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T, T^{-1} is not standard

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Abstract. A sequence of random variables, $Y_0, Y_1, Y_2, \ldots$, is called standard if there exists a one-sided isomorphism between it and a sequence of independent random variables. In this paper it is demonstrated that the sequence arising from the past of the T, T^{-1} map is not standard.

1. *Introduction*

Any sequence of random variables, $Y_0, Y_1, Y_2, \ldots$, defined on the space Y produces a decreasing sequence of σ-algebras $\mathcal{F}_n$, where $\mathcal{F}_n = \sigma(Y_n, Y_{n+1}, \ldots)$. The sequence Y_i is *exact* if $\cap \mathcal{F}_n = \emptyset$. An *isomorphism* between two such sequences $\{\mathcal{F}_n\}$ and $\{\mathcal{G}_n\}$ is a one-to-one measure-preserving map $\phi : \mathcal{F}_0 \to \mathcal{G}_0$ such that $\phi(\mathcal{F}_n) = \mathcal{G}_n \ \forall n$. A sequence of random variables Y_i is called *standard* if there exists an independent sequence of random variables X_i such that X_i is isomorphic to Y_i. An equivalent definition is that there exists a sequence of independent σ-algebras $\{\mathcal{I}_n\}$ such that $\mathcal{F}_n = \bigvee_{i=n}^{\infty} \mathcal{I}_i$.

Let T be any one-to-one map on $(Y, \mathcal{C}, \nu)$. Define T, T^{-1} on $(X \times Y, \mathcal{F}, \mu \times \nu)$ where $\mathcal{F} = \mathcal{B} \times \mathcal{C}$ by $T, T^{-1}(x, y) = (Sx, T^{x_0} y)$. If T is not specified then it is assumed to be the Bernoulli 2-shift. T, T^{-1} is two-to-one, since any point (x, y) has the preimages $(-1x, Ty)$ and $(1x, T^{-1}y)$. Mejilson proved that T, T^{-1} is exact whenever T^2 is ergodic [**3**].

In this paper a criterion developed by Vershik is used to demonstrate that the sequence of random variables generated from T, T^{-1} is not standard whenever T has positive entropy. This answers affirmatively a conjecture of Vershik in [**5**]. Vershik's manuscript contains a possible line of proof of the same fact. We have been told that Smorodinsky independently made the same conjecture.

2. *Notation*

In this section we introduce the terminology necessary to state the standardness criteria and also the terminology which is used in our proof. An *n branch* is an element of $\{-1, 1\}^n$. An *n tree* is a binary tree of height n consisting of 2^n branches. The top level

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is g_0 and the bottom level is g_{n-1}. Let $\mathcal{A}_n$ be the set of automorphisms of an n tree. A *labeled n tree* for a partition P over a point $y \in Y$ assigns to each branch g the label $P(T^{\sum g_i} y)$. The Hamming metric on labeled n trees if given by

$$d_n(W, W') = \frac{\#\text{ of branches on which the labels of } W \text{ and } W' \text{ disagree}}{2^n}$$

Fix P and let W and W' be labeled n trees over y and y' respectively. Define

$$v_n^P(y, y') = \inf_{a \in \mathcal{A}_n} d_n(aW, W').$$

In the case that $\{\mathcal{F}_n\}$ comes from T, T^{-1}, Vershik's standardness criterion is the following.

THEOREM 2.1. (Vershik [**4**]) *For every finite partition P,* $\int v_n^P(y, y')\, d\nu \times \nu \to 0$ *iff* $\{\mathcal{F}_n\}$ *is standard.*

Remark 2.1. A proof of this can also be found in [**1**].

Remark 2.2. With simple modifications our proof that T, T^{-1} is not standard works if T is any Bernoulli shift. Since any positive entropy T has an independent partition it follows that the corresponding T, T^{-1} is not standard.

For $m \leq n$ define an *m tree inside an n tree* to be a tree with 2^m branches such that the first $n - m$ coordinates all agree and the last m coordinates vary over all possibilities. The *C middle* of an m tree inside an n tree is the interval $[\sum_0^{n-m-1} g_i - C\sqrt{m}, \sum_0^{n-m-1} g_i - C\sqrt{m}]$ for any branch g of the m tree.

LEMMA 2.1. *For any collection $\mathcal{C}$ of m trees inside an n tree such that*

$$\#\mathcal{C} \geq \frac{2^{n-m+1} C\sqrt{m}}{\sqrt{n-m}},$$

there exists two whose C middles are disjoint.

Proof. This is true because the binomial coefficients are less than $2^{n-m}/\sqrt{n-m}$. □

LEMMA 2.2. *If* $4m < n$ *then the fraction of m trees whose C middles are contained in the C middle of the n tree is greater than* $1 - 4/C^2$.

Proof. This is by Chebychev's inequality and the fact that the variance of the distribution of $\sum_{i=0}^{n-m-1} g_i$ is $n - m$. □

3. T, T^{-1} *is not standard*

The following lemma was first mentioned to one of the authors by Dan Rudolph. A statement of it also appears in [**5**].

LEMMA 3.1. *Given any word,* $y_{-n}, y_{-n+2}, \ldots, y_{n-2}, y_n$*, of length* $n + 1$ *there is at most one word* $z = z_{-n}, z_{-n+2}, \ldots, z_{n-2}, z_n$ *such that* $z \neq y$ *and* $v_n^P(y, z) = 0$.

