Proceedings of the Royal Society of Edinburgh, **147A**, 1–23, *2017*
DOI:10.1017/S0308210516000081

A Hardy–Sobolev inequality for the twisted Laplacian

Adimurthi
Tata Institute of Fundamental Research, Centre for Applicable Mathematics, Sharada Nagar, Chikkabommasandra, Bangalore 560065, India (adiadimurthi@gmail.com; aditi@math.tifrbng.res.in)

P. K. Ratnakumar
Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211019, India (ratnapk@hri.res.in)
and
Homi Bhabha National Institute, Training School Complex, Anushakti Nagar, Mumbai 400085, India

Vijay Kumar Sohani
Department of Mathematics, Indian Institute of Science, Bangalore 560012, India (vijaykumarsohani@gmail.com)

(MS received 5 November 2014; accepted 5 October 2015)

We prove a strong optimal Hardy–Sobolev inequality for the twisted Laplacian on $\mathbb{C}^n$. The twisted Laplacian is the magnetic Laplacian for a system of n particles in the plane, corresponding to the constant magnetic field. The inequality we obtain is *strong optimal* in the sense that the weight cannot be improved. We also show that our result extends to a one-parameter family of weighted Sobolev spaces.

Keywords: Hardy–Sobolev inequality; fundamental solution; twisted Laplacian; special Hermite expansion

2010 *Mathematics subject classification:* Primary 35A23
Secondary 35A08; 33C50

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 3$. Then the classical Hardy inequality says that

$$\int_\Omega \frac{|u|^p}{|x|^p}\,\mathrm{d}x \leqslant \left(\frac{p}{n-p}\right)^p \int_\Omega |\nabla u|^p,$$

which is valid for all $u \in W_0^{1,p}(\Omega)$, and the constant $(p/(n-p))^p$ is the optimal one. This inequality arises in the analysis of partial differential equations (PDEs) involving the Laplace operator $\Delta = \sum_{i=1}^n \partial_{x_i}^2$. So it is natural to seek the validity of a similar inequality for general second-order operators on $\mathbb{R}^n$ of the form

$$L = \sum_{|\alpha|\leqslant 2} a_\alpha(x)\partial_x^\alpha.$$

In [2], Adimurthi and Sekar investigated the same problem using the fundamental solutions, and obtained optimal estimates. The techniques developed therein can be applied in a variety of contexts. In particular, Adimurthi and Sekar obtained, among other things, optimal results for the Laplace–Beltrami operator on certain manifolds and for the sub-Laplacian on the Heisenberg group.

The aim of this paper is to prove the optimal Hardy–Sobolev inequality for the twisted Laplacian on $\mathbb{C}^n$ given by

$$\mathcal{L} = \sum_{j=1}^{n} \left[\left(\mathrm{i}\partial_{x_j} + \frac{y_j}{2} \right)^2 + \left(\mathrm{i}\partial_{y_j} - \frac{x_j}{2} \right)^2 \right], \tag{1.1}$$

which when $n = 1$ represents the magnetic Laplacian for a single particle in the plane $\mathbb{R}^2 \approx \mathbb{C}$ for the magnetic vector potential $A(z) = \frac{1}{2}(-y, x)$, $z = x + \mathrm{i}y$, which corresponds to a constant magnetic field perpendicular to the plane. Similarly, for $n > 1$, the twisted Laplacian represents the magnetic Laplacian for a system of particles in the plane under the influence of a constant magnetic field perpendicular to the plane.

Setting $X_j = \partial_{x_j} - \frac{1}{2}\mathrm{i}y_j$ and $Y_j = \partial_{y_j} + \frac{1}{2}\mathrm{i}x_j$,[1] we write

$$\mathcal{L} = -\sum_{j=1}^{n} (X_j^2 + Y_j^2). \tag{1.2}$$

The twisted Laplacian can be stated explicitly as $\mathcal{L} = -\Delta + \frac{1}{4}(|x|^2 + |y|^2) + \mathrm{i}N$, where $\mathrm{i}N$ is the angular momentum operator given by

$$N = \sum_{j=1}^{n} (y_j \partial_{x_j} - x_j \partial_{y_j}). \tag{1.3}$$

Note that, for $n > 1$, $\mathrm{i}N$ represents the total angular momentum for a system of n non-interacting particles in the plane. Let $\nabla_{\mathcal{L}}$ denote the gradient operator associated with $\mathcal{L}$:

$$\nabla_{\mathcal{L}} u = (X_1 u, \dots, X_n u, Y_1 u, \dots, Y_n u).$$

Associated with the twisted Laplacian, there is an interesting convolution structure, on functions on $\mathbb{C}^n$: the so-called twisted convolution. Given two functions f and g on $\mathbb{C}^n$, their twisted convolution is defined by

$$\begin{aligned} f \times g(z) &= \int_{\mathbb{C}^n} f(z - w) g(w) \exp\{\tfrac{1}{2}\mathrm{i}\,\mathrm{Im}(z \cdot \bar{w})\}\,\mathrm{d}w \\ &= \int_{\mathbb{C}^n} f(w) g(z - w) \exp\{-\tfrac{1}{2}\mathrm{i}\,\mathrm{Im}(z \cdot \bar{w})\}\,\mathrm{d}w \end{aligned} \tag{1.4}$$

[1] In [13], Ratnakumar and Sohani considered the first-order Sobolev spaces defined using X_j and Y_j, in the study of global well-posedness, where the notation $\bar{L}_j$ and $\bar{M}_j$ was used for compatibility with notation in [12]. For notational convenience here we use X_j and Y_j.

whenever the integral converges. Unlike the usual convolution on $\mathbb{R}^n$, the twisted convolution has many interesting properties. For instance, if f and g are in $L^2(\mathbb{C}^n)$, then $f \times g \in L^2(\mathbb{C}^n)$, which is not the case with the usual convolution (see [14]). Also it is easy to verify that

$$S(f \times g) = f \times Sg \tag{1.5}$$

for $S = X_j$ and Y_j, $j = 1, 2, \dots, n$, as defined above.

As in the case of the Laplacian on $\mathbb{R}^n$, the solutions to the initial-value problem for the basic linear PDEs (for example, the heat, wave and Schrödinger equations) associated with the twisted Laplacian can be expressed as a twisted convolution.

The magnetic Laplacian naturally arises in the study of a system of particles in the presence of a magnetic field. There has recently been considerable interest in the study of the magnetic Laplacian. We refer the reader to the classic paper by Avron *et al*. [3], which discusses the magnetic Laplacian for a constant magnetic field, to a (relatively) new paper by Yajima [16] on the Schrödinger equation for the magnetic Laplacian and to [11–13], which treat the local and global well-posedness of the Schrödinger equation for the twisted Laplacian.

The Hardy–Sobolev-type inequality has been investigated for the magnetic Laplacian by many researchers (see, for example, [4, 5, 8]), as it plays an important role for Lieb–Thirring bounds and hence for the problem of the stability of matter. We refer the reader to the interesting paper by Lieb [10].

Most of the results that we have come across deal with magnetic fields decaying at infinity, such as magnetic fields in $L^2(\mathbb{R}^n)$ [4], or Aharanov–Bohm-type magnetic potential [5] (for which the magnetic field is zero in $\mathbb{R}^n \setminus \{0\}$). These inequalities are also non-optimal (except in [5]). As far as we are aware, there is no literature on the Hardy–Sobolev-type inequality for the magnetic Laplacian corresponding to a constant magnetic field.

Here we prove the optimal Hardy–Sobolev inequality for the twisted Laplacian, which represents a magnetic Laplacian corresponding to a constant magnetic field. Moreover, in contrast to the usual case, our result is also valid for the plane $\mathbb{R}^2 \approx \mathbb{C}^1$, i.e. $n = 1$, and the optimal weight arises naturally in logarithmic terms, owing to the view that the weight should be expressible in terms of the fundamental solution for the twisted Laplacian.

Laptev and Weidl [8] show that, for certain magnetic fields of Aharanov–Bohm type in the plane, the logarithmic term in the Hardy–Sobolev inequality can be dispensed with. Our result in the case $n = 1$ is in contrast to their result, where the logarithmic term is natural and cannot be avoided, in view of optimality.

The main result in this paper is the following identity, from which we deduce the Hardy–Sobolev inequality for the twisted Laplacian.

THEOREM 1.1. *Let E be the fundamental solution to the twisted Laplacian on $\mathbb{C}^n$. Then the identity*

$$\int_{\mathbb{C}^n} |\nabla_{\mathcal{L}} u|^2 + \operatorname{Im} \int_{\mathbb{C}^n} \bar{u} N u = \frac{1}{4} \int_{\mathbb{C}^n} \left(\frac{|\nabla_{\mathcal{L}} E|^2}{E^2} + \frac{|z|^2}{4} \right) |u|^2 + \int_{\mathbb{C}^n} E |\nabla v|^2 \tag{1.6}$$

is valid for all $u \in C_c^\infty(\mathbb{C}^n)$, the space of compactly supported smooth functions on $\mathbb{C}^n$, where $v = E^{-1/2} u$.

Note that if u is a real-valued function, then $\operatorname{Im}(\bar{u}Nu) = 0$. Also, the last integral on the right-hand side of the above identity is non-negative. Thus, (1.6) leads to the following Hardy–Sobolev-type inequality with weight

$$w(z) = \left(\frac{|\nabla_{\mathcal{L}} E|^2}{E^2} + \frac{|z|^2}{4}\right).$$

Corollary 1.2. *Let w be as above and N as in (1.3). Then the inequality*

$$\frac{1}{4}\int_{\mathbb{C}^n} |u|^2 w(z)\,\mathrm{d}z \leqslant \int_{\mathbb{C}^n} |\nabla_{\mathcal{L}} u|^2\,\mathrm{d}z$$

is valid for all real-valued functions u in $W^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$. The weight $w(z)$ is optimal and never achieved. In particular, the constant $\frac{1}{4}$ is the best for the above weight w, and never achieved.

Remark 1.3. The above corollary is also valid if $u \in W^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ is purely imaginary or a complex-valued poly-radial, i.e. of the form $u(|z_1|, \dots, |z_n|)$, as $\operatorname{Im}(\bar{u}Nu) = 0$ in these cases as well.

In § 5 we also prove a weighted version of the Hardy–Sobolev inequality, valid in a one-parameter family of weighted Sobolev spaces associated with the twisted Laplacian.

