\psi' - \psi'_\alpha|\,\mathrm{d}y'\mathrm{d}y\mathrm{d}t$$

for every $\alpha > 0$. A variant of lemma 4.3 (see, for example, [22, lemma A.1]) then ensures that the right-hand side of the above equation converges to zero as $\alpha \to 0$ and thus (4.12) holds true for any $\psi \in L^\infty((0,T)\times(0,R)^2)$. Now, for a given $\psi \in L^\infty((0,T)\times(0,R)^2)$, it follows from (3.2), (4.12) and lemma 4.3 that

$$\int_0^T\int_0^R f_\varepsilon\left(\int_0^R f'_\varepsilon\psi\,\mathrm{d}y'\right)\mathrm{d}y\mathrm{d}t \xrightarrow{\varepsilon\to 0} \int_0^T\int_0^R f\left(\int_0^R f'\psi\,\mathrm{d}y'\right)\mathrm{d}y\mathrm{d}t, \tag{4.14}$$

which exactly means that (4.7) holds. □

Step 2 (passing to the limit in (4.1)). Thanks to the previous analysis, we now have all the necessary tools to pass to the limit in equation (4.1). We consider $\varphi \in \mathcal{D}([0,+\infty)\times\mathbb{R}_+)$ with $\operatorname{supp}\varphi \subset [0, L-1]^2$ for some $L \geqslant 1$ and recall that $\varphi_\varepsilon(t)$ is the ε-step function defined by (1.10) for $t \geqslant 0$. Let $T \in \mathbb{R}_+$ and $R \in \mathbb{R}_+$. On the one hand, we have

$$\begin{aligned}\int_0^T\int_0^R\int_0^R &[a_\varepsilon f_\varepsilon f'_\varepsilon - b_\varepsilon T_\varepsilon(f_\varepsilon)][T_\varepsilon(\varphi_\varepsilon) - \varphi_\varepsilon - \varphi'_\varepsilon]\,\mathrm{d}y\mathrm{d}y'\mathrm{d}t\\ &\xrightarrow{\varepsilon\to 0} \int_0^T\int_0^R\int_0^R [aff' - bf''][\varphi'' - \varphi - \varphi']\,\mathrm{d}y\mathrm{d}y'\mathrm{d}t,\end{aligned} \tag{4.15}$$

where we have used (2.15) and lemmas 4.1, 4.4 and 4.3, with first $v_\varepsilon = f_\varepsilon f'_\varepsilon$, $w_\varepsilon = a_\varepsilon[T_\varepsilon(\varphi_\varepsilon) - \varphi_\varepsilon - \varphi'_\varepsilon]$ and next $v_\varepsilon = T_\varepsilon(f_\varepsilon)$, $w_\varepsilon = b_\varepsilon[T_\varepsilon(\varphi_\varepsilon) - \varphi_\varepsilon - \varphi'_\varepsilon]$.

On the other hand, for $R \geqslant L$,

$$\begin{aligned}&\left|\iint_{\mathbb{R}_+^2\setminus[0,R]^2} (b_\varepsilon T_\varepsilon(f_\varepsilon) - a_\varepsilon f_\varepsilon f'_\varepsilon)(\varphi_\varepsilon + \varphi'_\varepsilon)\,\mathrm{d}y\mathrm{d}y'\right|\\ &\quad\leqslant 2\int_0^L\left(\int_R^\infty a_\varepsilon f_\varepsilon f'_\varepsilon\varphi_\varepsilon\,\mathrm{d}y'\right)\mathrm{d}y + 2\int_0^L\left(\int_R^\infty b_\varepsilon T_\varepsilon(f_\varepsilon)\varphi_\varepsilon\,\mathrm{d}y'\right)\mathrm{d}y\\ &\quad\leqslant 2\sup_{y\leqslant L,\, y'\geqslant R}\left|\frac{a_\varepsilon(y,y')}{y'}\right|\|\varphi\|_{L^\infty}\int_0^\infty f_\varepsilon\,\mathrm{d}y\int_0^\infty y'f'_\varepsilon\,\mathrm{d}y'\\ &\qquad + 2\sup_{y\leqslant L,\, y'\geqslant R}\left|\frac{b_\varepsilon(y,y')}{y'}\right|\|\varphi\|_{L^\infty}\int_0^L\int_0^\infty y'T_\varepsilon(f_\varepsilon)\,\mathrm{d}y'\mathrm{d}y \to 0,\end{aligned} \tag{4.16}$$

as $R \to +\infty$ uniformly with respect to $\varepsilon \in (0,1)$, thanks to (2.14), (4.9) and lemmas 3.1 and 3.2. Note that a similar argument yields

$$\left|\iint_{\mathbb{R}_+^2\setminus[0,R]^2} (bf'' - aff')(\varphi + \varphi')\,\mathrm{d}y\mathrm{d}y'\right| \to 0 \tag{4.17}$$

as $R \to \infty$.

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Finally, for R large enough, we have

$$\begin{aligned}\int_0^\infty \int_0^\infty (a_\varepsilon f_\varepsilon f'_\varepsilon - b_\varepsilon T_\varepsilon(f_\varepsilon))(T_\varepsilon(\varphi_\varepsilon) - \varphi_\varepsilon - \varphi'_\varepsilon)\,\mathrm{d}y\mathrm{d}y' \\ = \int_0^R \int_0^R (a_\varepsilon f_\varepsilon f'_\varepsilon - b_\varepsilon T_\varepsilon(f_\varepsilon))(T_\varepsilon(\varphi_\varepsilon) - \varphi_\varepsilon - \varphi'_\varepsilon)\,\mathrm{d}y\mathrm{d}y' \\ + \int\int_{\mathbb{R}_+^2\setminus[0,R]^2} (b_\varepsilon T_\varepsilon(f_\varepsilon) - a_\varepsilon f_\varepsilon f'_\varepsilon)(\varphi_\varepsilon + \varphi'_\varepsilon)\,\mathrm{d}y\mathrm{d}y'. \qquad (4.18)\end{aligned}$$

As a conclusion, gathering (4.15), (4.16), (4.17) and (4.18), we are able to pass to the limit in (4.1) and obtain that f satisfies (2.18); f is thus a weak solution to the CCF equation (1.1)–(1.3) and the proof of theorem 2.3 is complete.

5. A time-explicit Euler scheme

We define an explicit time and size discretization for the CCF equation (1.1)–(1.3) and prove that the sequence of approximate solutions converges to a solution of the CCF equation (1.1)–(1.3).

