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DISTINCT SOLUTIONS TO GENERATED JACOBIAN EQUATIONS CANNOT INTERSECT

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Abstract

We prove that if two $C^{1,1}(\Omega)$ solutions of the second boundary value problem for the generated Jacobian equation intersect in Ω then they are the same solution. In addition, we extend this result to $C^2(\overline{\Omega})$ solutions intersecting on the boundary, via an additional convexity condition on the target domain.

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1. Introduction

The prescribed Jacobian equation coupled with the second boundary value problem arises in optimal transport and geometric optics. These equations, with their boundary condition, take the form

$$\det DY(\cdot, u, Du) = \frac{f(\cdot)}{f^*(Y(\cdot, u, Du))} \quad \text{in } \Omega, \tag{1.1}$$

$$Y(\cdot, u, Du)(\Omega) = \Omega^*, \tag{1.2}$$

where $Y : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, and the functions f, f^* are positive densities on the prescribed domains $\Omega, \Omega^* \subset \mathbf{R}^n$. Such equations have not been profitably studied without additional structure on Y. In this paper we require that Y arise from a generating function and thus work in the framework of generated Jacobian equations (GJE), which were introduced by Trudinger [10]. Since Y depends on u in an unknown way we no longer have uniqueness of solutions (even up to a constant). In this paper we prove a version of a uniqueness result: that distinct solutions cannot intersect at any point in the domain.

THEOREM 1.1. *Suppose that g is a generating function on Γ satisfying properties A1, A1* and A2, specified below, and f, $f^* > 0$ are C^1 and satisfy the mass balance condition (2.7). Suppose that $u, v \in C^{1,1}(\Omega)$ are g-convex generalised solutions of (1.1) subject to (1.2). If there is $x_0 \in \Omega$ such that $u(x_0) = v(x_0)$, then $u \equiv v$ in Ω.*

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Our plan is as follows. In Section 2 we introduce the theory of generating functions and the definitions required to understand the statement of Theorem 1.1. In Section 3 we prove, using a lemma of Alexandrov, that wherever solutions intersect they have the same gradient. We show in Section 4 a weak Harnack inequality that we use in Section 5 to prove that solutions intersecting in the interior of Ω are the same. Finally, in Section 6, we give conditions which yield the same result when $x_0 \in \partial\Omega$.

2. Generated Jacobian equations and g-convexity

The following framework is standard for GJE and mirrors [7]. Further details on GJE may also be found in [5, 6]. Let $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ be a domain for which the projections

$$I(x, y) := \{z \in \mathbf{R} : (x, y, z) \in \Gamma\},$$

are (possibly empty) open intervals. We consider a function $g \in C^4(\Gamma)$ which we assume satisfies the following properties.

A1 For each (x, u, p) in $\mathcal{U}$, which is defined by

$$\mathcal{U} = \{(x, g(x, y, z), g_x(x, y, z)) : (x, y, z) \in \Gamma\},$$

there exists a unique $(x, y, z) \in \Gamma$ such that

$$g(x, y, z) = u, \quad g_x(x, y, z) = p.$$

A1* For each fixed (y, z), the mapping

$$x \to \frac{-g_y}{g_z}(x, y, z),$$

is one to one.

A2 $g_z < 0$ and

$$E := g_{x,y} - (g_z)^{-1} g_{x,z} \otimes g_y,$$

satisfies $\det E \neq 0$.

Assumption A1 allows us to define mappings $Y : \mathcal{U} \to \mathbf{R}^n$ and $Z : \mathcal{U} \to \mathbf{R}$ by the requirement that they uniquely solve

$$g(x, Y(x, u, p), Z(x, u, p)) = u, \tag{2.1}$$

$$g_x(x, Y(x, u, p), Z(x, u, p)) = p. \tag{2.2}$$

Herein we assume that Y is the mapping appearing in (1.1) and (1.2). This assumption allows us to rewrite (1.1) as a Monge–Ampère type equation as follows. Setting $u = u(x)$, $p = Du(x)$ and differentiating (2.1) with respect to the jth coordinate yields

