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The size of the primes obstructing the existence of rational points

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The sequence of prime numbers p for which a variety over $\mathbb{Q}$ has no p-adic point plays a fundamental role in arithmetic geometry. This sequence is deterministic, however, we prove that if we choose a typical variety from a family then the sequence has random behaviour. We furthermore prove that this behaviour is modelled by a random walk in Brownian motion. This has several consequences, one of them being the description of the finer properties of the distribution of the primes in this sequence via the Feynman–Kac formula.

Keywords: Rational points; Brownian motion

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1. Introduction

1.1. Primes p for which typical smooth varieties have no p-adic point

The first step in checking whether a homogeneous Diophantine equation defined over the rational numbers has a non-trivial rational solution is to check whether it has non-trivial solutions in the p-adic completions of the rational numbers for primes p of bad reduction. It may be the case that the least prime p for which there is no p-adic solution is large compared to the coefficients of the equation. Therefore, a straightforward computational attempt to prove the non-existence of a $\mathbb{Q}$-point via p-adic checks that does not take into consideration the probable size of these primes p would fail if the running time is limited compared to the size of the coefficients of the equation. There are two basic questions one can ask for the (finite) sequence of primes p for which a typical smooth variety has no p-adic point:

Question 1.1. *Does this deterministic sequence behave in a random way as the variety varies in a family?*

Question 1.2. *If the behaviour can be modelled by that of a random, i.e. uniformly distributed sequence, what is the corresponding discrepancy?*

Naturally, these questions cannot be answered for any arbitrary variety over $\mathbb{Q}$, therefore, we restrict ourselves to statements that hold for 'almost all' members in general infinite collections of varieties. Our collections of varieties take the following shape. Let V be a proper, smooth irreducible algebraic projective variety over $\mathbb{Q}$ equipped with a dominant morphism $f : V \to \mathbb{P}^n$ with geometrically integral generic fibre. One can view V as a collection of infinitely many varieties, each variety being given by the fibre $f^{-1}(x)$ above a point $x \in \mathbb{P}^n(\mathbb{Q})$. This setting includes several situations of central importance to arithmetic geometry, see, for example, [**4**, **6**, **7**] and [**15**]. A natural question in this context is to study the density of fibres with a $\mathbb{Q}$-rational point. Serre [**25**] investigated this when every fibre of f is a conic and, in an important recent work, Loughran and Smeets [**21**, theorem 1.1] proved that 0% of the fibres of f have a $\mathbb{Q}$-rational point, as long as the fibre over some codimension one point of $\mathbb{P}^n$ is irreducible, but not geometrically integral. Both investigations proceeded by examining p-adic solubility for all primes p.

Associated to f is a non-negative number $\Delta(f)$ that depends on the geometry of the singular fibres of f. It was introduced by Loughran and Smeets [**21**, § 1] and it will frequently resurface throughout our work.

Definition 1.1 Loughran and Smeets. Let $f : V \to X$ be a dominant proper morphism of smooth irreducible varieties over a field k. For each (scheme-theoretic) point $x \in X$ with perfect residue field $\kappa(x)$, the absolute Galois group $\mathrm{Gal}(\overline{\kappa(x)}/\kappa(x))$ of the residue field acts on the irreducible components of $f^{-1}(x)_{\overline{\kappa(x)}} := f^{-1}(x) \times_{\kappa(x)} \overline{\kappa(x)}$ of multiplicity 1. We choose some finite group Γ_x through which this action factors. Then we define

$$\delta_x(f) = \frac{\sharp\left\{\gamma \in \Gamma_x : \begin{array}{l}\gamma \text{ fixes an irreducible component} \\ \text{of } f^{-1}(x)_{\overline{\kappa(x)}} \text{ of multiplicity 1}\end{array}\right\}}{\sharp\Gamma_x}$$

and

$$\Delta(f) = \sum_{D \in X^{(1)}} (1 - \delta_D(f)),$$

where $X^{(1)}$ denotes the set of codimension 1 points of X.

For $x \in \mathbb{P}^n(\mathbb{Q})$ we define the function

$$\omega_f(x) := \sharp\left\{\text{primes} p : f^{-1}(x)(\mathbb{Q}_p) = \emptyset\right\}. \tag{1.1}$$

Although we might have $\omega_f(x) = +\infty$ for certain $x \in \mathbb{P}^n(\mathbb{Q})$, note that the Lang–Weil estimates [**20**] and Hensel's lemma guarantee that $\omega_f(x) < +\infty$ when $f^{-1}(x)$ is geometrically integral. Let H denote the usual Weil height on $\mathbb{P}^n(\mathbb{Q})$. The case $r = 1$ of theorems 1.3 and 1.12 in the work of Loughran and Sofos [**22**] implies that

$$\limsup_{B \to +\infty} \frac{1}{\sharp\{x \in \mathbb{P}^n(\mathbb{Q}) : H(x) \leqslant B, f^{-1}(x) \text{ smooth}\}} \sum_{\substack{x \in \mathbb{P}^n(\mathbb{Q}), H(x) \leqslant B \\ f^{-1}(x) \text{ smooth}}} \omega_f(x)$$

is bounded if and only if $\Delta(f) = 0$. Put in simple terms, the condition $\Delta(f) = 0$ is equivalent to the generic variety $f^{-1}(x)$ having too few primes p for which there is

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no p-adic point. One example with $\Delta(f)=0$ is given by

$$V:\sum_{i=0}^{4} x_i y_i^2 = 0 \subset \mathbb{P}^4\times\mathbb{P}^4$$

and $f:V\to\mathbb{P}^4$ defined by $f(x,y)=x$. Here, for all $x\in\mathbb{P}^n(\mathbb{Q})$ with $f^{-1}(x)$ smooth we have $\omega_f(x)=0$, see [**26**, § 4.2.2, theorem 6(iv)]. To avoid such examples we shall study the statistics of the set of primes in (1.1) only when $\Delta(f)\neq 0$.

To state our results it will be convenient to use the following notation: for all $B\geqslant 1$ we introduce the set

$$\Omega_B := \{x\in\mathbb{P}^n(\mathbb{Q}) : H(x)\leqslant B\}$$

and let $\mathbf{P}_B$ be the uniform probability measure on Ω_B, that is for any set $S\subseteq\mathbb{P}^n(\mathbb{Q})$ we let

$$\mathbf{P}_B[S] := \frac{\sharp\{x\in\Omega_B : x\in S\}}{\sharp\Omega_B}.$$

Definition 1.2 The j-th smallest obstructing prime. For $x\in\mathbb{P}^n(\mathbb{Q})$ and $j\in\mathbb{Z}\cap[0,\omega_f(x)]$ we define $p_0(x):=-\infty$ and for $j\geqslant 1$ we define $p_j(x)$ to be the j-th smallest prime p such that $f^{-1}(x)$ has no p-adic point. If $j>\omega_f(x)$ we define $p_j(x):=+\infty$.

1.2. Distribution of the least obstructing prime

Before continuing with our discussion on the distribution of every element in the sequence $\{p_j(x)\}_{j\geqslant 1}$ we provide a result concerning the typical size of $p_1(x)$.

Theorem 1.3. *Assume that V is a smooth projective variety over $\mathbb{Q}$ equipped with a dominant morphism $f:V\to\mathbb{P}^n$ with geometrically integral generic fibre and $\Delta(f)\neq 0$. Let $\xi:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ be any function that satisfies $\lim_{B\to+\infty}\xi(B)=+\infty$ and which is bounded above by some polynomial. Then*

$$\frac{\sharp\{x\in\Omega_B : p_1(x)>\xi(B)\}}{\sharp\Omega_B} \ll \left(\frac{\log\log\xi(B)}{\log\xi(B)}\right)^{\Delta(f)}. \tag{1.2}$$

In particular,

$$\lim_{B\to+\infty}\mathbf{P}_B[\{x\in\mathbb{P}^n(\mathbb{Q}) : p_1(x)\leqslant\xi(H(x))\}]=1.$$

Hence, the value of $p_1(x)$ is typically small, for example, $p_1(x)\leqslant\log\log\log H(x)$ holds for almost all fibres $f^{-1}(x)$. The proof of theorem 1.3 is given in § 3.2.

1.3. Equidistribution of obstructing primes via moments

Let us now move to question 1.1. By the case $r=1$ of [**22**, theorem 1.3] we see that for $x\in\mathbb{P}^n(\mathbb{Q})$ with $H(x)\leqslant B$ and $f^{-1}(x)$ smooth the usual size of $\omega_f(x)$ is

$\Delta(f)\log\log B$. Furthermore, by lemma 3.1 we have $p_j(x) \leqslant B^{D_0+1}$ for all j and for some positive D_0 that only depends on f. Thus the points

$$\log\log p_1(x) < \log\log p_2(x) < \ldots < \log\log p_{\omega_f(x)}(x) \tag{1.3}$$

are approximately $\Delta(f)\log\log B$ in cardinality and they all lie in an interval whose shape is approximated by the interval $[0, \log\log B]$. Therefore, if the finite sequence (1.3) was equidistributed then the subset S of all $x \in \mathbb{P}^n(\mathbb{Q})$ for which

$$\log\log p_j(x) = \frac{j}{\Delta(f)}(1 + o(1)) \quad \text{for all } 1 \leqslant j \leqslant \omega_f(x) \tag{1.4}$$

would satisfy $\lim_{B\to+\infty} \mathbf{P}_B[S] = 1$. Our first result confirms this kind of equidistribution as long as j is not taken too small. Furthermore, it shows that the error in the approximation (1.4) follows a normal distribution.

Theorem 1.4. *Assume that V is a smooth projective variety over $\mathbb{Q}$ equipped with a dominant morphism $f : V \to \mathbb{P}^n$ with geometrically integral generic fibre and $\Delta(f) \neq 0$. Let $j : \mathbb{R}_{\geqslant 1} \to \mathbb{N}$ be any function with*

$$\lim_{B\to+\infty} j(B) = +\infty \quad \text{and} \quad \lim_{B\to+\infty} \frac{j(B) - \Delta(f)\log\log B}{\sqrt{\Delta(f)\log\log B}} = -\infty.$$

Then for any $r \in \mathbb{R}$ we have

$$\lim_{B\to\infty} \mathbf{P}_B\left[\left\{x \in \mathbb{P}^n(\mathbb{Q}) : \log\log p_{j(B)}(x) \leqslant \frac{j(B)}{\Delta(f)} + r\frac{j(B)^{1/2}}{\Delta(f)}\right\}\right]$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{r} \mathrm{e}^{-t^2/2}\,\mathrm{d}t.$$

An analogous result for the number of distinct prime divisors of a random integer was established by Galambos [**12**, theorem 2]. The proof of theorem 1.4 is given in § 3.5.

One of the simplest criteria for the randomness of a sequence is equidistribution, thus theorem 1.4 answers question 1.1 in an affirmative manner. We shall see in remark 3.14 that the second growth assumption placed on j is necessary for theorem 1.4 to hold. Theorem 1.4 gives an approximation to the size of $p_j(x)$ for a single value of j, therefore, it is natural to ask whether the main term in the approximation holds for several primes $p_j(x)$ simultaneously. This is indeed true as our next result shows.

Theorem 1.5. *Let V and f be as in theorem 1.4. Let $\varepsilon > 0, M > 0$ be arbitrary and let $\xi : [1,\infty) \to [1,\infty)$ be any function such that $\lim_{B\to+\infty} \xi(B) = +\infty$. Then*

$$\mathbf{P}_B\left[\left\{x \in \Omega_B : \xi(B) < j \leqslant \omega_f(x) \Rightarrow \left|\log\log p_j(x) - \frac{j}{\Delta(f)}\right| \leqslant j^{1/2+\varepsilon}\right\}\right] - 1$$
$$\ll_{f,\varepsilon,M} \frac{1}{\xi(B)^M},$$

where the implied constant depends at most on f, ε and M.

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The proof of theorem 1.5 is given in § 3.6 and it generalizes an analogous result given by Hall and Tenenbaum [**14**, theorem 10] regarding the number of distinct prime divisors $\omega(m)$ of a random integer m. One of the main steps in the proof of theorem 1.5 is the verification of theorem 3.10, where moments of arbitrary order of

$$\omega_f(x,T) := \sharp\big\{\text{primes } p \leqslant T : f^{-1}(x)(\mathbb{Q}_p) = \emptyset\big\}, \quad (x \in \mathbb{P}^n(\mathbb{Q}), T \geqslant 1), \tag{1.5}$$

are estimated asymptotically and uniformly in the parameter T. The arguments behind [**14**, theorem 10] rely on [**14**, theorem 010], whose proof makes use of the fact that for every $y > 0$ the function $y^{\omega(m)}$ is multiplicative. The function $\omega_f(x)$ does not have this property, which is why we have to resort to finding the moments of $\omega_f(x,T)$.

Recall that by [**22**, theorem 1.2] we have

$$\mathbf{P}_B\left[\left\{x \in \mathbb{P}^n(\mathbb{Q}) : \omega_f(x) \geqslant \frac{\Delta(f)}{2}\log\log B\right\}\right] = 1.$$

Therefore taking, for example, $\xi(B) := \log\log\log B$ in theorem 1.5 shows that the typical size of the j-th smallest prime p for which the variety $f^{-1}(x)$ has no p-adic point is doubly exponential in j for all *large* j, i.e.

$$p_j(x) \approx \exp\left(\exp\left(\frac{j}{\Delta(f)}\right)\right).$$

In particular, we conclude that the expected size of the obstructing large primes is independent of the variety.

Notation. All implied constants in the Landau/Vinogradov notation $O(\cdot), \ll$, depend at most on the fibration f, except where specified by the use of a subscript. The counting function of the distinct prime factors is denoted by $\omega(m) := \sharp\{p \text{ prime} : p \mid m\}$ and the standard Möbius function on the integers will be denoted by μ.

2. The connection with Brownian motion

One of the main results in the work of Loughran and Sofos [**22**, theorem 1.2] is that when $\Delta(f) \neq 0$ then for almost all $x \in \mathbb{P}^n(\mathbb{Q})$ we have

$$\omega_f(x) = \Delta(f)\log\log H(x) + \mathscr{Z}_x\sqrt{\Delta(f)\log\log H(x)},$$

where the function $\mathscr{Z}_x$ is distributed like a Gaussian random variable with mean 0 and variance 1, i.e.

$$\mathscr{Z}_x \sim \mathscr{N}(0,1).$$

One way to think of this result is as a Central Limit Theorem for a specific sequence of independently distributed random events; the probability space is to be thought as the set of all fibres $f^{-1}(x)$, the sequence is indexed by the primes p and the random event is the non-existence of p-adic points. Knowing the distribution of ω_f does not provide sufficient control over the distribution of the $p_j(x)$, which, as we already saw, corresponds to knowing the distribution of $\omega_f(x,T)$ for all

$1 \leqslant T \leqslant H(x)$. Indeed, it can be shown by the second part of lemma 3.2 that $\omega_f(x) = \omega_f(x, H(x)) + O(1)$, with an implied constant that only depends on f. Thus, $\omega_f(x)$ essentially coincides with $\omega_f(x, T)$ when T has size $H(x)$.

The analogy with the Central Limit Theorem above is useful due to the following fact: assume we have a sequence of independent, identically distributed random variables $X_i, i \geqslant 1$, each with mean 0 and variance 1. The Central Limit Theorem states that the random variable

$$Y(n) := \frac{1}{\sqrt{n}} \sum_{1 \leqslant j \leqslant n} X_j$$

is distributed like $\mathscr{N}(0,1)$ as $n \to +\infty$. For every $0 \leqslant T \leqslant 1$ one may also consider the averages

$$Y(n,T) := \frac{1}{\sqrt{n}} \sum_{1 \leqslant j \leqslant Tn} X_j.$$

As with $\omega_f(x, T)$, we have $Y(n,T) = Y(n)$ when $T = 1$. By the Central Limit Theorem we can see that, for fixed T and as $n \to +\infty$, $Y(n,T)$ is distributed like the normal distribution with mean 0 and variance T as $n \to +\infty$. However, the random variables $Y(n,T)$ have a richer structure than $Y(n)$, namely, Donsker's theorem [**9**] asserts that $Y(n,T)$ is distributed like a random walk in Brownian motion. Brownian motion is a subject that has been widely studied throughout the last 100 years and, in particular, there are many results regarding the distribution of these random walks.

Thus, if we showed an analogue of Donsker's theorem for $\omega_f(x, T)$, this would enable us to use the theory of Brownian motion to directly obtain distribution theorems for the sequence of primes $p_j(x), j \geqslant 1$. This is the main plan for the rest of this paper.

2.1. Paths associated to varieties

Let $B \geqslant 1$ and $x \in \mathbb{P}^n(\mathbb{Q})$ with $H(x) \leqslant B$. It turns out that the appropriate object that allows us to describe the location of the primes counted by $\omega_f(x)$ in (1.1) is $\omega_f(x, \exp(\log^t B))$ for $t \in [0,1]$. Note that as t grows from 0 to 1, this function grows gradually from being almost 0 to becoming almost $\omega_f(x)$. Taking $T = \exp(\log^t B)$ in theorem 3.10 shows that for fixed t and for $B \to +\infty$ the average of this function is approximated by

$$\Delta(f) \log\log(\exp(\log^t B)) = t\Delta(f) \log\log B.$$

This suggests the following normalization of $\omega_f(x, \exp(\log^t B))$.

Definition 2.1. Assume that V is a smooth projective variety over $\mathbb{Q}$ equipped with a dominant morphism $f : V \to \mathbb{P}^n$ with geometrically integral generic fibre and $\Delta(f) \neq 0$. For each $x \in \mathbb{P}^n(\mathbb{Q})$ and $B \in \mathbb{R}_{\geqslant 3}$ we define the function $X_B(\cdot, x) : [0,1] \to \mathbb{R}$ as follows,

$$t \mapsto X_B(t,x) := \frac{\omega_f(x, \exp(\log^t B)) - t\Delta(f) \log\log B}{(\Delta(f) \log\log B)^{1/2}}.$$

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Remark 2.2. We will later show that for most $x \in \mathbb{P}^n(\mathbb{Q})$ and when $B \to +\infty$, the function $X_B(\cdot, x)$ behaves like the function

$$t \mapsto Z_B(t,x) := \frac{1}{(\Delta(f)\log\log B)^{1/2}} \sum_{p \leqslant \exp(\log^t B)} \begin{cases} 1-\sigma_p, & \text{if } f^{-1}(x)(\mathbb{Q}_p)=\emptyset, \\ -\sigma_p, & \text{otherwise}, \end{cases} \tag{2.1}$$

where σ_p is given by

$$\sigma_p := \frac{\sharp\left\{x \in \mathbb{P}^n(\mathbb{F}_p) : f^{-1}(x) \text{ is non}-\text{split}\right\}}{\sharp\mathbb{P}^n(\mathbb{F}_p)}. \tag{2.2}$$

Here, a scheme over a field k is called split if it contains a geometrically integral open subscheme and is called non-split otherwise. The term was introduced by Skorobogatov [**27**, Definition 0.1]. The weight σ_p is $\Delta(f)/p$ on average over p, namely, it is shown by Loughran and Smeets [**21**, theorem 1.2] that

$$\Delta(f) = \lim_{B\to+\infty} \frac{\sum_{p\leqslant B} \sigma_p}{\sum_{p\leqslant B} \frac{1}{p}}. \tag{2.3}$$

For fixed $B \geqslant 3$ and $x \in \mathbb{P}^n(\mathbb{Q})$ we shall show that when (2.1) is thought as a function of t, it defines a right-continuous step function in the plane. This step function essentially behaves like a discontinuous random walk that moves upwards at primes p for which the fibre $f^{-1}(x)$ has no p-adic point and moves downwards at primes p for which the fibre has a p-adic point.

Let us now recall the definition of Brownian motion from [**2**, § 37]. First, a stochastic process is collection of random variables (on a probability space $(\Omega, \mathscr{F}, P)$) indexed by a parameter regarded as representing time. A *Brownian motion* or *Wiener process* is a stochastic process $\{B_\tau : \tau \geqslant 0\}$, on some probability space $(\Omega, \mathscr{F}, P)$, with the following properties:

- The process starts at 0 almost surely:

$$P[B_0 = 0] = 1.$$

- The increments are independent: If $0 \leqslant \tau_0 \leqslant \tau_1 \leqslant \ldots \leqslant \tau_k$, then for all intervals $H_i \subset \mathbb{R}$,

$$P[B_{\tau_i} - B_{\tau_{i-1}} \in H_i, i \leqslant k] = \prod_{i\leqslant k} P[B_{\tau_i} - B_{\tau_{i-1}} \in H_i].$$

- For $0 \leqslant \sigma < \tau$ the increment $B_\tau - B_\sigma$ is normally distributed with mean 0 and variance $\tau - \sigma$, i.e. for every interval $H \subset \mathbb{R}$,

$$P[B_\tau - B_\sigma \in H] = \frac{1}{\sqrt{2\pi(\tau-\sigma)}} \int_H \mathrm{e}^{-x^2/2(\tau-\sigma)}\, \mathrm{d}x.$$

- For each $\omega \in \Omega$, $B_\tau(\omega)$ is continuous in τ and $B_0(\omega) = 0$.