Throughout this section we assume that the kinetic coefficients $a, b \in \mathcal{C}(\mathbb{R}_+^2)$ satisfy (2.1), (2.2) and (2.3) and we fix an initial datum f^{in} satisfying (2.4). We put

$$M = \int_0^\infty (1+y) f^{\mathrm{in}}(y)\,\mathrm{d}y.$$

We will index our approximations by $k \in \mathbb{N}\setminus\{0\}$. For $k \geqslant 1$, we denote the time discretization step by Δ_k, the size discretization step by ε_k, the number of time-steps by N_k and the number of size cells by $J_k \geqslant 2$.

For $k \geqslant 1$, we put

$$a_{i,j}^k = \begin{cases} \varepsilon_k a(i\varepsilon_k, j\varepsilon_k) & \text{if } \max\{i,j\} \leqslant J_k, \\ 0 & \text{if } \max\{i,j\} > J_k \end{cases} \qquad (5.1)$$

and

$$b_{i,j}^k = \begin{cases} \varepsilon_k b(i\varepsilon_k, j\varepsilon_k) & \text{if } i+j \leqslant J_k, \\ 0 & \text{if } i+j > J_k. \end{cases} \qquad (5.2)$$

We define $c^{\mathrm{in},k} = (c_i^{\mathrm{in},k})$ by

$$c_i^{\mathrm{in},k} = \frac{1}{\varepsilon_k} \int_{\Lambda_i^{\varepsilon_k}} f^{\mathrm{in}}(y)\,\mathrm{d}y, \quad i \in \{1,\dots,2J_k\}, \qquad (5.3)$$

where $\Lambda_i^{\varepsilon_k} = [(i-\frac{1}{2})\varepsilon_k, (i+\frac{1}{2})\varepsilon_k)$, $i \geqslant 1$. If $f^{\mathrm{in}} \in \mathcal{C}(\mathbb{R}_+)$, we may also choose $c_i^{\mathrm{in},k} := f^{\mathrm{in}}(i\varepsilon_k)$ for $i \in \{1,\dots,2J_k\}$. In both cases, we have

$$f^{\mathrm{in},k} := \sum_{i=1}^{2J_k} c_i^{\mathrm{in},k} \chi_i^{\varepsilon_k}(y) \xrightarrow{k\to+\infty} f^{\mathrm{in}} \quad \text{in } L_1^1(\mathbb{R}_+)$$

as soon as $\varepsilon_k J_k \to +\infty$ as $k \to +\infty$.

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We next consider the following system of $2J_k$ difference equations,

$$\frac{c_i^{n+1,k}-c_i^{n,k}}{\Delta_k}=Q_i^k(c^{n,k}),\quad 0\leqslant n\leqslant N_k-1, \tag{5.4}$$

$$c_i^{0,k}=c_i^{\mathrm{in},k}, \tag{5.5}$$

for $i\in\{1,\dots,2J_k\}$, where we have set $c^{n,k}=(c_i^{n,k})_{i\geqslant 1}$, with $c_i^{n,k}\equiv 0$ for $i>2J_k$ and $Q_i^k(\cdot)$ defined as $Q_i(\cdot)$ with $(a_{i,j}^k)$, $(b_{i,j}^k)$ instead of $(a_{i,j})$, $(b_{i,j})$.

We finally put, for each $n\in\{0,\dots,N_k-1\}$,

$$f_k(t,y)=\sum_{i=1}^{2J_k}c_i^{n,k}\chi_i^{\varepsilon_k}(y)\quad \text{if } t\in[n\Delta_k,(n+1)\Delta_k). \tag{5.6}$$

With this notation, we may state our convergence result.

THEOREM 5.1. *There exists $\kappa=\kappa(A,M)$ such that, for any sequences (Δ_k), (ε_k), (J_k), (N_k) satisfying*

$$\Delta_k,\varepsilon_k\to 0,\qquad \varepsilon_kJ_k,\Delta_kN_k\to+\infty \tag{5.7}$$

and

$$\Delta_k(\varepsilon_kJ_k)^3,\Delta_k\varepsilon_kJ_k\exp(6AN_k\Delta_k)\leqslant\kappa, \tag{5.8}$$

the sequence (f_k) defined by (5.6) is a sequence of non-negative functions that lies in a weakly compact subset of $L^1((0,T)\times\mathbb{R}_+)$ and is bounded in $L^\infty(0,T;L^1_1(\mathbb{R}_+))$ for each $T\in\mathbb{R}_+$. In addition, up to the extraction of a subsequence, (f_k) converges weakly in $L^1((0,T)\times\mathbb{R}_+)$ towards a weak solution f to the CCF equation (1.1)–(1.3) for any $T\in\mathbb{R}_+$.

A possible choice is

$$\Delta_k=\frac{1}{k},\qquad \varepsilon_k=\frac{1}{k},\qquad J_k=k^{5/4},\qquad N_k=\frac{k\ln(k)}{10A}.$$

Observe that (5.8) strongly couples the admissible choices of the time and size discretizations and implies somehow that keeping a large number of equations (i.e. J_k large) requires a very small time-step Δ_k. This fact was already pointed out in [18], where some instabilities in the numerical simulations are reported when the time-step Δ_k is too large with respect to J_k. Theorem 5.1 thus provides quantitative information on this point. Let us further mention that, in [29], J_k is taken to be infinite, but the convergence of the scheme is restricted to bounded coagulation coefficients.

The proof of theorem 5.1 is very similar to the proof of theorem 2.3. We first establish the non-negativity of the function f_k for $k\geqslant 1$, together with the weak compactness in L^1 of the sequence (f_k). Without loss of generality, we may assume that $\varepsilon_kJ_k\geqslant 1$ for $k\geqslant 1$.

LEMMA 5.2. *The sequence (f_k) satisfies*

$$f_k(t,y)\geqslant 0\quad \textit{for } (t,y)\in\mathbb{R}_+^2$$

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and, for any $T > 0$,

$$\sup_{[0,T]} \int_0^\infty f_k(t,y)(1+y)\,\mathrm{d}y \leqslant C_T. \tag{5.9}$$

Proof. It follows from the symmetry property (2.1), (5.1), (5.2) and (5.4) that

$$\sum_{i=1}^{\infty} (c_i^{n+1,k} - c_i^{n,k})\varphi_i = \tfrac{1}{2}\Delta_k \sum_{i,j=1}^{\infty} (a_{i,j}^k c_i^{n,k} c_j^{n,k} - b_{i,j} c_{i+j}^{n,k})(\varphi_{i+j} - \varphi_i - \varphi_j) \tag{5.10}$$

for any sequence (φ_i) of real numbers. Taking $\varphi_i = i$, $i \geqslant 1$, we clearly get

$$\varepsilon_k^2 \sum_{i=1}^{\infty} i c_i^{n,k} = \varepsilon_k^2 \sum_{i=1}^{\infty} i c_i^{\mathrm{in},k} \leqslant 2 \int_0^\infty y f^{\mathrm{in}}(y)\,\mathrm{d}y. \tag{5.11}$$