$$g_{x_j} + g_{y_k} D_j Y^k + g_z D_j Z = D_j u,$$

and, since $g_x = Du$,

$$D_j Z = -\frac{1}{g_z} g_{y_k} D_j Y^k. \tag{2.3}$$

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Similarly differentiating (2.2) yields

$$g_{x_i,x_j} + g_{x_i,y_k} D_j Y^k + g_{x_i,z} D_j Z = D_{ij} u. \tag{2.4}$$

We substitute (2.3) into (2.4) and obtain

$$\Big(g_{x_i,y_k} - \frac{1}{g_z} g_{x_i,z} g_{y_k}\Big) D_j Y^k = D_{ij} u - g_{x_i x_j}.$$

Thus, with E as defined in A2,

$$DY(x, u, Du) = E^{-1}[D^2 u - g_{xx}(x, Y(x, u, Du), Z(x, u, Du))]$$

and we rewrite (1.1) as

$$\det[D^2 u - A(\cdot, u, Du)] = B(\cdot, u, Du), \tag{2.5}$$

where

$$A(\cdot, u, Du) = g_{xx}(\cdot, Y(\cdot, u, Du), Z(\cdot, u, Du)),$$
$$B(\cdot, u, Du) = \det E \frac{f(\cdot)}{f^*(Y(\cdot, u, Du))}.$$

The partial differential equation (PDE) (2.5) is degenerate elliptic when $D^2 u \geq g_{xx}$.

The assumptions on g allow for the introduction of a convexity theory where g plays the role of a supporting hyperplane. A function $u : \Omega \to \mathbf{R}$ is called g-convex if for every $x_0 \in \Omega$ there exists y_0, z_0 such that

$$u(x_0) = g(x_0, y_0, z_0), \tag{2.6}$$
$$u(x) \geq g(x, y_0, z_0),$$

for all $x \in \Omega$. We call $g(\cdot, y_0, z_0)$ a g-support at x_0.

Suppose that u is a differentiable g-convex function and $g(\cdot, y_0, z_0)$ is a g-support at x_0. Then $x \mapsto u(x) - g(x, y_0, z_0)$ has a minimum at x_0. Hence $Du(x_0) = g_x(x_0, y_0, z_0)$ which, with (2.6), implies via (2.1) and (2.2) that $y_0 = Y(x_0, u(x_0), Du(x_0))$. Furthermore, if u is C^2 then $D^2 u - g_{xx}$ is nonnegative definite and the equation is degenerate elliptic.

We work with generalised solutions. A definition of generalised solution exists for functions which are merely g-convex [10]. However. our results rely on differentiability so we give the definition of a differentiable g-convex generalised solution. A differentiable g-convex function $u : \Omega \to \mathbf{R}$ is called a generalised solution of (1.1) if, for every $E \subset \Omega$,

$$\int_{Y(\cdot,u,Du)(E)} f^*(y)\, dy = \int_E f(x)\, dx,$$

where f^* is extended to 0 outside Ω^*. If, in addition, $Y(\cdot, u, Du)(\Omega) \subset \overline{\Omega^*}$ we say that u is a generalised solution of (1.1) subject to (1.2), that is, a generalised solution of the second boundary value problem. Note that under the mass balance condition

$$\int_\Omega f = \int_{\Omega^*} f^*, \tag{2.7}$$

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which is necessary for classical solvability, generalised solutions of the second boundary value problem satisfy

$$Y(\cdot, u, Du)(\Omega) = \overline{\Omega^*} \setminus \mathcal{Z}, \tag{2.8}$$

for some set $\mathcal{Z}$ of Lebesgue measure 0.

Moreover, any generalised solution which is $C^{1,1}(\Omega)$ and thus twice differentiable almost everywhere satisfies both (1.1) and (2.5) almost everywhere in Ω.

3. Solutions have the same gradients where they intersect

In this section we show that generalised solutions of (1.1) subject to (1.2) satisfy $Du \equiv Dv$ on $\{x \in \Omega : u(x) = v(x)\}$. Our main tool is a lemma concerning arbitrary convex functions due to Alexandrov [1] and used by McCann [9, Lemma 13] in the Monge–Ampère case. We adapt McCann's proof to the g-convex case. We use the notation $Y_u(x) = Y(x, u(x), Du(x))$, and similarly for Y_v, Z_u, Z_v.