Wiener showed that such a process exists, see [**2**, theorem 37.1]. One can thus think of Ω as the space of continuous functions in $[0,\infty)$ and $\mathscr{F}$ as the σ-algebra generated by the open sets under the uniform topology in Ω.

Let D be the space of all real-valued right-continuous functions on $[0,1]$ that have left-hand limits, see [**3**, p. 121], and consider the Skorohod topology on D, see [**3**, p. 123]. For any $A \subset D$ we let $\partial A := \overline{A} \cap \overline{(D \setminus A)}$. We denote by $\mathscr{D}$ the Borel σ-algebra generated by the open subsets of D. As explained in [**3**, p.146], one can make $(D,\mathscr{D})$ into a probability space by extending the classical Wiener measure from the space of continuous functions equipped with the uniform topology to the space D. This measure will be denoted by W throughout this paper.

Note that for every $x \in \mathbb{P}^n(\mathbb{Q})$ the function $X_B(\cdot,x)$ is in D.

THEOREM 2.3. *Assume that V is a smooth projective variety over $\mathbb{Q}$ equipped with a dominant morphism $f: V \to \mathbb{P}^n$ with geometrically integral generic fibre and $\Delta(f) \neq 0$. Let S be any set in $\mathscr{D}$ with $W[\partial S] = 0$. Then*

$$\lim_{B \to +\infty} \mathbf{P}_B\left[\{x \in \mathbb{P}^n(\mathbb{Q}) : X_B(\cdot,x) \in S\}\right] = W[S].$$

This is proved in § 4.5. A similar result for strongly additive functions defined on the integers was established by Billingsley [**1**, § 4] and Philipp [**24**, theorem 2]. However, in our situation the relevant level of distribution is zero, while this is not true for the analogous problem over the integers, see remark 3.8. This necessitates the use of a truncated version of X_B [see (4.16)], which results in more technical arguments.

Wiener's measure gives a model for Brownian motion, hence, by remark 2.2, theorem 2.3 has the following interpretation: one has infinitely many random walks $X_B(\cdot,x)$ in $[0,1]\times\mathbb{R}$, each walk corresponding to every fibre $f^{-1}(x)$. The walk is traced out according to the existence of p-adic points on the variety $f^{-1}(x)$. Random walks and Brownian motion have been studied intensely in physics and probability theory, because they provide an effective way to predict the walk traced out by a particle in Brownian motion according to collision with molecules. As such, the underlying mathematical theory needed has been particularly enriched throughout the last century, see, for example, the book of Karatzas and Shreve [**19**]. In the next section we shall use parts of this theory to provide results that go beyond theorems 1.4 and 1.5.

2.2. Extreme values

We provide the first consequence of theorem 2.3. As one ranges over different values of T the function $\omega_f(x,T)$ takes into account the finer distribution of the primes p for which $f^{-1}(x)$ has no p-adic point. It is therefore important to know the maximal value of $\omega_f(x,T)$. This is answered by drawing upon results on the maximum value distribution of walks in Brownian motion.

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THEOREM 2.4. *Let V and f be as in theorem 1.4. For every $r \in \mathbb{R}_{>0}$ we have*

$$\lim_{B\to+\infty} \mathbf{P}_B \left[\left\{x \in \mathbb{P}^n(\mathbb{Q}) : \max_{\substack{p \text{ prime} \\ p \leqslant B}} \left\{\frac{\omega_f(x,T) - \Delta(f)\log\log p}{(\Delta(f)\log\log H(x))^{1/2}}\right\} \geqslant r\right\}\right]$$
$$= \frac{2}{\sqrt{2\pi}} \int_r^{+\infty} \mathrm{e}^{-t^2/2}\,\mathrm{d}t. \tag{2.4}$$

This will turn out to be a direct consequence of the reflection principle in Brownian motion. Taking $p = p_j(x)$ in theorem 2.4 leads to the following conclusion. Both theorem 2.4 and the next corollary are proved in § 5.1.

COROLLARY 2.5. *Let V and f be as in theorem 1.4. For every $r \in \mathbb{R}_{>0}$ we have*

$$\liminf_{B\to+\infty} \mathbf{P}_B \left[\left\{x \in \mathbb{P}^n(\mathbb{Q}) : 1 \leqslant j \leqslant \omega_f(x) \Rightarrow \log\log p_j(x) \geqslant \frac{j}{\Delta(f)}\right.\right.$$
$$\left.\left. - r\left(\frac{\log\log H(x)}{\Delta(f)}\right)^{1/2}\right\}\right] \geqslant 1 - \frac{2}{\sqrt{2\pi}} \int_r^{+\infty} \mathrm{e}^{-t^2/2}\,\mathrm{d}t.$$

Furthermore, for every function $\xi(B) : \mathbb{R}_{\geqslant 1} \to \mathbb{R}_{\geqslant 1}$ with $\lim_{B\to+\infty} \xi(B) = +\infty$ we have

$$\lim_{B\to\infty} \mathbf{P}_B \left[\left\{x \in \mathbb{P}^n(\mathbb{Q}) : 1 \leqslant j \leqslant \omega_f(x) \Rightarrow \log\log p_j(x)\right.\right.$$
$$\left.\left. \geqslant \frac{j}{\Delta(f)} - \xi(H(x))\,(\log\log H(x))^{1/2}\right\}\right] = 1.$$

In contrast to theorem 1.5 this result gives merely lower bounds for $p_j(x)$, however, it does apply to the whole range of j, in particular to those that are left uncovered by theorem 1.5.

2.3. Largest deviation

Our next result provides asymptotic estimates for the density with which $\omega_f(x,T)$ deviates from its expected value. Its analogue in Brownian motion regards random walks in the presence of absorbing barriers, see [**17**].

Let us define the function $\tau_\infty : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ via

$$\tau_\infty(r) := \frac{4}{\pi} \sum_{m=0}^{+\infty} \frac{(-1)^m}{2m+1} \exp\left\{-\frac{(2m+1)^2\pi^2}{8r^2}\right\}. \tag{2.5}$$

THEOREM 2.6. *Let V and f be as in theorem 1.4. For every $r \in \mathbb{R}_{>0}$ we have*

$$\lim_{B\to+\infty} \mathbf{P}_B \left[\left\{x \in \mathbb{P}^n(\mathbb{Q}) : \max_{\substack{p \text{ prime} \\ p \leqslant B}} \left|\frac{\omega_f(x,p) - \Delta(f)\log\log p}{(\Delta(f)\log\log H(x))^{1/2}}\right| \geqslant r\right\}\right] = 1 - \tau_\infty(r). \tag{2.6}$$

Theorem 2.6 and the next corollary are proved in § 5.2.

Corollary 2.7. *Let V and f be as in theorem 1.4. For every $r \in \mathbb{R}_{>0}$ the quantity*

$$\liminf_{B\to\infty} \mathbf{P}_B \left[\left\{ x \in \mathbb{P}^n(\mathbb{Q}) : 1 \leqslant j \leqslant \omega_f(x) \Rightarrow \left| \log\log p_j(x) - \frac{j}{\Delta(f)} \right| \leqslant r \left(\frac{\log\log H(x)}{\Delta(f)} \right)^{1/2} \right\} \right]$$

is at least as large as $\tau_\infty(r)$. Furthermore, the probability

$$\mathbf{P}_B \left[\left\{ x \in \mathbb{P}^n(\mathbb{Q}) : 1 \leqslant j \leqslant \omega_f(x) \Rightarrow \left| \log\log p_j(x) - \frac{j}{\Delta(f)} \right| \leqslant r\sqrt{\log\log H(x)} \right\} \right]$$

equals $1 + O_f\left((1+|r|)^{-2/3}\right)$, with an implied constant that depends at most on f.

It is useful to compare the second limit statement in corollary 2.7 with theorem 1.5. Choosing any function $\xi(B)$ with $\xi(B) = o((\log\log B)^{1/2})$ in theorem 1.5 will give a precise approximation for $\log\log p_j(x)$ in a range for j that is wider than the range in which the second limit statement in corollary 2.7 gives a precise approximation. However, the advantage of corollary 2.7 is that it gives a better error term in the estimate for $\mathbf{P}_B$ and, furthermore, it provides a better approximation to $\log\log p_j(x)$ than theorem 1.5 when

$$(\log\log B)^{1/2} \ll j \leqslant \omega_f(x) = \Delta(f)(\log\log B)(1+o(1)).$$

2.4. L_2-norm deviations

In statistical mechanics, the mean squared displacement (*MSD*) is a 'measure' of the deviation of the position of a particle with respect to a reference position over time. One of the fundamental results of the theory of Brownian motion is that the *MSD* of a free particle during a time interval t is proportional to t. It was studied via diffusion equations by Einstein and Langevin, see [**5**].

Let us now examine an analogous situation for p-adic solubility. Define for $y, q \in \mathbb{R}$,

$$\theta_1(y,q) := 2\sum_{m=0}^{\infty} (-1)^m q^{(2m+1)^2/4} \sin((2m+1)y),$$

let $\theta_2(y,q) := \frac{\partial}{\partial y}\theta_1(y,q)$ and for $r \geqslant 0$ set

$$\tau_2(r) := \frac{4}{\pi^{3/2}} \int_{0\leqslant u\leqslant r/2} \int_{0\leqslant t\leqslant \pi/2} \theta_2(t/2, \mathrm{e}^{-1/4u}) \frac{\mathrm{d}t}{(\cos t)^{1/2}} \frac{\mathrm{d}u}{u^{3/2}}.$$

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Theorem 2.8. *Let V and f be as in theorem 1.4. For every $r \in \mathbb{R}_{>0}$ we have*

$$\lim_{B\to+\infty} \mathbf{P}_B \left[\left\{ x \in \mathbb{P}^n(\mathbb{Q}) : \frac{1}{(\Delta(f)\log\log B)^2} \sum_{p\leqslant B} \sigma_p\Big(\omega_f(x,p) - \sum_{q\leqslant p}\sigma_q\Big)^2 < r \right\}\right] = \tau_2(r). \tag{2.7}$$

This is proved in § 5.4.

2.5. Concentration of obstructing primes

Let us now turn our attention to question 1.2. The results so far show that the elements in the sequence (1.3) are equidistributed, however, it may be that the set of primes p satisfying $f^{-1}(x)(\mathbb{Q}_p) = \emptyset$ is not fully equidistributed. This could be, for example, due to a possible clustering of some of its elements. To study the sparsity (or lack thereof) of such clusters we shall look into the following set: for $x \in \mathbb{P}^n(\mathbb{Q})$ we define

$$\mathscr{C}_f(x) := \left\{ p \text{ prime} : \omega_f(x,p) > \sum_{q\leqslant p}\sigma_q \right\}.$$

By lemma 3.4 and the case $r = 1$ of theorem 3.10 the expected value of $\omega_f(x,p)$ is

$$\Delta(f)\log\log p \approx \sum_{q\leqslant p}\sigma_q,$$

therefore, p is in $\mathscr{C}_f(x)$ exactly when there are 'many' primes ℓ with $f^{-1}(x)(\mathbb{Q}_\ell) = \emptyset$ that are concentrated below p. Let us note that for all $x \in \mathbb{P}^n(\mathbb{Q})$ outside a Zariski closed subset of $\mathbb{P}^n(x)$ this set is finite. This is because if $f^{-1}(x)$ is smooth then by lemma 3.1 we have $\omega_f(x) \ll_f \frac{\log H(x)}{\log\log H(x)}$ and therefore lemma 3.4 gives

$$p \in \mathscr{C}_f(x) \Rightarrow \log\log p \ll \sum_{q\leqslant p}\sigma_p < \omega_f(x,p) \leqslant \omega_f(x) \ll_f \frac{\log H(x)}{\log\log H(x)}. \tag{2.8}$$

We wish to study the distribution of $\sharp\mathscr{C}_f(x)$. It turns out that it is more convenient to do so for a version of $\sharp\mathscr{C}_f(x)$ where the primes are weighted appropriately. Recall (2.2) and let

$$\widehat{\mathscr{C}_f}(x) := \sum_{p\in\mathscr{C}_f(x)}\sigma_p.$$

For $x \in \mathbb{P}^n(\mathbb{Q})$ with $f^{-1}(x)$ smooth we can use (2.3) to get

$$\widehat{\mathscr{C}_f}(x) \leqslant \sum_{p\leqslant \max\{q:\ q\in\mathscr{C}_f(x)\}}\sigma_p \ll \log\log\max\{q : q \in \mathscr{C}_f(x)\},$$

hence, by (2.8) one has

$$\widehat{\mathscr{C}_f}(x) \ll_f \frac{\log H(x)}{\log\log H(x)}. \tag{2.9}$$

We shall see that this bound is best possible in § 5.6.

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Let us now turn our attention to the average order of magnitude of $\widehat{\mathscr{C}}_f(x)$. If $\mathscr{C}_f(x)$ consisted of all primes $p \leqslant H(x)$ then by (2.3) the order of magnitude of $\widehat{\mathscr{C}}_f(x)$ would be $\log\log H(x)$. The next result shows that there is, in fact, a distribution law for the corresponding ratio.

THEOREM 2.9. *Let V and f be as in theorem 1.4. For every $\alpha < \beta \in [0,1]$ we have*

$$\lim_{B\to+\infty} \mathbf{P}_B\left[\left\{x \in \mathbb{P}^n(\mathbb{Q}) : \frac{\widehat{\mathscr{C}}_f(x)}{\Delta(f)\log\log H(x)} \in (\alpha,\beta]\right\}\right] = \frac{1}{\pi}\int_\alpha^\beta \frac{\mathrm{d}u}{\sqrt{u(1-u)}}. \tag{2.10}$$

This can be viewed as a p-adic solubility analogue of Lévy's arcsine law that concerns the time that a random walk in Brownian motion spends above 0, see [**23**, § 5.4]. One consequence of theorem 2.9 is that, since the area computed by $\int_0^1 (u(1-u))^{-1/2}\,\mathrm{d}u$ is concentrated in the regions around $u=0$ and $u=1$, for most fibres $f^{-1}(x)$ the set of primes p without a p-adic point will be either very regularly or very irregularly spaced. Theorem 2.9 is proved in § 5.5.

2.6. The Feynman–Kac formula

The Feynman–Kac formula plays a major role in linking stochastic processes and partial differential equations, see the book of Karatzas and Schrieve [**19**, § 4.4] and the book of Mörters and Peres [**23**, § 7.4]. For its applications to other sciences see the book by Del Moral [**8**].

We shall use the formula to establish a link between p-adic solubility and differential equations. Our result will roughly say that in situations more general than those in theorems 2.4, 2.6, 2.8 and 2.9 the analogous distributions (such as those in the right side of (2.4),(2.6), (2.7) and (2.10)) are derived from equations similar to Schrödinger's equation in quantum mechanics. The following definition can be found in the work of Kac [**18**].

DEFINITION 2.10. Let $\mathscr{K} : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ be a non-negative bounded function. For $s,u \in \mathbb{R}$ with $s>0$ and $u>0$ we say that a solution $\Psi_{s,u}$ of the differential equation

$$\frac{1}{2}\frac{\mathrm{d}^2\Psi_{s,u}}{\mathrm{d}x^2} = (s + u\mathscr{K}(x))\Psi_{s,u}(x) \tag{2.11}$$

is fundamental if it satisfies the conditions

- $\lim_{|x|\to\infty} \Psi_{s,u}(x) = 0$,
- $\sup_{x\neq 0} |\Psi'_{s,u}(x)| < \infty$,
- $\Psi'_{s,u}(+0) - \Psi'_{s,u}(-0) = -2$.

Equation (2.11) is related to the heat equation, see, for example, § 7.4 in the book of Mörters and Peres [**23**]. The solution $\Psi_{s,u}(x)$ corresponds to the temperature at the place x for a heat flow with cooling at rate $-u\mathscr{K}(x)$.

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Influenced by the work of Feynman [11], Kac [18] proved that a fundamental solution exists, is unique, and, furthermore, that for every $s>0$ and $u>0$ it fulfils

$$\int_0^{+\infty} \mathrm{e}^{-st}\mathbb{E}^0\left(\exp\left\{-u\int_0^t \mathscr{K}(B_\tau)\,\mathrm{d}\tau\right\}\right)\mathrm{d}t = \int_{-\infty}^{+\infty} \Psi_{s,u}(x)\,\mathrm{d}x, \tag{2.12}$$

where $\mathbb{E}^0$ is taken over all Brownian motion paths $\{B_\tau : \tau \geqslant 0\}$ satisfying $B_0=0$ almost surely and with respect to the Wiener measure W. Kac then used this to calculate the distribution function

$$W\left[\int_0^t \mathscr{K}(B_\tau)\,\mathrm{d}\tau \leqslant z\right], \quad (t>0, z>0),$$

for various choices of $\mathscr{K}$. Thus, (2.12) employs differential equations in order to allow the use of appropriately general 'test functions' $\mathscr{K}$ that measure the evolution through time of the distance from the average position (i.e., $\tau=0$) of a Brownian motion path.

Recall the meaning of V, f and $\Delta(f)$ in § 1.1 and the definitions of $\omega_f(x,T)$ and σ_p in (1.5) and (2.2) respectively. We shall use theorem 2.3 and (2.12) to study the fluctuation of $\omega_f(x,p)$ as the prime p varies. For this, we define for every non-negative bounded function $\mathscr{K} : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ and every $B \in \mathbb{R}_{\geqslant 3}$ and $t \in [0,1]$ the function $\widetilde{\mathscr{K}_B}(\cdot,t) : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}$ given by

$$\widetilde{\mathscr{K}_B}(x,t) := \frac{1}{\Delta(f)\log\log B}\sum_{p\leqslant \exp(\log^t B)} \sigma_p \mathscr{K}\left(\frac{\omega_f(x,p)-\sum_{q\leqslant p}\sigma_q}{\sqrt{\Delta(f)\log\log B}}\right). \tag{2.13}$$

The choices $\mathscr{K}(x) := x^2$ and $\mathscr{K}(x) := \mathbf{1}_{[0,\infty)}(x)$ are relevant to theorems 2.8 and 2.9 respectively. Our next result allows general non-negative bounded 'test functions' $\mathscr{K}$, thus it provides a general method for dealing with question 1.2.

Theorem 2.11. *Assume that V is a smooth projective variety over $\mathbb{Q}$ equipped with a dominant morphism $f : V \to \mathbb{P}^n$ with geometrically integral generic fibre and $\Delta(f) \neq 0$. Let $\mathscr{K} : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ be a non-negative bounded function. Then for every $u>0$ and $t \in [0,1]$ the following limit exists,*

$$\widehat{\mathscr{K}}(u,t) := \lim_{B\to+\infty} \frac{1}{\sharp\Omega_B}\sum_{x\in\Omega_B} \exp\left(-u\widetilde{\mathscr{K}_B}(x,t)\right)$$

and for every $u>0$ and $s>0$ it satisfies

$$\int_0^{+\infty} \mathrm{e}^{-st}\widehat{\mathscr{K}}(u,t)\,\mathrm{d}t = \int_{-\infty}^{+\infty} \Psi_{s,u}(x)\,\mathrm{d}x, \tag{2.14}$$

where $\Psi_{s,u}$ is the fundamental solution of (2.11).

It is noteworthy that, for a fixed 'test function' $\mathscr{K}$, the left side of (2.14) is completely determined by the number-theoretic data associated to the fibration $f : V \to \mathbb{P}^n$, however, its right side is determined exclusively through differential equations. We are not aware of previous connections between the Feynman–Kac formula and arithmetic geometry. The proof of theorem 2.11 can be found in § 5.7.

3. Equidistribution

3.1. Auxiliary results from number theory

We begin by recalling some standard statements about the function $\omega_f(x)$ defined in (1.1). Firstly, we give a bound for $\omega_f(x)$ and a bound on the largest prime taken into account by $\omega_f(x)$.

LEMMA 3.1 lemma 3.1, [**22**]. *There exists $D_0 = D_0(f)$ such that if $x \in \mathbb{P}^n(\mathbb{Q})$ and $f^{-1}(x)$ is smooth then*

$$\omega_f(x) \ll \frac{\log H(x)}{\log\log H(x)} \quad \text{and} \quad \max\{p : f^{-1}(x)(\mathbb{Q}_p) = \emptyset\} \ll H(x)^{D_0}.$$

Secondly, we find an integer polynomial g such that primes taken into account by $\omega_f(x)$ must divide $g(x)$.

