Proceedings of the Royal Society of Edinburgh, **141A**, 865–880, *2011*

Non-existence of global solutions for a class of wave equations with nonlinear damping and source terms

Shun-Tang Wu
General Education Center, National Taipei University of Technology, Taipei 106, Taiwan (stwu@ntut.edu.tw)

(MS received 3 July 2010; accepted 16 November 2010)

An initial–boundary-value problem for a class of wave equations with nonlinear damping and source terms in a bounded domain is considered. We establish the non-existence result of global solutions with the initial energy controlled above by a critical value via the method introduced in a work by Autuori *et al.* in 2010. This improves the 2009 result of Liu and Wang.

1. Introduction

This paper is concerned with the initial–boundary-value problem for the following equation:

$$\left.\begin{aligned} (|u_t|^{l-2}u_t)_t - \Delta u_t - \operatorname{div}(a(x)|\nabla u|^{\alpha-2}\nabla u)& \\ -\operatorname{div}(|\nabla u_t|^{\beta-2}\nabla u_t) + Q(x,t,u_t) = f(x,u), &\quad x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), &\quad x \in \Omega, \\ u(x,t) = 0, &\quad x \in \partial\Omega, \quad t \geqslant 0, \end{aligned}\right\} \tag{1.1}$$

where $l, \alpha, \beta \geqslant 2$ and Ω is a bounded domain in $\mathbb{R}^N$, $N \geqslant 1$, with a smooth boundary $\partial\Omega$ so that the divergence theorem can be applied. Here ∇ denotes the gradient operator and $a(x)$, Q and f satisfy some conditions given in (A_1)–(A_3) below.

The equations in (1.1) form a class of essential nonlinear evolution equations used to describe longitudinal motion in viscoelasticity mechanics, and they can also be considered as a system governing the longitudinal motion of a viscoelastic configuration obeying the nonlinear Voight model [1–3, 9].

In the absence of viscosity and strong damping, equation (1.1) becomes

$$(|u_t|^{l-2}u_t)_t - \operatorname{div}(a(x)|\nabla u|^{\alpha-2}\nabla u) + Q(x,t,u_t) = f(x,u), \quad x \in \Omega, \ t > 0.$$

For $f = 0$, it is well known that the damping term ensures global existence and decay of the solution energy for arbitrary initial data [8, 10]. On the other hand, for $Q = 0$, the source term causes global non-existence of the solution and finite time blow-up in some cases [5, 7, 11, 12, 20]. While considering the interaction of

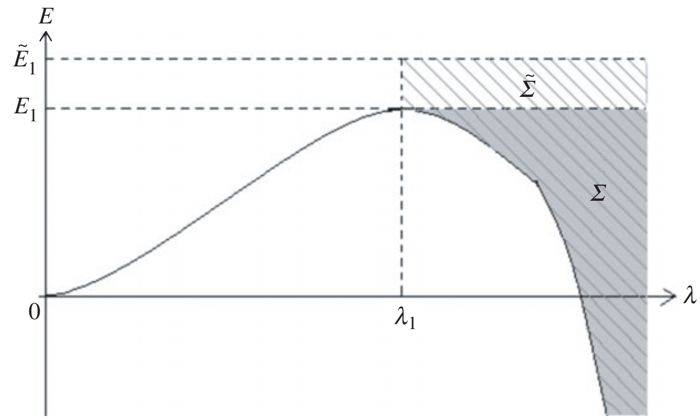

Figure 1. The phase plane (λ, E).

the nonlinear damping and source terms, there are also many results discussed by many authors (see [6, 9, 13–15, 18, 19, 22] and the references cited therein).

In the presence of viscosity and strong damping, Yang [24] obtained the non-existence properties of the global solutions when the initial energy is sufficiently negative with $l = 2$, $a(x) = 1$, $Q(x, t, u_t) = |u_t|^{m-2}u_t$, $m > 2$ and $f(x, u) = |u|^{p-2}u$, $p > 2$. Later, Messaoudi and Houari [17] extended the result of [24] to a situation when the initial energy is negative. Recently, following the technique used in [17,21], Liu and Wang [16] generalized the result of [17] to a more general model with small positive initial energy. Regarding equations of Kirchhoff type, Autuori *et al*. [4] investigated the following dissipative anisotropic non-homogeneous $p(x)$-Kirchhoff system

$$u_{tt} - M(\varphi u(t))\Delta_{p(x)}u + \mu|u|^{p(x)-2}u + Q(t, x, u, u_t) = f(t, x, u).$$

They established the new result of global non-existence for nonlinear Kirchhoff systems by a new approach of the classical potential well and concavity method.

Motivated by this research, we show in this study that the global non-existence results for equation (1.1) can be extended from the region

$$\Sigma = \{(\lambda, E) \mid \lambda > \lambda_1,\ E < E_1\}$$

to

$$\tilde{\Sigma} = \{(\lambda, E) \mid \lambda > \lambda_1,\ E < \tilde{E}_1\}$$

(see figure 1, from [4]), where λ_1, E_1 and $\tilde{E}_1$ are as given in § 3. To this end, we will improve the global non-existence results of [16] to a bigger region

$$\Sigma_1 = \{(\lambda, E) \mid \lambda > \lambda_1,\ E \leqslant E_1\}.$$

Our proof technique closely follows the arguments of [4, 16, 17], with some modifications being needed for our problem. The content of this paper is organized as follows. In § 2 we give some notation and assumptions and state the local existence result. In § 3 we state and prove the non-existence result of global solutions of (1.1) in theorems 3.6 and 3.9.

2. Preliminary and local existence results

In this section we give some notation and assumptions that will be used throughout this work. We denote by m' the Hölder conjugate of m, $\|u\|_p = \|u\|_{L^p(\Omega)}$, $\|u\|_{1,r} = \|u\|_{W^{1,r}(\Omega)}$, where $L^p(\Omega)$ and $W^{1,r}(\Omega)$ stand for Lebesgue spaces and classical Sobolev spaces, respectively. Now we make the following assumptions on a, Q and f as in [16].

(A_1) $a(x) \in L^\infty(\Omega)$ so that $a(x) \geqslant a_0 > 0$ almost everywhere (a.e.) in Ω.

(A_2) There are $m > 1$ and a measurable function $d = d(x,t)$ defined on $\Omega \times J$ such that $d(\cdot,t) \in L^{p/(p-m)}(\Omega)$ for a.e. $t \in J$ and

$$Q(x,t,v)v \geqslant 0, \qquad |Q(x,t,v)| \leqslant [d(x,t)]^{1/m}[Q(x,t,v)v]^{1/m'}, \tag{2.1}$$

for all values of x, t, v, where $J = [0,T)$, $0 < T \leqslant \infty$, and

$$d(x,t) \geqslant 0, \qquad \|d(\cdot,t)\|_{p/(p-m)} \in L^\infty_{\mathrm{loc}}(J). \tag{2.2}$$

(A_3) $f(x,u) \in C(\Omega \times \mathbb{R}^N; \mathbb{R}^N)$, $f(x,u) = \nabla_u F(x,u)$, with $F(x,0) = 0$. There are constants $d_1 > 0$, $p > \alpha$ and $0 < \mu < \mu_0 a_0$ such that, for $x \in \Omega$ and $u \in \mathbb{R}^N$,

$$|f(x,u)| \leqslant d_1|u|^{p-1} + \mu|u|^{\alpha-1}, \tag{2.3}$$

where μ_0 is the first eigenvalue of the nonlinear eigenvalue problem

$$-\operatorname{div}(|\nabla u|^{\alpha-2}\nabla u) = \tau|u|^{p-2}u \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

Moreover, there is $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$, there exists $d_2(\varepsilon) > 0$ such that

$$f(x,u)u - (p-\varepsilon)F(x,u) \geqslant d_2|u|^p \tag{2.4}$$

for all $x \in \Omega$.