Remark 1.4. The possibility of an easy derivation of the Hardy–Sobolev inequality for the twisted Laplacian, from the Hardy–Sobolev inequality for the Heisenberg group [2], was pointed out to us by the referee. Though it is true that a Hardy–Sobolev-type inequality can be deduced from the Heisenberg group case, it does not seem to be the case that the optimal inequality could be obtained that way (see § 6). Our approach is more direct, using the spectral theory of the twisted Laplacian, and yields the optimal result. We also follow the same philosophy as in [2], by presenting the weight in terms of the fundamental solution for the twisted Laplacian.

The plan of the paper is as follows. In the next section, we discuss the spectral theory of the twisted Laplacian $\mathcal{L}$ and the Sobolev space associated with it. In § 3, we derive a formula for the fundamental solution for the twisted Laplacian, and prove the corresponding Hardy–Sobolev inequality. Section 4 is devoted to the optimality question, which requires the study of the precise asymptotic properties of the fundamental solution and its gradient. Section 5 concerns the Hardy–Sobolev inequality in the weighted Sobolev spaces, and § 6 deals with the connection with the Heisenberg Laplacian.

2. Spectral theory of the twisted Laplacian

Here we give a brief description of the spectral theory of $\mathcal{L}$. For the materials discussed in this section we recommend the excellent books by Folland [7] and Thangavelu [14, 15] and the references therein.

The eigenfunctions of the twisted Laplacian $\mathcal{L}$ are called special Hermite functions, as they form a subclass of Hermite functions on $\mathbb{C}^n \approx \mathbb{R}^{2n}$: 'special' because they also span $L^2(\mathbb{C}^n)$ (see [14]). The special Hermite functions are defined in terms of the Fourier–Wigner transform. The Fourier–Wigner transform of a pair of functions $f, g \in L^2(\mathbb{R}^n)$ is a function on $L^2(\mathbb{C}^n)$ defined by

$$V(f,g)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} f(\xi + \tfrac{1}{2}y) g(\xi - \tfrac{1}{2}y)\,\mathrm{d}\xi,$$

where $z = x + \mathrm{i}y \in \mathbb{C}^n$. For any two multi-indices μ, ν, the special Hermite functions $\Phi_{\mu\nu}$ are given by

$$\Phi_{\mu\nu}(z) = V(h_\mu, h_\nu)(z),$$

where h_μ and h_ν are Hermite functions on $\mathbb{R}^n$. Recall that, for each non-negative integer k, the one-dimensional Hermite functions h_k are defined by

$$h_k(x) = \frac{(-1)^k}{\sqrt{2^k k! \sqrt{\pi}}} \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{e}^{-x^2} \right) \exp\{\tfrac{1}{2}x^2\}.$$

Now for each multi-index $\nu = (\nu_1, \dots, \nu_n)$, the n-dimensional Hermite functions are defined by the tensor product:

$$h_\nu(x) = \prod_{i=1}^{n} h_{\nu_i}(x_i), \quad x = (x_1, \dots, x_n).$$

A direct computation using the relations

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x} + x \right) h_k(x) = (2k+2)^{1/2} h_{k+1}(x),$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + x \right) h_k(x) = (2k)^{1/2} h_{k-1}(x)$$

satisfied by the Hermite functions h_k shows that $\mathcal{L}\Phi_{\mu\nu} = (2|\nu| + n)\Phi_{\mu\nu}$. Hence, $\Phi_{\mu\nu}$ are eigenfunctions of $\mathcal{L}$ with eigenvalue $2|\nu| + n$, and they also form a complete orthonormal system in $L^2(\mathbb{C}^n)$. Thus, every $f \in L^2(\mathbb{C}^n)$ has the expansion

$$f = \sum_{\mu,\nu} \langle f, \Phi_{\mu\nu} \rangle \Phi_{\mu\nu} \tag{2.1}$$

in terms of the eigenfunctions of $\mathcal{L}$. The above expansion may be written as

$$f = \sum_{k=0}^{\infty} P_k f, \tag{2.2}$$

where

$$P_k f = \sum_{\mu, |\nu| = k} \langle f, \Phi_{\mu\nu} \rangle \Phi_{\mu\nu}$$

is the spectral projection corresponding to the eigenvalue $2k + n$. Now, for any $f \in L^2(\mathbb{C}^n)$ such that $\mathcal{L}f \in L^2(\mathbb{C}^n)$, by the self-adjointness of $\mathcal{L}$ we have $P_k(\mathcal{L}f) = (2k+n)P_k f$. It follows that, for $f \in L^2(\mathbb{C}^n)$ with $\mathcal{L}f \in L^2(\mathbb{C}^n)$,

$$\mathcal{L}f = \sum_{k=0}^{\infty} (2k+n) P_k f. \tag{2.3}$$

https://doi.org/10.1017/S0308210516000081 Published online by Cambridge University Press

Note that the eigenspace corresponding to the eigenvalue $2k+n$ is infinite dimensional, as the eigenvalue depends only on the second index, ν. However, we can get a more compact representation for the above expansion. In fact, each $P_k f$ has the compact representation as a twisted convolution:

$$P_k f(z) = (2\pi)^{-n}(f \times \varphi_k)(z)$$

with the Laguerre functions $\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)\exp\{-\frac{1}{4}|z|^2\}$ (see [9, 14]). Hence, (2.2) becomes

$$f = (2\pi)^{-n}\sum_k f \times \varphi_k. \tag{2.4}$$

Note that φ_k is a Schwartz class function on $\mathbb{C}^n$, and so is the twisted translation $\tau_z\varphi_k(w) = \varphi_k(z-w)\exp\{-\frac{1}{2}\mathrm{i}\,\mathrm{Im}\, z\cdot\bar{w}\}$. Hence, $T \times \varphi_k(z) = T(\tau_z\varphi)$ makes sense for any tempered distribution T on $\mathbb{C}^n$. Since $\langle f \times \varphi_k, \psi\rangle = (f \times \varphi_k \times \psi)(0) = \langle f, \varphi_k \times \psi\rangle$, it is easy to see that (2.4) is also valid for any tempered distribution f, with convergence in the sense of tempered distribution.

Now we introduce the first-order Sobolev space $W_{\mathcal{L}}^{1,2}(\mathbb{C}^n)$ associated with the twisted Laplacian, which is the same as the Sobolev space considered in [13] for $p=2$, $m=1$.

Definition 2.1. $W_{\mathcal{L}}^{1,2}(\mathbb{C}^n) = \{f \in L^2(\mathbb{C}^n) \colon X_j f, Y_j f \in L^2(\mathbb{C}^n), 1 \leqslant j \leqslant n\}$, where X_j, Y_j are as in (1.2). $W_{\mathcal{L}}^{1,2}(\mathbb{C}^n)$ is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_{\mathcal{L}} = \sum_{j=1}^{n}(\langle X_j u, X_j v\rangle + \langle Y_j u, Y_j v\rangle).$$

3. The fundamental solution for the twisted Laplacian

In this section, we construct the fundamental solution for the twisted Laplacian $\mathcal{L}$. As in the case of the usual Laplacian on $\mathbb{R}^n$, the Green function for the twisted Laplacian is also obtained from the fundamental solution, by twisted translation: if $\mathcal{L}E = \delta_0$, then

$$K(z,w) = \tau_z(E(-w)) = E(z-w)\exp\{-\tfrac{1}{2}\mathrm{i}\,\mathrm{Im}(z.\bar{w})\}$$

is the Green function. In other words, the solution to the equation $\mathcal{L}u = f$ is given by the twisted convolution

$$u(z) = f \times E(z) = \int_{\mathbb{C}^n} f(w)E(z-w)\exp\{-\tfrac{1}{2}\mathrm{i}\,\mathrm{Im}(z.\bar{w})\}\,\mathrm{d}w.$$

This follows from the fact that $\mathcal{L}(f \times E) = f \times \mathcal{L}E$, as can be verified easily in view of (1.5) and (1.2).

We construct the fundamental solution by simple heuristic reasoning based on the series expansions (2.3) and (2.4): if a tempered distribution E satisfies $\mathcal{L}E = \delta_0$, then $E = \mathcal{L}^{-1}\delta_0$. Since δ_0 is a tempered distribution, we have the series expansion

$$\delta_0 = (2\pi)^{-n}\sum_{k=0}^{\infty}\delta_0 \times \varphi_k.$$

https://doi.org/10.1017/S0308210516000081 Published online by Cambridge University Press

Then E must be given by the tempered distribution $(2\pi)^{-n}\sum_k \mathcal{L}^{-1}(\delta_0 \times \varphi_k)$. By a formal computation

$$\mathcal{L}^{-1}(\delta_0 \times \varphi_k) = \delta_0 \times \mathcal{L}^{-1}\varphi_k = \frac{1}{2k+n}\delta_0 \times \varphi_k.$$

This prompts us to define

$$E(z) = E_n(z) = (2\pi)^{-n}\sum_k \frac{1}{2k+n}\varphi_k(z), \quad z \in \mathbb{C}^n. \tag{3.1}$$

Now we show that the function E_n defined above is indeed a fundamental solution to $\mathcal{L}$. By the self-adjointness of $\mathcal{L}$, for any $\psi \in \mathcal{S}(\mathbb{C}^n)$, we have $(\varphi_k, \mathcal{L}\psi) = (2k+n)(\varphi_k, \psi)$. Hence,

$$(\mathcal{L}E_n, \psi) = (E_n, \mathcal{L}\psi) = (2\pi)^{-n}\sum_k \frac{1}{2k+n}(\varphi_k, \mathcal{L}\psi) = (2\pi)^{-n}\sum_k (\varphi_k, \psi).$$

But $(\varphi_k, \psi) = \psi \times \varphi_k(0)$ and $\sum_k \psi \times \varphi_k(0) = \psi(0)$ as the special Hermite expansion (2.4) converges uniformly for Schwartz class functions. It follows that $(\mathcal{L}E_n, \psi) = \psi(0)$, i.e. $\mathcal{L}E_n = \delta_0$ as a tempered distribution, and hence verifies that E_n given by (3.1) is the fundamental solution for the twisted Laplacian on $\mathbb{C}^n$. Now we prove a compact representation for E_n.