LEMMA 3.2 lemma 3.2, [**22**]. *Let $g \in \mathbb{Z}[x_0, \dots, x_n]$ be a square-free form such that f is smooth away from the divisor $g(x) = 0 \subset \mathbb{P}^n_{\mathbb{Q}}$. Then there exists $A = A(f) > 0$ such that for all primes $p > A$ the following hold.*

(1) *The restriction of f to $\mathbb{P}^n_{\mathbb{F}_p}$ is smooth away from the divisor $g(x) = 0 \subset \mathbb{P}^n_{\mathbb{F}_p}$.*

(2) *If $x \in \mathbb{P}^n(\mathbb{Q})$ and $f^{-1}(x)(\mathbb{Q}_p) = \emptyset$ then $p \mid g(x)$.*

Recall the definition of σ_p in (2.2). Although it is not immediately obvious, σ_p is essentially the likelihood with which the prime p is taken into account by $\omega_f(x)$ for a randomly chosen element $x \in \Omega_B$. For this reason we need pointwise bounds and average-value results for σ_p. This is the subject of the next two lemmas.

LEMMA 3.3 lemma 3.3, [**22**]. *For all primes p we have*

$$\sigma_p \ll \frac{1}{p},$$

with an implied constant that depends at most on f.

LEMMA 3.4 proposition 3.6 [**22**]. *There exists a constant $\beta = \beta(f)$ such that for all $B \geqslant 3$ we have*

$$\sum_{p \leqslant B} \sigma_p = \Delta(f)(\log\log B) + \beta_f + O((\log B)^{-1}).$$

LEMMA 3.5. *There exists $A' > 0$ such that if $p > A'$ then $\sigma_p \leqslant 1/2$. Furthermore, there exists $\gamma_0 = \gamma_0(f, A') \in \mathbb{R}_{>0}$ such that*

$$\prod_{A' < p < T} (1 - \sigma_p)^{-1} = \gamma_0 (\log T)^{\Delta(f)} \left(1 + O\left(\frac{1}{\log T}\right)\right).$$

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Proof. By lemma 3.3 we have $\sigma_p \leqslant 1/2$ for all sufficiently large p. To deal with the product in the present lemma we use a Taylor expansion to obtain

$$\log \prod_{A'<p<T} (1-\sigma_p)^{-1} = \sum_{A'<p<T} \sigma_p + \sum_{k=2}^{\infty} \frac{1}{k} \sum_{A'<p<T} \sigma_p^k.$$

By lemma 3.3 we can now write for all $p > A'$,

$$\sigma_p^k \leqslant \sigma_p^2 2^{-k+2} \ll p^{-2} 2^{-k},$$

with an implied constant that is independent of p and k. This gives

$$\sum_{k=2}^{\infty} \frac{1}{k} \sum_{A'<p<T} \sigma_p^k - \sum_{k=2}^{\infty} \frac{1}{k} \sum_{A'<p} \sigma_p^k \ll \sum_{\substack{k\geqslant 2\\ p>T}} \frac{1}{k2^k} \frac{1}{p^2} \ll \sum_{\substack{k\geqslant 2\\ m\in\mathbb{N}, m>T}} \frac{1}{k2^k} \frac{1}{m^2}$$
$$\ll \sum_{k\geqslant 2} \frac{1}{k2^k} \frac{1}{T} \ll \frac{1}{T}.$$

We can now invoke lemma 3.4 to obtain

$$\log \prod_{A'<p<T} (1-\sigma_p)^{-1} = \Delta(f) \log\log T + \beta_f + O(1/\log T)$$
$$- \sum_{p\leqslant A'} \sigma_p + \sum_{k=2}^{\infty} \frac{1}{k} \sum_{p>A'} \sigma_p^k + O(1/T).$$

Letting $\gamma_0 := \mathrm{e}^{\lambda}$, where

$$\lambda := \beta_f - \sum_{p\leqslant A'} \sigma_p + \sum_{k=2}^{\infty} \frac{1}{k} \sum_{p>A'} \sigma_p^k,$$

concludes the proof. □

Let us now give the main number-theoretic input for the succeeding sections. It was verified in the proof of [**22**, proposition 3.9]. It essentially states that if we fix a finite set of primes then we can count the density of fibres that have no p-adic point at each of the primes p in the set. More importantly, we can do so with an error term that depends explicitly on the primes.

For a square-free integer Q define

$$\mathscr{A}_Q := \sharp\Big\{x \in \Omega_B : f^{-1}(x) \text{ smooth}, p \mid Q \Rightarrow f^{-1}(x)(\mathbb{Q}_p) = \emptyset\Big\}. \tag{3.1}$$

Lemma 3.6. *Assume that V is a smooth projective variety over $\mathbb{Q}$ equipped with a dominant morphism $f : V \to \mathbb{P}^n$ with geometrically integral generic fibre and $\Delta(f) \neq 0$. Fix any polynomial g as in lemma 3.2. Then there exist constants $A, d > 1$*

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that depend at most on f and g such that for each square-free integer Q with the property $p \mid Q \Rightarrow p > A$ and each $B \geqslant 1$ we have

$$\left|\mathscr{A}_Q - \mathscr{A}_1 \prod_{p|Q} \sigma_p\right| \ll d^{\omega(Q)} \left(\frac{B^{n+1}}{Q\min\{p : p \mid Q\}} + Q^{2n+1}B + QB^n(\log 2B)^{\lfloor 1/n \rfloor}\right),$$

where the implied constant depends at most on f and g.

Lemma 3.7. *Fix a positive integer r, let $\mathscr{C}, \varepsilon_1$ be any constants with*

$$\mathscr{C} > \frac{3r}{2} \text{ and } 0 < \varepsilon_1 \leqslant \min\left\{\frac{n-1/2}{2r(n+1)}, \frac{1}{4r}\right\}$$

and define the functions $t_0, t_1 : \mathbb{R}_{\geqslant 3} \to \mathbb{R}$ through

$$t_0(B) := (\log\log B)^{\mathscr{C}} \text{ and } t_1(B) = B^{\varepsilon_1}.$$

In the situation of lemma 3.6 we have

$$\sum_{\substack{Q\in\mathbb{N}, \omega(Q)\leqslant r \\ p|Q\Rightarrow p\in(t_0(B), t_1(B)]}} \mu(Q)^2 \left|\mathscr{A}_Q - \mathscr{A}_1 \prod_{p|Q}\sigma_p\right| \ll_{\mathscr{C},\varepsilon_1,r} B^{n+1}(\log\log B)^{r-1-\mathscr{C}},$$

where the implied constant depends at most on $f, g, \mathscr{C}, \varepsilon_1$ and r.

Remark 3.8. Lemma 3.6 may be viewed as a 'level of distribution' result in sieve theory. The main term $\mathscr{A}_1 \prod_{p|Q} \sigma_p$ essentially behaves like

$$\frac{B^{n+1}}{Q}$$

for most Q, while the error term contains the expression

$$\frac{B^{n+1}}{Q\min\{p : p \mid Q\}}.$$

Therefore, to get a power saving we need to assume that Q grows at least polynomially in terms of B. In sieve theory language this is phrased by saying that the exponent of the level of distribution is 0. As is surely familiar to sieve experts, such a bad level of distribution does not allow straightforward applications.

Let us now recall the *Fundamental lemma of sieve theory*, as given in [**16**, lemma 6.3].

Lemma 3.9. *Let $\kappa > 0$ and $y > 1$. There exist two sets of real numbers $\Lambda^+ = (\lambda_Q^+)_{Q\in\mathbb{N}}$ and $\Lambda^- = (\lambda_Q^-)_{Q\in\mathbb{N}}$ depending only on κ and y with the following* *properties*:

$$\lambda_1^{\pm} = 1, \tag{3.2}$$

$$|\lambda_Q^{\pm}| \leqslant 1 \text{ if } 1 < Q < y, \tag{3.3}$$

$$\lambda_Q^{\pm} = 0 \text{ if } Q \geqslant y, \tag{3.4}$$

and for any integer $n > 1$,

$$\sum_{Q|n} \lambda_Q^{-} \leqslant 0 \leqslant \sum_{Q|n} \lambda_Q^{+}. \tag{3.5}$$

Moreover, for any multiplicative function $g(d)$ *with* $0 \leqslant g(p) < 1$ *that satisfies*

$$\prod_{t_1 \leqslant p < t_2} (1 - g(p))^{-1} \leqslant \left(\frac{\log t_2}{\log t_1}\right)^{\kappa} \left(1 + \frac{K}{\log t_1}\right) \tag{3.6}$$

for all $2 \leqslant t_1 < t_2 \leqslant y$ *and some constant* K *that is independent of* t_1, t_2 *and* y, *we have*

$$\sum_{Q|P(z)} \lambda_Q^{\pm} g(Q) = \left(1 + O\left(\mathrm{e}^{-s}\left(1 + \frac{K}{\log z}\right)^{10}\right)\right) \prod_{p<z} (1 - g(p)), \tag{3.7}$$

where $P(z)$ *denotes the product of all primes* $p < z$ *and* $s = \log y / \log z$, *the implied constant depending only on* κ.

3.2. Proof of theorem 1.3

The strategy of the proof is to find an integer sequence such that the problem of counting $x \in \Omega_B$ for which every prime $p \leqslant \xi(B)$ satisfies $f^{-1}(x)(\mathbb{Q}_p) \neq \emptyset$ is essentially the same as the one of counting elements in the sequence that are coprime to every prime $p \leqslant \xi(B)$. Afterwards, sieving by the primes $p \leqslant \xi(B)$ will be performed via the application of lemma 3.9. Owing to the level-of-distribution problems explained in remark 3.8 it is necessary to introduce the parameter z_0 in what follows.

For the proof of theorem 1.3 we can clearly assume that $\xi(B) \leqslant B^{1/20}$. Let us now take

$$z_0 := (\log \xi(B))^{\Delta(f)+d}, z := \xi(B), y := B^{1/10},$$

where d is as in lemma 3.6. We take κ to be

$$\kappa := \Delta(f).$$

Letting

$$g(d) := \prod_{\substack{p|d \\ p > z_0}} \sigma_p,$$

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we can use lemma 3.5 to verify (3.6) in our setting. There are three cases, according to whether z_0 is in $(0, t_1), [t_1, t_2)$ or $[t_2, +\infty)$. In the first case we have

$$\prod_{t_1 \leqslant p < t_2} (1 - g(p))^{-1} = \prod_{t_1 \leqslant p < t_2} (1 - \sigma_p)^{-1}$$

and (3.6) follows directly from lemma 3.5. If $z_0 \in [t_1, t_2)$ then

$$\prod_{t_1 \leqslant p < t_2} (1 - g(p))^{-1} = \prod_{z_0 \leqslant p < t_2} (1 - \sigma_p)^{-1},$$

which, by lemma 3.5 equals

$$\left(\frac{\log t_2}{\log z_0}\right)^{\Delta(f)} \left(1 + O\left(\frac{1}{\log z_0}\right)\right) \leqslant \left(\frac{\log t_2}{\log t_1}\right)^{\Delta(f)} \left(1 + O\left(\frac{1}{\log t_1}\right)\right).$$

In the remaining case, $z_0 \in [t_2, +\infty)$, we have

$$\prod_{t_1 \leqslant p < t_2} (1 - g(p))^{-1} = 1,$$

which is clearly bounded by the right side of (3.6).

Let us now make a choice for the constant D_0 in lemma 3.1 and fix it for the rest of the proof of theorem 1.3. For $x \in \mathbb{P}^n(\mathbb{Q})$ we define the integer

$$F_x := \prod_{\substack{p \leqslant B^{D_0+1} \\ f^{-1}(x)(\mathbb{Q}_p) = \emptyset}} p.$$

This allows us to obtain

$$\begin{aligned}
&\sharp\{x \in \Omega_B : p_1(x) \geqslant \xi(B), f^{-1}(x) \text{ smooth}\} \\
&\quad \leqslant \sharp\{x \in \Omega_B : \gcd(F_x, \prod_{z_0 < p < z} p) = 1, f^{-1}(x) \text{ smooth}\} \\
&\quad = \sum_{\substack{x \in \Omega_B \\ f^{-1}(x) \text{ smooth}}} \sum_{\substack{Q \in \mathbb{N} \\ Q | \prod_{z_0 < p < z} p \\ Q | F_x}} \mu(Q) \leqslant \sum_{\substack{x \in \Omega_B \\ f^{-1}(x) \text{ smooth}}} \sum_{\substack{Q \in \mathbb{N} \\ Q | \prod_{z_0 < p < z} p \\ Q | F_x}} \lambda_Q^+ \\
&\quad = \sum_{\substack{Q \in \mathbb{N} \\ Q | \prod_{z_0 < p < z} p}} \lambda_Q^+ \mathscr{A}_Q,
\end{aligned}$$

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where $\mathscr{A}_Q$ was defined in (3.1) and we used the fact that $\mu(1) = 1 = \lambda_1^+$ and (3.5). Using lemma 3.6 this becomes

$$\sum_{\substack{Q\in\mathbb{N}\\ Q\mid\prod_{z_0<p<z} p}} \lambda_Q^+ \left(\mathscr{A}_1 \prod_{p\mid Q} \sigma_p \right.$$
$$\left. + O\left(\frac{d^{\omega(Q)} B^{n+1}}{Q \min\{p : p \mid Q\}} + d^{\omega(Q)} (Q^{2n+1} B + Q B^n (\log B)^{\lfloor 1/n \rfloor}) \right) \right).$$

Owing to $\mathscr{A}_q \ll B^{n+1}$ and the fact that every Q in the last sum is square-free, we see that the first error term is

$$\ll \frac{B^{n+1}}{z_0} \sum_{Q\mid\prod_{z_0<p<z} p} \frac{d^{\omega(Q)}}{Q} \leqslant \frac{B^{n+1}}{z_0} \prod_{z_0<p<z} \left(1 + \frac{d}{p}\right)$$
$$\ll \frac{B^{n+1}}{z_0} \left(\frac{\log z}{\log z_0} \right)^d = \frac{B^{n+1}}{(\log \xi(B))^{(\Delta(f)+d)}} \left(\frac{\log \xi(B)}{(\Delta(f)+d) \log\log \xi(B)} \right)^d$$
$$\ll \frac{B^{n+1}}{(\log \xi(B))^{\Delta(f)}}.$$

Using the bound $d^{\omega(Q)} \ll_\varepsilon Q^\varepsilon$, valid for all $\varepsilon > 0$, as well as that λ_Q^+ is supported on $[1, y]$, the second error term is

$$\ll_\varepsilon y^{\varepsilon+2n+1} B \sum_{Q<y} 1 \leqslant y^{\varepsilon+2n+2} B = B^{[(\varepsilon+2n+2)/10]+1} \leqslant B^{n+1/2}.$$

The third error term is

$$\ll_\varepsilon y^{1+\varepsilon} B^n (\log B)^{\lfloor 1/n \rfloor} \sum_{Q\leqslant y} 1 \ll y^{2+\varepsilon} B^{n+\varepsilon} \leqslant B^{n+1/2}.$$

Recalling that we have assumed $\xi(B) \leqslant B^{1/20}$ shows $B^{n+1}(\log \xi(B))^{-\Delta(f)} \gg B^{n+1/2}$, hence the estimate

$$\sharp\{x \in \Omega_B : p_1(x) \geqslant \xi(B), f^{-1}(x) \text{ singular}\} \ll B^n$$

shows that

$$\sharp\{x \in \Omega_B : p_1(x) \geqslant \xi(B), f^{-1}(x) \text{ smooth}\} \ll B^{n+1}\Upsilon + \frac{B^{n+1}}{(\log \xi(B))^{\Delta(f)}},$$

where

$$\Upsilon := \sum_{\substack{Q\in\mathbb{N}\\ Q\mid\prod_{z_0<p<z} p}} \lambda_Q^+ g(Q) = \sum_{\substack{Q\in\mathbb{N}\\ Q\mid\prod_{p<z} p}} \lambda_Q^+ g(Q).$$

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By (3.7) and lemma 3.5 we see that

$$\Upsilon \ll \prod_{p<z}(1-g(p)) = \prod_{z_0<p<z}(1-\sigma_p) \ll \left(\frac{\log z_0}{\log z}\right)^{\Delta(f)}$$
$$\ll \left(\frac{(\Delta(f)+d)\log\log\xi(B)}{\log\xi(B)}\right)^{\Delta(f)},$$

therefore

$$\sharp\{x\in\Omega_B : p_1(x)\geqslant \xi(B), f^{-1}(x) \text{ smooth}\} \ll B^{n+1}\left(\frac{\log\log\xi(B)}{\log\xi(B)}\right)^{\Delta(f)},$$

which concludes the proof. □

3.3. Equidistribution without probabilistic input

The main object of study in this section are moments involving the function $\omega_f(x,T)$ that is introduced in (1.5). For fibrations f as in § 1.1, any $B,T\geqslant 3$ and for $r\in\mathbb{Z}_{\geqslant 0}$, the r-th moment is defined by

$$\mathscr{M}_r(f,B,T) := \sum_{\substack{x\in\Omega_B\\ f^{-1}(x)\text{ smooth}}}\left(\frac{\omega_f(x,T)-\Delta(f)\log\log T}{\sqrt{\Delta(f)\log\log T}}\right)^r.$$

THEOREM 3.10. *Let V and f be as in theorem 1.4. Let c be a fixed positive constant, assume that $B\geqslant 9^{1/c}$ and let $T\in\mathbb{R}\cap[9,B^c]$. Then for every positive integer r we have*

$$\frac{\mathscr{M}_r(f,B,T)}{\sharp\Omega_B} = \mu_r + O_{f,c,r}\left(B^{n+1}\frac{\log\log\log\log T}{(\log\log T)^{1/2}}\right),$$

where μ_r is the r-th moment of the standard normal distribution and the implied constant depends at most on f, c and r but is independent of B and T.

The restriction $T\leqslant B^c$ is addressed in remark 3.11. Theorem 3.10 is proved in § 3.4. We will then use it to verify theorem 1.4 in § 3.5 and theorem 1.5 in § 3.6.

3.4. Proof of theorem 3.10

For a prime p we define the function $\theta_p:\mathbb{P}^n(\mathbb{Q})\to\{0,1\}$ via

$$\theta_p(x) := \begin{cases}1, & \text{if } f^{-1}(x)(\mathbb{Q}_p)=\emptyset,\\ 0, & \text{otherwise.}\end{cases} \tag{3.8}$$

Let

$$\varepsilon_r := \min\left\{\frac{n}{4r(n+1)},\frac{1}{4r}\right\}. \tag{3.9}$$

First we consider the case where

$$c\leqslant \varepsilon_r. \tag{3.10}$$

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Letting $T_0 := (\log\log T)^{3+3r}$ and $\omega_f^0(x,T) := \sum_{T_0<p\leqslant T}\theta_p(x)$ allows us to define $s(T)$ via

$$s(T)^2 := \sum_{T_0<p\leqslant T}\sigma_p(1-\sigma_p)$$

It is relatively straightforward to modify the proof of [22, proposition 3.9] to show that

$$\sum_{\substack{x\in\Omega_B\\ f^{-1}(x)\ \text{smooth}}}\left(\omega_f^0(x,T)-\sum_{p\in(T_0,T]}\sigma_p\right)^r \tag{3.11}$$

equals

$$\begin{cases} c_nB^{n+1}\mu_r s(T)^r + O_r\left(B^{n+1}(\log\log T)^{r/2-1}\right), & \text{if } 2\mid r,\\ O_r\left(B^{n+1}(\log\log T)^{r-1/2}\right), & \text{otherwise.}\end{cases}$$

To do so, one uses [22, lemma 3.8], with $\mathscr{A}, h(Q)$ and y having the same meaning as in the proof of [22, proposition 3.9], the only change being the replacement of the set $\mathscr{P}$ in the proof of [22, proposition 3.9] by $\{p\in(T_0,T] : p \text{ prime}\}$. Arguments that are completely identical to those found in the proof of [22, proposition 3.9] now lead to

$$\mathscr{E}(\mathscr{A},h,r) \ll_r B^{n+1}\frac{(\log\log T)^{r-1}}{T_0} + (T^{r(2n+1)}B + T^rB^n(\log B)^{\lfloor 1/n\rfloor})T^r,$$

where $\mathscr{E}(\mathscr{A},h,r)$ is given in [22, lemma 3.8]. The bound $T\leqslant B^c$ and (3.9)–(3.10) show that the two last terms in the right side contribute $\ll B^{n+1-\varpi}$ for some positive real $\varpi=\varpi(n,r)$. The first term is $\ll B^{n+1}(\log\log T)^{-2r-4}$ due to the choice of T_0. To conclude the stated estimate for (3.11) one follows verbatim the rest of the argument in the proof of [22, proposition 3.9].