Lemma 3.1. *Assume that $u, v : \Omega \to \mathbf{R}$ are g-convex and differentiable. Suppose for some $x_0 \in \Omega$ that $u(x_0) = v(x_0)$ and $Du(x_0) \neq Dv(x_0)$. With $\Omega' := \{x \in \Omega : u(x) > v(x)\}$ set $\Xi := Y_v^{-1}(Y_u(\Omega'))$. Then $\Xi \subset \Omega'$ and x_0 is a positive distance from Ξ.*

Proof. We begin by proving the subset assertion. Take $\xi \in \Xi$. The definition of Ξ implies there is $x \in \Omega'$ with $Y_v(\xi) = Y_u(x)$. We claim $Z_u(x) < Z_v(\xi)$. Indeed, were this not the case, $Z_u(x) \geq Z_v(\xi)$ which when combined with $Y_v(\xi) = Y_u(x)$ and $g_z < 0$ yields, for any z,

$$g(z, Y_u(x), Z_u(x)) \leq g(z, Y_v(\xi), Z_v(\xi)).$$

This would imply

$$u(x) = g(x, Y_u(x), Z_u(x)) \leq g(x, Y_v(\xi), Z_v(\xi)) \leq v(x),$$

where the final inequality is because $g(\cdot, Y_v(\xi), Z_v(\xi))$ is a g-support. Since $x \in \Omega'$ this contradiction establishes $Z_u(x) < Z_v(\xi)$. Using this and $g_z < 0$, for any z,

$$u(z) \geq g(z, Y_u(x), Z_u(x)) > g(z, Y_v(\xi), Z_v(\xi)). \tag{3.1}$$

For $z = \xi$ we obtain $u(\xi) > g(\xi, Y_v(\xi), Z_v(\xi)) = v(\xi)$, implying $\xi \in \Omega'$ and establishing the subset relation.

We move on to the distance claim. We suppose to the contrary that there exists a sequence $\{\xi_n\}_{n=1}^{\infty}$ in Ξ with $\xi_n \to x_0$. The definition of Ξ implies that for each ξ_n there exists an $x_n \in \Omega'$ with $Y_v(\xi_n) = Y_u(x_n)$. Now $Du(x_0) \neq Dv(x_0)$ implies in any neighbourhood of x_0 that there is a particular z for which

$$u(z) < g(z, Y_v(x_0), Z_v(x_0)). \tag{3.2}$$

For if not,

$$u(x_0) = v(x_0) = g(x_0, Y_v(x_0), Z_v(x_0)),$$
$$u(x) \geq g(x, Y_v(x_0), Z_v(x_0)) \text{ in a neighbourhood of } x_0.$$

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This implies that $u(\cdot) - g(\cdot, Y_v(x_0), Z_v(x_0))$ has a local minimum at x_0. Thus

$$Du(x_0) = g_x(x_0, Y_v(x_0), Z_v(x_0)) = Dv(x_0),$$

and this contradiction establishes (3.2).

Our derivation of (3.1) uses only that $x \in \Omega'$ and $\xi \in \Xi$ satisfies $Y_v(\xi) = Y_u(x)$, so (3.1) also holds for x_n and ξ_n. That is, for any z,

$$u(z) > g(z, Y_v(\xi_n), Z_v(\xi_n)). \tag{3.3}$$

Combining (3.2) and (3.3) gives

$$g(z, Y_v(x_0), Z_v(x_0)) > u(z) > g(z, Y_v(\xi_n), Z_v(\xi_n)),$$

which, on sending $\xi_n \to x_0$ yields a contradiction and completes the proof of Lemma 3.1. □

We use this lemma to show that solutions have the same gradient where they intersect.