Theorem 3.1. *The fundamental solution for the twisted Laplacian on* $\mathbb{C}^n$ *is given by*

$$E_n(z) = (4\pi)^{-n}\int_0^\infty [s(s+2)]^{n/2-1}\exp\{-\tfrac{1}{4}(1+s)|z|^2\}\,\mathrm{d}s.$$

Proof. We have already verified that E_n given by (3.1) is the fundamental solution for $\mathcal{L}$. The key idea in obtaining a compact representation is the following generating function identity for the Laguerre functions φ_k (see [14]):

$$\sum_k r^k \varphi_k(z) = (1-r)^{-n}\exp\left\{-\frac{1}{4}\frac{1+r}{1-r}|z|^2\right\}. \tag{3.2}$$

Writing

$$\frac{1}{2k+n} = \int_0^\infty \mathrm{e}^{-(2k+n)t}\,\mathrm{d}t$$

in (3.1) and using (3.2) we see that

$$\begin{aligned} E_n &= (2\pi)^{-n}\int_0^\infty \mathrm{e}^{-nt}\sum_0^\infty (\mathrm{e}^{-2t})^k \varphi_k\,\mathrm{d}t \\ &= (2\pi)^{-n}\int_0^\infty [2\sinh t]^{-n}\exp\{-\tfrac{1}{4}(\coth t)|z|^2\}\,\mathrm{d}t. \end{aligned}$$

Using the change of variable $v = \coth t$ and then $s = v - 1$, the above could be simplified further to an expression of the form

$$E_n(z) = (4\pi)^{-n}\int_0^\infty [s(s+2)]^{n/2-1}\exp\{-\tfrac{1}{4}(1+s)|z|^2\}\,\mathrm{d}s. \tag{3.3}$$

Hence, the theorem is proved. □

REMARK 3.2. Since E_n is a radial function, setting $E_n(z) = G_n(r)$ with $r = \frac{1}{2}|z|^2$, we see that the function G_n satisfies the recursion relation

$$16\pi^2 G_{n+2} = G_n'' - G_n \tag{3.4}$$

for $n \in \mathbb{N}$. Since

$$G_2(r) = (4\pi)^{-2}\frac{\mathrm{e}^{-r}}{r},$$

using the above recursion relation, we can show that when n is even E_n is of the form

$$P_n\left(\frac{4}{|z|^2}\right)\exp\left\{-\frac{|z|^2}{4}\right\},$$

where P_n is a polynomial of degree $n-1$ given by

$$P_n(x) = (4\pi)^{-n}\sum_{j=0}^{n/2-1}\frac{(\frac{1}{2}n-1)!(n-2-j)!}{(\frac{1}{2}n-1-j)!j!}2^j x^{n-1-j}.$$

Similarly, since G_1 is the special function given by

$$G_1(r) = (4\pi)^{-1}\int_0^\infty [s(s+2)]^{-1/2}\mathrm{e}^{-(1+s)r}\,\mathrm{d}s,$$

E_n for odd n could be given explicitly in terms of G_1 and its derivatives using the above recursion formula.

Equation (3.3) shows that E_n has a singularity at $z = 0$ and $E_n \in C^\infty(\mathbb{C}^n \setminus 0)$ with exponential decay near infinity. In fact, we compute the precise asymptotic behaviour of E_n in proposition 4.3. Interestingly, E_n has the same behaviour as the Euclidean Laplacian near the singularity $z = 0$. There is a distinction between the cases $n = 1$ and $n \geqslant 2$, as expected.

The identity in the next proposition is at the heart of the main theorem. For notational convenience we use E instead of E_n, except when the dependence on dimension needs to be highlighted.

PROPOSITION 3.3. *Let E be the fundamental solution to the twisted Laplacian on $\mathbb{C}^n$. Then the identity*

$$\begin{aligned}\int_{\mathbb{C}^n}|\nabla_{\mathcal{L}}u|^2 + \frac{1}{2}\int_{\mathbb{C}^n}(u_1Nu_2 - u_2Nu_1) \\ = \frac{1}{4}\int_{\mathbb{C}^n}\frac{|\nabla_{\mathcal{L}}E|^2}{E^2}|u|^2 + \int_{\mathbb{C}^n}E\sum_j\left[\left|\frac{\partial v}{\partial_{x_j}} - \mathrm{i}\frac{y_jv}{4}\right|^2 + \left|\frac{\partial v}{\partial_{y_j}} + \mathrm{i}\frac{x_jv}{4}\right|^2\right].\end{aligned} \tag{3.5}$$

is valid for all $u \in C_{\mathrm{c}}^\infty(\mathbb{C}^n)$, the space of compactly supported smooth functions on $\mathbb{C}^n$, where $v = E^{-1/2}u$.

Proof. We essentially follow the idea in [2] using the fundamental solution. Let E be the fundamental solution for the twisted Laplacian given by (3.3). For $u \in C_{\mathrm{c}}^\infty(\mathbb{C}^n)$, let v be the function defined by

$$u = E^{1/2}v. \tag{3.6}$$

https://doi.org/10.1017/S0308210516000081 Published online by Cambridge University Press

Then $v \in C_c^\infty(\mathbb{C}^n)$. To express the derivatives in terms of X_j and Y_j given in (1.2), the following observation will be useful:

$$\frac{\partial}{\partial_{x_j}}[\exp\{-\tfrac{1}{2}\mathrm{i}x\cdot y\}f(x,y)] = \exp\{-\tfrac{1}{2}\mathrm{i}x\cdot y\}X_j f(x,y),$$
$$\frac{\partial}{\partial_{y_j}}[\exp\{\tfrac{1}{2}\mathrm{i}x\cdot y\}f(x,y)] = \exp\{\tfrac{1}{2}\mathrm{i}x\cdot y\}Y_j f(x,y), \quad j = 1,2,\dots,n.$$

Thus, rewriting (3.6) as

$$\exp\{-\tfrac{1}{2}\mathrm{i}x\cdot y\}u = [\exp\{-\tfrac{1}{2}\mathrm{i}x\cdot y\}E]^{1/2}\exp\{-\tfrac{1}{4}\mathrm{i}x\cdot y\}v$$

and differentiating with respect to x_j, we get

$$\exp\{-\tfrac{1}{2}\mathrm{i}x\cdot y\}X_j u = \exp\{-\tfrac{1}{2}\mathrm{i}x\cdot y\}\left[\tfrac{1}{2}E^{-1/2}vX_jE + E^{1/2}\left(\frac{\partial v}{\partial_{x_j}} - \mathrm{i}\frac{y_j v}{4}\right)\right]. \quad (3.7)$$

Taking the square of the absolute value on both sides and replacing v by $E^{-1/2}u$ gives

$$|X_j u|^2 = \frac{1}{4}\frac{|X_jE|^2}{E^2}|u|^2 + E\left|\frac{\partial v}{\partial_{x_j}} - \mathrm{i}\frac{y_j v}{4}\right|^2 + \mathrm{Re}\left(X_jE\left[v\frac{\partial \bar{v}}{\partial_{x_j}} + \mathrm{i}\frac{y_j}{4}|v|^2\right]\right). \quad (3.8)$$

Since

$$v\frac{\partial \bar{v}}{\partial_{x_j}} = \frac{1}{2}\frac{\partial |v|^2}{\partial_{x_j}} + \mathrm{i}\,\mathrm{Im}\left(v\frac{\partial \bar{v}}{\partial_{x_j}}\right),$$

by writing

$$\frac{\partial |v|^2}{\partial_{x_j}} = \overline{X_j(|v|^2)} - \tfrac{1}{2}\mathrm{i}y_j|v|^2,$$

we see that

$$v\frac{\partial \bar{v}}{\partial_{x_j}} + \tfrac{1}{4}\mathrm{i}y_j|v|^2 = \tfrac{1}{2}\overline{X_j(|v|^2)} + \mathrm{i}\,\mathrm{Im}\left(v\frac{\partial \bar{v}}{\partial_{x_j}}\right).$$

Thus,

$$\mathrm{Re}\left(X_jE\left[v\frac{\partial \bar{v}}{\partial_{x_j}} + \mathrm{i}\frac{y_j|v|^2}{4}\right]\right) = \tfrac{1}{2}\,\mathrm{Re}[X_jE\overline{X_j(|v|^2)}] - \mathrm{Im}\left[X_jE\,\mathrm{Im}\left(v\frac{\partial \bar{v}}{\partial_{x_j}}\right)\right].$$

Using this in (3.8) and integrating, we see that

$$\int_{\mathbb{C}^n}|X_ju|^2 = \int_{\mathbb{C}^n}\frac{1}{4}\frac{|X_jE|^2}{E^2}|u|^2 + \int_{\mathbb{C}^n}E\left|\frac{\partial v}{\partial_{x_j}} - \mathrm{i}\frac{y_jv}{4}\right|^2$$
$$+ \tfrac{1}{2}\,\mathrm{Re}\int_{\mathbb{C}^n}X_jE\overline{X_j|v|^2} - \mathrm{Im}\int_{\mathbb{C}^n}\left[X_jE\,\mathrm{Im}\left(v\frac{\partial \bar{v}}{\partial_{x_j}}\right)\right]. \quad (3.9)$$

Again, on rewriting (3.6) as $\exp\{\frac{1}{2}\mathrm{i}x\cdot y\}u = [\exp\{\frac{1}{2}\mathrm{i}x\cdot y\}E]^{1/2}\exp\{\frac{1}{4}\mathrm{i}x\cdot y\}v$ and differentiating with respect to y_j, a similar procedure will lead to the identity

$$\int_{\mathbb{C}^n}|Y_ju|^2 = \int_{\mathbb{C}^n}\frac{1}{4}\frac{|Y_jE|^2}{E^2}|u|^2 + \int_{\mathbb{C}^n}E\left|\frac{\partial v}{\partial_{y_j}} + \mathrm{i}\frac{x_jv}{4}\right|^2 + \tfrac{1}{2}\operatorname{Re}\int_{\mathbb{C}^n}Y_jE\overline{Y_j|v|^2} - \operatorname{Im}\int_{\mathbb{C}^n}\left[Y_jE\operatorname{Im}\left(v\frac{\partial\bar{v}}{\partial_{y_j}}\right)\right]. \quad (3.10)$$

Since X_j and Y_j are skew-adjoint operators, we have

$$\int_{\mathbb{C}^n}X_jE\overline{X_j|v|^2} + \int_{\mathbb{C}^n}Y_jE\overline{Y_j|v|^2} = -\int_{\mathbb{C}^n}|v|^2(X_j^2+Y_j^2)E,$$

and hence

$$\sum_{j=1}^{n}\left(\int_{\mathbb{C}^n}X_jE\overline{X_j|v|^2} + \int_{\mathbb{C}^n}Y_jE\overline{Y_j|v|^2}\right) = \int_{\mathbb{C}^n}\mathcal{L}E\ |v|^2 = |v(0)|^2,$$

as $\mathcal{L}E = \delta_0$. Now, since v is a Schwartz class function and E has a singularity at the origin, (3.6) shows that $v(0)=0$.