It follows from lemma 3.4 and the definition of T_0 that

$$\sum_{T_0<p\leqslant T}\sigma_p = \Delta(f)\log\log T + O_r(\log\log\log\log T),$$

thus, writing $s(T)^r = (\Delta(f)\log\log T + O_r(\log\log\log\log T))^{r/2}$ we see that

$$s(T)^r = (\Delta(f)\log\log T)^{r/2} + O_r\left((\log\log T)^{(r/2)-1}\log\log\log\log T\right).$$

By lemma 3.2 there exists a homogeneous square-free polynomial $g\in\mathbb{Z}[x_0,\dots,x_n]$ such that if $x\in\mathbb{P}^n(\mathbb{Q})$ and $f^{-1}(x)(\mathbb{Q}_p)=\emptyset$ then $p\mid g(x)$. Thus, for $x\in\mathbb{P}^n(\mathbb{Q})$ with

$g(x) \neq 0$,

$$\omega_f^0(x,T) = \omega_f(x,T) + O\Big(\sum_{\substack{p \mid g(x) \\ p \leqslant T_0}} 1 \Big).$$

This shows that

$$\omega_f(x,T) - \Delta(f) \log\log T = \Big(\omega_f^0(x,T) - \sum_{T_0 < p \leqslant T} \sigma_p \Big) + O_r\Big(\log\log\log\log T + \sum_{\substack{p \mid g(x) \\ p \leqslant T_0}} 1 \Big). \tag{3.12}$$

It is easy to modify the proof of [**22**, lemma 3.10] in order to show that for every $B, z > 1$, $y \in (3, B^{1/2(r+1)}]$, $m \in \mathbb{Z}_{\geqslant 0}$ and a primitive homogeneous polynomial $G \in \mathbb{Z}[x_0, \ldots, x_n]$ one has

$$\sum_{\substack{x \in \Omega_B \\ G(x) \neq 0}} \Big(z + \sum_{\substack{p \mid G(x) \\ p \leqslant y}} 1 \Big)^m \ll_{F,m} B^{n+1} (z + \log\log y)^m$$

with an implied constant that is independent of y and z. Using this with (3.12) one can prove with arguments identical to the concluding arguments in the proof of [**22**, theorem 1.3] that

$$\mathscr{M}_r(f,B,T) = \sum_{\substack{x \in \Omega_B \\ f^{-1}(x) \text{ smooth}}} \left(\frac{\omega_f^0(x,T) - \sum_{T_0 < p \leqslant T} \sigma_p}{\sqrt{\Delta(f) \log\log T}} \right)^r + O_r\left(\frac{B^{n+1} \log\log\log\log T}{(\log\log T)^{1/2}} \right). \tag{3.13}$$

We have therefore shown that if (3.10) holds then for all $T \leqslant B^c$ one has

$$\frac{\mathscr{M}_r(f,B,T)}{\sharp \Omega_B} = \mu_r + O_r\left(\frac{B^{n+1} \log\log\log\log T}{(\log\log T)^{1/2}} \right). \tag{3.14}$$

It remains to prove this estimate for all $T \leqslant B^c$ in the remaining case $c > \varepsilon_r$. If $T \leqslant B^{\varepsilon_r}$ then it directly follows from (3.14). In the remaining cases we have $B^{\varepsilon_r} < T \leqslant B^c$. Then if $g(x) \neq 0$ we obtain

$$\omega_f(x,T) = \omega_f(x, B^{\varepsilon_r}) r + O_{c,r}(1)$$

because $\sum_{B^{\varepsilon_r} < p \leqslant T} \theta_p(x) \ll_{c,r} (\log T)/(\log(B^{\varepsilon_r}))$, that can be shown as in the proof of [**22**, theorem 1.3]. It is clear that we have $\log\log T = (\log\log B^{\varepsilon_r}) + O_{c,r}(1)$. Noting that the set $\{x \in \Omega_B : g(x) = 0\}$ has cardinality $\ll B^n$ and that if $f^{-1}(x)$

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is smooth then $\omega_f(x) \ll \log H(x)$ due to lemma 3.1, we obtain that

$$\sum_{\substack{x\in\Omega_B\\ f^{-1}(x)\ \text{smooth}}} \left(\frac{\omega_f(x,T)-\Delta(f)\log\log T}{\sqrt{\Delta(f)\log\log T}}\right)^r$$

equals

$$\sum_{\substack{x\in\Omega_B\\ f^{-1}(x)\ \text{smooth}}} \left(\frac{\omega_f(x,B^{\varepsilon_r})-\Delta(f)\log\log B^{\varepsilon_r}}{\sqrt{\Delta(f)\log\log B^{\varepsilon_r}}}\right)^r$$

up to an error term that is

$$\ll B^n(\log B)^r + \sum_{0\leqslant k\leqslant r-1} \sum_{\substack{x\in\Omega_B\\ f^{-1}(x)\ \text{smooth}}} \binom{r}{k} \left| \left(\frac{\omega_f(x,B^{\varepsilon_r})-\Delta(f)\log\log B^{\varepsilon_r}}{\sqrt{\Delta(f)\log\log B^{\varepsilon_r}}}\right)^k \right|.$$

Using (3.13) for $T = B^{\varepsilon_r}$ concludes the proof of theorem 3.10. □

REMARK 3.11. Note that some growth restriction on T is necessary in order for theorem 3.10 to hold. If, for example, it holds with $T \geqslant B^{\log B}$, then, $\log\log T \geqslant 2\log\log B$, hence the average of $\omega_f(x,T)$ would be at least $2\Delta(f)\log\log B$. According to lemma 3.1 there exist positive constants C, D_0 that depend only on f such that if $H(x) \leqslant B$ and $f^{-1}(x)$ is smooth then $\omega_f(x) = \omega_f(x, C_0 B^{D_0})$. We also know that the average value of $\omega_f(x)$ is $\Delta(f)\log\log B$, thus one would get a contradiction because $\Delta(f) \neq 0$.

COROLLARY 3.12. *Let V and f be as in theorem 1.4. Let c be a fixed positive constant, assume that $B \geqslant 3^{1/c}$ and let $T : \mathbb{R}_{\geqslant 3} \to \mathbb{R}_{\geqslant 3}$ be any function with*

$$\lim_{B\to+\infty} T(B) = +\infty \ \text{and} T(B) \leqslant B^c \ \text{forall} B \geqslant 1.$$

Then for any interval $\mathscr{J} \subset \mathbb{R}$ we have

$$\lim_{B\to\infty} \mathbf{P}_B\left[\left\{x\in\mathbb{P}^n(\mathbb{Q}) : \frac{\omega_f(x,T(B))-\Delta(f)\log\log T(B)}{\sqrt{\Delta(f)\log\log T(B)}} \in \mathscr{J}\right\}\right] = \frac{1}{\sqrt{2\pi}}\int_{\mathscr{J}} \mathrm{e}^{-t^2/2}\,\mathrm{d}t.$$

Proof. The proof uses the moment estimates provided by theorem 3.10 and is based on the fact that the standard normal distribution is characterized by its moments. It is identical to the proof of [**22**, theorem 1.2] that is given in [**22**, § 3.5] and is thus not repeated here. □

REMARK 3.13. Let D_0 be any constant as in lemma 3.1. The special choice $T(B) = B^{1+D_0}$ of corollary 3.12 is equivalent to [**22**, theorem 1.2].

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3.5. Proof of theorem 1.4

We consider $r \in \mathbb{R}$ to be fixed throughout this proof. Defining $K : \mathbb{R}_{\geqslant 3} \to \mathbb{R}$ via

$$K(B) := \exp\left(\exp\left(\frac{j(B) + r\sqrt{j(B)}}{\Delta(f)}\right)\right)$$

makes clear that

$$j(B) = \Delta(f) \log\log K(B) - r\sqrt{j(B)} \tag{3.15}$$

and

$$\sqrt{j(B)} = \frac{-r + \sqrt{r^2 + 4\Delta(f)\log\log K(B)}}{2}.$$

This provides us with $\sqrt{j(B)} = \sqrt{\Delta(f)\log\log K(B)} + O_r(1)$, which, when combined with (3.15), shows that

$$j(B) = \Delta(f)\log\log K(B) - r\sqrt{\Delta(f)\log\log K(B)} + O_r(1). \tag{3.16}$$

By the assumptions of theorem 1.4 regarding $j(B)$ one can see that for all sufficiently large B the inequality $j(B) \leqslant \Delta(f)\log\log B - |r|\sqrt{\Delta(f)\log\log B}$ holds. This shows that

$$\frac{j(B) + r\sqrt{j(B)}}{\Delta(f)} \leqslant \log\log B,$$

thus, $K(B) \leqslant B$. This allows us to use corollary 3.12 with $T(B) := K(B)$ to obtain

$$\lim_{B\to\infty} \mathbf{P}_B\left[\left\{x \in \mathbb{P}^n(\mathbb{Q}) : \frac{\omega_f(x, K(B)) - \Delta(f)\log\log K(B)}{\sqrt{\Delta(f)\log\log K(B)}} < -r\right\}\right]$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-r} \mathrm{e}^{-t^2/2}\,\mathrm{d}t. \tag{3.17}$$

For any $B, u \in \mathbb{R}_{\geqslant 3}$ and $\ell \in \mathbb{N}$ it is clear that $p_\ell(x) > u$ is equivalent to $\omega_f(x, u) < \ell$. Using this with $u = K(B)$ and $\ell = j(B)$ gives

$$\mathbf{P}_B\left[\left\{x \in \mathbb{P}^n(\mathbb{Q}) : \log\log p_{j(B)}(x) > \frac{j(B)}{\Delta(f)} + r\frac{\sqrt{j(B)}}{\Delta(f)}\right\}\right]$$
$$= \mathbf{P}_B\left[\{x \in \mathbb{P}^n(\mathbb{Q}) : \omega_f(x, K(B)) < j(B)\}\right],$$

which, when (3.16) is invoked, gives

$$\mathbf{P}_B\Big[\Big\{x \in \mathbb{P}^n(\mathbb{Q}) : \omega_f(x, K(B)) < \Delta(f)\log\log K(B)$$
$$-r\sqrt{\Delta(f)\log\log K(B)} + O_r(1)\Big\}\Big].$$

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Alluding to (3.17) shows that

$$\lim_{B\to+\infty} \mathbf{P}_B\left[\left\{x\in\mathbb{P}^n(\mathbb{Q}) : \log\log p_{j(B)}(x) > \frac{j(B)}{\Delta(f)} + r\frac{\sqrt{j(B)}}{\Delta(f)}\right\}\right]$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-r} \mathrm{e}^{-t^2/2}\,\mathrm{d}t$$
$$= 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{r} \mathrm{e}^{-t^2/2}\,\mathrm{d}t,$$

which is clearly sufficient for theorem 1.4. □

REMARK 3.14. Let us note that the assumption

$$\lim_{B\to+\infty}\frac{j(B)-\Delta(f)\log\log B}{\sqrt{\Delta(f)\log\log B}} = -\infty \tag{3.18}$$

of theorem 1.4 is unpleasant because it does not allow its application when $j(B)$ is close to its maximal value $\omega_f(x)$, which, by [22, theorem 1.2] can be as large as

$$\Delta(f)\log\log B + t\sqrt{\Delta(f)\log\log B},$$

where t is any fixed positive constant. The assumption is, however, necessary. Indeed, if theorem 1.4 holds without the assumption (3.18) then we are allowed to take

$$j_0(B) = \Delta(f)\log\log B.$$

Let us see how this leads to a contradiction. Indeed, with this choice of $j_0(B)$ we deduce that by theorem 1.4 with $r = \sqrt{\Delta(f)}$ that

$$\lim_{B\to\infty} \mathbf{P}_B\left[\left\{x\in\mathbb{P}_n(\mathbb{Q}) : \log\log p_{j_0(B)}(x) > \frac{j_0(B)}{\Delta(f)} + \sqrt{\frac{j_0(B)}{\Delta(f)}}\right\}\right]$$
$$= \frac{1}{\sqrt{2\pi}}\int_{\sqrt{\Delta(f)}}^{\infty} \mathrm{e}^{-t^2/2}\,\mathrm{d}t.$$

The value of the integral is strictly positive, hence there are arbitrarily large $B \geqslant 3$ with the following property: there exists $x = x(B) \in \Omega_B$ with $f^{-1}(x)$ smooth and a prime $p = p(x)$ with $f^{-1}(x)(\mathbb{Q}_p) = \emptyset$ and $\log\log p > \log\log B + \sqrt{\log\log B}$. However, by lemma 3.1 any such p must satisfy $\log\log p \leqslant O(1) + \log\log H(x) \leqslant O(1) + \log\log B$, therefore giving a contradiction.

3.6. Proof of theorem 1.5

We shall use the approach in the proof of [14, theorem 10], where a similar result is proved for the number of prime divisors of an integer in place of ω_f. The approach must be altered somewhat because it is difficult to prove for ω_f a statement that is analogous to the exponential decay bound in [14, theorem 010] which is used in the proof of [14, theorem 10], the reason being that for any $A, T > 0$ the function

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$A^{\sharp\{p\leqslant T: p\mid m\}}$ is a multiplicative function of the integer m, while this is not true for $A^{\omega_f(x,T)}$. To prove theorem 1.5 it is clearly sufficient to restrict to the cases with

$$\xi(B) \leqslant (\log\log B)^{1/2}, 0<\varepsilon<1/2$$

and we shall assume that both inequalities hold during the rest of the proof. By lemma 3.1 there exist $C, D_0>0$ that only depend on f such that if $x\in\mathbb{P}^n(\mathbb{Q})$ is such that $H(x)\leqslant B$ and $f^{-1}(x)$ has no p-adic point then $p\leqslant CB^{D_0}$. Fixing any $\psi>1+D_0$ with the property $CB^{D_0}\leqslant B^\psi$ for all $B\geqslant 2$ and letting $\chi(B):=2\xi(B)/\Delta(f)$ we shall define the set

$$\begin{aligned}\mathscr{A} = \Big\{ & x\in\mathbb{P}^n(\mathbb{Q}) : t\in\mathbb{R}\cap\left(\mathrm{e}^{\mathrm{e}^{\chi(B)}}, B^\psi\right] \Rightarrow |\omega_f(x,t)-\Delta(f)\log\log t| \\ & \leqslant \frac{1}{2}(\Delta(f)\log\log t)^{1/2+\varepsilon/2}\Big\}.\end{aligned}$$

This set is well-defined because $\mathrm{e}^{\mathrm{e}^{\chi(B)}}<B^\psi$ is implied by our assumption $\xi(B)\leqslant(\log\log B)^{1/2}$ for all large enough B. Let us now prove that

$$\mathbf{P}_B\left[\mathscr{A}\right] = 1 + O\left(\frac{1}{\xi(B)^M}\right). \tag{3.19}$$

Note that for this it suffices to show

$$\mathbf{P}_B\left[\left\{x\in\mathbb{P}^n(\mathbb{Q}) : x\in\mathscr{A}, f^{-1}(x)\text{ smooth}\right\}\right] = 1 + O_{\varepsilon,M}\left(\frac{1}{\xi(B)^M}\right).$$

For $k\in\mathbb{N}$ we let $t_k:=\mathrm{e}^{\mathrm{e}^k}$ and we find the largest $k_0=k_0(B)$ and the smallest $k_1=k_1(B)$ with

$$\left(\mathrm{e}^{\mathrm{e}^{\chi(B)}}, B^\psi\right] \subseteq \bigcup_{k=k_0}^{k_1}(t_k, t_{k+1}].$$

Thus we deduce that if $H(x)\leqslant B$ is such that $x\notin\mathscr{A}$ then there exists $k\in[k_0,k_1)$ and $t\in\mathbb{R}$ having the properties $t\in(t_k,t_{k+1}]$ and $|\omega_f(x,t)-\Delta(f)\log\log t|\leqslant\frac{1}{2}(\Delta(f)\log\log t)^{1/2+\varepsilon/2}$. The last inequality implies that either

$$\begin{aligned}\omega_f(x,t_{k+1}) \geqslant \omega_f(x,t) &\geqslant \Delta(f)\log\log t - \frac{1}{2}(\Delta(f)\log\log t)^{1/2+\varepsilon/2} \\ &\geqslant \Delta(f)k - \frac{1}{2}(\Delta(f)(k+1))^{1/2+\varepsilon/2} \\ &\geqslant \Delta(f)(k+1) - (\Delta(f)(k+1))^{1/2+\varepsilon}\end{aligned}$$

or

$$\begin{aligned}\omega_f(x,t_k) \leqslant \omega_f(x,t) &\leqslant \Delta(f)\log\log t + \frac{1}{2}(\Delta(f)\log\log t)^{1/2+\varepsilon/2} \\ &\leqslant \Delta(f)(k+1) + \frac{1}{2}(\Delta(f)(k+1))^{1/2+\varepsilon/2} \\ &\leqslant \Delta(f)k + (\Delta(f)k)^{1/2+\varepsilon}.\end{aligned}$$

Letting ℓ denote $k+1$ or k respectively, we have shown that the cardinality of $x \notin \mathscr{A}$ with $f^{-1}(x)$ smooth is at most

$$\sum_{\substack{\ell \in \mathbb{N} \\ k_0 \leqslant \ell \leqslant 1+k_1}} \sharp\{x \in \Omega_B : f^{-1}(x) \text{ smooth}, |\omega_f(x, t_\ell) - \Delta(f)\ell| > (\Delta(f)\ell)^{1/2+\varepsilon}\}.$$

Note that the inequalities $t_{1+k_0} > \mathrm{e}^{\mathrm{e}^{\chi(B)}}$ and $t_{k_1} \leqslant B^\psi$ imply that $k_0 > -1 + \chi(B)$ and $t_{1+k_1} = t_{k_1}^{\mathrm{e}} \leqslant B^{\mathrm{e}\psi}$. Therefore the sum above is at most

$$\sum_{\substack{\ell \in \mathbb{N} \\ -1+\chi(B) < \ell \leqslant 1+(\log \psi)+(\log\log B)}} \sharp\{x \in \Omega_B : f^{-1}(x) \text{ smooth}, |\omega_f(x, t_\ell) - \Delta(f)\ell| > (\Delta(f)\ell)^{1/2+\varepsilon}\}.$$

Letting $m = m(\varepsilon)$ be the least integer with $2m\varepsilon \geqslant M + 1$ and using Chebychev's inequality we see that the sum is at most

$$\sum_{\substack{\ell \in \mathbb{N} \\ -1+\chi(B) < \ell \leqslant 1+(\log \psi)+(\log\log B)}} \frac{1}{(\Delta(f)\ell)^{2m\varepsilon}} \sum_{\substack{x \in \Omega_B \\ f^{-1}(x) \text{ smooth}}} \left(\frac{\omega_f(x, t_\ell) - \Delta(f)\ell}{\sqrt{\Delta(f)\ell}} \right)^{2m}.$$

Let us now apply theorem 3.10 with $r = 2m, c = \mathrm{e}\psi$ and $T = t_\ell \leqslant B^{\mathrm{e}\psi} = B^c$. We obtain that the expression above is

$$\ll_{m,\psi} \sum_{\ell > -1+\chi(B)} \frac{\sharp\Omega_B}{\ell^{2M\varepsilon}},$$

which is $O(\xi(B)^{-M} \sharp\Omega_B)$ because $2M\varepsilon \geqslant M + 1$. This concludes the proof of (3.19).