Remark 2.1. We note that when

$$Q(x,t,u_t) = b(1+t)^\rho|u_t|^{m-2}u_t, \quad -\infty < \rho \leqslant m-1,$$

condition (A_2) holds.

Let

$$U = L^\infty([0,T), W_0^{1,\alpha}(\Omega)) \cap W^{1,\infty}([0,T), L^2(\Omega)) \\ \cap W^{1,\beta}([0,T), W_0^{1,\beta}(\Omega)) \cap W^{1,m}([0,T), L^m(\Omega)),$$

where $T > 0$ is a real number. Now we are in a position to state the local existence result that can be obtained by combing arguments in [23, 25].

Theorem 2.2. *Assume that* (A_1), (A_2) *and* (A_3) *hold. Suppose* $2 < \alpha < p < p^*$ *and* $u_0 \in W_0^{1,\alpha}(\Omega)$, $u_1 \in L^2(\Omega)$. *Then there exists a unique local solution* u *of (1.1) satisfying* $u \in U$, *where* $p^* = N\alpha/(N-\alpha)$ *if* $N > \alpha$ *and* $p^* = \infty$ *if* $N \leqslant \alpha$.

3. Non-existence of global solutions

In this section we shall discuss the non-existence of global solutions of problem (1.1). For this purpose, we first define the energy function associated with a solution u of (1.1) by

$$E(t) = \frac{l-1}{l}\|u_t\|_l^l + \mathcal{A}(u) - \mathcal{F}(u) \quad \text{for } t \geqslant 0, \tag{3.1}$$

where

$$\mathcal{A}(u) = \frac{1}{\alpha}\int_\Omega a(x)|\nabla u|^\alpha \,\mathrm{d}x - \frac{\mu}{\alpha}\|u\|_\alpha^\alpha \tag{3.2}$$

and

$$\mathcal{F}(u) = \int_\Omega F(x,u)\,\mathrm{d}x - \frac{\mu}{\alpha}\|u\|_\alpha^\alpha. \tag{3.3}$$

We also set

$$\lambda_1 = \left(a_0 - \frac{\mu}{\mu_0}\right)^{1/(p-\alpha)} (d_1 B_1^p)^{-1/(p-\alpha)}, \tag{3.4}$$

$$\begin{aligned} E_1 &= \left(\frac{1}{\alpha} - \frac{1}{p}\right)\left(a_0 - \frac{\mu}{\mu_0}\right)^{p/(p-\alpha)} (d_1 B_1^p)^{-\alpha/(p-\alpha)} \\ &= \left(1 - \frac{\alpha}{p}\right) w_1, \end{aligned} \tag{3.5}$$

where

$$w_1 = \frac{1}{\alpha}\left(a_0 - \frac{\mu}{\mu_0}\right)\lambda_1^\alpha$$

and B_1 is the best constant of the Sobolev embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^p(\Omega)$ given by

$$B_1^{-1} = \inf\{\|\nabla u\|_\alpha^\alpha \colon u \in W_0^{1,\alpha}(\Omega),\ \|u\|_p^p = 1\}.$$

Remark 3.1. From (3.2), (A_1) and the definition of μ_0 by (A_3), we observe that

$$\begin{aligned} \mathcal{A}(u) &= \frac{1}{\alpha}\int_\Omega a(x)|\nabla u|^\alpha \,\mathrm{d}x - \frac{\mu}{\alpha}\|u\|_\alpha^\alpha \\ &\geqslant \frac{1}{\alpha}\left(a_0 - \frac{\mu}{\mu_0}\right)\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x \end{aligned} \tag{3.6}$$

and since $f(x,u) = \nabla_u F(x,u)$, it follows from (3.3), (2.3) and the Sobolev embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^p(\Omega)$ that

$$\begin{aligned} \mathcal{F}(u) &= \int_\Omega F(x,u)\,\mathrm{d}x - \frac{\mu}{\alpha}\|u\|_\alpha^\alpha \\ &= \int_\Omega \int_0^1 f(x,\tau u)u \,\mathrm{d}\tau\,\mathrm{d}x - \frac{\mu}{\alpha}\|u\|_\alpha^\alpha \\ &\leqslant \frac{d_1}{p}\|u\|_p^p \leqslant \frac{d_1 B_1^p}{p}\|\nabla u\|_\alpha^p. \end{aligned} \tag{3.7}$$

Then, as in [16], we have the following results.

https://doi.org/10.1017/S0308210510001125 Published online by Cambridge University Press

LEMMA 3.2 (Liu and Wang [16]). *Suppose that* (A_1)–(A_3) *hold, that*

$$u_0 \in W_0^{1,\alpha}(\Omega), \qquad u_1 \in L^2(\Omega)$$

and let u *be a solution of (1.1). Then* $E(t)$ *is a non-increasing function on* $[0,T]$ *and*

$$E'(t) = -\int_\Omega |\nabla u_t|^2 \,\mathrm{d}x - \int_\Omega |\nabla u_t|^\beta \,\mathrm{d}x - \int_\Omega Q(x,t,u_t)u_t \,\mathrm{d}x \leqslant 0. \tag{3.8}$$

THEOREM 3.3 (Liu and Wang [16]). *Suppose that*

$$l,\alpha,\beta,m,p \geqslant 2, \qquad \max\{l,\beta,m\} < \alpha < p < p^*, \qquad (\|\nabla u_0\|_\alpha, E(0)) \in \Sigma$$

that (A_1), (A_2) *and* (A_3) *hold, and that* $u \in U$ *is a solution to (1.1) on* $[0,T]$. *Then* T *is necessarily finite, i.e.* u *cannot be continued for all* $t > 0$.

REMARK 3.4. It follows from (3.1), (3.6), (3.8) and $0 < \mu < \mu_0 a_0$ by (A_3) that

$$\begin{aligned}\mathcal{F}(u) &= \frac{l-1}{l}\|u_t\|_l^l + \mathcal{A}(u) - E(t) \\ &\geqslant \frac{1}{\alpha}\left(a_0 - \frac{\mu}{\mu_0}\right)\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x - E(0) \\ &\geqslant -E(0)\end{aligned}$$

for all $t \geqslant 0$. In other words, $\mathcal{F}(u)$ is bounded below for all $t \geqslant 0$ along any solution $u \in U$.