We next claim that

$$c_i^{n,k} \geqslant 0 \quad \text{for any } n \in \{0,\dots,N_k\} \text{ and } i \geqslant 1 \tag{5.12}$$

and

$$\varepsilon_k \sum_{i=1}^{\infty} c_i^{n,k} \leqslant C(A,M) \exp\{6An\Delta_k\}. \tag{5.13}$$

We proceed by induction. First, equations (5.12) and (5.13) are obviously true for $n = 0$. We next assume that (5.12) and (5.13) hold true for $n' \in \{0,\dots,n\}$ for some $n \in \{0,\dots,N_k - 1\}$. We fix $m_k \geqslant 1$ such that $1 \in \Lambda_{m_k}^{\varepsilon_k}$ and proceed as in the proof of lemma 3.2 to obtain that

$$\sum_{i=1}^{m_k+1} \varepsilon_k (c_i^{n'+1,k} - c_i^{n',k}) \leqslant 6A\Delta_k \left(\sum_{i=1}^{m_k+1} \varepsilon_k c_i^{n',k} + 2M \right)$$

for any $n' \leqslant n$, where we have used (2.2), (5.11) and the non-negativity of $c_i^{n',k}$ for $0 \leqslant n' \leqslant n$. The discrete Gronwall lemma then yields

$$\sum_{i=1}^{m_k+1} \varepsilon_k c_i^{n',k} \leqslant C(A,M) \exp\{6An'\Delta_k\}$$

for any $n' \leqslant n+1$, whence (5.13) for $n' \in \{0,\dots,n+1\}$.

On the other hand, either $c_i^{n+1,k} = 0$ if $i > 2J_k$ or $i \leqslant 2J_k$ and (5.4) and (5.12) for n yield

$$c_i^{n+1,k} \geqslant \left\{ 1 - \Delta_k \left(\frac{1}{2} \sum_{j=1}^{i-1} b_{i-j,j}^k + \sum_{j=1}^{\infty} a_{i,j}^k c_j^{n,k} \right) \right\} c_i^{n,k},$$

with

$$\frac{1}{2} \sum_{j=1}^{i-1} b_{i-j,j}^k \leqslant A(1 + 2J_k\varepsilon_k)^2 J_k \varepsilon_k \leqslant 9A(J_k\varepsilon_k)^3$$

https://doi.org/10.1017/S0308210500002080 Published online by Cambridge University Press

by (2.2) and

$$\sum_{j=1}^{\infty} a_{i,j}^k c_j^{n,k} \leqslant (1+2J_k\varepsilon_k)\sum_{j=1}^{\infty}(1+\varepsilon_k j)\varepsilon_k c_j^{n,k}$$
$$\leqslant 3AJ_k\varepsilon_k(C(A,M)\exp\{6An\Delta_k\}+2M)$$
$$\leqslant C(A,M)J_k\varepsilon_k\exp\{6AN_k\Delta_k\}$$

by (2.2), (5.11) and (5.13). Consequently,

$$c_i^{n+1,k} \geqslant \{1 - C(A,M)\Delta_k((J_k\varepsilon_k)^3 + J_k\varepsilon_k\exp\{6AN_k\Delta_k\})\} \geqslant 0$$

thanks to (5.8) and the induction argument is complete. We have thus proved that (f_k) satisfies $f_k \geqslant 0$ and

$$\sup_{[0,T]}\int_0^\infty f_k(t,y)(1+y)\,\mathrm{d}y \leqslant C_T$$

for any $k \geqslant 1$. □

In order to establish that (f_k) lies in a weakly compact subset of $L^1((0,T)\times\mathbb{R}_+)$, we proceed as in lemma 3.3. We first recall that (2.4) and a refined version of the de la Vallée–Poussin theorem [9, 23] ensure that there exists a function $\varPhi$ fulfilling the assumptions of lemma 3.3 and such that $\varPhi(u)/u \to +\infty$ as $u \to +\infty$ and

$$\int_{\mathbb{R}_+} \varPhi(f^{\mathrm{in}})\,\mathrm{d}y < +\infty.$$

LEMMA 5.3. *For every $T \in \mathbb{R}_+$ and $R \in \mathbb{R}_+$, there exists k_T such that*

$$\sup_{[0,T]}\int_0^R \varPhi(f_k(t,y))\,\mathrm{d}y \leqslant C(T,R) \quad \textit{for } k \geqslant k_T. \tag{5.14}$$

Proof. We omit the index k for simplicity. Let m be the integer such that $m\varepsilon < R \leqslant (m+1)\varepsilon$. Since $\varPhi$ is convex, we infer from (5.4) and the non-negativity of $(a_{i,j})$, $(b_{i,j})$ and (c_i) that

$$\sum_{i=1}^{m+1}\varepsilon(\varPhi(c_i^{n+1}) - \varPhi(c_i^n))$$
$$\leqslant \varepsilon\sum_{i=1}^{m+1}(c_i^{n+1}-c_i^n)\varPhi'(c_i^{n+1})$$
$$\leqslant \tfrac{1}{2}\Delta\varepsilon\sum_{i=1}^{m+1}\sum_{j=1}^{i-1} a_{j,i-j}c_{i-j}^n c_j^n\varPhi'(c_i^{n+1}) + \Delta\varepsilon\sum_{i=1}^{m+1}\sum_{j=1}^{\infty} b_{i,j}c_{i+j}^n\varPhi'(c_i^{n+1}).$$

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On the one hand, thanks to (3.6), (5.9) and the monotonicity of Φ, we have

$$\begin{aligned}\sum_{i=1}^{m+1}\sum_{j=1}^{i-1} & a_{j,i-j}c_{i-j}^n c_j^n \Phi'(c_i^{n+1}) \\ & \leqslant A\sum_{i=1}^{m+1}\sum_{j=1}^{i-1}\varepsilon(1+\varepsilon j)c_j^n(1+\varepsilon(i-j))c_{i-j}^n\Phi'(c_i^{n+1}) \\ & \leqslant A\sum_{i=1}^{m+1}\sum_{j=1}^{i-1}\varepsilon(1+\varepsilon j)c_j^n\{\Phi(c_i^{n+1})+\Phi((1+\varepsilon(i-j))c_{i-j}^n)\} \\ & \leqslant C(T)\biggl\{\sum_{i=1}^{m+1}\Phi(c_i^{n+1})+\sum_{i=1}^{m+1}\Phi((R+2)c_i^n)\biggr\}.\end{aligned}$$