Corollary 3.2. *Assume the conditions of Theorem 1.1. Then $Du \equiv Dv$ on the set $\{x \in \Omega : u(x) = v(x)\}$.*

Proof. Suppose otherwise. Then there is $x_0 \in \Omega$ with $u(x_0) = v(x_0)$ and $Du(x_0) \neq Dv(x_0)$. This implies that any neighbourhood of x_0 contains a z with $u(z) > v(z)$, which is to say $x_0 \in \partial\Omega' \cap \Omega$. By the previous lemma, for ε sufficiently small, $B_\varepsilon(x_0) \cap \Xi = \emptyset$ and thus $\Xi \subset \Omega' \setminus B_\varepsilon(x_0)$. On the other hand, since $x_0 \in \partial\Omega'$ and u is continuous, $|B_\varepsilon(x_0) \cap \Omega'| > 0$. Hence

$$|Y_v^{-1}(Y_u(\Omega'))| = |\Xi| \leq |\Omega' \setminus B_\varepsilon(x_0)| < |\Omega'|$$

and, since f^* is bounded below, this implies

$$\int_{Y_v^{-1}(Y_u(\Omega'))} f^*(Y_v) \det DY_v \, dx < \int_{\Omega'} f^*(Y_v) \det DY_v \, dx. \tag{3.4}$$

The change of variables formula holds for the mappings Y_u and Y_v even though they may not be diffeomorphisms. The reasoning here is the same reasoning which yields the change of variables formula for the gradient of $C^{1,1}$ convex functions and uses assumption A1* (see [3, Theorem A.31] and [10, Section 4]). In light of this, (3.4) yields the following contradiction:

$$\begin{aligned}
\int_{\Omega'} f(x)\,dx &= \int_{Y_u(\Omega')} f^*(y)\,dy && (3.5)\\
&= \int_{Y_v(Y_v^{-1}(Y_u(\Omega')))} f^*(y)\,dy && (3.6)\\
&= \int_{Y_v^{-1}(Y_u(\Omega'))} f^*(Y_v) \det DY_v \, dy\\
&< \int_{\Omega'} f^*(Y_v) \det DY_v \, dy = \int_{\Omega'} f(x)\,dx.
\end{aligned}$$

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Here the equality between (3.5) and (3.6) follows from the generalised boundary condition in conjunction with (2.8) from which we deduce

$$Y_v(Y_v^{-1}(Y_u(\Omega'))) = Y_v(\Omega) \cap Y_u(\Omega') = Y_u(\Omega') \setminus \mathcal{Z},$$

for some set $\mathcal{Z}$ with Lebesgue measure 0, so that the integrals over these sets are equal. □

4. A weak Harnack inequality

Proposition 4.1. *Suppose that $u, v \in C^{1,1}(\Omega)$ satisfy (1.1) almost everywhere and $u \geq v$ in Ω. Then for any $\tilde{\Omega} \subset \Omega$ there exist $p, C > 0$ such that*

$$\left(\frac{1}{|\tilde{\Omega}|}\int_{\tilde{\Omega}} (u-v)^p\right)^{1/p} \leq C \inf_{\tilde{\Omega}}(u-v). \tag{4.1}$$

Proof. Provided we are able to show that $u - v$ is a supersolution of a homogeneous linear elliptic PDE, this is a consequence of the weak Harnack inequality [4, Theorem 9.22] and a covering argument. To apply the Harnack inequality to $u - v$ we recall that $C^{1,1}(\Omega) \subset W^{2,\infty}_{\text{loc}}(\Omega)$ [2, Theorem 4.5]. We now show that $w := u - v$ satisfies

$$Lw := a^{ij}D_{ij}w + b^k D_k w + cw \leq 0, \tag{4.2}$$

where

$$\begin{aligned} a^{ij} &= [D^2u - A(\cdot, u, Du)]^{ij}, \\ b^i &= -a^{ij}(A_{ij})_{p_k} - \tilde{B}_{p_k}, \\ c &= -a^{ij}(A_{ij})_u - \tilde{B}_u, \end{aligned}$$

and $\tilde{B} = \log B$. From (2.5), almost everywhere,

$$\begin{aligned} 0 = \log\det[D^2v - A(\cdot, v, Dv)] &- \log\det[D^2u - A(\cdot, u, Du)] \\ &+ \tilde{B}(\cdot, u, Du) - \tilde{B}(\cdot, v, Dv). \end{aligned} \tag{4.3}$$