LEMMA 3.5. *If* u *is a solution of (1.1), then*

$$w_2 = \inf_{t \in R_0^+} \mathcal{F}(u) > -\infty.$$

Furthermore, if $E(0) < (p/\alpha - 1)w_2 = \tilde{E}_1$, *then*

$$w_2 > 0 \quad \textit{and} \quad (\|\nabla u(t)\|_\alpha, E(t)) \in \tilde{\Sigma} = \{(\lambda, E) \mid \lambda > \lambda_1,\ E < \tilde{E}_1\}$$

for all $t \in R_0^+$.

Proof. From remark 3.4 and $E(0) < (p/\alpha - 1)w_2 = \tilde{E}_1$, we have

$$\begin{aligned} w_2 &\geqslant \frac{1}{\alpha}\left(a_0 - \frac{\mu}{\mu_0}\right)\|\nabla u\|_\alpha^\alpha - E(0) \\ &> \frac{1}{\alpha}\left(a_0 - \frac{\mu}{\mu_0}\right)\|\nabla u\|_\alpha^\alpha - \left(\frac{p}{\alpha} - 1\right)w_2, \end{aligned}$$

which implies

$$w_2 > \frac{1}{p}\left(a_0 - \frac{\mu}{\mu_0}\right)\|\nabla u\|_\alpha^\alpha \geqslant 0$$

due to $0 < \mu < \mu_0 a_0$, and so $w_2 > 0$. □

By (3.8), it is clear that $E(t) \leqslant E(0) < \tilde{E}_1$ for all $t \in R_0^+$. Suppose that there exists $t_1 \in R_0^+$ such that $\|\nabla u(t_1)\|_\alpha \leqslant \lambda_1$. Then, by (3.7), (3.1) and (3.6),

$$\begin{aligned}\left(\frac{p}{\alpha}-1\right)\frac{d_1 B^p}{p}\|\nabla u(t_1)\|_\alpha^p &\geqslant \left(\frac{p}{\alpha}-1\right)\mathcal{F}(u)\\ &\geqslant \left(\frac{p}{\alpha}-1\right)w_2\\ &= \tilde{E}_1 > E(0)\\ &\geqslant \mathcal{A}(u(t_1)) - \mathcal{F}(u(t_1))\\ &\geqslant \frac{1}{\alpha}\left(a_0 - \frac{\mu}{\mu_0}\right)\|\nabla u(t_1)\|_\alpha^\alpha - \frac{d_1 B_1^p}{p}\|\nabla u(t_1)\|_\alpha^p.\end{aligned}$$

This implies that

$$\|\nabla u(t_1)\|_\alpha > \left(a_0 - \frac{\mu}{\mu_0}\right)^{1/(p-\alpha)}(d_1 B_1^p)^{-1/(p-\alpha)} = \lambda_1,$$

which is a contradiction. Thus

$$\|\nabla u(t)\|_\alpha > \lambda_1 \quad \text{and} \quad (\|\nabla u(t)\|_\alpha, E(t)) \in \tilde{\Sigma} \quad \text{for all } t \in R_0^+.$$

THEOREM 3.6. *Let* $l, \alpha, \beta, m, p \geqslant 2$ *and* $\max\{l, \beta, m\} < \alpha < p < p^*$. *Assume that* (A_1), (A_2) *and* (A_3) *hold, and that* $u_0 \in W_0^{1,\alpha}(\Omega)$, $u_1 \in L^2(\Omega)$. *Then any solution of (1.1) with initial data satisfying* $E(0) < \tilde{E}_1$ *cannot be continued for all* $t > 0$.

Proof. We will prove this theorem by contradiction, so we suppose that a solution u of (1.1) is admitted on $[0, \infty)$ and set

$$H(t) = E_2 - E(t), \quad \forall t \geqslant 0, \tag{3.9}$$

where $E_2 > 0$, $E_2 \in (E(0), \tilde{E}_1)$. By (3.8), we get $H'(t) \geqslant 0$. Thus, we obtain

$$H(t) \geqslant H(0) = E_2 - E(0) > 0, \quad \forall t \geqslant 0. \tag{3.10}$$

In addition, by the choice of E_2, (3.1), the definition of $\tilde{E}_1$ and the definition of w_2, we have

$$\begin{aligned}H(t) = E_2 - E(t) &< \tilde{E}_1 + \mathcal{F}(u)\\ &= \left(\frac{p}{\alpha}-1\right)w_2 + \mathcal{F}(u)\\ &\leqslant \frac{p}{\alpha}\mathcal{F}(u).\end{aligned}$$

Hence, by (3.10) and (3.7), we get

$$0 < H(0) \leqslant H(t) \leqslant \frac{p}{\alpha}\mathcal{F}(u) \leqslant \frac{d_1}{\alpha}\|u\|_p^p. \tag{3.11}$$

Let

$$\Phi(t) = \int_\Omega u|u_t|^{l-2}u_t \,\mathrm{d}x. \tag{3.12}$$

https://doi.org/10.1017/S0308210510001125 Published online by Cambridge University Press

Taking the derivative of $\Phi(t)$ and using (1.1) yield

$$\Phi'(t) = \|u_t\|_l^l - \int_\Omega \nabla u \nabla u_t \,\mathrm{d}x - \int_\Omega a(x)|\nabla u|^\alpha \,\mathrm{d}x - \int_\Omega |\nabla u_t|^{\beta-2}\nabla u_t \nabla u \,\mathrm{d}x - \int_\Omega Q(x,t,u_t)u \,\mathrm{d}x + \int_\Omega f(x,u)u \,\mathrm{d}x. \tag{3.13}$$

In correspondence to (A_3), there exists $\varepsilon_0 > 0$ such that (A_3) holds true. Without loss of generality, we take ε_0 so small that

$$\varepsilon_0 w_2 \leqslant (p-\alpha)w_2 - \alpha E_2, \tag{3.14}$$

which is possible since $w_2 > 0$ and $\tilde{E}_1 > E_2$. Fix $\varepsilon \in (0, \varepsilon_0)$. Then, via (3.1), (3.13) becomes

$$\begin{aligned}\Phi'(t) = a_1\|u_t\|_l^l &- \int_\Omega \nabla u \nabla u_t \,\mathrm{d}x - \int_\Omega |\nabla u_t|^{\beta-2}\nabla u_t \nabla u \,\mathrm{d}x + \int_\Omega f(x,u)u \,\mathrm{d}x \\ &+ (p-\varepsilon-\alpha)\mathcal{A}(u) - \mu \int_\Omega |u|^\alpha \,\mathrm{d}x - \int_\Omega Q(x,t,u_t)u \,\mathrm{d}x \\ &- (p-\varepsilon)E(t) - (p-\varepsilon)\mathcal{F}(u),\end{aligned}$$

here $a_1 = 1 + (l-1)(p-\varepsilon)/l > 0$. On the other hand, we note from (3.3) and (2.4) that

$$\begin{aligned}&\int_\Omega f(x,u)u \,\mathrm{d}x - (p-\varepsilon)\mathcal{F}(u) - \mu \int_\Omega |u|^\alpha \,\mathrm{d}x \\ &\quad = \int_\Omega f(x,u)u \,\mathrm{d}x - \mu\|u\|_\alpha^\alpha - (p-\varepsilon)\left(\int_\Omega F(x,u) \,\mathrm{d}x - \frac{\mu}{\alpha}\|u\|_\alpha^\alpha\right) \\ &\quad \geqslant d_2\|u\|_p^p + \frac{\mu(p-\varepsilon-\alpha)}{\alpha}\|u\|_\alpha^\alpha.\end{aligned}$$