But the concavity of Φ' entails that $\Phi'((R+2)u)\leqslant(R+2)\Phi'(u)$ for $u\geqslant 0$, whence $\Phi((R+2)u)\leqslant(R+2)^2\Phi(u)$. Consequently,

$$\sum_{i=1}^{m+1}\sum_{j=1}^{i-1}a_{j,i-j}c_{i-j}^n c_j^n\Phi'(c_i^{n+1})\leqslant C(T)\sum_{i=1}^{m+1}\Phi(c_i^{n+1})+C(T,R)\sum_{i=1}^{m+1}\Phi(c_i^n).$$

On the other hand, combining the above argument with the one used in the proof of lemma 3.3, we obtain

$$\sum_{i=1}^{m+1}\sum_{j=1}^{\infty}b_{i,j}c_{i+j}^n\Phi'(c_i^{n+1})\leqslant C(T)\sum_{i=1}^{m+1}\Phi(c_i^{n+1})+\frac{C(T,R)}{\varepsilon}.$$

Therefore,

$$\begin{gathered}\sum_{i=1}^{m+1}\varepsilon\Phi(c_i^{n+1})\leqslant\Delta C(T)\sum_{i=1}^{m+1}\varepsilon\Phi(c_i^{n+1})+\Delta C(T,R)+(1+\Delta C(T,R))\sum_{i=1}^{m+1}\varepsilon\Phi(c_i^n), \\ \sum_{i=1}^{m+1}\varepsilon\Phi(c_i^{n+1})\leqslant\frac{1+\Delta C(T,R)}{1-\Delta C(T)}\sum_{i=1}^{m+1}\varepsilon\Phi(c_i^n)+\frac{\Delta C(T,R)}{1-\Delta C(T)},\end{gathered}$$

and we have $\Delta_k C(T)<1$ for k large enough by (5.7). We may then use the discrete Gronwall lemma and argue as in the proof of lemma 3.3 to conclude that (5.14) holds true. □

Finally, the functions f_k being discontinuous with respect to time, a weaker version of lemma 3.5 is available, namely,

$$\int_0^R f_k(\cdot,y)\psi(y)\,\mathrm{d}y\quad\text{is bounded in } BV(0,T)$$

for any $\psi\in L^\infty(0,R)$, $R\in\mathbb{R}_+$ and $T\in\mathbb{R}_+$.

We are now in a position to proceed as in § 4 to pass to the limit as $k\to+\infty$ in the equation satisfied by f_k, which reads, for $\varphi\in\mathcal{D}([0,+\infty)\times\mathbb{R}_+)$ and k large

https://doi.org/10.1017/S0308210500002080 Published online by Cambridge University Press

enough,

$$\int_{\Delta_k}^T \int_0^\infty f_k \frac{\varphi_{\varepsilon_k} - \tau_{\Delta_k}\varphi_{\varepsilon_k}}{\Delta_k}\,\mathrm{d}y\mathrm{d}t + \int_0^{\Delta_k}\int_0^\infty f^{\mathrm{in},k}\varphi_{\varepsilon_k}(t,\cdot)\,\mathrm{d}y\mathrm{d}t \\ + \frac{1}{2}\int_0^T\int_0^\infty\int_0^\infty (a_k f_k f_k' - b_k T_{\varepsilon_k}(f_k))(T_{\varepsilon_k}(\varphi_{\varepsilon_k}) - \varphi_{\varepsilon_k} - \varphi_{\varepsilon_k}')\,\mathrm{d}y\mathrm{d}y'\mathrm{d}t = 0,$$

where

$$a_k(y,y') = \sum_{i,j=1}^\infty \frac{a_{i,j}^k}{\varepsilon_k}\chi_i^{\varepsilon_k}(y)\chi_j^{\varepsilon_k}(y'), \qquad b_k(y,y') = \sum_{i,j=1}^\infty \frac{b_{i,j}^k}{\varepsilon_k}\chi_i^{\varepsilon_k}(y)\chi_j^{\varepsilon_k}(y'),$$

and $\varphi_{\varepsilon_k}(t)$ is the ε_k-step function defined by (1.10), $\tau_{\Delta_k}\varphi_{\varepsilon_k}(t,y) = \varphi_{\varepsilon_k}(t-\Delta_k, y)$ for $(t,y,y') \in \mathbb{R}_+^3$.

6. Other kinetic coefficients

In this section we discuss several extensions of theorem 2.3 under various assumptions on the coagulation and fragmentation rates and first consider the case of sublinear coagulation coefficients as described in theorem 2.5. In that case, the kinetic coefficients do not necessarily satisfy the growth condition (2.3), which is used to control the behaviour of f_ε for large values of y in the proof of theorem 2.3. Fortunately, it turns out that the assumption (2.22) provides such a control and the following lemma is actually the only new ingredient needed for the approach developed previously to work.

LEMMA 6.1. *Let $\Phi \in \mathcal{C}^1([0,+\infty))$ be a non-negative and piecewise $\mathcal{C}^2$-smooth convex function such that $\Phi(0) = 0$, $\Phi'(0) \geqslant 0$ and Φ' is concave. Under the assumptions of theorem 2.5, if*

$$M_\Phi := \int_0^\infty \Phi(y) f^{\mathrm{in}}(y)\,\mathrm{d}y < \infty, \tag{6.1}$$

there holds

$$\int_0^\infty \Phi(y) f_\varepsilon(t,y)\,\mathrm{d}y \leqslant C(T), \quad t \in [0,T], \tag{6.2}$$

where $C(T)$ depends only on A, M, Φ, M_Φ and T.

Proof. It follows from (6.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^\infty \Phi(i\varepsilon)c_i^\varepsilon = \frac{1}{2}\sum_{i,j=1}^\infty a_{i,j}^\varepsilon c_i^\varepsilon c_j^\varepsilon(\Phi((i+j)\varepsilon) - \Phi(i\varepsilon) - \Phi(j\varepsilon)) \\ - \frac{1}{2}\sum_{i,j=1}^\infty b_{i,j}^\varepsilon c_{i+j}^\varepsilon(\Phi((i+j)\varepsilon) - \Phi(i\varepsilon) - \Phi(j\varepsilon)).$$

Observe first that the convexity of Φ and $\Phi(0) = 0$ ensure that

$$\Phi((i+j)\varepsilon) \geqslant \Phi(i\varepsilon) + \Phi(j\varepsilon), \quad i,j \geqslant 1,$$

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and recall that the properties enjoyed by Φ warrant that Φ satisfies

$$(y+y')(\Phi(y+y')-\Phi(y)-\Phi(y')) \leqslant 2(y\Phi(y')+y'\Phi(y)) \tag{6.3}$$

for $(y,y') \in \mathbb{R}_+^2$ [20, lemma A.2]. In addition, equation (2.22) yields

$$a_{i,j}^{\varepsilon} \leqslant A\varepsilon(1+(i+j)\varepsilon), \quad i,j \geqslant 1.$$