Thus, summing over $j = 1, 2, \dots, n$, (3.9) and (3.10) give

$$\int_{\mathbb{C}^n}|\nabla_{\mathcal{L}}u|^2 = \frac{1}{4}\int_{\mathbb{C}^n}\frac{|\nabla_{\mathcal{L}}E|^2}{E^2}|u|^2 + \int_{\mathbb{C}^n}E\sum_j\left[\left|\frac{\partial v}{\partial_{x_j}} - \mathrm{i}\frac{y_jv}{4}\right|^2 + \left|\frac{\partial v}{\partial_{y_j}} + \mathrm{i}\frac{x_jv}{4}\right|^2\right] - \operatorname{Im}\int_{\mathbb{C}^n}\sum_j\left[X_jE\operatorname{Im}\left(v\frac{\partial\bar{v}}{\partial_{x_j}}\right) + Y_jE\operatorname{Im}\left(v\frac{\partial\bar{v}}{\partial_{y_j}}\right)\right]. \quad (3.11)$$

Now, writing $v = v_1 + \mathrm{i}v_2$, and using the fact that $v = E^{-1/2}u$, we see that

$$\begin{aligned}\operatorname{Im}(v\partial_{x_j}\bar{v}) &= v_2\partial_{x_j}v_1 - v_1\partial_{x_j}v_2\\ &= E^{-1}(u_2\partial_{x_j}u_1 - u_1\partial_{x_j}u_2).\end{aligned} \quad (3.12)$$

Similarly,

$$\begin{aligned}\operatorname{Im}(v\partial_{y_j}\bar{v}) &= v_2\partial_{y_j}v_1 - v_1\partial_{y_j}v_2\\ &= E^{-1}(u_2\partial_{y_j}u_1 - u_1\partial_{y_j}u_2).\end{aligned} \quad (3.13)$$

Since $\operatorname{Im}X_jE = -\frac{1}{2}y_jE$ and $\operatorname{Im}Y_jE = \frac{1}{2}x_jE$, it follows that

$$\operatorname{Im}\sum_j\left[X_jE\operatorname{Im}\left(v\frac{\partial\bar{v}}{\partial_{x_j}}\right) + Y_jE\operatorname{Im}\left(v\frac{\partial\bar{v}}{\partial_{y_j}}\right)\right] = -\tfrac{1}{2}(u_2Nu_1 - u_1Nu_2), \quad (3.14)$$

with N given by (1.3). Thus, (3.11) reads as

$$\int_{\mathbb{C}^n} |\nabla_{\mathcal{L}} u|^2 + \frac{1}{2}\int_{\mathbb{C}^n} (u_1 N u_2 - u_2 N u_1) = \frac{1}{4}\int_{\mathbb{C}^n} \frac{|\nabla_{\mathcal{L}} E|^2}{E^2}|u|^2 + \int_{\mathbb{C}^n} E \sum_j \left[\left|\frac{\partial v}{\partial_{x_j}} - \mathrm{i}\frac{y_j v}{4}\right|^2 + \left|\frac{\partial v}{\partial_{y_j}} + \mathrm{i}\frac{x_j v}{4}\right|^2\right]. \quad (3.15)$$

This completes the proof of proposition 3.3. □

We now deduce the proof of theorem 1.1 from the identity (3.15).

Proof of theorem 1.1. We first express the last integral in (3.15) in terms of u. Setting $v = v_1 + \mathrm{i}v_2$, a straightforward computation shows that

$$\sum_j \left[\left|\frac{\partial v}{\partial_{x_j}} - \mathrm{i}\frac{y_j v}{4}\right|^2 + \left|\frac{\partial v}{\partial_{y_j}} + \mathrm{i}\frac{x_j v}{4}\right|^2\right] = |\nabla v|^2 + \tfrac{1}{16}|z|^2|v|^2 + \tfrac{1}{2}(v_2 N v_1 - v_1 N v_2). \quad (3.16)$$

Note that the operator N given by (1.3) is a derivation, and $N\psi = 0$ if ψ is a poly-radial function, i.e. a function of $|z_1|, \dots, |z_n|$. Thus, since $v = E^{-1/2}u$, and E is radial, it follows that $Nv_i = E^{-1/2}Nu_i,\ i = 1, 2$. Hence,

$$(v_2 N v_1 - v_1 N v_2) = E^{-1}(u_2 N u_1 - u_1 N u_2). \quad (3.17)$$

Since $\mathrm{Im}\,(\bar{u}Nu) = u_1 N u_2 - u_2 N u_1$, and in view of the above two identities, (3.15) may be rewritten as

$$\int_{\mathbb{C}^n} |\nabla_{\mathcal{L}} u|^2 + \mathrm{Im} \int_{\mathbb{C}^n} \bar{u} N u = \frac{1}{4}\int_{\mathbb{C}^n} \left(\frac{|\nabla_{\mathcal{L}} E|^2}{E^2} + \frac{|z|^2}{4}\right)|u|^2 + \int_{\mathbb{C}^n} E|\nabla v|^2,$$

which is (1.6), thus proving theorem 1.1. □

Proof of corollary 1.2. Note that $\mathrm{Im}(\bar{u}Nu) = 0$ if u is a real-valued or purely imaginary function. Also if u is complex valued and poly-radial on $\mathbb{C}^n$, then $Nu = 0$. Hence, in these cases, the second term on the left-hand side is zero. Thus, since the last term on right-hand side is non-negative, the proof follows from the above identity. □

In the next section, we discuss the optimality and show that the weight we obtained is also optimal.

4. The optimality of the results

In this section, following [1], we show that the weight

$$w(z) = \left(\frac{|\nabla_{\mathcal{L}} E|^2}{E^2} + \frac{|z|^2}{4}\right)$$

appearing in corollary 1.2 is the best possible and never achieved. In particular, the constant $\frac{1}{4}$ for the above weight w is the optimal one and never achieved. We start with the following.

https://doi.org/10.1017/S0308210516000081 Published online by Cambridge University Press

Definition 4.1. We say that the weight w in the corollary 1.2 is optimal if a weight $\tilde{w}$ satisfies the inequalities

$$\frac{1}{4}\int_{\mathbb{C}^n} w(z)|u|^2\,\mathrm{d}z \leqslant \frac{1}{4}\int_{\mathbb{C}^n} \tilde{w}(z)|u|^2\,\mathrm{d}z \leqslant \int_{\mathbb{C}^n} |\nabla_{\mathcal{L}} u(z)|^2\,\mathrm{d}z \tag{4.1}$$

for all $u \in W^{1,2}_{\mathcal{L}}$. Then $w = \tilde{w}$.

Definition 4.2. We say that the weight w in the corollary 1.2 is achieved if the equality holds for some non-zero function u with

$$\int_{\mathbb{C}^n} |\nabla_{\mathcal{L}} u|^2\,\mathrm{d}z < \infty.$$

Now we prove the following asymptotic estimates for E_n and ∇E_n, a crucial ingredient for the proof of optimality. We write $f(z) \approx g(z)$ to indicate that $C_1 f(z) \leqslant g(z) \leqslant C_2 f(z)$ for some positive constants C_1 and C_2.

Proposition 4.3. *Let $E_n(z)$ be as in (3.3). Then E_n satisfies the following:*

$$E_n(z) \approx [|z|^{-n} + |z|^{-2n+2}]\exp\{-\tfrac{1}{4}|z|^2\} \quad \text{for } n \geqslant 2, \tag{4.2}$$

$$E_1(z) \approx \left[\left(\frac{1}{|z|}\right)\chi_{|z|\geqslant 1/2} + \log\left(\frac{1}{|z|}\right)\chi_{|z|\leqslant 1/2}\right]\exp\{-\tfrac{1}{4}|z|^2\} \tag{4.3}$$

for $z \in C^n$, $n \geqslant 1$.

Proof. Setting $\alpha = \frac{1}{4}|z|^2$ with a change of variable $s \to \alpha s$ in (3.3) shows that

$$E_n(z) = (4\pi)^{-n}\alpha^{-n+1}\mathrm{e}^{-\alpha}\int_0^\infty [s(s+2\alpha)]^{n/2-1}\mathrm{e}^{-s}\,\mathrm{d}s. \tag{4.4}$$

First, we consider the case when $n \geqslant 2$. Since $(s+2\alpha)^{n/2-1} \approx s^{n/2-1} + \alpha^{n/2-1}$ in this case, we see that

$$\begin{aligned}|E_n(z)| &\approx \alpha^{-n+1}\mathrm{e}^{-\alpha}\int_0^\infty (s^{n-2} + s^{n/2-1}\alpha^{n/2-1})\mathrm{e}^{-s}\,\mathrm{d}s\\ &\approx \alpha^{-n+1}\mathrm{e}^{-\alpha}(1+\alpha^{n/2-1})\\ &\approx (|z|^{-2n+2} + |z|^{-n})\exp\{-\tfrac{1}{4}|z|^2\},\end{aligned}$$

which proves (4.2).

The case $n = 1$ is more delicate. We need to consider the case of small and large α separately. For small α, namely $0 < \alpha < \frac{1}{16}$ (which corresponds to $|z| < \frac{1}{2}$), we split the integral into three parts and estimate each of them.

(i) For $0 < s \leqslant \alpha$, we have $s + 2\alpha \approx \alpha$ and $\mathrm{e}^{-s} \approx 1$ as $0 \leqslant \alpha < \frac{1}{16}$. Hence,

$$\int_0^\alpha [s(s+2\alpha)]^{-1/2}\mathrm{e}^{-s}\,\mathrm{d}s \approx \alpha^{-1/2}\int_0^\alpha s^{-1/2}\,\mathrm{d}s \approx 1. \tag{4.5}$$

(ii) For $\alpha < s < 1$, we have $s + 2\alpha \approx s$ and $\mathrm{e}^{-s} \approx 1$ as $s \in (0,1)$. Hence,

$$\int_\alpha^1 [s(s+2\alpha)]^{-1/2}\mathrm{e}^{-s}\,\mathrm{d}s \approx \int_\alpha^1 s^{-1}\,\mathrm{d}s \approx \log\left(\frac{1}{\alpha}\right) \approx \log\left(\frac{1}{|z|}\right). \tag{4.6}$$

(iii) Also for $s \geqslant 1$, $s + 2\alpha \approx s$ as $0 \leqslant \alpha < \frac{1}{16}$, which leads to

$$\int_1^\infty [s(s+2\alpha)]^{-1/2} \mathrm{e}^{-s} \, \mathrm{d}s \approx \int_1^\infty s^{-1} \mathrm{e}^{-s} \, \mathrm{d}s \approx 1. \tag{4.7}$$

From (4.5)–(4.7), we see that

$$E_1(z) \approx 1 + \log\left(\frac{1}{|z|}\right) \approx \log\left(\frac{1}{|z|}\right) \quad \text{for } |z| \leqslant \frac{1}{2}. \tag{4.8}$$

For $\alpha \geqslant \frac{1}{16}$, we split the integral in (4.4) into two parts:

$$\begin{aligned} E_1(z) &= \frac{1}{4\pi} \mathrm{e}^{-\alpha} \int_0^\alpha [s(s+2\alpha)]^{-1/2} \mathrm{e}^{-s} \, \mathrm{d}s + \mathrm{e}^{-\alpha} \int_\alpha^\infty [s(s+2\alpha)]^{-1/2} \mathrm{e}^{-s} \, \mathrm{d}s \\ &\approx \mathrm{e}^{-\alpha} \alpha^{-1/2} \int_0^\alpha s^{-1/2} \mathrm{e}^{-s} \, \mathrm{d}s + \mathrm{e}^{-\alpha} \int_\alpha^\infty s^{-1} \mathrm{e}^{-s} \, \mathrm{d}s, \end{aligned} \tag{4.9}$$

where we used the facts that $s+2\alpha \approx \alpha$ for $s \in (0, \alpha)$ and $s+2\alpha \approx s$ for $s \in (\alpha, \infty)$.