Thus,

$$\begin{aligned}\Phi'(t) \geqslant a_1\|u_t\|_l^l + d_2\|u\|_p^p &- \int_\Omega \nabla u \nabla u_t \,\mathrm{d}x + (p-\varepsilon-\alpha)(\mathcal{A}(u) - E(t)) \\ &- \alpha E(t) - \int_\Omega |\nabla u_t|^{\beta-2}\nabla u_t \nabla u \,\mathrm{d}x - \int_\Omega Q(x,t,u_t)u \,\mathrm{d}x.\end{aligned}$$

Therefore, by using (3.1) again, the definition of w_2 and $E(t) = E_2 - H(t)$ by (3.9), we see that

$$\begin{aligned}\Phi'(t) &\geqslant a_2\|u_t\|_l^l + d_2\|u\|_p^p - \int_\Omega \nabla u \nabla u_t \,\mathrm{d}x + (p-\varepsilon-\alpha)\mathcal{F}(u) \\ &\quad - \alpha E(t) - \int_\Omega |\nabla u_t|^{\beta-2}\nabla u_t \nabla u \,\mathrm{d}x - \int_\Omega Q(x,t,u_t)u \,\mathrm{d}x \\ &\geqslant a_2\|u_t\|_l^l + d_2\|u\|_p^p - \int_\Omega \nabla u \nabla u_t \,\mathrm{d}x + (p-\varepsilon-\alpha)w_2 \\ &\quad + \alpha H(t) - \alpha E_2 - \int_\Omega |\nabla u_t|^{\beta-2}\nabla u_t \nabla u \,\mathrm{d}x - \int_\Omega Q(x,t,u_t)u \,\mathrm{d}x,\end{aligned} \tag{3.15}$$

where $a_2 = 1 + \alpha(l-1)/l$. Furthermore, since $\varepsilon w_2 < \varepsilon_0 w_2 \leqslant (p-\alpha)w_2 - \alpha E_2$ by (3.14), we derive from (3.15) that

$$\Phi'(t) \geqslant a_2\|u_t\|_l^l + d_2\|u\|_p^p + \alpha H(t) - \int_\Omega \nabla u \nabla u_t \,\mathrm{d}x - \int_\Omega |\nabla u_t|^{\beta-2}\nabla u_t \nabla u \,\mathrm{d}x - \int_\Omega Q(x,t,u_t)u \,\mathrm{d}x. \quad (3.16)$$

Next, we want to estimate the last three terms of the right-hand side of (3.16) as in [16]. Using (A_2), the Hölder inequality and Young's inequality, we obtain

$$\begin{aligned}\int_\Omega Q(x,t,u_t)u \,\mathrm{d}x &\leqslant \frac{\delta^m}{m}\int_\Omega |u|^m \,\mathrm{d}(x,t)\,\mathrm{d}x + \frac{m-1}{m}\delta^{-m/(m-1)}\int_\Omega Q(x,t,u_t)u_t \,\mathrm{d}x \\ &\leqslant \frac{\delta^m}{m}\|u\|_p^m \|d(\cdot,t)\|_{p/(p-m)} + \frac{m-1}{m}\delta^{-m/(m-1)}\int_\Omega Q(x,t,u_t)u_t \,\mathrm{d}x \\ &\leqslant \frac{\delta^m a_3}{m}\|u\|_p^m + \frac{m-1}{m}\delta^{-m/(m-1)}\int_\Omega Q(x,t,u_t)u_t \,\mathrm{d}x, \end{aligned} \quad (3.17)$$

$$\int_\Omega \nabla u \nabla u_t \,\mathrm{d}x \leqslant \frac{1}{4\mu_1}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \mu_1 \int_\Omega |\nabla u_t|^2 \,\mathrm{d}x \quad (3.18)$$

and

$$\int_\Omega |\nabla u_t|^{\beta-2}\nabla u_t \nabla u \,\mathrm{d}x \leqslant \frac{\mu_2^\beta}{\beta}\int_\Omega |\nabla u|^\beta \,\mathrm{d}x + \frac{\beta-1}{\beta}\mu_2^{-\beta/(\beta-1)}\int_\Omega |\nabla u_t|^\beta \,\mathrm{d}x, \quad (3.19)$$

where δ, μ_1, μ_2 and a_3 are some positive constants. A substitution of (3.17)–(3.19) into (3.16) gives

$$\begin{aligned}\Phi'(t) \geqslant{}& a_2\|u_t\|_l^l + d_2\|u\|_p^p + \alpha H(t) - \frac{1}{4\mu_1}\int_\Omega |\nabla u|^2 \,\mathrm{d}x \\ &- \mu_1\int_\Omega |\nabla u_t|^2 \,\mathrm{d}x - \frac{\mu_2^\beta}{\beta}\int_\Omega |\nabla u|^\beta \,\mathrm{d}x - \frac{\beta-1}{\beta}\mu_2^{-\beta/(\beta-1)}\int_\Omega |\nabla u_t|^\beta \,\mathrm{d}x \\ &- \frac{\delta^m a_3}{m}\|u\|_p^m - \frac{m-1}{m}\delta^{-m/(m-1)}\int_\Omega Q(x,t,u_t)u_t \,\mathrm{d}x. \end{aligned} \quad (3.20)$$

At this point, we choose δ, μ_1, μ_2 so that

$$\delta^{-m/(m-1)} = M_1 H^{-r}(t), \qquad \mu_1 = M_2 H^{-r}(t), \qquad \mu_2^{-\beta/(\beta-1)} = M_3 H^{-r}(t),$$

for M_1, M_2 and M_3 to be specified later and

$$0 < r < \min\left\{\frac{\alpha-2}{p},\ \frac{\alpha-\beta}{p(\beta-1)},\ \frac{\alpha-m}{p(m-1)},\ \frac{\alpha-l}{\alpha l}\right\}. \quad (3.21)$$