Owing to (6.3), the symmetry of $a_{i,j}^{\varepsilon}$ and the non-negativity of $b_{i,j}^{\varepsilon}$ and c^{ε}, we deduce

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty}\Phi(i\varepsilon)c_i^{\varepsilon}\varepsilon &\leqslant \frac{1}{2}\sum_{i,j=1}^{\infty}\frac{a_{i,j}^{\varepsilon}}{(i+j)}c_i^{\varepsilon}c_j^{\varepsilon}(j\varepsilon\Phi(i\varepsilon)+i\varepsilon\Phi(j\varepsilon)) \\ &\leqslant A\biggl(\sum_{i=1}^{\infty}\varepsilon(1+i\varepsilon)c_i^{\varepsilon}\biggr)\biggl(\sum_{i=1}^{\infty}\Phi(i\varepsilon)c_i^{\varepsilon}\varepsilon\biggr). \end{aligned}$$

Thanks to (3.1) and (3.2), we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty}\Phi(i\varepsilon)c_i^{\varepsilon}\varepsilon \leqslant C(T)\sum_{i=1}^{\infty}\Phi(i\varepsilon)c_i^{\varepsilon}\varepsilon,$$

whence, by Gronwall's lemma,

$$\sum_{i=1}^{\infty}\Phi(i\varepsilon)c_i^{\varepsilon}(t)\varepsilon \leqslant C(T)\sum_{i=1}^{\infty}\Phi(i\varepsilon)c_i^{\mathrm{in},\varepsilon}\varepsilon, \quad t\in[0,T]. \tag{6.4}$$

Now the convexity of Φ and the concavity of Φ' entail that

$$\Phi(u+v) \leqslant \Phi(u)+v(\Phi'(0)+(u+v)\Phi''(0)), \quad u,v \geqslant 0, \tag{6.5}$$

and thus, since Φ is increasing, we have

$$\Phi(y) \leqslant \Phi(i\varepsilon+\tfrac{1}{2}\varepsilon) \leqslant \Phi(i\varepsilon)+C(\Phi)(1+\varepsilon i) \tag{6.6}$$

and

$$\Phi(i\varepsilon) \leqslant \Phi(y+\tfrac{1}{2}\varepsilon) \leqslant \Phi(y)+C(\Phi)(1+\varepsilon i) \tag{6.7}$$

for any $y \in \Lambda_i^{\varepsilon}$. Consequently,

$$\begin{aligned} \int_0^{\infty}\Phi(y)f_{\varepsilon}(t,y)\,\mathrm{d}y &\leqslant \varepsilon\sum_{i=1}^{\infty}c_i^{\varepsilon}(\Phi(i\varepsilon)+C(\Phi)(1+\varepsilon i)) \\ &\leqslant C(T,\Phi)\varepsilon\sum_{i=1}^{\infty}c_i^{\mathrm{in},\varepsilon}(1+\varepsilon i+\Phi(i\varepsilon)) \\ &\leqslant C(T,\Phi)\int_0^{\infty}f^{\mathrm{in}}(y)(1+y+\Phi(y))\,\mathrm{d}y, \end{aligned}$$

which completes the proof of lemma 6.1. □

The above computation is formal, as the series might not have the required convergence properties. A rigorous justification may be performed by replacing Φ by Φ_R, defined by

$$\Phi_R(y) = \begin{cases} \Phi(y) & \text{if } y \in [0,R], \\ \Phi'(R)(y-R) + \Phi(R) & \text{if } y \in [R,+\infty), \end{cases}$$

which enjoys the same properties, and then pass to the limit as $R \to +\infty$.

Proof of theorem 2.5. Since $y \mapsto y f^{\mathrm{in}}(y)$ belongs to $L^1(\mathbb{R}_+)$ by (2.4), we use once more a refined version of the de la Vallée–Poussin theorem [9, 23] to deduce that there exists Φ satisfying the requirements of lemma 6.1 and such that $\Phi(u)/u \to +\infty$ as $u \to +\infty$ and

$$\int_0^\infty \Phi(y) f^{\mathrm{in}}(y)\,\mathrm{d}y < +\infty.$$

On the one hand, the estimates of § 3 yields the weak compactness of (f_ε) in $L^1((0,T) \times \mathbb{R}_+)$ for each $T \in \mathbb{R}_+$. On the other hand, the superlinearity of Φ at infinity and lemma 6.1 imply the uniform integrability at infinity of (f_ε) in $L^1(\mathbb{R}_+; y\,\mathrm{d}y)$. Therefore, (f_ε) is weakly compact in $L^1((0,T)\times\mathbb{R}_+;(1+y)\,\mathrm{d}t\mathrm{d}y)$ for each $T \in \mathbb{R}_+$, whence (2.24). This, in turn, allows us to proceed as in § 4 to perform the limit $\varepsilon \to 0$. In addition, since c^ε satisfies (2.23), we have

$$\int_0^\infty f_\varepsilon(t,y) y\,\mathrm{d}y = \sum_{i=1}^\infty \varepsilon^2 i c_i^\varepsilon(t) = \sum_{i=1}^\infty \varepsilon^2 i c_i^{\mathrm{in}} = \int_0^\infty f^{\mathrm{in}}(y) \mathcal{I}^\varepsilon(y)\,\mathrm{d}y,$$

with

$$\mathcal{I}^\varepsilon(y) = \sum_{i=1}^\infty i \chi_i^\varepsilon(y).$$

Since $|\mathcal{I}^\varepsilon(y) - y| \leqslant \frac{1}{2}\varepsilon \to 0$ for any $y \geqslant 0$, the assertion (2.25) readily follows. □

Another situation in which theorem 2.5 holds is the so-called *strong fragmentation* case [7, 16]. More precisely, when the kinetic coefficients satisfy

$$a(y,y') \leqslant A(1+y)^\alpha (1+y')^\alpha \quad \text{and} \quad b(y,y') \geqslant B(1+y+y')^{-\beta}$$

for some $\alpha \in [0,1]$ and $\beta < 2(1-\alpha)$, the existence of a solution to the DCF and CCF equations, satisfying (2.23) and (2.25), respectively, follows from [7] and [16], respectively. Combining the moment estimates in [7, 16] and the proof of theorem 2.3 ensures that theorem 2.5 holds true in that case, too.