Since $\alpha \geqslant \frac{1}{16}$, the second integral is at most

$$\alpha^{-1/2} \int_\alpha^\infty s^{-1/2} \mathrm{e}^{-s} \, \mathrm{d}s.$$

Hence, from (4.9) we see that

$$E_1(z) \lesssim \mathrm{e}^{-\alpha} \alpha^{-1/2} \left(\int_0^\alpha s^{-1/2} \mathrm{e}^{-s} \, \mathrm{d}s + \int_\alpha^\infty s^{-1/2} \mathrm{e}^{-s} \, \mathrm{d}s \right) \lesssim \mathrm{e}^{-\alpha} \alpha^{-1/2}$$

as

$$\int_0^\infty s^{-1/2} \mathrm{e}^{-s} \, \mathrm{d}s < \infty.$$

From (4.9), we also see that

$$E_1(z) \gtrsim \mathrm{e}^{-\alpha} \alpha^{-1/2} \int_0^\alpha s^{-1/2} \mathrm{e}^{-s} \, \mathrm{d}s \gtrsim \mathrm{e}^{-\alpha} \alpha^{-1/2} \int_0^{1/16} s^{-1/2} \mathrm{e}^{-s} \, \mathrm{d}s.$$

The above two inequalities shows that $E_1(z) \approx \mathrm{e}^{-\alpha} \alpha^{-1/2}$ for $\alpha \geqslant \frac{1}{16}$, i.e.

$$E_1(z) \approx |z|^{-1} \exp\{-\tfrac{1}{4}|z|^2\} \quad \text{for } |z| \geqslant \tfrac{1}{2}. \tag{4.10}$$

Equation (4.8), together with (4.10), gives (4.3). □

We also need the following gradient estimate for E_n to prove the optimality of our result. Interestingly, the asymptotic behaviour of the gradient is uniform in all dimensions.

PROPOSITION 4.4. *Let $E_n(z)$ be as in (3.3). Then gradient of E_n has the following asymptotic behaviour:*

$$|\nabla E_n(z)| \approx (|z|^{-n+1} + |z|^{-2n+1}) \exp\{-\tfrac{1}{4}|z|^2\} \quad \textit{for all } n \geqslant 1. \tag{4.11}$$

Proof. Differentiating (3.3) and setting $\alpha = \frac{1}{4}|z|^2$, with a simple change of variable as before, yields

$$|\nabla E_n(z)| \approx |z| \exp\{-\tfrac{1}{4}|z|^2\}\alpha^{-n} \int_0^\infty s^{n/2-1}[(s+2\alpha)^{n/2-1}(s+\alpha)]\mathrm{e}^{-s}\,\mathrm{d}s. \quad (4.12)$$

Since $s + 2\alpha \approx s + \alpha$ and $(s+\alpha)^{n/2} \approx s^{n/2} + \alpha^{n/2}$, we have

$$\begin{aligned}\alpha^{-n} \int_0^\infty s^{n/2-1}[(s+2\alpha)^{n/2-1}(s+\alpha)]\mathrm{e}^{-s}\,\mathrm{d}s \\ &\approx \alpha^{-n} \int_0^\infty (s^{n-1} + s^{n/2-1}\alpha^{n/2})\mathrm{e}^{-s}\,\mathrm{d}s \\ &\approx \alpha^{-n} + \alpha^{-n/2} \\ &\approx (|z|^{-2n} + |z|^{-n}).\end{aligned}$$

Thus, (4.11) follows from (4.12). □

The above two propositions yield the following.

Corollary 4.5. *Let $E_n(z)$ be as in (3.3). Then $\nabla E_n/E_n$ has the following behaviour:*

$$\left|\frac{\nabla E_n(z)}{E_n(z)}\right| \approx |z| + |z|^{-1} \quad \text{for } n \geqslant 2,$$

$$\left|\frac{\nabla E_1(z)}{E_1(z)}\right| \approx \frac{1}{|z|\log(1/|z|)}\chi_{|z|\leqslant 1/2} + |z|\chi_{|z|\geqslant 1/2}.$$

Now we proceed to show that the weight w in corollary 1.2 is optimal. For this we first produce a sequence $\{u_\varepsilon\}$ of smooth functions with compact support, such that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{C}^n} E|\nabla v_\varepsilon|^2 = 0,$$

where $v_\varepsilon = E^{-1/2}u_\varepsilon$.

For this we choose a smooth function ϕ on $\mathbb{R}_+ = (0,\infty)$ such that

$$\phi(x) = \begin{cases} 0 & \text{if } x \leqslant 1, \\ 1 & \text{if } x \geqslant 2, \end{cases} \qquad 0 \leqslant \phi'(x) \leqslant M,$$

for some constant M. For example, ϕ may be chosen as

$$\phi(x) = C\int_1^x \exp\left\{-\frac{1}{x-1}\right\}\exp\left\{-\frac{1}{2-x}\right\}\mathrm{d}x,$$

on the interval $[1,2]$, for an appropriate constant C for which $\phi(2) = 1$. Note that $\operatorname{supp}\phi \subset [1,\infty)$ and $\operatorname{supp}\phi' \subset [1,2]$.

Now choose $\delta = \delta(\varepsilon)$ such that $\delta(\varepsilon) \to 0$, $\varepsilon^\delta \to 0$ and $\log(1/\varepsilon)\varepsilon^\delta \to 0$ as $\varepsilon \to 0$. For example,

$$\delta = \frac{2\log(\log(1/\varepsilon))}{\log(1/\varepsilon)}$$

is one such function. For $\varepsilon > 0$, define

$$\psi_\varepsilon(x) = x^\delta \phi\left(\frac{x}{\varepsilon}\right)\phi\left(\frac{1}{\varepsilon x}\right). \tag{4.13}$$

Clearly, $\psi_\varepsilon \in C_c^\infty(\mathbb{R}_+)$, with $\operatorname{supp}\psi_\varepsilon \subset [\varepsilon, 1/\varepsilon]$. Also, since $\psi_\varepsilon \equiv x^\delta$ on $[2\varepsilon, 1/2\varepsilon]$ and since $\delta \to 0$ as $\varepsilon \to 0$, we have $\psi_\varepsilon(x) \to 1$ as $\varepsilon \to 0$ for $x \in \mathbb{R}_+$.

Proposition 4.6. *Let ψ_ε be as above and define $v_\varepsilon(z) = \psi_\varepsilon(|z|)$. Then $v_\varepsilon \in C_c^\infty(\mathbb{C}^n)$, and*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{C}^n} E|\nabla v_\varepsilon(z)|^2 \,\mathrm{d}z = 0.$$

Proof. Note that $v_\varepsilon \in C_c^\infty(\mathbb{C}^n)$ since $\psi_\varepsilon \in C_c^\infty(\mathbb{C}^n)$.

$$\begin{aligned}|\nabla v_\varepsilon(z)| \lesssim \delta|z|^{\delta-1}\phi\left(\frac{|z|}{\varepsilon}\right)\phi\left(\frac{1}{\varepsilon|z|}\right)\\ + \frac{|z|^\delta}{\varepsilon}\phi'\left(\frac{|z|}{\varepsilon}\right)\phi\left(\frac{1}{\varepsilon|z|}\right) + \frac{|z|^\delta}{\varepsilon|z|^2}\phi\left(\frac{|z|}{\varepsilon}\right)\phi'\left(\frac{1}{\varepsilon|z|}\right).\end{aligned}$$

Since ϕ and ϕ' are uniformly bounded, using the support properties of ϕ and ϕ' we see that

$$\begin{aligned}\int_{\mathbb{C}^n} E|\nabla v_\varepsilon(z)|^2 \,\mathrm{d}z \lesssim \delta^2 \int_{\varepsilon<|z|<1/\varepsilon} E(z)|z|^{2\delta-2}\,\mathrm{d}z + \varepsilon^{-2}\int_{\varepsilon<|z|<2\varepsilon} E(z)|z|^{2\delta}\,\mathrm{d}z\\ + \varepsilon^{-2}\int_{1/2\varepsilon<|z|<1/\varepsilon} E(z)|z|^{2\delta-4}\,\mathrm{d}z. \end{aligned}\tag{4.14}$$

We show, using the asymptotic properties of E_n given by proposition 4.3, that each of the terms on right-hand side of the above inequality tends to zero as $\varepsilon \to 0$.

First, we consider the case $n \geqslant 2$. In this case

$$\begin{aligned}\int_{\varepsilon<|z|<1/\varepsilon} E(z)|z|^{2\delta-2}\,\mathrm{d}z &\lesssim \int_{\varepsilon<|z|<1/\varepsilon} (|z|^{-n} + |z|^{2-2n})\exp\{-\tfrac{1}{4}|z|^2\}|z|^{2\delta-2}\,\mathrm{d}z\\ &\lesssim \int_\varepsilon^{1/\varepsilon} (r^{-n} + r^{2-2n})r^{2\delta-2}r^{2n-1}\exp\{-\tfrac{1}{4}r^2\}\,\mathrm{d}r\\ &\lesssim \int_0^\infty (r^{(n/2)+\delta-2} + r^{\delta-1})\mathrm{e}^{-r}\,\mathrm{d}r\\ &\lesssim \Gamma(\tfrac{1}{2}n + \delta - 1) + \Gamma(\delta),\end{aligned}$$

where we used a change of variable in the third inequality. Since $\delta^2\Gamma(\delta) \to 0$ as $\delta \to 0$, it follows that the first term on right-hand side of (4.14) tends to zero as $\varepsilon \to 0$.