As a last step in our proof we shall deduce theorem 1.5 from (3.19). Setting $t = p_j(x)$ in (3.19) shows that for all $x \in \Omega_B$, except at most $\ll B^{n+1}/\psi(B)^M$, one has

$$\mathrm{e}^{\mathrm{e}^{\chi(B)}} < p_j(x) \leqslant B^\psi \Rightarrow |j - \Delta(f) \log\log p_j(x)| \leqslant \frac{1}{2}(\Delta(f) \log\log p_j(x))^{1/2+\varepsilon/2}.$$

Recalling that $\varepsilon < 1/2$ the last inequality implies that $\Delta(f) \log\log p_j(x) \leqslant 2j$. Therefore the inequality $\mathrm{e}^{\mathrm{e}^{\chi(B)}} < p_j(x)$ implies that

$$\frac{2\xi(B)}{\Delta(f)} = \chi(B) < \log\log p_j(x) \leqslant \frac{2j}{\Delta(f)},$$

hence $\xi(B) \leqslant j$. Finally, by the definition of ψ we have that the inequality $p_j(x) \leqslant B^\psi$ is equivalent to $p_j(x) \leqslant \omega_f(x)$. Owing to $\Delta(f) \log\log p_j(x) \leqslant 2j$ one can see that for all sufficiently large B and all $j \geqslant \xi(B)$ one has

$$\frac{1}{2}(\Delta(f)\log\log p_j(x))^{1/2+\varepsilon/2} \leqslant \frac{1}{2}(2j)^{1/2+\varepsilon/2} \leqslant \Delta(f)j^{1/2+\varepsilon}.$$

This shows that for all $x \in \mathbb{P}^n(\mathbb{Q})$ with $H(x) \leqslant B$, except at most $\ll B^{n+1}/\psi(B)^M$, one has

$$\xi(B) < j \leqslant \omega_f(x) \Rightarrow \left|\log\log p_j(x) - \frac{j}{\Delta(f)}\right| < j^{1/2+\varepsilon/2},$$

thereby finishing the proof of theorem 1.5. □

4. Modelling by Brownian motion

The main result in this section is theorem 2.3, which proves that certain paths related to the sequence (1.3) are distributed according to Brownian motion. To prove theorem 2.3 we begin by proving theorem 4.2 in § 4.1. It is a generalization of the work of Granville and Soundararajan [**13**] that allows us to estimate correlations that are more involved than the moments in theorem 3.10. We use theorem 4.2 to verify proposition 4.8 in § 4.3 and proposition 4.12 in § 4.4. These two propositions are then combined in § 4.5 to prove theorem 2.3.

4.1. An extension of work by Granville and Soundararajan

Assume that we are given a finite set $\mathscr{A}$ and that for each $a \in \mathscr{A}$ we have a sequence of real numbers $\{c_n(a)\}_{n\in\mathbb{N}}$ with the property that $\sum_{n=1}^{\infty} c_n(a)$ converges absolutely for every $a \in \mathscr{A}$. A central object of study in analytic number theory are the moments

$$\sum_{a\in\mathscr{A}}\Big(\sum_{n\in J} c_n(a)\Big)^k, k \in \mathbb{N}, \tag{4.1}$$

where $J \subset \mathbb{R}$ is an interval. In this paper we shall need the following generalization.

Definition 4.1 Interval correlation. Let $\mathscr{A}$ and $\{c_n(a)\}_{a\in\mathscr{A}}$ be as above and assume that $J_1, \ldots, J_m \subset \mathbb{R}$ are m pairwise disjoint intervals. For $\mathbf{k} \in \mathbb{N}^m$ the $\mathbf{k}$-th interval correlation of the sequence $\{c_n(a) : n \in \mathbb{N}, a \in \mathscr{A}\}$ is defined as

$$\sum_{a\in\mathscr{A}}\Big(\sum_{n\in J_1} c_n(a)\Big)^{k_1}\cdots\Big(\sum_{n\in J_m} c_n(a)\Big)^{k_m}. \tag{4.2}$$

These moments record how the values of $c_n(a)$ for n in an interval affect the values of $c_n(a)$ for n in a different interval.

The work of Granville and Soundararajan [**13**, proposition 3] provides accurate estimates for the moments in (4.1) when the sequence $\{c_n(a) : n \in \mathbb{N}, a \in \mathscr{A}\}$ has a specific number-theoretic structure and our aim in this section is to use their method to provide estimates for the interval correlations in (4.2).

Assume that $\mathscr{P}$ is a finite set of primes and that $\mathscr{A} := \{a_1, \ldots, a_y\}$ is a multiset of y natural numbers. For $Q \in \mathbb{N}$ let $\mathscr{A}_Q := \sharp\{m \leqslant y : Q \mid a_m\}$ and let h be a real-valued, non-negative multiplicative function such that for square-free Q we have

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$0 \leqslant h(Q) \leqslant Q$. Whenever a square-free positive integer Q satisfies $p \mid Q \Rightarrow p \in \mathscr{P}$ we define

$$\mathscr{W}(Q) := \sharp \mathscr{A}_Q - \frac{h(Q)}{Q} y$$

and for any $\mathscr{P}_i \subset \mathscr{P}$ for $1 \leqslant i \leqslant m$ and $\mathbf{k} \in \mathbb{N}^m$ we let

$$\mathscr{E}_{\mathscr{P}_1,\ldots,\mathscr{P}_m}(\mathscr{A}, h, \mathbf{k}) := \sum_{\substack{\mathbf{Q} \in \mathbb{N}^m \\ \forall i: \omega(Q_i) \leqslant k_i \\ \forall i: p \mid Q_i \Rightarrow p \in \mathscr{P}_i}} \left| \mathscr{W}(Q_1 \cdots Q_m) \right| \prod_{i=1}^m \mu(Q_i)^2. \tag{4.3}$$

Note that, setting $Q := Q_1 \cdots Q_m$ provides us with

$$\mathscr{E}_{\mathscr{P}_1,\ldots,\mathscr{P}_m}(\mathscr{A}, h, \mathbf{k}) \leqslant \sum_{\substack{Q \in \mathbb{N} \\ \omega(Q) \leqslant k_1 + \cdots + k_m \\ p \mid Q \Rightarrow p \in \mathscr{P}}} \mu(Q)^2 \left| \mathscr{W}(Q) \right|. \tag{4.4}$$

Furthermore, for any $r \in \mathbb{N}$ we let $C_r := \Gamma(r+1)/(2^{r/2}\Gamma(1+r/2))$, where Γ is the Euler gamma function. For any $\mathscr{R} \subset \mathscr{P}$ we define

$$\mu_{\mathscr{R}} := \sum_{p \in \mathscr{R}} \frac{h(p)}{p}, \ \sigma_{\mathscr{R}} := \left(\sum_{p \in \mathscr{R}} \frac{h(p)}{p} \left(1 - \frac{h(p)}{p}\right) \right)^{1/2} \tag{4.5}$$

and for $a \in \mathscr{A}$ we define $\omega_{\mathscr{R}}(a) := \sharp\{p \in \mathscr{R} : p \mid a\}$.

THEOREM 4.2. *Assume that $\mathscr{P}_1, \ldots, \mathscr{P}_m$ are disjoint subsets of $\mathscr{P}$. Then for any $\mathbf{k} \in \mathbb{N}^m$ with $k_i \leqslant \sigma_{\mathscr{P}_i}^{2/3}$ for all $1 \leqslant i \leqslant m$ we have*

$$\sum_{a \in \mathscr{A}} \prod_{i=1}^m (\omega_{\mathscr{P}_i}(a) - \mu_{\mathscr{P}_i})^{k_i} = y \prod_{i=1}^m \left(C_{k_i} \sigma_{\mathscr{P}_i}^{k_i} \left(1 + O\left(\frac{k_i^3}{\sigma_{\mathscr{P}_i}^2}\right)\right)\right) + O\left(\frac{\mathscr{E}_{\mathscr{P}_1,\ldots,\mathscr{P}_m}(\mathscr{A}, h, \mathbf{k})}{\prod_{i=1}^m (1 + \mu_{\mathscr{P}_i})^{-1}} \right) \tag{4.6}$$

if k_i is even for every $1 \leqslant i \leqslant m$, and

$$\sum_{a \in \mathscr{A}} \prod_{i=1}^m (\omega_{\mathscr{P}_i}(a) - \mu_{\mathscr{P}_i})^{k_i} \ll y \left(\prod_{i=1}^m C_{k_i} \sigma_{\mathscr{P}_i}^{k_i} \right) \left(\prod_{\substack{1 \leqslant i \leqslant m \\ k_i \text{ odd}}} \frac{k_i^{3/2}}{\sigma_{\mathscr{P}_i}} \right) + \frac{\mathscr{E}_{\mathscr{P}_1,\ldots,\mathscr{P}_m}(\mathscr{A}, h, \mathbf{k})}{\prod_{i=1}^m (1 + \mu_{\mathscr{P}_i})^{-1}} \tag{4.7}$$

if there exists $1 \leqslant i \leqslant m$ such that k_i is odd. The implied constants depend at most on m.

Proof. As in the proof of [**13**, proposition 3] we can write

$$\sum_{a\in\mathscr{A}}\prod_{i=1}^{m}\left(\omega_{\mathscr{P}_i}(a)-\mu_{\mathscr{P}_i}\right)^{k_i}=\sum_{\forall i:p_{1,i},\cdots,p_{k_i,i}\in\mathscr{P}_i}\sum_{a\in\mathscr{A}}\prod_{i=1}^{m}f_{r_i}(a), \tag{4.8}$$

where $r_i:=\prod_{1\leqslant j\leqslant k_i}p_{j,i}$ and

$$f_r(a):=\prod_{p\mid r}\begin{cases}1-\frac{h(p)}{p}, & \text{if } p\mid a,\\ -\frac{h(p)}{p}, & \text{otherwise.}\end{cases}$$

Since $\mathscr{P}_j\cap\mathscr{P}_{j'}=\emptyset$ whenever $j\neq j'$, we have $\gcd(r_j,r_{j'})=1$ for $j\neq j'$. This allows us to write $\prod_{i=1}^{m}f_{r_i}(a)=f_{r_1\cdots r_m}(a)$. Let $\mathrm{rad}(t)$ be the radical of a natural number t, i.e. the largest square-free integer dividing t. We can then employ the estimate [**13**, equation (13)] to obtain

$$\sum_{a\in\mathscr{A}}\prod_{i=1}^{m}f_{r_i}(a)=yG(r_1\cdots r_m)+\sum_{t\mid\mathrm{rad}(r_1\cdots r_m)}\mathscr{W}(r_1\cdots r_m)E(r_1\cdots r_m,t), \tag{4.9}$$

where the quantities G,E are introduced in [**13**, equations (14)–(15)] through

$$G(r):=\prod_{p\mid r}\left(\frac{h(p)}{p}\Big(1-\frac{h(p)}{p}\Big)^{\nu_p(r)}+\Big(\frac{-h(p)}{p}\Big)^{\nu_p(r)}\Big(1-\frac{h(p)}{p}\Big)\right)$$

and for $r,t\in\mathbb{N}$ with $t\mid\mathrm{rad}(r)$,

$$E(r,t):=\prod_{\substack{p\mid r\\ p\mid t}}\left(\Big(1-\frac{h(p)}{p}\Big)^{\nu_p(r)}-\Big(\frac{-h(p)}{p}\Big)^{\nu_p(r)}\right)\prod_{\substack{p\mid r\\ p\mid\mathrm{rad}(r)/t}}\Big(\frac{-h(p)}{p}\Big)^{\nu_p(r)}.$$

The function G is multiplicative, therefore using that the r_i are coprime in pairs it is evident that the contribution of the first term in the right-hand side in (4.9) towards (4.8) is

$$\prod_{i=1}^{m}\left(\sum_{p_{1,i},\ldots,p_{k_i,i}\in\mathscr{P}_i}G(p_{1,i}\cdots p_{k_i,i})\right).$$

As shown in [**13**, p. 22], one has the following estimate whenever $k\leqslant\sigma_{\mathscr{P}_i}^{2/3}$,

$$\sum_{p_1,\ldots,p_k\in\mathscr{P}_i}G(p_1\cdots p_k)=\begin{cases}C_k\sigma_{\mathscr{P}_i}^{k}(1+O(k^3\sigma_{\mathscr{P}_i}^{-2})), & \text{if } 2\mid k,\\ O(C_k\sigma_{\mathscr{P}_i}^{k-1}k^{3/2}), & \text{otherwise,}\end{cases}$$

which concludes the analysis of the main term in theorem 4.2.

It remains to study the contribution of the sum over t in (4.9) towards (4.8) and for this we first use the coprimality of r_i to rewrite it as

$$\sum_{\substack{\mathbf{t}\in\mathbb{N}^m\\ \forall i: t_i|\mathrm{rad}(r_i)}} \mathscr{W}(r_1\cdots r_m)E(r_1\cdots r_m, t_1\cdots t_m).$$

We then use the obvious estimate $|E(r,t)| \leqslant \prod_{p\nmid t} h(p)/p$ to see that the said contribution is

$$\sum_{\substack{\boldsymbol{\ell}\in\mathbb{N}^m\\ \forall i: 1\leqslant \ell_i\leqslant k_i}} \sum_{\substack{\mathbf{t}\in\mathbb{N}^m\\ \forall i: t_i = q_{1,i}\cdots q_{\ell_i,i}\\ q_{1,i}<q_{2,i}<\cdots<q_{\ell_i,i}\in\mathscr{P}_i}} |\mathscr{W}(t_1\cdots t_m)| \sum_{\substack{\forall i: p_{1,i},\cdots,p_{k_i,i}\in\mathscr{P}_i\\ \forall i: t_i|p_{1,i}\cdots p_{k_i,i}}} \prod_{\substack{1\leqslant j\leqslant k_i\\ p_{j,i}\nmid t_i}} \frac{h(p_j)}{p_j}.$$

The proof is then concluded by alluding to the estimate

$$\sum_{\substack{p_1,\cdots,p_k\in\mathscr{P}_i\\ t|p_1\cdots p_k}} \prod_{\substack{1\leqslant j\leqslant k\\ p_j\nmid t}} \frac{h(p_j)}{p_j} \ll \mu_{\mathscr{P}_i}^k$$

that is proved in [**13**, p.23]. □

4.2. Auxiliary facts from probability theory

In this section we recall some necessary notions from probability theory.

Firstly, we need the following notion from [**3**, p. 20]. Let X, Y be two metric spaces and denote the corresponding σ-algebras by $\mathscr{X}$ and $\mathscr{Y}$. Assume that we are given a function $h : X \to Y$ such that if $A \in \mathscr{Y}$ then $\{x \in X : h(x) \in A\} \in \mathscr{X}$. If ν is a probability measure on $(X, \mathscr{X})$ then we can define a probability measure on $(Y, \mathscr{Y})$ (that is denoted by νh^{-1}) as follows: for any $A \in \mathscr{Y}$ we let

$$(\nu h^{-1})[A] := \nu(x \in X : h(x) \in A). \tag{4.10}$$

We will later need the following result from [**3**, theorem 29.4].

Lemma 4.3 Crámer–Wold. *For random vectors*

$$\mathbf{X}_m = (X_{m,1}, \ldots, X_{m,k}) \text{ and } \mathbf{Y}_m = (Y_{m,1}, \ldots, Y_{m,k}),$$

a necessary and sufficient condition for the convergence in distribution of $\mathbf{X}_m$ *to* $\mathbf{Y}$ *is that*

$$\sum_{i=1}^k a_i X_{m,i}$$

converges in distribution to $\sum_{i=1}^k a_i Y_i$ *for each* $\mathbf{a} \in \mathbb{R}^k$.

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Let $\mathbf{t} \in [0,1]^k$. Recalling the meaning of $(D, \mathscr{D})$ in § 2.1 allows us to consider the function $\pi_{\mathbf{t}} : D \to \mathbb{R}^k$ that is defined through

$$\pi_{(t_1,\dots,t_k)}(y) := (y(t_1), \dots, y(t_k)).$$

According to [3, p. 138], if $\mathbf{P}$ is a probability measure on $(D, \mathscr{D})$ then the set

$$T_{\mathbf{P}} := \{0,1\} \cap \big\{t \in (0,1) : \mathbf{P}[x \in D : x(t) \neq \lim_{\substack{s \to t \\ s < t}} x(s)] = 0\big\}$$

has complement in $[0,1]$ that is countable. Next, we shall need the definition in [3, equation (12.27)]. Namely, for a function $u : [0,1] \to \mathbb{R}$ and any $\delta > 0$ we define

$$w''(\delta, u) := \sup_{\substack{t_1, t, t_2 \in [0,1] \\ t_1 \leqslant t \leqslant t_2 \\ t_2 - t_1 \leqslant \delta}} \min \big\{|u(t) - u(t_1)|, |u(t_2) - u(t)|\big\}.$$

The following result can be found in [3, theorem 13.3].

LEMMA 4.4. *Suppose that P and $(P_m)_{m\in\mathbb{N}}$ are probability measures on $(D, \mathscr{D})$. If*

$$P_m \pi_{\mathbf{t}}^{-1} \text{ converges in probability to } P\pi_{\mathbf{t}}^{-1} \text{ whenever } \mathbf{t} \in T_P^k, \tag{4.11}$$

$$\text{for every } \varepsilon > 0 \text{ we have } \lim_{\delta \to 0} P\big[u \in D : |u(1) - u(1-\delta)| \geqslant \varepsilon\big] = 0 \tag{4.12}$$

and for each $\varepsilon, \eta > 0$ there exists $\delta \in (0,1), m_0 \in \mathbb{N}$ such that for all $m \geqslant m_0$ we have

$$P_m\big[u \in D : w''(\delta, u) \geqslant \varepsilon\big] \leqslant \eta, \tag{4.13}$$

then P_m converges in probability to P.

Recall that D is a metric space whose metric is given by

$$d(X,Y) := \inf_{\lambda \in \Lambda} \max \Big\{ \sup\{|\lambda(t) - t| : t \in [0,1]\}, \sup\{|X(t) - Y(\lambda(t))| : t \in [0,1]\} \Big\} \tag{4.14}$$

whenever $X, Y \in D$ and where Λ denotes the set of all strictly increasing, continuous maps $\lambda : [0,1] \to [0,1]$, see, for example [3, equation (12.13)].

To verify (4.13) in a specific situation we shall later need the following two results.

LEMMA 4.5 theorem 11.3, [1]. *Let P be any probability measure on $(D, \mathscr{D})$. Assume that $0 = s_0 < s_1 \cdots < s_k = 1$ and $s_i - s_{i-1} \geqslant \delta$, $i = 1, \cdots, k$, then*

$$P\left[u \in D : w''(u,\delta) > \varepsilon\right] \leqslant \sum_{i=0}^{k-2} P\left[u \in D : \varepsilon < \sup_{\substack{t_1, t, t_2 \in [0,1]^3 \\ s_i \leqslant t_1 \leqslant t \leqslant t_2 \leqslant s_{i+2}}} \min\{|u(t) - u(t_1)|, |u(t_2) - u(t)|\}\right].$$

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The second result corresponds to the case with $\alpha = 1 = \beta$ of [3, theorem 10.1]. Let $\xi_1, \ldots, \xi_N$ be random variables on a probability space (Ω_1, P_1) and define

$$m_{ijk} := \min\left\{\left|\sum_{h=i+1}^{j} \xi_h\right|, \left|\sum_{h=j+1}^{k} \xi_h\right|\right\}, 0 \leqslant i \leqslant j \leqslant k \leqslant N.$$

LEMMA 4.6. *Suppose that $u_1, \ldots, u_N$ are non-negative numbers with*

$$P_1\left[m_{ijk} \geqslant \lambda\right] \leqslant \frac{1}{\lambda^4}\left(\sum_{i<l\leqslant k} u_l\right)^2, \quad 0 \leqslant i \leqslant j \leqslant k \leqslant N,$$

for $\lambda > 0$. Then, for $\lambda > 0$,

$$P_1\left[m_{ijk} \geqslant \lambda\right] \ll \frac{1}{\lambda^4}\left(\sum_{0<l\leqslant N} u_l\right)^2,$$

where the implied constant is absolute.

4.3. Pointwise convergence

Define $\psi : \mathbb{R}_{\geqslant 3} \to \mathbb{R}$ through

$$\psi(B) := (\log\log B)^{-\frac{1}{4}}. \tag{4.15}$$

For $x \in \mathbb{P}^n(\mathbb{Q})$ and $B \in \mathbb{R}_{\geqslant 3}$ we bring into play the function $Y_B(\cdot, x) : [0,1] \to \mathbb{R}$ given by

$$t \mapsto Y_B(t,x) := \frac{1}{(\Delta(f)\log\log B)^{1/2}} \sum_{\substack{p\leqslant \exp(\log^t B)\\ \log B<p\leqslant B^{\psi(B)}}} \begin{cases} 1-\sigma_p, & \text{if } f^{-1}(x)(\mathbb{Q}_p) = \emptyset,\\ -\sigma_p, & \text{otherwise.}\end{cases} \tag{4.16}$$

This is a truncated version of the function in (2.1). The truncation is introduced for technical reasons.