Then, using the fact that

$$H'(t) = -E'(t) = \int_\Omega |\nabla u_t|^2 \,\mathrm{d}x + \int_\Omega |\nabla u_t|^\beta \,\mathrm{d}x + \int_\Omega Q(x,t,u_t)u_t \,\mathrm{d}x$$

https://doi.org/10.1017/S0308210510001125 Published online by Cambridge University Press

by (3.9) and (3.8), we deduce from (3.20) that

$$\begin{aligned}
\Phi'(t) \geqslant{}& a_2\|u_t\|_l^l + d_2\|u\|_p^p + \alpha H(t) - \frac{1}{4M_2}H^r(t)\int_\Omega |\nabla u|^2\,\mathrm{d}x \\
&- \frac{M_3^{-(\beta-1)}}{\beta}H^{r(\beta-1)}(t)\int_\Omega |\nabla u|^\beta\,\mathrm{d}x - \frac{M_1^{1-m}a_3}{m}H^{r(m-1)}(t)\|u\|_p^m \\
&- \bigg[M_2\int_\Omega |\nabla u_t|^2\,\mathrm{d}x + \frac{(\beta-1)M_3}{\beta}\int_\Omega |\nabla u_t|^\beta\,\mathrm{d}x \\
&\qquad + \frac{(m-1)M_1}{m}\int_\Omega Q(x,t,u_t)u_t\,\mathrm{d}x\bigg]H^{-r}(t) \\
\geqslant{}& a_2\|u_t\|_l^l + d_2\|u\|_p^p + \alpha H(t) - \frac{1}{4M_2}H^r(t)\int_\Omega |\nabla u|^2\,\mathrm{d}x \\
&- \frac{M_3^{-(\beta-1)}}{\beta}H^{r(\beta-1)}(t)\int_\Omega |\nabla u|^\beta\,\mathrm{d}x - \frac{M_1^{1-m}a_3}{m}H^{r(m-1)}(t)\|u\|_p^m \\
&- MH^{-r}(t)H'(t),
\end{aligned} \tag{3.22}$$

where

$$M = M_2 + \frac{(\beta-1)M_3}{\beta} + \frac{(m-1)M_1}{m}.$$

Since $\alpha > \beta \geqslant 2$, we have

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x \leqslant C(\Omega)\bigg(\int_\Omega |\nabla u|^\alpha\,\mathrm{d}x\bigg)^{2/\alpha},$$
$$\int_\Omega |\nabla u|^\beta\,\mathrm{d}x \leqslant C(\Omega)\bigg(\int_\Omega |\nabla u|^\alpha\,\mathrm{d}x\bigg)^{\beta/\alpha},$$

where $C(\Omega)$ is some positive constant depending on Ω only. We then use the embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^p(\Omega)$ and (3.11) to obtain

$$H^r(t)\int_\Omega |\nabla u|^2\,\mathrm{d}x \leqslant C(\Omega)\bigg(\frac{B_1^p\,\mathrm{d}_1}{\alpha}\bigg)^r\bigg(\int_\Omega |\nabla u|^\alpha\,\mathrm{d}x\bigg)^{(pr+2)/\alpha}, \tag{3.23}$$

$$H^{r(\beta-1)}(t)\int_\Omega |\nabla u|^\beta\,\mathrm{d}x \leqslant C(\Omega)\bigg(\frac{B_1^p d_1}{\alpha}\bigg)^{r(\beta-1)}\bigg(\int_\Omega |\nabla u|^\alpha\,\mathrm{d}x\bigg)^{(pr(\beta-1)+\beta)/\alpha}, \tag{3.24}$$

$$H^{r(m-1)}(t)\|u\|_p^m \leqslant C(\Omega)\bigg(\frac{B_1^p d_1}{\alpha}\bigg)^{r(m-1)}B_1^m\bigg(\int_\Omega |\nabla u|^\alpha\,\mathrm{d}x\bigg)^{(pr(m-1)+m)/\alpha}. \tag{3.25}$$

Now, exploiting the relation in [17],

$$z^\xi \leqslant (z+1) \leqslant \bigg(1+\frac{1}{\eta}\bigg)(z+\eta), \tag{3.26}$$

which holds for all $z \geqslant 0$, $0 < \xi \leqslant 1$ and $\eta > 0$, then, taking $\eta = H(0)$ and using (3.10) and (3.21), we have the following:

$$\left(\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x\right)^{(pr+2)/\alpha} \leqslant a_4\left(\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x + H(t)\right), \tag{3.27}$$

$$\left(\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x\right)^{(pr(\beta-1)+\beta)/\alpha} \leqslant a_4\left(\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x + H(t)\right), \tag{3.28}$$

$$\left(\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x\right)^{(pr(m-1)+m)/\alpha} \leqslant a_4\left(\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x + H(t)\right), \tag{3.29}$$

for all $t \geqslant 0$ and $a_4 = 1 + 1/H(0)$. Inserting (3.23)–(3.25) and (3.27)–(3.29) into (3.22), we see that

$$\begin{aligned}\Phi'(t) \geqslant a_2\|u_t\|_l^l + d_2\|u\|_p^p &+ \left(\alpha - \frac{C_1}{M_1^{m-1}} - \frac{C_2}{M_2} - \frac{C_3}{M_3^{\beta-1}}\right)H(t) \\ &- \left(\frac{C_1}{M_1^{m-1}} + \frac{C_2}{M_2} + \frac{C_3}{M_3^{\beta-1}}\right)\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x - MH^{-r}(t)H'(t),\end{aligned} \tag{3.30}$$

where

$$C_1 = \frac{a_3 a_5}{m}\left(\frac{B_1^p \,\mathrm{d}_1}{\alpha}\right)^{r(m-1)} B_1^m, \qquad C_2 = \frac{a_5}{4}\left(\frac{B_1^p \,\mathrm{d}_1}{\alpha}\right)^{r}, \qquad C_3 = \frac{a_5}{\beta}\left(\frac{B_1^p \,\mathrm{d}_1}{\alpha}\right)^{r(\beta-1)}$$

and

$$a_5 = a_4 C(\Omega).$$

Now, we define

$$L(t) = H(t)^{1-r} + \delta_1\Phi(t), \quad t \geqslant 0, \tag{3.31}$$

where δ_1 is a positive constant to be chosen later. Differentiating (3.31) and then using (3.30), (3.1), $E(t) = E_2 - H(t)$ by (3.9), (3.6) and (3.7) to obtain

$$\begin{aligned}L'(t) &\geqslant (1 - r - \delta_1 M)H^{-r}(t)H'(t) + \left(\delta_1 a_2 + \frac{k(l-1)}{l}\right)\|u_t\|_l^l + \delta_1 \,\mathrm{d}_2\|u\|_p^p \\ &\quad - kE_2 + k\mathcal{A}(u) - k\mathcal{F}(u) + \delta_1\left(\frac{k}{\delta_1} + \alpha - \left(\frac{C_1}{M_1^{m-1}} + \frac{C_2}{M_2} + \frac{C_3}{M_3^{\beta-1}}\right)\right)H(t) \\ &\quad - \delta_1\left(\frac{C_1}{M_1^{m-1}} + \frac{C_2}{M_2} + \frac{C_3}{M_3^{\beta-1}}\right)\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x \\ &\geqslant (1 - r - \delta_1 M)H^{-r}(t)H'(t) + \left(\delta_1 a_2 + \frac{k(l-1)}{l}\right)\|u_t\|_l^l \\ &\quad + \left(\delta_1 \,\mathrm{d}_2 - \frac{kd_1}{p}\right)\|u\|_p^p + \delta_1\left(\frac{k}{\delta_1} + \alpha - \left(\frac{C_1}{M_1^{m-1}} + \frac{C_2}{M_2} + \frac{C_3}{M_3^{\beta-1}}\right)\right)H(t) \\ &\quad + \delta_1\left(\frac{k}{\alpha\delta_1}\left(a_0 - \frac{\mu}{\mu_0}\right) - \left(\frac{C_1}{M_1^{m-1}} + \frac{C_2}{M_2} + \frac{C_3}{M_3^{\beta-1}}\right)\right)\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x - kE_2,\end{aligned}$$

where k is some positive constant. We also observe from the definition of $\tilde{E_1}$ and the definition of w_2 and (3.7) that

$$E_2 < \tilde{E_1} = \left(\frac{p}{\alpha} - 1\right) w_2 \leqslant \left(\frac{p}{\alpha} - 1\right) \mathcal{F}(u) \leqslant \left(\frac{p}{\alpha} - 1\right) \frac{d_1}{p} \|u\|_p^p.$$