Similarly, using the asymptotic behaviour of E_n near zero, we see that

$$\int_{|z|<2\varepsilon} E(z)|z|^{2\delta}\,\mathrm{d}z \lesssim \int_{\varepsilon<|z|<2\varepsilon} |z|^{2-2n+2\delta}\,\mathrm{d}z \lesssim \int_\varepsilon^{2\varepsilon} r^{1+2\delta}\,\mathrm{d}r \lesssim \varepsilon^{2\delta+2}.$$

Hence, the second term on the right-hand side of (4.14) also goes to zero as $\varepsilon \to 0$.

Again, using the behaviour of E_n near infinity, we see that

$$\begin{aligned}\int_{1/2\varepsilon<|z|<1/\varepsilon} E(z)|z|^{2\delta-4} &\lesssim \int_{1/2\varepsilon<|z|<1/\varepsilon} |z|^{-n+2\delta-4}\exp\{-\tfrac{1}{4}|z|^2\}\,\mathrm{d}z \\ &\lesssim \int_{1/2\varepsilon}^{1/\varepsilon} r^{n+2\delta-5}\exp\{-\tfrac{1}{4}r^2\}\,\mathrm{d}r \\ &\lesssim \exp\left\{-\frac{1}{16\varepsilon^2}\right\}\int_{1/2\varepsilon}^{1/\varepsilon} r^{n+2\delta-5}\,\mathrm{d}r \\ &\lesssim \exp\left\{-\frac{1}{16\varepsilon^2}\right\}\varepsilon^{-n-2\delta+4},\end{aligned}$$

which shows that the third term on the right-hand side of (4.14) also goes to zero as $\varepsilon \to 0$. Hence, this proves the proposition for $n \geqslant 2$.

Now we consider the case $n = 1$. By proposition 4.3 $E_1 \approx \log(1/|z|)$ near the origin and $E_1 \approx \exp\{-\frac{1}{4}|z|^2\}/|z|$ away from the origin. Thus, we split the first integral on the right-hand side of (4.14) as

$$\int_{|z|>1/2} E(z)|z|^{2\delta-2}\,\mathrm{d}z + \int_{\varepsilon<|z|<1/2} E(z)|z|^{2\delta-2}\,\mathrm{d}z$$

and consider each term separately. Since $E_1 \approx \exp\{-\frac{1}{4}|z|^2\}/|z|$ away from the origin, as in the higher-dimensional case, we have

$$\begin{aligned}\delta^2\int_{|z|>1/2} E(z)|z|^{2\delta-2}\,\mathrm{d}z &\lesssim \delta^2\int_{|z|>1/2} E(z)|z|^{2\delta-2}\,\mathrm{d}z \\ &\lesssim \delta^2\int_{|z|>1/2} \exp\{-\tfrac{1}{4}|z|^2\}|z|^{-1}|z|^{2\delta-2}\,\mathrm{d}z \\ &\lesssim \delta^2\int_{1/2}^{\infty} \exp\{-\tfrac{1}{4}r^2\}r^{-1}r^{2\delta-2}r\,\mathrm{d}r \\ &\lesssim \delta^2\int_{1/2}^{\infty} \exp\{-\tfrac{1}{4}r^2\}r^{2\delta-2}\,\mathrm{d}r \to 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Since $E_1 \approx \log(1/|z|)$ near the origin, we see that

$$\begin{aligned}\delta^2\int_{\varepsilon<|z|<1/2} E(z)|z|^{2\delta-2}\,\mathrm{d}z &\lesssim \delta^2\int_{\varepsilon<|z|<1/2} \log\left(\frac{1}{|z|}\right)|z|^{2\delta-2}\,\mathrm{d}z \\ &\lesssim \delta^2\int_{\varepsilon}^{1/2} \log\left(\frac{1}{r}\right)r^{2\delta-2}r\,\mathrm{d}r \\ &\lesssim \delta^2\int_{\varepsilon}^{1} \log\left(\frac{1}{r}\right)r^{2\delta-1}\,\mathrm{d}r \\ &\lesssim \delta^2\int_{\log(1/\varepsilon)}^{\infty} s\mathrm{e}^{-2\delta s}\,\mathrm{d}s \\ &\lesssim \int_{\log(1/\varepsilon^{\delta})}^{\infty} s\mathrm{e}^{-2s}\,\mathrm{d}s.\end{aligned}$$

https://doi.org/10.1017/S0308210516000081 Published online by Cambridge University Press

Since se^{-2s} is integrable on $(0,\infty)$ and $\varepsilon^{\delta}\to 0$ as $\varepsilon\to 0$, it follows that the above integral converges to zero as ε tends to zero. Hence, the first term on right-hand side of (4.14) tends to zero as $\varepsilon\to 0$.

Similarly,

$$\begin{aligned}\int_{\varepsilon<|z|<2\varepsilon} E(z)\frac{|z|^{2\delta}}{\varepsilon^2}\,\mathrm{d}z &\lesssim \varepsilon^{-2}\int_{\varepsilon<|z|<2\varepsilon}\log\left(\frac{1}{|z|}\right)|z|^{2\delta}\,\mathrm{d}z\\ &\lesssim \varepsilon^{-2}\int_{\varepsilon}^{2\varepsilon}\log\left(\frac{1}{r}\right)r^{1+2\delta}\,\mathrm{d}r\\ &\lesssim \varepsilon^{-2}\log\left(\frac{1}{\varepsilon}\right)\int_{\varepsilon}^{2\varepsilon}r^{1+2\delta}\,\mathrm{d}r\\ &\lesssim \log\left(\frac{1}{\varepsilon}\right)\varepsilon^{2\delta}.\end{aligned}$$

Since $\varepsilon^{\delta}\to 0$ and $\log(1/\varepsilon)\varepsilon^{2\delta}\to 0$, the above integral converges to zero as $\varepsilon\to 0$, and hence settles the case of the second term. The third term is similar to the higher-dimensional case:

$$\varepsilon^{-2}\int_{1/2\varepsilon<|z|<1/\varepsilon} E(z)|z|^{2\delta-4}\to 0 \quad \text{as } \varepsilon\to 0.$$

Hence, this also proves the case $n=1$. □

The next proposition shows that the weight w in corollary 1.2 is optimal.

PROPOSITION 4.7. *The weight w in corollary 1.2 cannot be improved. In particular, the constant $\frac{1}{4}$ in corollary 1.2 is the best.*

Proof. Let $\tilde{w}$ be a weight satisfying the inequalities in definition 4.1. Then we first observe that $w\leqslant\tilde{w}$ almost everywhere (a.e.). In fact, since $C_c^{\infty}\subset W_{\mathcal{L}}^{1,2}$, the inequalities in definition 4.1 imply that both w and $\tilde{w}$ are locally integrable. Also, since

$$\int_{\mathbb{C}^n}(\tilde{w}-w)|u|^2\,\mathrm{d}z\geqslant 0$$

for all $u\in C_c^{\infty}(\mathbb{C}^n)$, it follows, using the regularity of the Lebesgue measure and the smooth version of Urysohn's lemma, that $\tilde{w}-w\geqslant 0$ a.e. Note that the same conclusion holds if the above inequality holds for all poly-radial $u\in C_c^{\infty}(\mathbb{C}^n)$.

Now we show $\tilde{w}-w\leqslant 0$ using proposition 4.6. Let v_ε be as in proposition 4.6 and set $u_\varepsilon=E^{1/2}v_\varepsilon$. Note that $\lim_{\varepsilon\to 0}u_\varepsilon(z)=E^{1/2}(z)$ as $\lim_{\varepsilon\to 0}v_\varepsilon(z)=1$.

Hence, by Fatou's lemma and (4.1), we see that

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{C}^n}(\tilde{w}-w)E(z)\,\mathrm{d}z &\leqslant \liminf_{\varepsilon\to 0}\frac{1}{4}\int_{\mathbb{C}^n}(\tilde{w}-w)|u_\varepsilon|^2\,\mathrm{d}z\\ &\leqslant \limsup_{\varepsilon\to 0}\int_{\mathbb{C}^n}[|\nabla_{\mathcal{L}}u_\varepsilon|^2-\tfrac{1}{4}w|u_\varepsilon|^2]\,\mathrm{d}z\\ &= \limsup_{\varepsilon\to 0}\int_{\mathbb{C}^n}E|\nabla v_\varepsilon|^2\,\mathrm{d}z\end{aligned}$$

by the identity in theorem 1.1, and the fact that the u_ε are poly-radial. By proposition 4.6 the above limit is zero, and hence

$$\frac{1}{4}\int_{\mathbb{C}^n}(\tilde{w}-w)E(z)\,\mathrm{d}z \leqslant 0.$$

Since E is non-negative, this gives $\tilde{w}-w \leqslant 0$, and hence the proof. □

The fact that the weight w in corollary 1.2 is never achieved follows from the next result.

PROPOSITION 4.8. *The equality in corollary 1.2 holds if and only if $u = 0$.*

Proof. Suppose that there exists some function $u \in W^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ for which the equality in corollary 1.2 holds. Since $C^\infty_{\mathrm{c}}(\mathbb{C}^n)$ is dense in $W^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$, we can choose a sequence $\tilde{u}_\varepsilon$ of C^∞_{c} functions such that $\tilde{u}_\varepsilon \to u$ in $W^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$, i.e. $\nabla_{\mathcal{L}}(\tilde{u}_\varepsilon - u) \to 0$ in L^2. Note that u_ε is real valued, purely imaginary or poly-radial, depending on u. Hence, by corollary 1.2 and remark 1.3, we have

$$\lim_{\varepsilon\to 0}\frac{1}{4}\int_{\mathbb{C}^n} w(z)|\tilde{u}_\varepsilon - u|^2\,\mathrm{d}z \leqslant \lim_{\varepsilon\to 0}\int_{\mathbb{C}^n}|\nabla_{\mathcal{L}}(\tilde{u}_\varepsilon - u)|^2\,\mathrm{d}z = 0.$$

Since $|z|^2 \lesssim w(z)$, the above inequality shows that $|z|\tilde{u}_\varepsilon(z) \to |z|u(z)$ in $L^2(\mathbb{C}^n)$ as $\varepsilon \to 0$. Thus, in view of the inequality $|\nabla u(z)| \lesssim |\nabla_{\mathcal{L}} u(z)| + |z|\,|u(z)|$, we also see that $\nabla\tilde{u}_\varepsilon(z) \to \nabla u(z)$ in $L^2(\mathbb{C}^n)$ as $\varepsilon \to 0$.