For $r \in \mathbb{Z}_{\geqslant 0}$ we denote the r-th moment of the standard normal distribution by

$$M_r := \begin{cases} \frac{1}{2^{r/2}}\frac{r!}{(r/2)!}, & r \text{ even},\\ 0, & r \text{ odd}.\end{cases}$$

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LEMMA 4.7. *Let V and f be as in theorem 1.4. For every $B \geqslant 3$, $m \in \mathbb{N}$, $\mathbf{k} \in \mathbb{Z}^m_{\geqslant 0}$, $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{t} \in [0,1]^m$ with $0 \leqslant t_1 < \ldots < t_m \leqslant 1$ we consider the sum*

$$\sum_{\substack{x\in\Omega_B\\ f^{-1}(x)\ smooth}} \prod_{i=1}^{m} \Bigg(\sum_{\substack{\log B < p \leqslant B^{\psi(B)}\\ \exp(\log^{t_i} B) < p \leqslant \exp(\log^{t_{i+1}} B)}} \begin{cases} 1-\sigma_p, & \text{if } f^{-1}(x)(\mathbb{Q}_p) = \emptyset, \\ -\sigma_p, & \text{otherwise} \end{cases} \Bigg)^{k_i},$$

where by convention we set $0^0 := 1$. Letting $r := k_1 + \cdots + k_m$, the sum equals

$$\sharp\Omega_B \left(\prod_{i=1}^{m} M_{k_i}(t_{i+1}-t_i)^{k_i/2} \right) (\Delta(f)\log\log B)^{r/2} + O_{\mathbf{a},\mathbf{k},\mathbf{t},m}\Big(\sharp\Omega_B (\log\log B)^{r-1/2}\Big).$$

Proof. We shall assume that $t_1 = 0$ and $t_m = 1$, an obvious modification of our arguments makes available the proof when $(t_1, t_m) \neq (0, 1)$. Let us define the multiset

$$\mathscr{A} := \Big\{ a_x := \prod_{\substack{p \text{ prime}\\ f^{-1}(x)(\mathbb{Q}_p)=\emptyset}} p : x \in \Omega_B, f^{-1}(x) \text{ smooth} \Big\},$$

the sets of primes

$$\mathscr{P} := \Big\{ p \text{ prime} : \log B < p \leqslant B^{\psi(B)} \Big\},$$
$$\mathscr{P}_i := \Big\{ p \in \mathscr{P} : \exp(\log^{t_i} B) < p \leqslant \exp(\log^{t_{i+1}} B) \Big\}, \quad (1 \leqslant i \leqslant m)$$

and introduce the multiplicative function $h : \mathbb{N} \to \mathbb{R}$ as $h(Q) := Q \prod_{p\mid Q} \sigma_p$. In the terminology of § 4.1 the sum in lemma 4.7 takes the shape

$$\sum_{a\in\mathscr{A}} \prod_{i=1}^{m} (\omega_{\mathscr{P}_i}(a) - \mu_{\mathscr{P}_i})^{k_i}.$$

Recalling (4.5) and using lemma 3.4 we see that

$$\mu_{\mathscr{P}_i} = \begin{cases} t_2\Delta(f)\log\log B + O_{\mathbf{a},\mathbf{t}}(\log\log\log B), & \text{if } i = 1, \\ (t_{i+1}-t_i)\Delta(f)\log\log B + O_{\mathbf{a},\mathbf{t}}(1), & \text{if } 1 < i < m-1, \\ (1-t_{m-1})\Delta(f)\log\log B + O_{\mathbf{a},\mathbf{t}}(\log\log\log B), & \text{if } i = m-1, \end{cases}$$

which can be written as

$$\mu_{\mathscr{P}_i} = (t_{i+1}-t_i)\Delta(f)\log\log B + O_{\mathbf{a},\mathbf{t}}(\log\log\log B), \quad (1 \leqslant i \leqslant m-1).$$

By (4.5) and lemma 3.3 we have $\sigma^2_{\mathscr{P}_i} = \mu_{\mathscr{P}_i} + O(1)$, hence

$$\sigma_{\mathscr{P}_i} = ((t_{i+1}-t_i)\Delta(f)\log\log B)^{1/2} + O_{\mathbf{a},\mathbf{t}}(\log\log\log B), \quad (1 \leqslant i \leqslant m-1).$$

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This allows us to deduce that the product in the right side of (4.6) equals

$$\begin{aligned}&\sharp\{x \in \Omega_B : f^{-1}(x) \text{ smooth}\} \prod_{i=1}^{m} \Big(M_{k_i}\sigma_{\mathscr{P}_i}^{k_i}\Big(1 + O_{\mathbf{a},\mathbf{k},\mathbf{t}}\Big(\frac{1}{\log\log B}\Big)\Big)\Big)\\ &= \frac{\sharp\{x \in \Omega_B : f^{-1}(x) \text{ smooth}\}}{(\Delta(f)\log\log B)^{-r/2}} \left(\prod_{i=1}^{m} M_{k_i}(t_{i+1}-t_i)^{k_i/2}\right)\\ &\quad \times \left(1 + O_{\mathbf{a},\mathbf{k},\mathbf{t}}\Big(\frac{\log\log\log B}{\log\log B}\Big)\right).\end{aligned}$$

Similarly, the product in the right side of (4.7) is $\ll_{\mathbf{a},\mathbf{k},\mathbf{t}} B^{n+1}(\log\log B)^{r-1/2}$. Using the estimate $\sharp\{x \in \Omega_B : f^{-1}(x)\text{smooth}\} = \sharp\Omega_B + O(B^{-1})$ we can put both formulas in the succinct form

$$\sharp\Omega_B \left(\prod_{i=1}^{m} M_{k_i}(t_{i+1}-t_i)^{k_i/2}\right)(\Delta(f)\log\log B)^{r/2} + O_{\mathbf{a},\mathbf{k},\mathbf{t}}\Big(\sharp\Omega_B(\log\log B)^{r-1/2}\Big).$$

Therefore, theorem 4.2 shows that the sum in our lemma equals

$$\begin{aligned}\sharp\Omega_B &= \left(\prod_{i=1}^{m} M_{k_i}(t_{i+1}-t_i)^{k_i/2}\right)(\Delta(f)\log\log B)^{r/2}\\ &\quad + O_{\mathbf{a},\mathbf{k},\mathbf{t}}\Big(\sharp\Omega_B(\log\log B)^{r-1/2} + (\log\log B)^m \mathscr{E}_{\mathscr{P}_1,\ldots,\mathscr{P}_m}(\mathscr{A}, h, \mathbf{k})\Big). \quad (4.17)\end{aligned}$$

It remains to bound the quantity $\mathscr{E}$ above. By (4.4) it is at most

$$\sum_{\substack{Q\in\mathbb{N},\omega(Q)\leqslant r\\ p|Q\Rightarrow p\in\mathscr{P}}} \mu(Q)^2\big|\mathscr{W}(Q)\big|.$$

Now define the functions $t_0(B) := (\log\log B)^{\mathscr{C}}$ and $t_1(B) = B^{\varepsilon_1}$, where $\mathscr{C} := 2r + m$ and $\varepsilon_1 := (8r(n+1))^{-1}$. We certainly have $t_0(B) < \log B < B^{\psi(B)} < t_1(B)$ for all sufficiently large B, thus the last sum over Q is at most

$$\sum_{\substack{Q\in\mathbb{N},\omega(Q)\leqslant r\\ p|Q\Rightarrow p\in(t_0(B),t_1(B)]}} \mu(Q)^2\big|\mathscr{W}(Q)\big|.$$

This quantity occurs also in the proof of [**22**, proposition 3.9], where it is shown to be

$$\ll_{r,\mathscr{C},\varepsilon_1} B^{n+1}(\log\log B)^{r-1-\mathscr{C}}.$$

This yields immediately $(\log\log B)^m\mathscr{E}_{\mathscr{P}_1,\ldots,\mathscr{P}_m}(\mathscr{A},h,\mathbf{k}) \ll_{\mathbf{a},\mathbf{k},\mathbf{t}} \sharp\Omega_B(\log\log B)^{r-1/2}$, which, in light of (4.17), is sufficient for our proof. □

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PROPOSITION 4.8. *Let V and f be as in theorem 1.4. Let $\mathbf{t} \in [0,1]^m$ with*

$$0 \leqslant t_1 < \ldots < t_m \leqslant 1$$

and assume that $S_1, \ldots, S_m$ are Lebesgue-measurable subsets of $\mathbb{R}$. Then

$$\lim_{B\to+\infty} \mathbf{P}_B\left[\left\{x \in \Omega_B : 1 \leqslant i \leqslant m \Rightarrow Y_B(t_i, x) \in S_i\right\}\right] = \prod_{\substack{1\leqslant i\leqslant m\\ t_i\neq 0}} \int_{S_i} \frac{\exp(-\theta^2/2t_i)}{(2\pi t_i)^{1/2}}\,\mathrm{d}\theta.$$

Proof. We assume that $t_1 > 0$ but the proof can be easily modified when $t_1 = 0$. Let us now assume that $Z_0, Z_1, \ldots, Z_m$ are random variables on a probability space (Ω, P) such that they are independent in pairs, that for every $1 \leqslant i \leqslant m$ the random variable Z_i follows the normal distribution with mean 0 and variance t_i and that Z_0 assumes the value 0 with probability 1. Therefore, for any S_i as in the statement of the proposition we have

$$P[\mathbf{Z} \in S_1 \times \cdots \times S_m] = \prod_{1\leqslant i\leqslant m} \int_{S_i} \frac{\exp(-\theta^2/2t_i)}{(2\pi t_i)^{1/2}}\,\mathrm{d}\theta.$$

By lemma 4.3 it is sufficient to show that for every $\mathbf{a} \in \mathbb{R}^m$ the random variable

$$\sum_{i=1}^m a_i Y_B(t_i, x)$$

defined on $(\Omega_B, \mathbf{P}_B)$ converges in distribution to $\sum_{1\leqslant i\leqslant m} a_i Z_i$ as $B \to +\infty$. Let

$$\Omega_B^* := \{x \in \Omega_B : f^{-1}(x) \text{ smooth}\}$$

and denote the indicator function of a set S by $\mathbf{1}_S$. The estimate $\mathbf{P}_B[\Omega_B \setminus \Omega_B^*] \ll B^{-1}$ shows that it suffices to show that

$$\mathbf{1}_{\Omega_B^*}(x) \sum_{i=1}^m a_i Y_B(t_i, x)$$

defined on $(\Omega_B, \mathbf{P}_B)$ converges in distribution to $\sum_{1\leqslant i\leqslant m} a_i Z_i$. We will do so by using the method of moments (see [**2**, theorem 30.2]), thus, proposition 4.8 would follow from verifying

$$\frac{1}{\sharp\Omega_B} \lim_{B\to+\infty} \sum_{x\in\Omega_B^*} \left(\sum_{i=1}^m a_i Y_B(t_i, x)\right)^r = \int_{\mathbb{R}} \theta^r P\left(\sum_{i=1}^m a_i Z_i \leqslant \theta\right) \mathrm{d}\theta, \quad (r \in \mathbb{Z}_{\geqslant 0}). \tag{4.18}$$

We begin by simplifying the right side of (4.18). Whenever $1 \leqslant i \leqslant m-1$ we define $b_i := a_i + a_{i+1} + \cdots + a_k$ so that for every $1 \leqslant i \leqslant m-1$ we can write $a_i = b_{i+1} - b_i$. Thus

$$\sum_{i=1}^m a_i Z_i = b_1 Z_0 + \sum_{i=1}^m b_i (Z_i - Z_{i-1}),$$

from which we deduce that $\sum_{i=1}^m a_i Z_i$ is a random variable that follows the normal distribution with mean 0 and variance $\sum_{1\leqslant i\leqslant m} b_i^2 (t_i - t_{i-1})$. This immediately yields

$$\int_{\mathbb{R}} \theta^r P\left(\sum_{i=1}^m a_i Z_i \leqslant \theta\right) \mathrm{d}\theta = M_r \left(\sum_{i=1}^m b_i^2 (t_{i+1}-t_i)\right)^{r/2}. \tag{4.19}$$

We continue with the treatment of the left side of (4.18). Let $\mathscr{P}$ denote the set of all primes in the interval $(\log B, B^{\psi(B)}]$ and set

$$\mathscr{P}_1 := \mathscr{P} \cap (1, \exp(\log^{t_1} B)], \mathscr{P}_i := \mathscr{P} \cap (\exp(\log^{t_{i-1}} B), \exp(\log^{t_i} B)], \quad (2 \leqslant i \leqslant m).$$

We see that

$$\sum_{i=1}^m a_i Y_B(t_i, x) = \frac{1}{(\Delta(f)\log\log B)^{1/2}} \sum_{i=1}^m b_i \sum_{p \in \mathscr{P}_i} \begin{cases} 1-\sigma_p, & \text{if } f^{-1}(x)(\mathbb{Q}_p) = \emptyset, \\ -\sigma_p, & \text{otherwise.} \end{cases}$$

Thus the multinomial theorem yields

$$\begin{aligned} &\sum_{\substack{x \in \Omega_B \\ f^{-1}(x) \text{ smooth}}} \left(\sum_{i=1}^m a_i Y_B(t_i, x)\right)^r \\ &= \sum_{\substack{\mathbf{k} \in \mathbb{Z}_{\geqslant 0}^{m-1} \\ k_1+\cdots+k_m = r}} \frac{r!}{k_1! \cdots k_m!} \frac{b_1^{k_1} \cdots b_m^{k_m}}{(\Delta(f)\log\log B)^{r/2}} \\ &\quad \times \sum_{\substack{x \in \Omega_B \\ f^{-1}(x) \text{ smooth}}} \prod_{i=1}^m \left(\sum_{p \in \mathscr{P}_i} \begin{cases} 1-\sigma_p, & \text{if } f^{-1}(x)(\mathbb{Q}_p) = \emptyset, \\ -\sigma_p, & \text{otherwise} \end{cases}\right)^{k_i}, \end{aligned}$$

where by convention we set $0^0 := 1$. Invoking lemma 4.7 shows that this is

$$\sharp\Omega_B \left(\sum_{\substack{\mathbf{k} \in \mathbb{Z}_{\geqslant 0}^m \\ k_1+\cdots+k_m = r}} \frac{r! b_1^{k_1} \cdots b_m^{k_m}}{k_1! \cdots k_m!} \prod_{i=1}^m M_{k_i} (t_{i+1}-t_i)^{k_i/2}\right) + O_{\mathbf{a},\mathbf{t},m,r}\left(\frac{\sharp\Omega_B}{(\log\log B)^{1/2}}\right).$$

Recalling that M_{k_i} vanishes if k_i is odd shows that the sum over $\mathbf{k}$ zero if r is odd. If r is even we let $r = 2s$ and $k_i = 2u_i$ to write the sum over $\mathbf{k}$ as

$$\sum_{\substack{\mathbf{u} \in \mathbb{Z}_{\geqslant 0}^m \\ u_1+\cdots+u_m = s}} \frac{(2s)! b_1^{2u_1} \cdots b_m^{2u_m}}{(2u_1)! \cdots (2u_m)!} \prod_{i=1}^m \frac{(2u_i)!}{u_i! 2^{u_i}} (t_{i+1}-t_i)^{u_i} = M_r \left(\sum_{i=1}^m b_i^2 (t_{i+1}-t_i)\right)^{r/2}.$$

Using this with (4.19) verifies (4.18), which completes our proof. □

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4.4. Tightness

Our aim in this section is to prove proposition 4.12, which is one of the main ingredients in the proof of theorem 2.3.

Recall the definition of θ_p in (3.8).

LEMMA 4.9. *Let V and f be as in theorem 1.4. Then for all $\mathbf{y} \in \mathbb{R}^3_{\geqslant 1}$ with $y_1 \leqslant y_2 \leqslant y_3$ the following bound holds with an implied constant depending at most on f,*

$$\sum_{\substack{x \in \Omega_B \\ f^{-1}(x) \text{ smooth}}} \prod_{i=1}^{2} \left(\sum_{\substack{y_i < p \leqslant y_{i+1} \\ \log B < p \leqslant B^{\psi(B)}}} (\theta_p(x) - \sigma_p) \right)^2 \ll B^{n+1} \left(1 + \sum_{\substack{y_1 < p \leqslant y_3 \\ \log B < p \leqslant B^{\psi(B)}}} \frac{1}{p} \right)^2 .$$

Proof. We will make use of theorem 4.2 with $m = 2 = k_1 = k_2$,

$$\mathscr{P} := \{p \text{ prime} : \log B < p \leqslant B^{\psi(B)}\}, \mathscr{P}_i := \{p \in \mathscr{P} : y_i < p \leqslant y_{i+1}\}, \quad (i = 1, 2),$$

and with $\mathscr{A}$, $h(p)$ being as in the proof of lemma 4.7. According to (4.5) we have

$$\sigma^2_{\mathscr{P}_i} \leqslant \mu_{\mathscr{P}_i} = \sum_{\substack{y_i < p \leqslant y_{i+1} \\ \log B < p \leqslant B^{\psi(B)}}} \sigma_p, \quad (i = 1, 2).$$

Therefore, $\sigma^2_{\mathscr{P}_i}(1 + O(\sigma^{-2}_{\mathscr{P}_i})) \ll 1 + \mu_{\mathscr{P}_i}$ with an absolute implied constant. Injecting this into (4.6) we obtain that the sum over x in our lemma is

$$\ll (1 + \mu_{\mathscr{P}_1})(1 + \mu_{\mathscr{P}_2}) \left(B^{n+1} + \mathscr{E}_{\mathscr{P}_1, \mathscr{P}_2}(\mathscr{A}, h, (2, 2)) \right) .$$

We can bound the quantity $\mathscr{E}$ above as in the proof of lemma 4.7. This gives

$$\begin{aligned} |\mathscr{E}_{\mathscr{P}_1, \mathscr{P}_2}(\mathscr{A}, h, (2, 2))| &\leqslant \sum_{Q \in \mathbb{N}, \omega(Q) \leqslant 4 \ \ p | Q \Rightarrow p \in ((\log \log B)^{\mathscr{C}}, B^{\varepsilon_1}]} \mu(Q)^2 \left| \mathscr{W}(Q) \right| \\ &\ll_{r, \mathscr{C}, \varepsilon_1} B^{n+1} (\log \log B)^{3 - \mathscr{C}}, \end{aligned}$$

so that, taking $\mathscr{C} = 3$ we get

$$\sum_{\substack{x \in \Omega_B \\ f^{-1}(x) \text{ smooth}}} \prod_{i=1}^{2} \left(\sum_{\substack{y_i < p \leqslant y_{i+1} \\ \log B < p \leqslant B^{\psi(B)}}} (\theta_p(x) - \sigma_p) \right)^2 \ll B^{n+1} (1 + \mu_{\mathscr{P}_1})(1 + \mu_{\mathscr{P}_2}).$$

Using lemma 3.3 shows that

$$\mu_{\mathscr{P}_i} = \sum_{\substack{p \in (y_i, y_{i+1}] \\ \log B < p \leqslant B^{\psi(B)}}} \sigma_p \ll \sum_{p \in (y_i, y_{i+1}]} \frac{1}{p},$$

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where the implied constant depends only on f. Thus,

$$\sum_{\substack{x\in\Omega_B\\ f^{-1}(x)\text{ smooth}}}\prod_{i=1}^{2}\left(\sum_{\substack{y_i<p\leqslant y_{i+1}\\ \log B<p\leqslant B^{\psi(B)}}}(\theta_p(x)-\sigma_p)\right)^2 \ll B^{n+1}\prod_{i=1}^{2}\left(1+\sum_{y_i<p\leqslant y_{i+1}}\frac{1}{p}\right).$$

Using the inequality $(1+\varepsilon_1)(1+\varepsilon_2)\leqslant(1+\varepsilon_1+\varepsilon_2)^2$, valid whenever both ε_i are non-negative, concludes the proof. □

Define for $y_1,y_2,y_3\in[0,1]$ with $y_1\leqslant y_2\leqslant y_3$, $B\geqslant 3$ and $x\in\mathbb{P}^n(\mathbb{Q})$ the function

$$\Psi_{\mathbf{y}}(x,B):=\min\left\{\left|\sum_{\substack{y_i<p\leqslant y_{i+1}\\ \log B<p\leqslant B^{\psi(B)}}}\frac{(\theta_p(x)-\sigma_p)}{(\Delta(f)\log\log B)^{1/2}}\right|:i=1,2\right\}.$$

Lemma 4.10. *Let V and f be as in theorem 1.4. Then for all $\lambda>0$ and $\mathbf{y}\in\mathbb{R}^3_{\geqslant 1}$ with $y_1\leqslant y_2\leqslant y_3$ the following holds with an implied constant depending at most on f,*

$$\mathbf{P}_B[\{x\in\mathbb{P}^n(\mathbb{Q}):\Psi_{\mathbf{y}}(x,B)\geqslant\lambda\}]\ll\frac{1}{(\log\log B)^2\lambda^4}\left(1+\sum_{\substack{y_1<p\leqslant y_3\\ \log B<p\leqslant B^{\psi(B)}}}\frac{1}{p}\right)^2.$$

Proof. The bound $\mathbf{P}_B[\{x\in\Omega_B:f^{-1}(x)\text{ singular}\}]\ll B^{-1}$ shows that

$$\mathbf{P}_B[\{x\in\mathbb{P}^n(\mathbb{Q}):\Psi_{\mathbf{y}}(x,B)\geqslant\lambda\}]-\mathbf{P}_B[\{x\in\mathbb{P}^n(\mathbb{Q}):f^{-1}(x)\text{ smooth},\Psi_{\mathbf{y}}(x,B)\geqslant\lambda\}]\ll\frac{1}{B}. \tag{4.20}$$

Note that if $\Psi_{\mathbf{y}}(x,B)\geqslant\lambda$ then

$$\lambda^2\leqslant\prod_{i=1}^{2}\left|\sum_{\substack{y_i<p\leqslant y_{i+1}\\ \log B<p\leqslant B^{\psi(B)}}}\frac{(\theta_p(x)-\sigma_p)}{(\Delta(f)\log\log B)^{1/2}}\right|.$$

Thus, the entity $\mathbf{P}_B[.]$ on the right side of (4.20) is bounded by the following quantity due to Chebychev's inequality,

$$\frac{1}{\lambda^4\sharp\Omega_B}\sum_{\substack{x\in\Omega_B\\ f^{-1}(x)\text{ smooth}}}\prod_{i=1}^{2}\left(\sum_{\substack{y_i<p\leqslant y_{i+1}\\ \log B<p\leqslant B^{\psi(B)}}}\frac{(\theta_p(x)-\sigma_p)}{(\Delta(f)\log\log B)^{1/2}}\right)^2.$$

Alluding to lemma 4.9 concludes the proof. □

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Recall (4.16) and for $\lambda > 0, B \geqslant 3$ and $s, s' \in [0,1]$ with $s \leqslant s'$ define $\Gamma_{\lambda,B}(s,s')$ as

$$\mathbf{P}_B \left[\left\{ x \in \mathbb{P}^n(\mathbb{Q}) : \lambda < \sup_{\substack{t_1,t,t_2 \in [0,1] \\ s \leqslant t_1 \leqslant t \leqslant t_2 \leqslant s'}} \min \left\{ |Y_B(t,x) - Y_B(t_1,x)|, |Y_B(t_2,x) - Y_B(t,x)| \right\} \right\} \right].$$

Lemma 4.11. *Let V and f be as in theorem 1.4. For all $\lambda > 0$ and any $s, s' \in [0,1]$ with $s < s'$ there exists B_0 that depends at most on f and $s' - s$ such that if $B \geqslant B_0$ then*

$$\Gamma_{\lambda,B}(s,s') \ll \frac{(s'-s)^2}{\lambda^4}$$

with an implied constant that depends at most on f.