As a final example, let us consider the case of multiplicative coagulation coefficients, which includes, in particular, the case $a(y,y') = yy'$. More precisely, assume that a satisfies

$$r(y)r(y') \leqslant a(y,y') \leqslant A r(y) r(y'), \quad (y,y') \in \mathbb{R}_+^2, \tag{6.8}$$

for some non-negative function r and there is a positive function $\beta \in L^\infty(\mathbb{R}_+)$ such that $\beta(y) \to 0$ as $y \to +\infty$ and

$$b(y,y') \leqslant \beta(y+y'), \quad (y,y') \in \mathbb{R}_+^2. \tag{6.9}$$

https://doi.org/10.1017/S0308210500002080 Published online by Cambridge University Press

In that case, theorem 2.3 is still valid with the same proof except that one has to control rf_ε for large values of y, since a does not necessarily satisfy (2.3). Such a control is supplied by the multiplicative structure (6.8) of a as there holds

$$\int_0^T \biggl(\int_{R+1}^\infty r^\varepsilon(y) f_\varepsilon(t,y)\,\mathrm{d}y\biggr)^2 \mathrm{d}t \leqslant C(T)\Bigl(R^{-1} + \sup_{\{y\geqslant R-1\}}\{\beta(y)\}\Bigr)$$

for $T \in \mathbb{R}_+$ and $R \geqslant 1$ with

$$r^\varepsilon = \sum_{i=1}^\infty r_i^\varepsilon \chi_i^\varepsilon \quad \text{and} \quad r_i^\varepsilon = \frac{1}{\varepsilon}\int_{\Lambda_i^\varepsilon} r(y)\,\mathrm{d}y, \quad i \geqslant 1.$$

7. Spatially non-homogeneous models

As already mentioned, the convergence of the family of solutions of the DCF equations to a solution of the CCF equation relies on a weak stability principle. The above convergence result introduced in a spatially homogeneous setting is thus likely to be extended to a spatially non-homogeneous setting in a framework where such a stability result is available. Such a theory has been developed recently in [22] for the diffusive continuous coagulation–fragmentation equation and we present now, without proof, an example of an available convergence result. We refer to [22] for details, as well as for other assumptions on the coagulation and fragmentation coefficients (such as a detailed balance condition) for which convergence from the discrete to the continuous coagulation–fragmentation equations could also be obtained. The well posedness of the diffusive CCF equation is also investigated in [2] with a different approach.

In the non-homogeneous setting considered here, the clusters are assumed to move in an open bounded subset Ω of $\mathbb{R}^N$, $N \geqslant 1$, with smooth boundary $\partial\Omega$, according to Brownian movement or diffusion (thermal coagulation). The diffusion coefficient $d = d(y) \in \mathcal{C}(\mathbb{R}_+) > 0$ is assumed to be only size dependent and the diffusive CCF equation reads

$$\partial_t f - d(y)\Delta_x f = Q(f), \qquad (t,x,y) \in (0,+\infty)\times\Omega\times\mathbb{R}_+, \tag{7.1}$$

$$\partial_n f = 0, \qquad (t,x,y) \in (0,+\infty)\times\partial\Omega\times\mathbb{R}_+, \tag{7.2}$$

$$f(0,x,y) = f^{\mathrm{in}}(x,y), \qquad (x,y) \in \Omega\times\mathbb{R}_+, \tag{7.3}$$

where $Q(f)$ is still defined by (1.3).

Assume now that, in addition to the symmetry and growth conditions (2.1), (2.2) and (2.3), the following monotonicity condition holds,

$$a(y', y-y') \leqslant a(y', y) \quad \text{for } y \geqslant y' \geqslant 0, \tag{7.4}$$

and that the coagulation process dominates the fragmentation process in the following sense,

$$b(y', y-y') \leqslant Aa(y', y) + B(y') \quad \text{for any } y \geqslant y' \geqslant 0, \tag{7.5}$$

where A is a non-negative constant and B is a non-negative function such that

$$B \in L^1(\mathbb{R}_+) \quad \text{and} \quad y \mapsto yB(y) \in L^\infty(\mathbb{R}_+). \tag{7.6}$$

We finally consider an initial datum f^{in} satisfying

$$f^{\text{in}} \in L^1(\Omega \times \mathbb{R}_+; (1+y)\,\mathrm{d}x\mathrm{d}y) \quad \text{is non-negative a.e.} \tag{7.7}$$

The above assumptions on d, a, b and f^{in} then ensure the existence of a weak solution to (7.1)–(7.3) [22, theorem 2.6].

We next introduce a sequence of diffusive DCF equations as follows. For $\varepsilon \in (0,1)$, the kinetic coefficients are still given by (2.5) and we put $d_i^\varepsilon = d(i\varepsilon)$,

$$c_i^{\text{in},\varepsilon}(x) = \frac{1}{\varepsilon}\int_{\Lambda_i^\varepsilon} f^{\text{in}}(x,y)\,\mathrm{d}y, \quad x \in \Omega,$$

with $\Lambda_i^\varepsilon = [(i-\frac{1}{2})\varepsilon, (i+\frac{1}{2})\varepsilon)$ for $i \geqslant 1$. We then denote by $c^\varepsilon = (c_i^\varepsilon)_{i\geqslant 1}$ a solution to the diffusive DCF equations

$$\partial_t c_i^\varepsilon - d_i^\varepsilon \Delta_x c_i^\varepsilon = Q_i^\varepsilon(c^\varepsilon) \quad \text{in } (0,+\infty)\times\Omega, \tag{7.8}$$

$$\partial_n c_i^\varepsilon = 0 \quad \text{on } (0,+\infty)\times\partial\Omega, \tag{7.9}$$

$$c_i^\varepsilon(0) = c_i^{\text{in},\varepsilon} \quad \text{in } \Omega \tag{7.10}$$

for $i \geqslant 1$, where $c_i^\varepsilon = c_i^\varepsilon(t,x) \geqslant 0$ denotes the local concentration of clusters of size i. The existence of c^ε follows from, for example, [21] (see also the references therein for a more precise account of the existence results for the diffusive DCF equations). We finally put

$$f^\varepsilon(t,x,y) := \sum_{i=1}^{\infty} c_i^\varepsilon(t,x)\chi_i^\varepsilon(y), \quad (t,x,y) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}_+.$$

THEOREM 7.1. *Under the above assumptions, the family (f_ε) lies in a weakly compact subset of $L^1((0,T)\times\Omega\times\mathbb{R}_+)$ for each $T \in \mathbb{R}_+$ and, up to the extraction of a subsequence, $f_\varepsilon \rightharpoonup f$ weakly in $L^1((0,T)\times\Omega\times\mathbb{R}_+)$, where f is a weak solution to the diffusive CCF (7.1)–(7.3).*

Acknowledgments

S.M. gratefully acknowledges the partial support of the TMR project 'Asymptotic Methods in Kinetic Theory' ERB FMRX CT97 0157 during this work.

Appendix A. Weak and mild solutions

Assume that the kinetic coefficients a and b enjoy the symmetry condition (2.1) and the growth condition (2.2) and consider an initial datum satisfying (2.4).