We can also assume that $\tilde{u}_\varepsilon \to u$ and $\nabla\tilde{u}_\varepsilon \to \nabla u$ a.e., by taking a subsequence if necessary. Then $\tilde{v}_\varepsilon \to v$ and $\nabla\tilde{v}_\varepsilon \to \nabla v$ a.e. on $\mathbb{C}^n$, where $\tilde{v}_\varepsilon = E^{-1/2}\tilde{u}_\varepsilon$ and $v = E^{-1/2}u$. Also, in view of theorem 1.1 and the equality for u in corollary 1.2,

$$\int_{\mathbb{C}^n} E(z)|\nabla\tilde{v}_\varepsilon(z)|^2\,\mathrm{d}z \to 0$$

as $\varepsilon \to 0$.

In view of Fatou's lemma, we see that

$$0 \leqslant \int_{\mathbb{C}^n} E(z)|\nabla v(z)|^2\,\mathrm{d}z \leqslant \lim_{\varepsilon\to 0}\int_{\mathbb{C}^n} E(z)|\nabla\tilde{v}_\varepsilon(z)|^2\,\mathrm{d}z = 0.$$

Therefore, $\nabla v(z) = 0$, and $v(z)$ is constant for almost every $z \in \mathbb{C}^n$. By corollary 4.5 $E^{1/2} \notin W^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ and $u = E^{1/2}v$, which forces $u = 0$. This completes the proof. □

5. The Hardy–Sobolev inequality in the weighted Sobolev space

In this section, we show that our method also gives the Hardy–Sobolev inequality that is valid in a one-parameter family of weighted Sobolev spaces associated with the twisted Laplacian. For $\alpha > 0$, let $L_{2,\alpha}(\mathbb{C}^n)$ denote the weighted L^2 space on $\mathbb{C}^n$ with weight $E^{1-2\alpha}(z)$, where E is the fundamental solution for $\mathcal{L}$ given by theorem 3.1, and let $\mathcal{S}'$ denote the space of tempered distributions on $\mathbb{C}^n$. We define the weighted Sobolev space $W_{\mathcal{L},\alpha}$ by

$$W_{\mathcal{L},\alpha}(\mathbb{C}^n) = \{f \in \mathcal{S}' \colon X_j f, Y_j f \in L_{2,\alpha}(\mathbb{C}^n),\ j = 1,2,\ldots,n\}.$$

For a given $u \in C_c^\infty(\mathbb{C}^n)$ and $\alpha > 0$, we introduce the function $v = v_\alpha = E^{-\alpha}u$, as in (3.6). A simple calculation shows that

$$E^{1-2\alpha}|X_j u|^2 = \alpha^2|u|^2\frac{|X_jE|^2}{E^2}E^{1-2\alpha} + E\left|\partial_{x_j}v + \mathrm{i}\frac{\alpha-1}{2}y_jv\right|^2 + \alpha\,\mathrm{Re}(X_jE\overline{X_j(|v|^2)}) - \tfrac{1}{4}\alpha(2\alpha-1)y_j^2E|v|^2 + \alpha y_j\,\mathrm{Im}(v\partial_{x_j}\bar{v})E, \tag{5.1}$$

$$E^{1-2\alpha}|Y_j u|^2 = \alpha^2|u|^2\frac{|Y_jE|^2}{E^2}E^{1-2\alpha} + E\left|\partial_{y_j}v - \mathrm{i}\frac{\alpha-1}{2}x_jv\right|^2 + \alpha\,\mathrm{Re}(Y_jE\overline{Y_j(|v|^2)}) - \tfrac{1}{4}\alpha(2\alpha-1)x_j^2E|v|^2 - \alpha x_j\,\mathrm{Im}(v\partial_{y_j}\bar{v})E. \tag{5.2}$$

Adding the above two identities, integrating and summing over j yields

$$\begin{aligned}\int_{\mathbb{C}^n}|\nabla_{\mathcal{L}}u|^2E^{1-2\alpha}\,\mathrm{d}z &= \alpha^2\int_{C^n}|u|^2|\nabla_{\mathcal{L}}E|^2E^{-1-2\alpha}\,\mathrm{d}z \\ &\quad + \int_{\mathbb{C}^n}E\sum_{j=1}^n\left\{\left|\partial_{x_j}v + \mathrm{i}\frac{\alpha-1}{2}y_jv\right|^2 + \left|\partial_{y_j}v - \mathrm{i}\frac{\alpha-1}{2}x_jv\right|^2\right\}\mathrm{d}z \\ &\quad + \alpha\sum_{j=1}^n\int_{\mathbb{C}^n}\mathrm{Re}\{X_jE\overline{X_j|v|^2} + Y_jE\overline{Y_j|v|^2}\}\,\mathrm{d}z \\ &\quad - \frac{\alpha(2\alpha-1)}{4}\int_{\mathbb{C}^n}|z|^2|u|^2E^{1-2\alpha}\,\mathrm{d}z \\ &\quad + \alpha\sum_{j=1}^n\int_{\mathbb{C}^n}E(z)[y_j\,\mathrm{Im}(v\partial_{x_j}\bar{v}) - x_j\,\mathrm{Im}(v\partial_{y_j}\bar{v})]\,\mathrm{d}z.\end{aligned} \tag{5.3}$$

Writing $v = v_1 + \mathrm{i}v_2$, we see that $\mathrm{Im}(v\partial_{x_j}\bar{v}) = v_2\partial_{x_j}v_1 - v_1\partial_{x_j}v_2$. Now, setting $v_j = E^{-\alpha}u_j$, this leads to the identities

$$\mathrm{Im}(v\partial_{x_j}\bar{v}) = E^{-2\alpha}(u_2\partial_{x_j}u_1 - u_1\partial_{x_j}u_2),$$
$$\mathrm{Im}(v\partial_{y_j}\bar{v}) = E^{-2\alpha}(u_2\partial_{y_j}u_1 - u_1\partial_{y_j}u_2).$$

It follows that the integrand in the last term on right-hand side of (5.3) is

$$E\sum_{j=1}^n[y_j\,\mathrm{Im}(v\partial_{x_j}\bar{v}) - x_j\,\mathrm{Im}(v\partial_{y_j}\bar{v})] = E^{1-2\alpha}(u_2Nu_1 - u_1Nu_2). \tag{5.4}$$

The integrand in the second term on the right-hand side is same as

$$E\{|\nabla v|^2 + \tfrac{1}{4}(\alpha-1)^2|z|^2|v|^2 + (1-\alpha)E^{-2\alpha}(u_2Nu_1 - u_1Nu_2)\}, \tag{5.5}$$

and the third term on the right-hand side is

$$\int_{\mathbb{C}^n}\mathcal{L}E|v|^2 = |v(0)|^2 = 0,$$

as $\mathcal{L}E = \delta_0$.

Thus, (5.3) leads to the identity

$$\int_{\mathbb{C}^n} |\nabla_{\mathcal{L}} u|^2 E^{1-2\alpha}\,\mathrm{d}z = \int_{C^n} \left[\alpha^2 \frac{|\nabla E|^2}{E^2} + (1-\alpha)\frac{|z|^2}{4}\right] |u|^2 E^{1-2\alpha}\,\mathrm{d}z + \int_{\mathbb{C}^n} E|\nabla v|^2\,\mathrm{d}z + \int_{\mathbb{C}^n} E^{1-2\alpha}(u_2 N u_1 - u_1 N u_2)\,\mathrm{d}z, \tag{5.6}$$

where we have used the fact that $|\nabla_{\mathcal{L}} E|^2 = |\nabla E|^2 + \frac{1}{4}|z|^2$.

Setting

$$\omega_\alpha(z) = \alpha^2 \frac{|\nabla E|^2}{E^2} + (1-\alpha)\tfrac{1}{4}|z|^2,$$

the identity (5.6) leads to a family of Hardy–Sobolev-type inequalities valid in the weighted Sobolev spaces $W_{\mathcal{L},\alpha}(\mathbb{C}^n)$.

Theorem 5.1. *Let ω_α be as above and let N be as in (1.3). Then the inequality*

$$\int_{C^n} |u|^2 \omega_\alpha(z) E^{1-2\alpha}\,\mathrm{d}z \leqslant \int_{\mathbb{C}^n} |\nabla_{\mathcal{L}} u|^2 E^{1-2\alpha}\,\mathrm{d}z \tag{5.7}$$

is valid for functions u in the weighted Sobolev space $W_{\mathcal{L},\alpha}(\mathbb{C}^n)$, $0 \leqslant \alpha \leqslant 1$, that is, real-valued, purely imaginary or complex-valued poly-radial functions. Moreover, for each of the weighted Sobolev space $W_{\mathcal{L},\alpha}(\mathbb{C}^n)$, the weight ω_α is optimal and never achieved.

Note that, from (5.4), it is clear that $(u_2 N u_1 - u_1 N u_2) = 0$ if u is real or purely imaginary, and $Nu = 0$ when u is poly-radial. Also the weight ω_α is non-negative for $0 \leqslant \alpha \leqslant 1$. The fact that the weight is optimal and never achieved follows by arguments similar to those in the proof of corollary 1.2.

6. Connection with the Heisenberg Laplacian

The twisted Laplacian $\mathcal{L}$ is closely connected to the Heisenberg Laplacian

$$\mathcal{L}_{\mathrm{H}} = -\sum_{j=1}^{n} (\tilde{X}_j^2 + \tilde{Y}_j^2),$$

defined in terms of the generators of the Heisenberg Lie algebra:

$$\tilde{X}_j = \partial_{x_j} + \tfrac{1}{2} y_j \partial_t, \quad \tilde{Y}_j = \partial_{y_j} - \tfrac{1}{2} x_j \partial_t, \quad j = 1, \dots, n.$$

More explicitly, we have $-\mathcal{L}_{\mathrm{H}} = \Delta_z + \frac{1}{4}|z|^2 \partial_t^2 + N\partial_t$, with N is as in (1.3).

Taking functions of the form $u(x, y, t) = f(x, y)\mathrm{e}^{-\mathrm{i}t}$ on the Heisenberg group H^n identified with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, we get

$$\tilde{X}_j u(x, t) = \mathrm{e}^{-\mathrm{i}t} X_j f(x, y) \quad \text{and} \quad \tilde{Y}_j u(x, t) = \mathrm{e}^{-\mathrm{i}t} Y_j f(x, y). \tag{6.1}$$

This leads to the identity $\mathcal{L}_{\mathrm{H}} u(x, y, t) = \mathrm{e}^{-\mathrm{i}t} \mathcal{L} f(x, y)$ for u as above.