Proof. Order all primes in $\{p : \mathrm{e}^{\log^s B} < p \leqslant \mathrm{e}^{\log^{s'} B}, \log B < p \leqslant B^{\psi(B)}\}$ as $p_1 < \ldots < p_N$, with the convention that $N = 0$ if the set is empty. For every $1 \leqslant i \leqslant N$ we define the random variable ξ_i on the probability space $(\Omega_B, \mathbf{P}_B)$ through

$$\xi_i(x) := \frac{(\theta_{p_i}(x) - \sigma_{p_i})}{(\Delta(f) \log\log B)^{1/2}}, \quad x \in \Omega_B.$$

For any i, j, k with $0 \leqslant i \leqslant j \leqslant k \leqslant N$, any $B \geqslant 3$ and $x \in \Omega_B$ let us bring into play

$$m_{ijk}(x) := \min \left\{ \frac{|\sum_{h=i+1}^{j} (\theta_{p_h}(x) - \sigma_{p_h})|}{(\Delta(f) \log\log B)^{1/2}}, \frac{|\sum_{h=j+1}^{k} (\theta_{p_h}(x) - \sigma_{p_h})|}{(\Delta(f) \log\log B)^{1/2}} \right\}.$$

In particular, one has

$$\Gamma_{\lambda,B}(s,s') = \mathbf{P}_B \left[\left\{ x \in \mathbb{P}^n(\mathbb{Q}) : \lambda < \max_{0 \leqslant i \leqslant j \leqslant k \leqslant N} m_{ijk}(x) \right\} \right].$$

Note that lemma 4.10 allows us to apply lemma 4.6 with $P_1 = \mathbf{P}_B$ and $u_l = 1/p_l$. Thus

$$\mathbf{P}_B \left[\left\{ x \in \mathbb{P}^n(\mathbb{Q}) : \lambda \leqslant \max_{0 \leqslant i \leqslant j \leqslant k \leqslant N} m_{ijk}(x) \right\} \right] \ll \frac{1}{(\log\log B)^2 \lambda^4} \left(\sum_{\substack{\log B < p \leqslant B^{\psi(B)} \\ \mathrm{e}^{\log^s B} < p \leqslant \mathrm{e}^{\log^{s'} B}}} \frac{1}{p} \right)^2.$$

Ignoring the condition $\log B < p \leqslant B^{\psi(B)}$ we see that Mertens' theorem on the asymptotics of $\sum_{p \leqslant t} 1/p$ implies in particular that the sum over p is at most $2(s' - s) \log\log B$ for B large enough, which completes the proof. □

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Proposition 4.12. *Let V and f be as in theorem 1.4. There exists $K>0$ that depends at most on f such that for every $\lambda>0$ and $0<\delta<1$ there exists $B_0 = B_0(f,\delta,\lambda)>0$ with*

$$B \geqslant B_0 \Rightarrow \mathbf{P}_B\big[\big\{x\in\mathbb{P}^n(\mathbb{Q}) : w''(\delta, Y_B(\cdot,x)) \geqslant \lambda\big\}\big] \leqslant \frac{K\delta}{\lambda^4}.$$

Proof. Let $k=k(\delta)$ be the largest positive integer satisfying $\delta k<1$. Define $h : \Omega_B \to D$ through $h(x) := Y_B(\cdot,x)$ and in the terminology of (4.10) define $P_2 := \mathbf{P}_B h^{-1}$. We use lemma 4.5 with $P=P_2$ and

$$s_i := \begin{cases} i\delta, & \text{if } i=0,1,\dots,k-1,\\ 1, & \text{if } i=k.\end{cases}$$

We obtain that

$$\mathbf{P}_B\big[\big\{x\in\mathbb{P}^n(\mathbb{Q}) : w''(\delta, Y_B(\cdot,x)) \geqslant \lambda\big\}\big] \leqslant \sum_{i=0}^{k-2}\Gamma_{\varepsilon,B}(s_i,s_{i+2}).$$

Using lemma 4.11 we obtain B_0 that depends at most on δ such that if $B\geqslant B_0$ then the sum over i is at most $K'\lambda^{-4}\sum_{i=0}^{k-2}(s_{i+2}-s_i)^2$. For every $i\neq k-2$ we have $s_{i+2}-s_i = 2\delta^{-1}$. Note that by the definition of k we have $(k+1)\delta\geqslant 1$, therefore $s_{k-2} = (k+1)\delta - 3\delta \geqslant 1-3\delta$. We obtain $s_k - s_{k-2} \leqslant 3\delta$. This gives

$$\sum_{i=0}^{k-2}(s_{i+2}-s_i)^2 \leqslant (k-2)\delta^2 + 9\delta^2 \leqslant 9k\delta^2 < 9\delta,$$

which concludes the proof. □

4.5. Proof of theorem 2.3

We modify the argument behind the analogous statement for completely additive functions defined on the integers, see the work of Billingsley [**1**, theorem 4.1]. Technical difficulties arise owing to the comments in remark 3.8. While our level of distribution is 0, the level of distribution in Billingsley's proof is at a sharp contrast, namely, it attains its maximum value, 1. To see this, note that the related estimate in his proof is

$$\sharp\{m\in\mathbb{N}\cap[1,n] : m\equiv 0 \,(\mathrm{mod}\ Q)\} = \frac{n}{Q} + O(1)$$

and clearly the error term is dominated by the main as long as $Q\leqslant n^{1-\varepsilon}$, where $\varepsilon>0$ is arbitrary.

We begin by estimating the approximation of $X_B(\cdot,x)$ by $Y_B(\cdot,x)$. Recall the definition of the Skorohod metric in (4.14) and the function $Y_B(\cdot,x)$ in (4.16).

Lemma 4.13. *Let V and f be as in theorem 1.4. For every $\varepsilon>0$ we have*

$$\mathbf{P}_B\Big[\Big\{x\in\mathbb{P}^n(\mathbb{Q}) : d(X_B(\cdot,x),Y_B(\cdot,x)) \geqslant \varepsilon\Big\}\Big] \ll_\varepsilon (\log\log B)^{-1/4}.$$

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Proof. Let $m(B,t) := \min\{\exp(\log^t B), B^{\psi(B)}\}$ and

$$M_1(B,t) := \sum_{p \leqslant \log B} \begin{cases} 1, & \text{if } f^{-1}(x)(\mathbb{Q}_p) = \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$M_2(B,t,x) := \sum_{m(B,t) < p \leqslant \exp(\log^t B)} \begin{cases} 1, & \text{if } f^{-1}(x)(\mathbb{Q}_p) = \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$M_3(B,t) := -t\Delta(f) \log\log B + \sum_{\log B < p \leqslant m(B,t)} \sigma_p,$$

where empty sums are set equal to zero. A moment's thought allows one to see that

$$(X_B(t,x) - Y_B(t,x)) \left(\Delta(f) \log\log B\right)^{1/2} = M_1(B,t) + M_2(B,t,x) + M_3(B,t). \tag{4.21}$$

According to lemma 3.2, if $g(x) \neq 0$ then

$$|M_1(B,t)| \leqslant \sum_{p \mid g(x), p \leqslant \log B} 1. \tag{4.22}$$

Similarly, if $g(x) \neq 0$ and $H(x) \leqslant B$ then lemma 3.2 ensures that

$$|M_2(B,t,x)| \leqslant \sum_{\substack{m(B,t) < p \leqslant \exp(\log^t B) \\ p \mid g(x)}} 1.$$

If $m(B,t) = \exp(\log^t B)$ then this sum is empty and if $m(B,t) = B^{\psi(B)}$ then

$$|M_2(B,t,x)| \leqslant \sum_{\substack{p > B^{\psi(B)} \\ p \mid g(x)}} 1 \ll \frac{\log |g(x)|}{\log(B^{\psi(B)})} \ll \frac{\log(B^{\deg(g)})}{\log(B^{\psi(B)})} \ll (\log\log B)^{1/4} \tag{4.23}$$

because a non-zero integer m can have at most $\log |m| / \log M$ prime divisors in the range $p > M$. To bound $M_3(B,t)$ when $m(B,t) = \exp(\log^t B)$ we invoke lemma 3.4 to obtain

$$|M_3(B,t)| \leqslant \left| -t\Delta(f) \log\log B + \sum_{p \leqslant \exp(\log^t B)} \sigma_p \right| + \left| \sum_{p \leqslant \log B} \sigma_p \right| \ll 1 + \log\log\log B.$$

In the remaining case $m(B,t) = B^{\psi(B)}$ we note that $\leqslant B^{\psi(B)} \leqslant \exp(\log^t B) \leqslant B$ implies

$$-\log \psi(B) + \log\log B \leqslant t \log\log B \leqslant \log\log B$$

and therefore $t \log\log B = \log\log B + O(\log\log\log B)$ with an absolute implied constant. Thus, lemma 3.4 shows that $M_3(B,t)$ equals

$$-\log\log B + O(\log\log\log B) + \sum_{\log B < p \leqslant B^{\psi(B)}} \sigma_p \ll \log\log\log B.$$

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This shows that for all $x \in \Omega_B$ with $g(x) \neq 0$ one has

$$|M_3(B,t)| \ll \log\log\log B. \tag{4.24}$$

Injecting (4.22), (4.23) and (4.24) into (4.21) shows that if $H(x) \leqslant B$ and $g(x) \neq 0$ then

$$\big|X_B(t,x) - Y_B(t,x)\big| \ll (\log\log B)^{-1/2}\Big((\log\log B)^{1/4} + \sum_{p\mid g(x), p\leqslant \log B} 1\Big), \tag{4.25}$$

where the implied constant is independent of t and B. We may now take $\lambda(t) := t$ in (4.14) to see that $d(X,Y) \leqslant \sup\{|X(t) - Y(t)| : t \in [0,1]\}$, therefore

$$d(X_B(\cdot,x), Y_B(\cdot,x)) \ll (\log\log B)^{-1/2}\Big((\log\log B)^{1/4} + \sum_{p\mid g(x), p\leqslant \log B} 1\Big). \tag{4.26}$$

Note that since g is not identically zero we have $\mathbf{P}_B[\{x \in \mathbb{P}^n(\mathbb{Q}) : g(x) = 0\}] \ll B^{-1}$. This shows that the quantity $\mathbf{P}_B$ in the statement of our lemma equals

$$O(B^{-1}) + \mathbf{P}_B\left[\{x \in \mathbb{P}^n(\mathbb{Q}) : g(x) \neq 0, d(X_B(\cdot,x), Y_B(\cdot,x)) \geqslant \varepsilon\}\right]$$

and by Markov's inequality this is

$$\ll B^{-1} + \frac{1}{\varepsilon B^{n+1}} \sum_{\substack{x\in\Omega_B \\ g(x)\neq 0}} d(X_B(\cdot,x), Y_B(\cdot,x)).$$

Using (4.26) and [**22**, lemma 3.10] for $z(B) = (\log\log B)^{1/4}, y(B) := \log B$ yields the bound

$$\ll_\varepsilon B^{-1} + (\log\log B)^{-1/2}\left((\log\log B)^{1/4} + \log\log\log B\right),$$

which concludes our proof. □

By [**3**, theorem 3.1] and lemma 4.13 we see that theorem 2.3 holds as long as we prove it with Y_B in place of X_B. We shall do so by using lemma 4.4 with P being the Wiener measure W and $P_B := \mathbf{P}_B Y_B^{-1}$. The latter measure is defined on $(D, \mathscr{D})$ via (4.10) with

$$(X, \mathscr{X}) = (\Omega_B, \{A : A \subset \Omega_B\}), Y := (D, \mathscr{D}), \nu := \mathbf{P}_B$$

and $h : (\Omega_B, \mathbf{P}_B) \to (D, \mathscr{D})$ being given by $x \mapsto Y_B(\cdot,x)$. In particular, for every $B \geqslant 1$ and every $\delta, \varepsilon > 0$ we can write

$$P_B\big[\big\{u \in D : w''(\delta,u) \geqslant \varepsilon\big\}\big] = \mathbf{P}_B\big[\big\{x \in \mathbb{P}^n(\mathbb{Q}) : w''(\delta, Y_B(\cdot,x) \geqslant \varepsilon\big\}\big]. \tag{4.27}$$

Now fix any $\mathbf{t} \in [0,1]^m$. To rephrase (4.12) we use (4.10) with

$$(X, \mathscr{X}) := (D, \mathscr{D}), (Y, \mathscr{Y}) := (\mathbb{R}, \mathscr{B}(\mathbb{R})), \nu := P_B$$

and $h : D \to \mathbb{R}^m$ defined by $u \mapsto \pi_{\mathbf{t}}(u)$. Here, $\mathscr{B}(\mathbb{R})$ is the standard Borel σ-algebra in the real line. This shows that $P_B\pi_{\mathbf{t}}^{-1}$ is a measure on $(\mathbb{R}^m, \mathscr{B}(\mathbb{R}^m))$ and, in

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particular, if $S_1 \times \cdots \times S_m \in \mathscr{B}(\mathbb{R})^m$ then

$$P_B \pi_{\mathbf{t}}^{-1}[S_1 \times \cdots \times S_m] = P_B[\{u \in D : 1 \leqslant i \leqslant m \Rightarrow u(t_i) \in S_i\}],$$

which, as explained above, equals $\mathbf{P}_B[\{x \in \mathbb{P}^n(\mathbb{Q}) : 1 \leqslant i \leqslant m \Rightarrow Y_B(t_i, x) \in S_i\}]$. A similar construction with ν replaced by W shows that

$$P \pi_{\mathbf{t}}^{-1}[S_1 \times \cdots \times S_m] = W[\{u \in D : 1 \leqslant i \leqslant m \Rightarrow u(t_i) \in S_i\}].$$

Recall that part of the definition of the Wiener measure is that this equals

$$\prod_{\substack{1 \leqslant i \leqslant m \\ t_i \neq 0}} \int_{S_i} \frac{\exp(-\theta^2/2t_i)}{(2\pi t_i)^{1/2}} \, \mathrm{d}\theta.$$

This can be seen by taking $(s,t) = (0, t_i)$ in [**2**, equation (37.4)]. Therefore, in our setting, (4.11) is equivalent to proposition 4.8.

Let us now see why (4.12) is automatically satisfied when P is the Wiener measure. Alluding to [**3**, equation (8.4)] we have for $\varepsilon, \delta > 0$ that

$$\begin{aligned} W[\{u \in D : |u(1) - u(1-\delta)| \geqslant \varepsilon\}] &= \frac{1}{\sqrt{2\pi\delta}} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \exp(-\theta^2/(2\delta)) \, \mathrm{d}\theta \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}\setminus(-\varepsilon/\sqrt{\delta}, \varepsilon/\sqrt{\delta})} \exp(-\theta'^2/2) \, \mathrm{d}\theta'. \end{aligned}$$

For fixed ε and for $\delta \to 0$ the last expression is the tail of a convergent integral, thus it converges to zero.

To complete the proof of theorem 2.3 via lemma 4.4 it remains to verify (4.13). Owing to (4.27) we see that (4.13) can be reformulated equivalently as follows: for each $\varepsilon, \eta > 0$ there exists $\delta \in (0,1), B_0 \in \mathbb{N}$ such that for all $B \geqslant B_0$ we have

$$\mathbf{P}_B\big[\big\{x \in \mathbb{P}^n(\mathbb{Q}) : w''(\delta, Y_B(\cdot, x)) \geqslant \varepsilon\big\}\big] \leqslant \eta.$$

The fact that this holds is verified in proposition 4.12. This completes the proof of theorem 2.3. □

5. Consequences of the Brownian model

In this section we give some number theoretic consequences of the fact that p-adic solubility can be modelled by Brownian motion.

5.1. Proof of theorem 2.4 and corollary 2.5

Define

$$S := \{u \in D : r \leqslant \max_{0 \leqslant t \leqslant 1} u(t)\}.$$

Owing to the reflection principle this set has Wiener measure given by

$$W[S] = \frac{2}{\sqrt{2\pi}} \int_r^{+\infty} \mathrm{e}^{-t^2/2} \, \mathrm{d}t,$$

see [**23**, theorem 2.21]. An application of theorem 2.3 concludes the proof. □

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5.2. Proof of theorem 2.6 and corollary 2.7

The set

$$S := \{u \in D : r \leqslant \max_{0 \leqslant t \leqslant 1} |u(t)|\}$$

has Wiener measure $W[S] = 1 - \tau_\infty(r)$ owing to Donsker's theorem and [**10**, II,p. 292]. An application of theorem 2.3 concludes the proof of theorem 2.6. To prove corollary 2.7 we only have to show that $\tau_\infty(r) = 1 + O(|r|^{-2/3})$. For $M := 1 + |r|^{2/3}$ we see that the series in (2.5) is alternating, thus its tail is bounded by

$$\sum_{m>M} \frac{(-1)^m}{2m+1} \exp\left\{ -\frac{(2m+1)^2\pi^2}{8r^2} \right\} \ll \frac{1}{2M+1} \exp\left\{ -\frac{(2M+1)^2\pi^2}{8r^2} \right\} \ll \frac{1}{M}.$$

By the Taylor expansion $\exp(y) = 1 + O(y)$, valid when $|y| \ll 1$, we get

$$\sum_{0 \leqslant m \leqslant M} \frac{(-1)^m}{2m+1} \exp\left\{ -\frac{(2m+1)^2\pi^2}{8r^2} \right\} = \sum_{0 \leqslant m \leqslant M} \frac{(-1)^m}{2m+1} + O\left(\frac{M^2}{r^2}\right),$$

owing to $\sum_{m \leqslant M} m \ll M^2$. The last sum over m can be completed by introducing an error term of size $\ll 1/M$, thus giving

$$\sum_{0 \leqslant m \leqslant M} \frac{(-1)^m}{2m+1} \exp\left\{ -\frac{(2m+1)^2\pi^2}{8r^2} \right\} = \frac{\pi}{4} + O\left(\frac{1}{M} + \frac{M^2}{r^2}\right).$$

This completes the proof. □

5.3. A variant of the path X_B

For a prime $p \geqslant 3$ define p_- to be the greatest prime strictly smaller than p and let $2_- := 1$. Recall the definition of $\theta_p(x)$ in (3.8). Before proceeding to the proof of the rest of our results it is necessary to approximate the path $X_B(\cdot, x)$ in definition 2.1 by the following variant: for each $x \in \mathbb{P}^n(\mathbb{Q})$ and $B \in \mathbb{R}_{\geqslant 3}$ we define the function $Z_B(\cdot, x) : [0,1] \to \mathbb{R}$ as follows,

$$t \mapsto Z_B(t, x) := \frac{1}{(\Delta(f) \log\log B)^{1/2}} \sum\nolimits^*_{p \leqslant B} (\theta_p(x) - \sigma_p),$$

where the sum $\sum^*$ is taken over all primes p satisfying

$$\sum_{q \leqslant p_-} \sigma_q \leqslant \Delta(f) t \log\log B.$$

Therefore, labelling all primes in ascending order as $q_1 = 2, q_2 = 3, \ldots$, and letting

$$T_i(B, x) := \left\{ t : \sum_{q \text{ prime } \leqslant q_i} \sigma_q < \Delta(f) t \log\log B \leqslant \sum_{q \text{ prime } \leqslant q_{i+1}} \sigma_q \right\},$$

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we infer

$$\text{meas}\,(T_i(B,x)) = \frac{\sigma_{q_{i+1}}}{\Delta(f)\log\log B} \tag{5.1}$$

and

$$t \in T_i(B,x) \Rightarrow Z_B(t,x) = \frac{\omega_f(x,q_{i+1}) - \sum_{q \leqslant q_{i+1}} \sigma_q}{(\Delta(f)\log\log B)^{1/2}}. \tag{5.2}$$

Recall the definition of $\psi(B)$ in (4.15).