LEMMA A.1. *For any weak solution f of (1.1)–(1.3) (in the sense of definition 2.1), there exists $\tilde{f} \in \mathcal{C}^{0,1/3}([0,T]; L^1)$ for any $T \in \mathbb{R}_+$ such that $\tilde{f} = f$ a.e. in $\mathbb{R}_+^2$ and $\tilde{f}$ satisfies $\tilde{f}(0) = f^{\text{in}}$ and, for any $t_1 > t_0 \geqslant 0$,*

$$\tilde{f}(t_1,y) - \tilde{f}(t_0,y) = \int_{t_0}^{t_1} Q(\tilde{f})(s,y)\,\mathrm{d}s \quad \textit{for a.e. } y \in \mathbb{R}_+. \tag{A 1}$$

In other words, $\tilde{f}$ is a mild solution to (1.1)–(1.3) in the sense of definition 2.2.

Proof. We fix $\varepsilon > 0$ and a non-negative function $\varrho \in \mathcal{D}(\mathbb{R}_+)$ such that $\|\varrho\|_{L^1} = 1$. We next define

$$\rho_\varepsilon(t) = \frac{1}{\varepsilon}\rho\left(\frac{t}{\varepsilon}\right), \qquad f_\varepsilon = f *_t \rho_\varepsilon, \qquad Q_\varepsilon = Q(f) *_t \rho_\varepsilon. \tag{A 2}$$

It readily follows from (1.1) that f_ε satisfies

$$\frac{\partial}{\partial t} f_\varepsilon = Q_\varepsilon \quad \text{in } \mathcal{D}'([0,+\infty) \times \mathbb{R}_+). \tag{A 3}$$

Furthermore, the bounds on f and (2.2) imply that, for any $T \in \mathbb{R}_+$ and $R \in \mathbb{R}_+$, we have

$$Q_k(f) \in L^\infty((0,T); L^1(0,R)). \tag{A 4}$$

We deduce from (A 3) and (A 4) that, for any $\psi \in L^\infty(\mathbb{R}_+)$ with compact support in $[0,+\infty)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty f_\varepsilon \psi \,\mathrm{d}y = \int_0^\infty Q_\varepsilon \psi \,\mathrm{d}y \quad \text{in } \mathcal{D}'([0,+\infty)). \tag{A 5}$$

But since

$$t \mapsto \int_0^\infty f_\varepsilon \psi \,\mathrm{d}y = \left(\int_0^\infty f\psi \,\mathrm{d}y\right) *_t \varrho_\varepsilon \in \mathcal{C}([0,+\infty)), \tag{A 6}$$

we also have, for any $t_1 > t_0 \geqslant 0$,

$$\int_0^\infty f_\varepsilon(t_1)\psi \,\mathrm{d}y - \int_0^\infty f_\varepsilon(t_0)\psi \,\mathrm{d}y = \int_{t_0}^{t_1}\int_0^\infty Q_\varepsilon \psi \,\mathrm{d}y\mathrm{d}t. \tag{A 7}$$

Consider now $T \geqslant t_1 > t_0 \geqslant 0$ and $R > 0$. On the one hand, we choose

$$\psi(y) = \mathrm{sgn}(f_\varepsilon(t_1,y) - f_\varepsilon(t_0,y))\mathbf{1}_{[0,R]}$$

in (A 7). This gives

$$\int_0^R |f_\varepsilon(t_1) - f_\varepsilon(t_0)| \,\mathrm{d}y \leqslant |t_1 - t_0| \|\varrho_\varepsilon\|_{L^1} \sup_{[0,T]} \int_0^R |Q(f)| \,\mathrm{d}y,$$

and we notice that $\|\varrho_\varepsilon\|_{L^1} = 1$, while definition 2.1 and (2.2) yield

$$\begin{aligned}\int_0^R |Q(f)| \,\mathrm{d}y &\leqslant \frac{3}{2}\bigg\{\int_0^R\int_0^\infty a(y,y')f(t,y)f(t,y') \,\mathrm{d}y\mathrm{d}y' \\ &\qquad + \int_0^\infty f(t,z)\int_0^{\min(z,R)} b(y,z-y) \,\mathrm{d}y\mathrm{d}z\bigg\} \\ &\leqslant \tfrac{3}{2}A\{\|f(t)\|_{L^1_1}(\|f(t)\|_{L^1_1} + (1+R)^2)\} \\ &\leqslant C(T)(1+R)^2\end{aligned}$$

for $t \in [t_0,t_1]$. Therefore,

$$\begin{aligned}\int_0^\infty |f_\varepsilon(t_1) - f_\varepsilon(t_0)| \,\mathrm{d}y &\leqslant \int_0^R |f_\varepsilon(t_1) - f_\varepsilon(t_0)| \,\mathrm{d}y + \int_R^\infty |f_\varepsilon(t_1) - f_\varepsilon(t_0)| \,\mathrm{d}y \\ &\leqslant C(T)|t_1 - t_0|(1+R^2) + \frac{2}{R}\sup_{[0,T]} \|f(t)\|_{L^1_1},\end{aligned}$$

https://doi.org/10.1017/S0308210500002080 Published online by Cambridge University Press

and the choice $R = |t_1 - t_0|^{-1/3}$ leads to

$$\|f_\varepsilon(t_1) - f_\varepsilon(t_0)\|_{L^1} \leqslant C(T)|t_1 - t_0|^{1/3}. \tag{A 8}$$

Now the definition of f_ε warrants that (f_ε) converges towards f in $L^1((0,T)\times\mathbb{R}_+)$. Consequently, there is a subsequence of (f_ε) (not relabelled) such that

$$f_\varepsilon(t,\cdot) \to f(t,\cdot) \quad \text{in } L^1(\mathbb{R}_+)$$

for every $t \in [0,T] \setminus \mathcal{Z}$, where $\mathcal{Z}$ is a subset of $[0,T]$ with zero measure. Putting $\tilde{f}(t,\cdot) = f(t,\cdot)$ for $t \in [0,T] \setminus \mathcal{Z}$, we infer from (A 8) that

$$\|\tilde{f}(t_1) - \tilde{f}(t_0)\|_{L^1} \leqslant C(T)|t_1 - t_0|^{1/3}$$

for any $t_0 \in [0,T]\setminus\mathcal{Z}$ and $t_1 \in [0,T]\setminus\mathcal{Z}$ and we may thus extend $\tilde{f}$ by continuity to a function of $\mathcal{C}^{0,1/3}([0,T];L^1(\mathbb{R}_+))$, which we still denote by $\tilde{f}$. Clearly, $\tilde{f} = f$ a.e. in $\mathbb{R}_+^2$ and it follows from definition 2.1 that $\tilde{f}$ also satisfies (2.18). The continuity of $\tilde{f}$ and (2.18) then allow us to conclude that $\tilde{f}(0) = f^{\text{in}}$ and $\tilde{f}$ satisfies (2.20). □

References

1 M. Aizenman and T. A. Bak. Convergence to equilibrium in a system of reacting polymers. *Commun. Math. Phys.* **65** (1979), 203–230.