In view of these relations, it is possible to obtain a Hardy–Sobolev inequality for the twisted Laplacian $\mathbb{C}^n$ from the Hardy–Sobolev inequality for the sub-Laplacian

$\mathcal{L}_{\mathrm{H}}$ on the Heisenberg group H_n, but not the optimal one: for $u \in C_{\mathrm{c}}^{\infty}(\mathbb{C}^n)$ and $\eta \in C_{\mathrm{c}}^{\infty}(\mathbb{R})$, the function

$$v_R(x,y,t) = \eta\left(\frac{t}{R}\right) u(x,y)\mathrm{e}^{-\mathrm{i}t} \in C_{\mathrm{c}}^{\infty}(H_n)$$

for each $R > 0$. Now, the Hardy–Sobolev inequality on the Heisenberg group [2] gives the inequality

$$n^2 \int_{H^n} |v_R|^2 \frac{|z|^2}{|z|^4 + t^2} \,\mathrm{d}z\,\mathrm{d}t \leqslant \int_{H^n} |\nabla_{\mathrm{H}}(v_R)|^2 \,\mathrm{d}z\,\mathrm{d}t. \tag{6.2}$$

Since

$$\nabla_{\mathrm{H}} v_R(z,t) = \eta\left(\frac{t}{R}\right)\mathrm{e}^{-\mathrm{i}t}\nabla_{\mathcal{L}} u(x,y) + (\tfrac{1}{2}y, -\tfrac{1}{2}x)u(x,y)\frac{1}{R}\eta'\left(\frac{t}{R}\right)\mathrm{e}^{-\mathrm{i}t},$$

using the convenient notation $v(z,t) = v(x,y,t)$ we see that

$$\begin{aligned}|\nabla_{\mathrm{H}} v_R(z,t)|^2 = |\nabla_{\mathcal{L}} u(z)|^2 \left|\eta\left(\frac{t}{R}\right)\right|^2 + |z|^2|u(z)|^2\frac{1}{4R^2}\left|\eta'\left(\frac{t}{R}\right)\right|^2 \\ - \frac{1}{R}\eta\left(\frac{t}{R}\right)\eta'\left(\frac{t}{R}\right)\operatorname{Re}\left(\bar{u}(z)\sum_{j=1}^{n}[x_j Y_j u(z) - y_j X_j u(z)]\right).\end{aligned}$$

Now, since

$$\int_{\mathbb{R}} \eta(t)\eta'(t)\,\mathrm{d}t = 0,$$

(6.2) reads as follows:

$$\begin{aligned}\int_{\mathbb{C}^n\times\mathbb{R}} \left|u(z)\eta\left(\frac{t}{R}\right)\right|^2 \frac{n^2|z|^2}{|z|^4+t^2}\,\mathrm{d}z\,\mathrm{d}t \\ \leqslant \int_{\mathbb{C}^n\times\mathbb{R}} |\nabla_{\mathcal{L}} u(z)|^2\left|\eta\left(\frac{t}{R}\right)\right|^2\mathrm{d}z\,\mathrm{d}t + \int_{\mathbb{C}^n\times\mathbb{R}} |z|^2|u(z)|^2\frac{1}{4R^2}\left|\eta'\left(\frac{t}{R}\right)\right|^2\mathrm{d}z\,\mathrm{d}t.\end{aligned} \tag{6.3}$$

By a change of variable in t, and choosing η with $\int_{\mathbb{R}} |\eta(t)|^2\,\mathrm{d}t = 1$, (6.3) leads to an inequality of the form

$$\int_{\mathbb{C}^n} |u(z)|^2 \omega_R(z)\,\mathrm{d}z \leqslant \int_{\mathbb{C}^n} |\nabla_{\mathcal{L}} u(z)|^2\,\mathrm{d}z \tag{6.4}$$

with weight ω_R given by

$$\omega_R(z) = \omega_{\eta,R}(z) = |z|^2 \int_{\mathbb{R}} \left[\frac{n^2|\eta(t)|^2}{|z|^4 + R^2t^2} - \frac{|\eta'(t)|^2}{4R^2}\right]\mathrm{d}t.$$

Now we show that we can make the weight ω_R non-negative by a suitable choice of R. First, observe that

$$\lim_{R\to 0}\omega_R(z) = -\infty \quad \text{and} \quad \lim_{R\to\infty}\omega_R(z) = 0.$$

https://doi.org/10.1017/S0308210516000081 Published online by Cambridge University Press

It is easy to see that $R \to \omega_R(z)$ has a positive maximum on $(0, \infty)$. For this we set $\varphi(R) = \omega_{R^{1/2}}(z)$, and the critical point is given by

$$0 = \varphi'(R) = \frac{|z|^2}{4R^2}\left(C - \int_{\mathbb{R}} \frac{4n^2t^2|\eta(t)|^2}{[(|z|^4/R) + t^2]^2}\,\mathrm{d}t\right),$$

where

$$C = \int_{\mathbb{R}} |\eta'(t)|^2\,\mathrm{d}t.$$

Note that the critical point is unique, since the expression inside the bracket is a monotone function of R, and hence gives the maximum, which is positive. Thus, for each z, there exists an $R = R(\eta, z)$ for which $\omega_R(z) = \omega_{R(\eta,z)}(z) \geqslant 0$ and (6.4) yields a Hardy–Sobolev-type inequality.

However, the Hardy–Sobolev inequality obtained in this way has a weight that depends on the particular choice of η, and hence it is unclear whether one can obtain the optimal Hardy–Sobolev inequality in this way.

The correspondence (6.1) also leads to a deduction of the fundamental solution for $\mathcal{L}$ from the fundamental solution E_{H} for the sub-Laplacian $\mathcal{L}_{\mathrm{H}}$. In fact, taking the partial Fourier transform in the t-variable in the relation $\mathcal{L}_{\mathrm{H}} E_{\mathrm{H}}(z,t) = \delta(z,t) = \delta(z)\delta(t)$ leads to the identity

$$(-\Delta + \tfrac{1}{4}\lambda^2|z|^2 + \mathrm{i}\lambda N)E_{\mathrm{H}}^{\lambda}(z) = \delta(z), \tag{6.5}$$

where E_{H}^{λ} denotes the partial Fourier transform in the t-variable, evaluated at λ, of the tempered distribution $E_{\mathrm{H}}(z,t)$, which was determined explicitly by Folland in [6]. Note that

$$E_{\mathrm{H}}(z,t) = \frac{c_n}{(|z|^4 + 16t^2)^{n/2}}$$

for the vector field that we consider, with the same constant c_n as in [6]. Taking $\lambda = 1$, this gives the following alternative representation of the fundamental solution to $\mathcal{L}$:

$$E(z) = E_{\mathrm{H}}^{1}(z) = \int_{\mathbb{R}} E_{\mathrm{H}}(z,t)\mathrm{e}^{-\mathrm{i}t}\,\mathrm{d}t = c_n \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mathrm{i}t}}{(|z|^4 + 16t^2)^{n/2}}\,\mathrm{d}t. \tag{6.6}$$

Acknowledgements

This work was initiated during the visit of Adimurthi to the Harish-Chandra Research Institute (HRI). He wishes to thank HRI for the research facilities provided and also for the hospitality. This work was completed during the National Board for Higher Mathematics (NBHM) postdoctoral position of Vijay Kumar Sohani, who thanks the NBHM, Department of Atomic Energy and the Department of Science and Technology, Government of India, for financial support via a J. C. Bose fellowship. The authors also thank the referee for many useful suggestions, particularly for pointing out the connection between the fundamental solution for the twisted Laplacian and the fundamental solution for the sub-Laplacian on the Heisenberg group. The subsequent modification improved the clarity of the main result in this direction.

https://doi.org/10.1017/S0308210516000081 Published online by Cambridge University Press

References

1 Adimurthi. Best constants and Pohozaev identity for Hardy–Sobolev type operators. *Commun. Contemp. Math.* **15** (2013), 1250050.

2 Adimurthi and A. Sekar. Role of the fundamental solution in Hardy–Sobolev-type inequalities. *Proc. R. Soc. Edinb.* A **136** (2006), 1111–1130.

3 J. Avron, I. Herbst and B. Simon. Schrödinger operator with magnetic fields. I. General interactions. *Duke Math. J.* **45** (1978), 847–883.

4 A. Balinsky, A. Laptev and A. V. Sobolev. Generalised Hardy inequality for the magnetic Dirichlet form. *J. Statist. Phys.* **116** (2004), 507–521.

5 C. Bennewitz and W. D. Evans. Inequalities associated with magnetic fields. *J. Comput. Appl. Math.* **171** (2004), 59–72.

6 G. B. Folland. A fundamental solution for a subelliptic operator. *Bull. Am. Math. Soc.* **79** (1973), 373–376.

7 G. B. Folland. *Harmonic analysis in phase space*. Annals of Mathematics Studies, vol. 122 (Princeton University Press, 1989).

8 A. Laptev and T. Weidl. Hardy inequalities for magnetic Dirichlet forms. In *Mathematical results in quantum mechanics*. Operator Theory: Advances and Applications, vol. 108, pp. 299–305 (Basel: Birkhäuser, 1999).

9 N. N. Lebedev. *Special functions and their applications* (New York: Dover, 1972).

10 E. H. Lieb. The stability of matter: from atoms to stars. *Bull. Am. Math. Soc.* **22** (1990), 1–49.

11 P. K. Ratnakumar. On Schrödinger propagator for the special Hermite operator. *J. Fourier Analysis Applic.* **14** (2008), 286–300.

12 P. K. Ratnakumar and V. K. Sohani. Nonlinear Schrödinger equation for the twisted Laplacian. *J. Funct. Analysis* **265** (2013), 1–27.

13 P. K. Ratnakumar and V. K. Sohani. Nonlinear Schrödinger equation and the twisted Laplacian: global well posedness. *Math. Z.* **280** (2015), 583–605.

14 S. Thangavelu. *Lectures on Hermite and Laguerre expansions*. Mathematical Notes, vol. 42 (Princeton University Press, 1993).

15 S. Thangavelu. *Harmonic analysis on the Heisenberg group*. Progress in Mathematics, vol. 154 (Birkhäuser, 1998).

16 K. Yajima. Schrödinger evolution equations with magnetic fields. *J. Analysis Math.* **56** (1991), 29–76.