Lemma 5.1. *For $x \in \mathbb{P}^n(\mathbb{Q})$ and $B \in \mathbb{R}_{\geqslant 3}$ we define $Z'_B(\cdot,x) : [0,1] \to \mathbb{R}$ given by*

$$t \mapsto Z'_B(t,x) := \frac{1}{(\Delta(f)\log\log B)^{1/2}} \sum\nolimits^*_{\log B < p \leqslant B^{\psi(B)}} (\theta_p(x) - \sigma_p),$$

where the sum $\sum^*$ is taken over all primes p satisfying

$$\sum_{q \leqslant p_-} \sigma_q \leqslant \Delta(f) t \log\log B.$$

Then for every $\varepsilon > 0$ we have

$$\mathbf{P}_B\Big[\Big\{x \in \mathbb{P}^n(\mathbb{Q}) : d(Z_B(\cdot,x), Z'_B(\cdot,x)) \geqslant \varepsilon\Big\}\Big] \ll_\varepsilon (\log\log B)^{-1/4}.$$

Proof. Ignoring the condition in $\sum^*$ gives

$$|Z_B(t,x) - Z'_B(t,x)|(\Delta(f)\log\log B)^{1/2} \leqslant \sum_{p \leqslant \log B} (\theta_p(x) + \sigma_p) + \sum_{B^{\psi(B)} < p \leqslant B} (\theta_p(x) + \sigma_p).$$

By lemma 3.4 the σ_p terms contribute

$$\ll \log\log\log B + \log \frac{\log B}{\log B^{\psi(B)}} \ll \log\log\log B.$$

As in the proof of lemma 4.13, if $g(x) \neq 0$ and $H(x) \leqslant B$ then the remaining terms are

$$\ll \sum_{\substack{p \mid g(x) \\ B^{\psi(B)} < p \leqslant B}} 1 + \sum_{\substack{p \mid g(x) \\ p \leqslant \log B}} 1 \ll \frac{\log|g(x)|}{\log B^{\psi(B)}} + \sum_{\substack{p \mid g(x) \\ p \leqslant \log B}} 1 \ll \frac{1}{\psi(B)} + \sum_{\substack{p \mid g(x) \\ p \leqslant \log B}} 1,$$

hence by (4.15) we obtain

$$|Z_B(t,x) - Z'_B(t,x)| \ll (\log\log B)^{-1/2}\Big((\log\log B)^{1/4} + \sum_{p \mid g(x), p \leqslant \log B} 1\Big).$$

The right side coincides with that in (4.25), and the rest of the proof can now be completed as in the proof of lemma 4.13. □

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Recall the definition of $Y_B(\cdot,x)$ in (4.15).

LEMMA 5.2. *For every $\varepsilon>0$ we have*

$$\mathbf{P}_B\Big[\Big\{x\in\mathbb{P}^n(\mathbb{Q}): d(Y_B(\cdot,x),Z'_B(\cdot,x))\geqslant\varepsilon\Big\}\Big]\ll_\varepsilon(\log\log B)^{-1/2}.$$

Proof. Let $S_1:=\{p\leqslant\exp\left((\log B)^t\right)\}$ and

$$S_2:=\left\{p\leqslant B:\sum_{q\leqslant p_-}\sigma_p\leqslant\Delta(f)t\log\log p\right\}.$$

We infer that

$$|Y_B(t,x)-Z'_B(t,x)|(\Delta(f)\log\log B)^{1/2}\leqslant\sum_{\substack{p\in S_2\setminus S_1\\ \log B<p\leqslant B^{\psi(B)}}}(\theta_p+\sigma_p)+\sum_{\substack{p\in S_1\setminus S_2\\ \log B<p\leqslant B^{\psi(B)}}}(\theta_p+\sigma_p).$$

We will deal with the sum over $p\in S_2\setminus S_1$ since the other sum can be treated similarly. For a prime p not in S_1 we have

$$\sum_{q\leqslant p_-}\sigma_q\leqslant\Delta(f)t\log\log B,$$

hence by $\sigma_p\leqslant 1$ we have

$$\sum_{q\leqslant p}\sigma_q\leqslant 1+\Delta(f)t\log\log B.$$

By lemma 3.4 there exists a constant $C_1=C_1(f)$ such that

$$(\Delta(f)\log\log p)-C_1\leqslant 1+\Delta(f)t\log\log B,$$

hence $\log\log p\leqslant C_2+t\log\log B$ for some $C_2=C_2(f)$. Let us now define z_1 and z_2 through

$$\log\log z_1=t\log\log B\ \text{ and }\ \log\log z_2=C_2+t\log\log B$$

and observe that if $p\in S_2\setminus S_1$ then $z_1<p\leqslant z_2$. By lemma 3.3 we have

$$\sum_{\substack{p\in S_2\setminus S_1\\ \log B<p\leqslant B^{\psi(B)}}}\sigma_p\ll\sum_{z_1<p\leqslant z_2}\frac{1}{p}=o(1)+(\log\log z_2)-(\log\log z_1)\ll 1,\tag{5.3}$$

with an implied constant depending at most on f. We furthermore have

$$\sum_{\substack{p\in S_2\setminus S_1\\ \log B<p\leqslant B^{\psi(B)}}}\theta_p(x)\leqslant\sum_{\substack{z_2<p\leqslant z_1\\ \log B<p\leqslant B^{\psi(B)}}}\theta_p(x),$$

hence

$$\sum_{\substack{x\in\Omega_B\\ f^{-1}(x)\ \text{smooth}}}\ \sum_{\substack{p\in S_2\setminus S_1\\ \log B<p\leqslant B^{\psi(B)}}}\theta_p(x)\leqslant \sum_{\substack{z_2<p\leqslant z_1\\ \log B<p\leqslant B^{\psi(B)}}}\mathscr{A}_p,$$

where $\mathscr{A}_p$ is as in (3.1). Using lemma 3.6 and the bound $\mathscr{A}_1 \ll B^{n+1}$ shows that this is

$$\ll \sum_{\substack{z_2<p\leqslant z_1\\ \log B<p\leqslant B^{\psi(B)}}}\left(B^{n+1}\sigma_p+\frac{B^{n+1}}{p\log B}+p^{2n+1}B+pB^n(\log B)^{\lfloor 1/n\rfloor}\right).$$

Invoking (5.3) the first term is $\ll B^{n+1}$. The second term is

$$\ll \frac{B^{n+1}}{\log B}\sum_{p\leqslant B}\frac{1}{p}\ll B^{n+1}.$$

The third term is

$$\ll B\sum_{p\leqslant B^{\psi(B)}}p^{2n+1}\ll_{\varepsilon} B^{1+\varepsilon},$$

valid for all $\varepsilon>0$. The fourth term can be bounded by

$$\ll B^n(\log B)^{\lfloor 1/n\rfloor}\sum_{p\leqslant B^{\psi(B)}}p\ll_{\varepsilon} B^{n+\varepsilon}.$$

We have thus shown that

$$\sum_{\substack{x\in\Omega_B\\ f^{-1}(x)\ \text{smooth}}}\ \sum_{\substack{p\in S_2\setminus S_1\\ \log B<p\leqslant B^{\psi(B)}}}(\theta_p+\sigma_p)\ll B^{n+1},$$

from which we can obtain

$$\sum_{\substack{x\in\Omega_B\\ f^{-1}(x)\ \text{smooth}}}|Y_B(t,x)-Z'_B(t,x)|\ll B^{n+1}(\log\log B)^{-1/2}.$$

An application of Markov's inequality as in the last stage of the proof of lemma 4.13 concludes the proof. □

Remark 5.3. The statement of theorem 2.3 remains valid when $X_B(\cdot,x)$ is replaced by any of the functions $Y_B(\cdot,x), Z'_B(\cdot,x)$ or $Z_B(\cdot,x)$. This can be seen by bringing together lemmas 4.13, 5.1 and 5.2.

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5.4. Proof of theorem 2.8

Letting

$$S := \left\{ u \in D : r > \int_0^1 u(t)^2 \, \mathrm{d}t \right\}$$

and combining Donsker's theorem with the result of Erdős and Kac [10, III] we obtain

$$W[S] = \tau_2(r).$$

By remark 5.3 we can use theorem 2.3 with $X_B(\cdot, x)$ replaced by $Z_B(\cdot, x)$. This yields

$$\lim_{B \to +\infty} \mathbf{P}_B \left[\{ x \in \mathbb{P}^n(\mathbb{Q}) : Z_B(\cdot, x) \in S \} \right] = \tau_2(r).$$

To complete the proof it remains to analyse the condition $Z_B(\cdot, x) \in S$. Labelling all primes in ascending order as $q_1 = 2, q_2 = 3, \ldots,$ we see that the condition is equivalent to

$$\begin{aligned} z > \int_0^1 Z_B(\cdot, x)^2 &= \sum_{q_{i+1} \leqslant B} \left(\frac{\omega_f(x, q_{i+1}) - \sum_{q \leqslant q_{i+1}} \sigma_q}{(\Delta(f) \log \log B)^{1/2}} \right)^2 \operatorname{meas}\left(T_i(B, x)\right) \\ &= \frac{1}{(\Delta(f) \log \log B)^2} \sum_{3 \leqslant p \leqslant B} \sigma_p \Big(\omega_f(x, p) - \sum_{q \leqslant p} \sigma_q \Big)^2, \end{aligned}$$

by (5.1) and (5.2). This concludes the proof because the contribution of the prime $p = 2$ in the last sum is $O((\log \log B)^{-2}) = o(1)$. □

5.5. Proof of theorem 2.9

Let us now proceed to the proof of theorem 2.9. For $0 \leqslant \alpha \leqslant \beta \leqslant 1$ define

$$S := \{ u \in D : \alpha < \operatorname{meas}(0 \leqslant t \leqslant 1 : u(t) > 0) \leqslant \beta \}.$$

By [23, theorem 5.28] we have

$$W[S] = \frac{1}{\pi} \int_\alpha^\beta \frac{\mathrm{d}\theta}{\sqrt{\theta(1-\theta)}},$$

thus, by theorem 2.3 and remark 5.3 we obtain

$$\lim_{B \to +\infty} \mathbf{P}_B \left[\{ x \in \mathbb{P}^n(\mathbb{Q}) : Z_B(\cdot, x) \in S \} \right] = \frac{1}{\pi} \int_\alpha^\beta \frac{\mathrm{d}\theta}{\sqrt{\theta(1-\theta)}}.$$

We have $Z_B(\cdot, x) \in S$ if and only if

$$\alpha < \operatorname{meas}\left(0 \leqslant t \leqslant 1 : Z_B(\cdot, x) > 0\right) \leqslant \beta.$$

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Labelling all primes in ascending order as $q_1 = 2, q_2 = 3, \ldots$, and alluding to (5.1) and (5.2) we obtain

$$\begin{aligned}\operatorname{meas}(0 \leqslant t \leqslant 1 : Z_B(\cdot, x) > 0) &= \sum_{\substack{i \geqslant 1 \\ \omega_f(x, q_{i+1}) > \sum_{q \leqslant q_{i+1}} \sigma_q}} \operatorname{meas}(T_i(B, x)) \\ &= \sum_{\substack{i \geqslant 1 \\ q_{i+1} \in \mathscr{C}_f(x)}} \frac{\sigma_{q_{i+1}}}{\Delta(f) \log\log B} \\ &= \frac{\widehat{\mathscr{C}_f}(x) - c\sigma_2}{\Delta(f) \log\log B},\end{aligned}$$

where the term c equals 1 if $\omega_f(x, 2) > \sigma_2$ and is 0 otherwise. If $B^{1/2} < H(x) \leqslant B$ then $-1 + \log\log B \leqslant \log\log H(x) \leqslant \log\log B$, hence for 100% of all $x \in \mathbb{P}^n(\mathbb{Q})$ one has

$$\frac{\widehat{\mathscr{C}_f}(x) - c\sigma_2}{\Delta(f) \log\log B} = \frac{\widehat{\mathscr{C}_f}(x)}{\Delta(f) \log\log H(x)} + O\left(\frac{1}{\log\log B}\right).$$

This concludes the proof of theorem 2.9. □

5.6. Lower bounds for $\widehat{\mathscr{C}_f}$

Let us provide an example which shows that (2.9) is best possible. Let V be the conic bundle $x_0^2 + x_1^2 = stx_2^2$ and define $f : V \to \mathbb{P}^1$ through $f(x_0, x_1, x_2, s, t) := (s, t)$. It is easy to see that $\Delta(f) = 1$ and that

$$\sigma_p = \begin{cases} \frac{2}{p+1}, & \text{if } p \equiv 3 \pmod 4, \\ 0, & \text{if } p \equiv 1 \pmod 4. \end{cases}$$

Label all primes $q \equiv 3 \pmod 4$ in ascending order by $q_1 < q_2 < \ldots$ and for each $N \in \mathbb{N}$ define

$$x_N := \left[1 : \prod_{i=1}^{N} q_i\right] \in \mathbb{P}^1(\mathbb{Q}).$$

One can use Hilbert symbols (see [**26**, Ch.III, theorem 1]) to show that

$$\left\{p \text{ prime} : f^{-1}(x_N)(\mathbb{Q}_p) = \emptyset\right\} = \left\{q_i : 1 \leqslant i \leqslant N\right\}.$$

Next, note that for any prime $p \leqslant q_N$ we have

$$\omega_f(x_N, p) = \sharp\{q \equiv 3 \pmod 4 : q \leqslant p\} \sim \frac{p}{2\log p}, \text{ as } p \to +\infty,$$

due to the prime number theorem for arithmetic progressions. Clearly this is greater than the quantity $\sum_{q \leqslant p} 1/q$ for all sufficiently large p, therefore $\mathscr{C}_f(x)$ contains all

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primes p in the range $1 \ll p \leqslant q_N$, with an absolute implied constant. Letting p' be the largest prime with $\log\log p' < N$ we obtain that whenever $p \in (q_N, p']$ then

$$\omega_f(x_N, p) = \omega_f(x_N) = N > \log\log p' \geqslant \log\log p',$$

therefore $\mathscr{C}_f(x_N)$ contains all primes p in the range $(q_N, p']$. We obtain that

$$\widehat{\mathscr{C}}_f(x_N) \geqslant \sum_{\substack{1 \ll p \leqslant p' \\ p \equiv 3 (\mathrm{mod}\ 4)}} \frac{2}{p+1} \gg \log\log(p'+1) \geqslant N.$$

The prime number theorem for arithmetic progressions shows that

$$\log H(x_N) = \sum_{\substack{p \leqslant q_N \\ p \equiv 3 (\mathrm{mod}\ 4)}} \log p \sim \frac{q_N}{2} \sim N \log N, \text{ as } N \to +\infty,$$

therefore

$$\widehat{\mathscr{C}}_f(x_N) \gg N \gg \frac{\log H(x_N)}{\log\log H(x_N)}$$

for all sufficiently large $N \in \mathbb{N}$.

5.7. Proof of theorem 2.11

By theorem 2.3 and remark 5.3 the random function $Z_B(\cdot, x)$ converges in distribution to the standard Wiener process. Fix t and u as in the statement of theorem 2.11. Letting $h : D \to \mathbb{R}$ be given by

$$h(u) := \exp\left(-u \int_0^t \mathscr{K}(u(\tau))\, \mathrm{d}\tau\right),$$

we obtain

$$\lim_{B \to +\infty} \mathbb{E}_{x \in \Omega_B}\left[h\left(Z_B(\cdot, x)\right)\right] = \mathbb{E}^0\left[\exp\left\{-u \int_0^t \mathscr{K}(B_\tau)\, \mathrm{d}\tau\right\}\right], \tag{5.4}$$

where $\mathbb{E}^0$ is taken over all Brownian motion paths $\{B_\tau : \tau \geqslant 0\}$ satisfying $B_0 = 0$ almost surely and with respect to the Wiener measure W. We have

$$\mathbb{E}_{x \in \Omega_B}\left[h\left(Z_B(\cdot, x)\right)\right] = \frac{1}{\sharp\Omega_B} \sum_{x \in \Omega_B} \exp\left(-u \int_0^t \mathscr{K}(Z_B(\tau, x))\, \mathrm{d}\tau\right) \tag{5.5}$$

and it thus remains to analyse the last integral. Labelling all primes in ascending order as $q_1 = 2, q_2 = 3, \ldots$ and using (5.2) gives us

$$\int_0^t \mathscr{K}(Z_B(\tau, x))\, \mathrm{d}\tau = \sum_{i \geqslant 1} \mathscr{K}\left(\frac{\omega_f(x, q_{i+1}) - \sum_{q \leqslant q_{i+1}} \sigma_q}{(\Delta(f) \log\log B)^{1/2}}\right) \mathrm{meas}\left(T_i(B, x) \cap [0, t]\right). \tag{5.6}$$

Note that if

$$j(= j(t,x,B)) := \max\left\{ i \geqslant 1 : \sum_{q\leqslant q_j} \sigma_q \leqslant \Delta(f) t \log\log B \right\},$$

then the sum in (5.6) includes all terms with $i \leqslant j-1$ and does not include any term with $i \geqslant j+1$. Hence by (5.1) the sum equals

$$\frac{1}{\Delta(f)\log\log B} \sum_{p\leqslant q_j} \sigma_p \mathscr{K}\left(\frac{\omega_f(x,p) - \sum_{q\leqslant p} \sigma_q}{(\Delta(f)\log\log B)^{1/2}} \right) + O\left(\frac{1}{\log\log B}\right),$$

where we have set $p = q_{i+1}$ and the error term is due to the term with $i = j$ and the fact that $\mathscr{K}$ is bounded and non-negative. Furthermore, the implied constant depends at most on f. The definition of j implies that

$$\sum_{p\leqslant q_j} \sigma_p \leqslant \Delta(f) t \log\log B < \sum_{p\leqslant q_{j+1}} \sigma_p$$

and therefore by lemma 3.4 there exist non-negative constants c_0, c_1 such that

$$-c_0 + t\log\log B < \log\log q_j \leqslant c_1 + t\log\log B.$$

Using the fact that $\mathscr{K}$ is bounded shows that the difference

$$\sum_{p\leqslant q_j} \sigma_p \mathscr{K}\left(\frac{\omega_f(x,p) - \sum_{q\leqslant p} \sigma_q}{(\Delta(f)\log\log B)^{1/2}} \right) - \sum_{p\leqslant \exp(\log^t B)} \sigma_p \mathscr{K}\left(\frac{\omega_f(x,p) - \sum_{q\leqslant p} \sigma_q}{(\Delta(f)\log\log B)^{1/2}} \right)$$

has modulus

$$\ll \sum \{\sigma_p : \log\log p \in (-c_0 + t\log\log B, c_1 + t\log\log B]\} \ll 1,$$

with an implied constant. depending at most on f. Recalling (2.13) gives

$$\int_0^t \mathscr{K}(Z_B(\tau,x))\,\mathrm{d}\tau = \widetilde{\mathscr{K}_B}(x,t) + O\left(\frac{1}{\log\log B}\right), \tag{5.7}$$

with an implied constant depending at most on f. Combining (5.4), (5.5), (5.7) and (2.12) concludes the proof. □

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