2 H. Amann. Coagulation–fragmentation processes. *Arch. Ration. Mech. Analysis* **151** (2000), 339–366.

3 H. Babovsky. On a Monte Carlo scheme for Smoluchowski's coagulation equation. *Monte Carlo Meth. Appl.* **5** (1999), 1–18.

4 J. M. Ball and J. Carr. The discrete coagulation–fragmentation equations: existence, uniqueness and density conservation. *J. Stat. Phys.* **61** (1990), 203–234.

5 I. Bouzoubaa, K. Hamdache and A. Noussair. A global weak solution via transport-projection scheme to a thin film growth kinetic model with nonlocal boundary conditions. *SIAM J. Numer. Analysis* **39** (2001), 1810–1834.

6 O. Bruno, A. Friedman and F. Reitich. Asymptotic behavior for a coalescence problem. *Trans. Am. Math. Soc.* **338** (1993), 133–158.

7 F. P. da Costa. Existence and uniqueness of density conserving solutions to the coagulation–fragmentation equations with strong fragmentation. *J. Math. Analysis Applic.* **192** (1995), 892–914.

8 M. Deaconu, N. Fournier and E. Tanré. Study of a stochastic particle system associated with the Smoluchowski coagulation equation (2001). (Preprint.)

9 C. Dellacherie and P. A. Meyer. *Probabilités et potentiel*, ch. I and IV (Paris: Hermann, 1975).

10 P. Deuflhard and M. Wulkow. Computational treatment of polyreaction kinetics by orthogonal polynomials of a discrete variable. *Impact Comput. Sci. Engng* **1** (1989), 269–301.

11 R. L. Drake. A general mathematical survey of the coagulation equation. In *Topics in current aerosol research, part 2*. International Reviews in Aerosol Physics and Chemistry, pp. 203–376 (Oxford: Pergamon, 1972).

12 P. B. Dubovskiĭ and I. W. Stewart. Existence, uniqueness and mass conservation for the coagulation–fragmentation equation. *Math. Meth. Appl. Sci.* **19** (1996), 571–591.

13 A. Eibeck and W. Wagner. An efficient stochastic algorithm for studying coagulation dynamics and gelation phenomena. *SIAM J. Sci. Comput.* **22** (2000), 802–821.

14 A. Eibeck and W. Wagner. Stochastic particle approximations for Smoluchowski's coagulation equation. *Ann. Appl. Prob.* **11** (2001), 1137–1165.

15 M. Escobedo, S. Mischler and B. Perthame. Gelation in coagulation and fragmentation models. *Commun Math. Phys.* (In the press.)

16 M. Escobedo, Ph. Laurençot, S. Mischler and B. Perthame. Gelation and mass conservation in coagulation–fragmentation models. (Preprint.)

17 F. Guiaş. A Monte Carlo approach to the Smoluchowski equations. *Monte Carlo Meth. Appl.* **3** (1997), 313–326.

18 D. S. Krivitsky. Numerical solution of the Smoluchowski kinetic equation and asymptotics of the distribution function. *J. Phys.* A **28** (1995), 2025–2039.

19 Ph. Laurençot. On a class of continuous coagulation–fragmentation equations. *J. Diff. Eqns* **167** (2000), 245–274.

20 Ph. Laurençot. The Lifshitz–Slyozov equation with encounters. *Math. Models Meth. Appl. Sci.* **11** (2001), 731–748.

21 Ph. Laurençot and S. Mischler. Global existence for the discrete diffusive coagulation–fragmentation equation in L^1. *Rev. Mat. Iberoamericana*. (In the press.)

22 Ph. Laurençot and S. Mischler. The continuous coagulation–fragmentation equation with diffusion. *Arch. Ration. Mech. Analysis* **162** (2002), 45–99.

23 Lê Châu-Hoàn. Etude de la classe des opérateurs m-accrétifs de $L^1(\Omega)$ et accrétifs dans $L^\infty(\Omega)$. PhD thesis, Université de Paris VI (1977).

24 M. H. Lee. On the validity of the coagulation equation and the nature of runaway growth. *Icarus* **143** (2000), 74–86.

25 S. Mischler. Convergence of discrete-velocity schemes for the Boltzmann equation. *Arch. Ration. Mech. Analysis* **140** (1997), 53–77.

26 H. Müller. Zur allgemeinen Theorie der raschen Koagulation. *Kolloidchemische Beihefte* **27** (1928), 223–250.

27 E. Otto and H. Fissan. Brownian coagulation of submicron particles. *Adv. Powder Technol.* **10** (1999), 1–20.

28 A. Palczewski and J. Schneider. Existence, stability, and convergence of solutions of discrete velocity models to the Boltzmann equation. *J. Stat. Phys.* **91** (1998), 307–326.

29 K. K. Sabelfeld and A. A. Kolodko. Monte Carlo simulation of the coagulation processes governed by Smoluchowski equation with random coefficients. *Monte Carlo Meth. Appl.* **3** (1997), 275–311.

30 M. Smoluchowski. Drei Vorträge über Diffusion, Brownsche Molekularbewegung und Koagulation von Kolloidteilchen. *Phys. Z.* **17** (1916), 557–599.

31 M. Smoluchowski. Versuch einer mathematischen Theorie der Koagulationskinetik kolloider Lösungen. *Z. Phys. Chem.* **92** (1917), 129–168.

32 J. L. Spouge. An existence theorem for the discrete coagulation–fragmentation equations. *Math. Proc. Camb. Phil. Soc.* **96** (1984), 351–357.

33 I. W. Stewart. A global existence theorem for the general coagulation–fragmentation equation with unbounded kernels. *Math. Meth. Appl. Sci.* **11** (1989), 627–648.

34 I. W. Stewart. A uniqueness theorem for the coagulation–fragmentation equation. *Math. Proc. Camb. Phil. Soc.* **107** (1990), 573–578.

35 I. I. Vrabie. *Compactness methods for nonlinear evolutions*, 2nd edn. Pitman Monographs and Surveys on Pure and Applied Mathematics, vol. 75 (New York: Longman, 1995).

36 R. M. Ziff and E. D. McGrady. The kinetics of cluster fragmentation and depolymerisation. *J. Phys.* A **18** (1985), 3027–3037.

(*Issued Publication date 2002*)