Thus, by choosing $k = \alpha d_2 \delta_1 / 2d_1$, we note that

$$\begin{aligned}\left(\delta_1 d_2 - \frac{kd_1}{p}\right)\|u\|_p^p - kE_2 &> \left(\delta_1\, \mathrm{d}_2 - \frac{kd_1}{p}\right)\|u\|_p^p - \frac{k(p-\alpha)}{\alpha}\frac{d_1}{p}\|u\|_p^p \\ &= \tfrac{1}{2} d_2 \delta_1 \|u\|_p^p \\ &\geqslant 0.\end{aligned}$$

Therefore,

$$\begin{aligned} L'(t) \geqslant{} & (1 - r - \delta_1 M) H^{-r}(t) H'(t) + a_6 \|u_t\|_l^l \\ & + \delta_1 \left(\frac{d_2 \alpha}{2d_1} + \alpha - \left(\frac{C_1}{M_1^{m-1}} + \frac{C_2}{M_2} + \frac{C_3}{M_3^{\beta-1}}\right)\right) H(t) \\ & + \delta_1 \left(\frac{d_2(a_0 - \mu/\mu_0)}{2d_1} - \left(\frac{C_1}{M_1^{m-1}} + \frac{C_2}{M_2} + \frac{C_3}{M_3^{\beta-1}}\right)\right) \int_\Omega |\nabla u|^\alpha \,\mathrm{d}x, \end{aligned}$$

where $a_6 = \delta_1 a_2 + d_2 \delta_1 \alpha (l-1)/2d_1 l$. Now, we take M_1, M_2 and M_3 large enough such that

$$L'(t) \geqslant (1 - r - \delta_1 M) H^{-r}(t) H'(t) + a_7 \delta_1 \left(H(t) + \|u_t\|_l^l + \int_\Omega |\nabla u|^\alpha \,\mathrm{d}x\right),$$

where a_7 is a positive constant. Once M_1, M_2 and M_3 are fixed, pick δ_1 sufficiently small such that

$$1 - r - \delta_1 M > 0$$

and

$$L(0) = H^{1-r}(0) + \delta_1 \int_\Omega u_0 |u_1|^{l-2} u_1 \,\mathrm{d}x > 0.$$

Hence

$$L'(t) \geqslant a_7 \delta_1 \left(H(t) + \|u_t\|_l^l + \int_\Omega |\nabla u|^\alpha \,\mathrm{d}x\right) > 0 \tag{3.32}$$

and

$$L(t) > L(0) > 0 \quad \text{for all } t \geqslant 0.$$

Setting $\theta_1 = 1/(1-r) > 1$ by (3.21), it is evident that, by Young's inequality and the Hölder inequality, (3.31) takes the form

$$\begin{aligned} L(t)^{\theta_1} &\leqslant 2^{\theta_1 - 1}\left[H(t) + \left(\delta_1 \int_\Omega u |u_t|^{l-2} u_t \,\mathrm{d}x\right)^{\theta_1}\right] \\ &\leqslant 2^{\theta_1 - 1}[H(t) + \delta_1^{\theta_1} \|u\|_l^{\theta_1} \|u_t\|_l^{(l-1)\theta_1}] \\ &\leqslant a_8 [H(t) + \|u\|_l^{\theta_1 \mu} + \|u_t\|_l^{(l-1)\theta_1 \nu}], \end{aligned}$$

where a_8 is some positive constant and $1/\mu+1/\nu=1$. Taking $(l-1)\theta_1\nu=l$ (hence $\mu=(1-r)l/(1-lr)$) to give

$$L(t)^{\theta_1}\leqslant a_8[H(t)+\|u\|_l^{l/(1-lr)}+\|u_t\|_l^l]. \tag{3.33}$$

Using

$$z^\eta\leqslant(z+1)\leqslant(1+1/\eta)(z+\eta)$$

once more, with $z=\|\nabla u\|_\alpha^\alpha$, $\xi=l/((1-lr)\alpha)<1$ by (3.21) and $\eta=H(0)$, we obtain

$$\begin{aligned}\|u\|_l^{l/(1-lr)}&\leqslant B_1^{l/(1-lr)}(\|\nabla u\|_\alpha^\alpha)^{l/((1-lr)\alpha)}\\&\leqslant B_1^{l/(1-lr)}\left(1+\frac{1}{H(0)}\right)(\|\nabla u\|_\alpha^\alpha+H(0))\\&\leqslant a_9(\|\nabla u\|_\alpha^\alpha+H(t)),\end{aligned}$$

where $a_9=B_1^{l/(1-lr)}(1+1/H(0))$. Consequently, (3.33) becomes

$$L(t)^{\theta_1}\leqslant a_{10}\left(H(t)+\|u_t\|_l^l+\int_\Omega|\nabla u|^\alpha\,\mathrm{d}x\right), \tag{3.34}$$

where a_{10} is some positive constant. Combining (3.32) and (3.34), we have

$$L'(t)\geqslant a_{11}L(t)^{\theta_1},\quad t\geqslant 0, \tag{3.35}$$

where $a_{11}=a_7\delta_1/a_{10}$. An integration of (3.35) over $(0,t)$ yields

$$L(t)\geqslant(L(0)^{1-\theta_1}-a_{11}(\theta_1-1)t)^{-1/(\theta_1-1)}. \tag{3.36}$$

Since $L(0)>0$, (3.36) shows that L cannot be global. This completes the proof. □

Remark 3.7. If $E(0)<\tilde{E}_1$, then, by (3.6), lemma 3.5 and the definition of w_1 by (3.5),

$$\begin{aligned}\mathcal{A}(u)&\geqslant\frac{1}{\alpha}\left(a_0-\frac{\mu}{\mu_0}\right)\int_\Omega|\nabla u|^\alpha\,\mathrm{d}x\\&>\frac{1}{\alpha}\left(a_0-\frac{\mu}{\mu_0}\right)\lambda_1^\alpha\\&=w_1.\end{aligned} \tag{3.37}$$

Hence, using (3.1) and since $E(t)$ is non-increasing by (3.8), we have

$$\mathcal{F}(u)\geqslant\mathcal{A}(u)-E(0)>w_1-\tilde{E}_1.$$

This yields that

$$\tilde{E}_1=\left(\frac{p}{\alpha}-1\right)w_2>\left(1-\frac{\alpha}{p}\right)w_1=E_1,$$

thus, we improve the non-existence result of [16] from the region

$$\Sigma=\{(\lambda,E)\mid\lambda>\lambda_1,\ E<E_1\}\quad\text{to}\quad\tilde{\Sigma}=\{(\lambda,E)\mid\lambda>\lambda_1,\ E<\tilde{E}_1\}$$

(see figure 1).