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Proof. By applying the automorphism that sends g to $-g$ to the tree over y we obtain the tree over the reflection of y, that is the word $y_n, y_{n-2}, \ldots, y_{-n+2}, y_{-n}$. A word is of *period 2* if $y_j = y_{j+4}\ \forall j, -n \le j \le n-4$. If y is of period 2 it is possible to obtain the tree over the translate of y, $y_{-n+2}, y_{-n+4}, \ldots, y_{n-2}, y_n, y_{n-2}$, by the automorphism that sends $(g_0, g_1, g_2, \ldots, g_n)$ to $(-g_0, g_1, g_2, \ldots, g_n)$. If y is of period 2 and n is even then y is its own reflection; if n is odd then its reflection is the same as its translate. These are the only possibilities.

The proof is by induction. The base case is true because there are only two automorphisms of a tree of height 1. Suppose this lemma is true for $n-1$. An n tree has two $n-1$ subtrees inside of it. Any automorphism acting on the whole tree must give the tree of a word of length $n-1$ when restricted to each of these subtrees. Thus there are at most eight possibilities for words. They arise from combinations of interchanging the two $n-1$ subtrees and whether the automorphism on the two $n-1$ trees is the identity or not. We leave it to the reader to check the possibilities. □

THEOREM 3.1. *T, T^{-1} is not standard.*

Proof. The proof is by induction and models Kalikow's proof that the T, T^{-1} transformation is not loosely Bernoulli [**2**]. Pick P to be the partition into two sets of the zeroth coordinate. It suffices to find $\{n_k\} \to \infty$, $\epsilon_k \to \epsilon > 0$, $\alpha_k \to 0$, and $\{C_k\}$ such that if we define

$$\Theta_k^y = \{y' \mid \exists y'' \text{ such that } (y'')_i = (y')_i\ \forall |i| \le C_k\sqrt{n_k} \text{ and } v_{n_k}^P(y, y'') < \epsilon_k\}$$

then for all y and k, $\mu(\Theta_k^y) \le \alpha_k$. Set:

1. $n_0 = 40000$;
2. $\epsilon_0 = 2^{-n_0}$;
3. $\alpha_0 = 2^{-3\sqrt{n_0}}$;
4. $n_k = (k+3)^6 n_{k-1}$;
5. $\epsilon_k = (1 - 8/(k+3)^2)\epsilon_{k-1}$;
6. $\alpha_k = (n_k)^4(\alpha_{k-1})^2$; and
7. $C_k = k+3$.

Since $\sum 8/(k+3)^2 < \infty$, $\epsilon_k \to \epsilon > 0$. By a minor variant of a computation in [**2**], $\alpha_k \to 0$.

The base case is to show that $\mu(\Theta_0^y) \le \alpha_0$ for all y. From the way ϵ_0 was chosen the labelled n_0 trees over y and y^μ must agree on every symbol after the application of a tree automorphism to y. Lemma 3.1 says that there are at most two possibilities for y''. The measure of y' such that y' agrees with one of these two words for all even i, $|i| \le 3\sqrt{n_0}$, is at most $2^{-3\sqrt{n_0}}$. Hence the first step of the induction is true.

For the kth step of the induction, fix y and $y' \in \Theta_k^y$. There is an appropriate y'' such that $v_{n_k}^P(y'', y) < \epsilon_k$. Fix an automorphism a that attains the minimum in $v_{n_k}^P(y'', y)$. Call an n_{k-1} tree inside of the n_k tree over y'' *good* if the number of errors (after the automorphism was applied to the tree over y) on that tree is less than $\epsilon_k 2^{n_{k-1}}$. Let r_k be the fraction of good n_{k-1} trees. Thus

$$r_k \ge 1 - \frac{\epsilon_k}{\epsilon_{k-1}} = \frac{8}{(k+3)^2}.$$

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Combining this with Lemma 2.2, the fraction of n_{k-1} trees that are good, and whose C_{k-1} middle lie in the C_k middle of the n_k tree, is at least $4/(k+3)^2$. It follows from Lemma 2.1 and the following calculation

$$\begin{aligned}\frac{2^{n_k-n_{k-1}+1}C_{k-1}\sqrt{n_{k-1}}}{\sqrt{n_k-n_{k-1}}} &< \frac{2^{n_k-n_{k-1}+3/2}C_{k-1}\sqrt{n_{k-1}}}{\sqrt{n_k}}\\ &< \frac{2^{n_k-n_{k-1}+2}}{(k+3)^2}\\ &< 2^{n_k-n_{k-1}}\frac{4}{(k+3)^2}\end{aligned}$$

thus there are at least two good n_{k-1} trees whose C_{k-1} middles are disjoint and lie in the C_k middle of the n_{k-1} tree.

To estimate $\mu(\Theta_k^y)$ notice the following. There are two n_{k-1} tress which are good, and whose disjoint C_{k-1} middles are in the C_k middle of the n_k tree over y''. Thus there exists l_1 and l_2 such that $T^{l_1}y' \in \Theta_{k-1}^{T^{l_2}(y)}$, and l_3 and l_4 such that $T^{l_3}y' \in \Theta_{k-1}^{T^{l_4}(y)}$. Since the C_{k-1} middles of the n_{k-1} trees are disjoint, $|l_1 - l_3|$ is large enough so that the above events are independent. Hence $\mu(\Theta_k^y) \leq (\alpha_{k-1})^2(n_k)^4 = \alpha_k$. □

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