Lemma 3.8. *If $u \in U$ is a global solution of (1.1) with $E(0) \leqslant E_1$, then $w_2 \leqslant (\alpha/p)w_1$.*

Proof. If not, then $w_2 > (\alpha/p)w_1$ and so

$$\tilde{E_1} = \left(\frac{p}{\alpha} - 1\right)w_2 > \left(1 - \frac{\alpha}{p}\right)w_1 = E_1 \geqslant E(0).$$

Thus, by theorem 3.6, u could not be a global solution of (1.1). □

To state our next result, dealing with the case $E(0) = E_1$, we need a supplementary assumption:

(A_4) there exists $t^* > 0$ such that $\phi \in U$ and

$$\int_\Omega Q(x, t, \phi_t)\phi_t \, dx = 0 \quad \text{in } [0, t^*]$$

implies that $\phi_t(t, \cdot) = 0$ for all $t \in [0, t^*]$.

Theorem 3.9. *Let u be a solution of (1.1) and suppose that* (A_1)–(A_4) *hold. Then, if the initial data satisfy $\|\nabla u(0)\|_\alpha > \lambda_1$ and $E(0) = E_1$, the solution u cannot be continued for all $t > 0$.*

Proof. Assume by contradiction that $u \in U$ is a global solution of (1.1) in $R_0^+ \times \Omega$. Then, by lemma 3.8, we get $w_2 \leqslant (\alpha/p)w_1$. First, we claim that $w_2 < (\alpha/p)w_1$ cannot occur. Otherwise there exists t_0 such that $\mathcal{F}(u(t_0)) < (\alpha/p)w_1$. Hence, by (3.1) and (3.6), we have

$$\begin{aligned} w_1 - \mathcal{F}(u(t_0)) > \left(1 - \frac{\alpha}{p}\right)w_1 &= E_1 = E(0) \\ &\geqslant E(t_0) \geqslant \mathcal{A}(u(t_0)) - \mathcal{F}(u(t_0)) \\ &\geqslant \frac{1}{\alpha}\left(a_0 - \frac{\mu}{\mu_0}\right)\|\nabla u(t_0)\|_\alpha^\alpha - \mathcal{F}(u(t_0)), \end{aligned}$$

which implies $\|\nabla u(t_0)\|_\alpha < \lambda_1$. Thus, $t_0 > 0$ and, by continuity of $\|\nabla u(t)\|_\alpha$, there exists $s \in (0, t_0)$ such that $\|\nabla u(s)\|_\alpha = \lambda_1$. The above argument and (3.5)–(3.7) show that

$$\begin{aligned} E_1 = E(0) \geqslant E(s) &\geqslant \mathcal{A}(u(s)) - \mathcal{F}(u(s)) \\ &\geqslant \frac{1}{\alpha}\left(a_0 - \frac{\mu}{\mu_0}\right)\|\nabla u(s)\|_\alpha^\alpha - \frac{d_1 B^p}{p}\|\nabla u(s)\|_\alpha^p \\ &= E_1, \end{aligned}$$

thus $E(s) = E_1$. On the other hand, it follows from (3.8) that

$$\begin{aligned} E(s) = E(0) - \int_0^s \int_\Omega |\nabla u_t|^2 \, dx \, dt - \int_0^s \int_\Omega |\nabla u_t|^\beta \, dx \, dt \\ - \int_0^s \int_\Omega Q(x, t, u_t)u_t \, dx \, dt. \end{aligned} \tag{3.38}$$

https://doi.org/10.1017/S0308210510001125 Published online by Cambridge University Press

This implies that

$$\int_0^s \int_\Omega Q(x,t,u_t)u_t \,\mathrm{d}x\,\mathrm{d}t = 0$$

due to $E_1 = E(0) = E(s)$ and so, by (2.1) and (A_4), $u_t(t,\cdot) = 0$, for all $t \in [0, s_0]$, $s_0 = \min\{t^*, s\}$. Thus u is constant with respect to t in $[0, s_0]$ and so $u(t,x) = u_0(x)$, for all $t \in [0, s_0]$. Multiplying (1.1) by $u_0(x)$ and integrating it over $(0,t) \times \Omega$, $t \in [0, s_0]$, we obtain

$$\int_0^t \int_\Omega \operatorname{div}(a(x)|\nabla u_0|^{\alpha-2}\nabla u_0)u_0 \,\mathrm{d}x\,\mathrm{d}t \\ - \int_0^t \int_\Omega Q(x,t,0)u_0 \,\mathrm{d}x\,\mathrm{d}t + \int_0^t \int_\Omega f(x,u_0)u_0 \,\mathrm{d}x\,\mathrm{d}t = 0,$$

where we use $u(t,x) = u_0(x)$, for all $t \in [0, s_0]$. Yet $Q(x,t,0) = 0$ for all $t \in [0, s_0]$ by (2.1), and thus

$$\int_\Omega a(x)|\nabla u_0|^\alpha \,\mathrm{d}x = \int_\Omega f(x,u_0)u_0 \,\mathrm{d}x,$$

for all $t \in [0, s_0]$. After that, employing (3.2), (2.4) and (3.3), we obtain

$$\begin{aligned} \alpha\mathcal{A}(u_0) &= \int_\Omega a(x)|\nabla u_0|^\alpha \,\mathrm{d}x - \mu\|u_0\|_\alpha^\alpha \\ &= \int_\Omega f(x,u_0)u_0 \,\mathrm{d}x - \mu\|u_0\|_\alpha^\alpha \\ &\geqslant (p-\varepsilon)\int_\Omega F(x,u_0)\,\mathrm{d}x + d_2\|u_0\|_p^p - \mu\|u_0\|_\alpha^\alpha \\ &= (p-\varepsilon)\biggl(\int_\Omega F(x,u_0)\,\mathrm{d}x - \frac{\mu}{\alpha}\|u_0\|_\alpha^\alpha\biggr) + \biggl(\frac{p-\varepsilon}{\alpha} - 1\biggr)\mu\|u_0\|_\alpha^\alpha + d_2\|u_0\|_p^p \\ &\geqslant (p-\varepsilon)\mathcal{F}(u_0), \end{aligned}$$

for some small $\varepsilon \in (0, \varepsilon_0)$. Hence,

$$\begin{aligned} E_1 = E(0) &= \mathcal{A}(u_0) - \mathcal{F}(u_0) \\ &\geqslant \mathcal{A}(u_0) - \frac{\alpha}{p}\mathcal{A}(u_0) \\ &> E_1, \end{aligned}$$

because of $u_t(0,\cdot) = 0$, $\mathcal{A}(u_0) > 1/(a_0 - \mu/\mu_0)\lambda_1^\alpha = w_1$ by (3.37) and $(1-\alpha/p)w_1 = E_1$. This is a contradiction. Thus, $w_2 = (\alpha/p)w_1$. In particular, $\mathcal{F}(u(t)) \geqslant (\alpha/p)w_1$ for all $t \in R_0^+$. We assert that the equality cannot occur at a finite time. Indeed, if there is s such that $\mathcal{F}(u(s)) = (\alpha/p)w_1$, then, by (3.7),

$$\frac{\alpha}{p}w_1 = \mathcal{F}(u(s)) \leqslant \frac{d_1 B_1^p}{p}\|\nabla u(s)\|_\alpha^p,$$

and so $\|\nabla u(s)\|_\alpha \geqslant \lambda_1$. But $\|\nabla u(s)\|_\alpha > \lambda_1$ would imply

$$\begin{aligned} E(0) &\geqslant \mathcal{A}(u(s)) - \mathcal{F}(u(s)) \\ &\geqslant \frac{1}{\alpha}\left(a_0 - \frac{\mu}{\mu_0}\right)\|\nabla u(s)\|_\alpha^\alpha - \frac{\alpha w_1}{p} \\ &> E_1. \end{aligned}$$

This contradicts $E(0) = E_1$. Hence $(\alpha/p)w_1 = \mathcal{F}(u(s))$, $\|\nabla u(s)\|_\alpha = \lambda_1$ and so $E(s) = E_1$. We can repeat the argument above in correspondence at such s and assumption (A_4) to get contradiction again.

It therefore remains to consider the case where

$$w_2 = (\alpha/p)w_1, \qquad \mathcal{F}(u(t)) > w_2, \qquad \|\nabla u(s)\|_\alpha > \lambda_1$$

for all $t \in R_0^+$. A continuity shows that

$$\liminf_{t\to\infty} \mathcal{F}(u(t)) = w_2.$$

On the other hand, by (3.1) and (3.8), we have

$$w_1 - \mathcal{F}(u) < E(t) \leqslant E(0) = E_1,$$

so that $\limsup_{t\to\infty} E(t) = E_1$. Hence,

$$\int_0^\infty \int_\Omega Q(x,t,u_t)u_t \,\mathrm{d}x\,\mathrm{d}t = 0$$

by (3.38). In particular,

$$\int_\Omega Q(x,t,u_t)u_t \,\mathrm{d}x = 0 \quad \text{in } R_0^+,$$

which is again impossible by using the argument already produced. This completes the proof. □

Acknowledgements

The authors thank the anonymous referees for their valuable comments and useful suggestions on this work.

References

1 G. Andrews. On the existence of solutions to the equation $u_{tt} - u_{xtx} = \sigma(u_x)_x$. *J. Diff. Eqns* **35** (1980), 200–231.

2 G. Andrews and J. M. Ball. Asymptotic behavior and changes in phase in one-dimensional nonlinear viscoelasticity. *J. Diff. Eqns* **44** (1982), 306–341.

3 D. D. Ang and A. P. N. Dinh. Strong solutions of quasilinear wave equation with nonlinear damping. *SIAM J. Math. Analysis* **19** (1988), 337–347.

4 G. Autori, P. Pucci and M. C. Salvatori. Global non-existence for nonlinear Kirchhoff systems. *Arch. Ration. Mech. Analysis* **196** (2010), 489–516.

5 J. M. Ball. Finite time blow-up in nonlinear problems. In *Nonlinear evolution equations* (ed. M. G. Grandall), pp. 189–205 (New York: Academic Press).

6 V. Georgiev and D. Todorova. Existence of solutions of the wave equations with nonlinear damping and source terms. *J. Diff. Eqns* **109** (1994), 295–308.

7 R. T. Glassey. Blow-up theorems for nonlinear wave equations. *Math. Z.* **132** (1973), 183–203.

8 A. Haraux and E. Zuazua. Decay estimates for some semilinear damped hyperbolic problems. *Arch. Ration. Mech. Analysis* **150** (1988), 191–206.

9 S. Kavashima and Y. Shibata. Global existence and exponential stability of small solutions to nonlinear viscoelasticity. *Commun. Math. Phys.* **148** (1992), 189–208.

10 M. Kopackova. Remarks on bounded solutions of a semilinear dissipative hyperbolic equation. *Commentat. Math. Univ. Carolinae* **30** (1989), 713–719.

11 H. A. Levine. Instability and non-existence of global solutions of nonlinear wave equation of the form $Du_{tt} = Au + F(u)$. *Trans. Am. Math. Soc.* **192** (1974), 1–21.

12 H. A. Levine. Some additional remarks on the non-existence of global solutions to nonlinear wave equations. *SIAM J. Math. Analysis* **5** (1974), 138–146.

13 H. A. Levine and J. Serrin. Global non-existence theorems for quasilinear evolutions equations with dissipation. *Arch. Ration. Mech. Analysis* **137** (1997), 341–361.

14 W. J. Liu. Partial exact controllability for the linear thermo-viscoelastic model. *Elec. J. Diff. Eqns* **17** (1998), 1–11.

15 W. J. Liu and M. X. Wang. Blow-up of solutions for a p-Laplacian equation with positive initial energy. *Acta Appl. Math.* **103** (2008), 141–146.

16 W. J. Liu and M. X. Wang. Global non-existence of solutions with positive initial energy for a class of wave equations. *Math. Meth. Appl. Sci.* **32** (2009), 594–605.

17 S. A. Messaoudi and B. S. Houari. Global non-existence of solutions of a class of wave equations with nonlinear damping and source terms. *Math. Meth. Appl. Sci.* **27** (2004), 1687–1696.

18 F. Q. Sun and M. X. Wang. Non-existence of global solutions for nonlinear strongly damped hyperbolic systems. *Discrete Contin. Dynam. Syst.* **12** (2005), 949–958.

19 F. Q. Sun and M. X. Wang. Global and blow-up solutions for a system of nonlinear hyperbolic equations with dissipative terms. *Nonlin. Analysis* **64** (2006), 739–761.

20 M. Tsutsumi. On solutions of semilinear differential equations in a Hilbert space. *Math. Japon.* **17** (1972), 173–193.

21 E. Vitillaro. Global non-existence theorems for a class of evolution equations with dissipation. *Arch. Ration. Mech. Analysis* **149** (1999), 155–182.

22 Y. Yamada. Some remarks on the equation $Y_{tt} - \sigma(Y_x)Y_{xx} - Y_{xtx} = f$. *Osaka J. Math.* **17** (1980), 303–323.

23 Z. J. Yang. Existence and asymptotic behaviour of solutions for a class of quasilinear evolution equations with nonlinear damping and source terms. *Math. Meth. Appl. Sci.* **25** (2002), 795–814.

24 Z. J. Yang. Blowup of solutions for a class of nonlinear evolution equations with nonlinear damping and source terms. *Math. Meth. Appl. Sci.* **25** (2002), 825–833.

25 Z. J. Yang and G. W. Chen. Global existence of solutions for quasilinear wave equations with viscous damping. *J. Math. Analysis Applic.* **285** (2003), 604–618.

(*Issued 5 August 2011*)