A Taylor series for

$$h(t) := \log\det[t(D^2v - A(\cdot, v, Dv)) + (1-t)(D^2u - A(\cdot, u, Du))]$$

yields

$$h(1) - h(0) = h'(0) + \tfrac{1}{2}h''(\tau),$$

for some τ in $[0, 1]$. Concavity of log det implies $h''(\tau) \leq 0$ and thus, on computing $h'(0)$, we obtain

$$\begin{aligned} \log\det[D^2v - A(\cdot, v, Dv)] &- \log\det[D^2u - A(\cdot, u, Du)] \\ &\leq a^{ij}D_{ij}(v-u) + a^{ij}(A_{ij}(\cdot, u, Du) - A_{ij}(\cdot, v, Dv)), \end{aligned} \tag{4.4}$$

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where $a^{ij} = [D^2u - A(\cdot, u, Du)]^{ij}$. Substituting (4.4) into (4.3) implies

$$0 \le a^{ij}D_{ij}(v-u) + a^{ij}(A_{ij}(\cdot, u, Du) - A_{ij}(\cdot, v, Dv)) + \tilde{B}(\cdot, u, Du) - \tilde{B}(\cdot, v, Dv). \tag{4.5}$$

The mean value theorem yields

$$A_{ij}(\cdot, u, Du) - A_{ij}(\cdot, v, Dv) = A_u(\cdot, w, p)(u-v) + A_{p_k}(\cdot, w, p)D_k(u-v),$$

for some $w = t_1v + (1-t_1)u$ and $p = t_2Dv + (1-t_2)Du$, and there is a similar result for $\tilde{B}(\cdot, u, Du) - \tilde{B}(\cdot, v, Dv)$. Thus (4.5) becomes

$$0 \le a^{ij}D_{ij}(v-u) - \Big(a^{ij}(A_{ij})_{p_k} + \frac{B_{p_k}}{B}\Big)D_k(v-u) - \Big(a^{ij}(A_{ij})_u + \frac{B_u}{B}\Big)(v-u),$$

which is (4.2). (Multiply by -1 since $w = u - v$.) □

5. Solutions intersecting on the interior are the same

Here we provide the proof of Theorem 1.1. The Harnack inequality implies that one solution cannot touch another from above. Now we show that, given two distinct solutions, since their derivatives are equal where they intersect, their maximum is a $C^{1,1}(\Omega)$ solution touching from above—a contradiction.

Proof of Theorem 1.1. At the outset we fix $\tilde{\Omega} \subset \Omega$ containing x_0. Since u, v are g-convex, the same is true for $w := \max\{u, v\}$. Furthermore, $Du \equiv Dv$ on $\{u = v\}$, implies that w is $C^{1,1}(\Omega)$. Thus w solves (2.5) almost everywhere. Hence we may apply our weak Harnack inequality (4.1) to $w - v$ to obtain $w \equiv v$ in $\tilde{\Omega}$. The same argument yields $w \equiv u$ in $\tilde{\Omega}$ and hence $u \equiv v$ in Ω via continuity. □

6. Solutions intersecting on the boundary are the same

We conclude by proving that if solutions intersect on the boundary then they are the same throughout the domain. We require a convexity assumption on the target domain Ω^*. We say that a C^2 connected domain Ω^* is Y^* convex with respect to $x \in \Omega$ and $u \in \mathbf{R}$ provided there exists a defining function $\varphi^* \in C^2(\overline{\Omega^*})$ satisfying

$$\varphi^* < 0 \text{ in } \Omega^* \quad \varphi = 0 \text{ on } \partial\Omega^*$$
$$D_p^2\varphi^*(Y(x, u, p)) \ge 0 \quad |D\varphi| \ne 0 \text{ on } \partial\Omega^*.$$

For a comparison between this and other definitions of domain convexity see [8, Section 2.2]. In the same paper Liu and Trudinger prove that for $C^2(\overline{\Omega})$ solutions and

$$G(x, u, p) := \varphi^*(Y(x, u, p))$$

the boundary condition

$$G(\cdot, u, Du) = 0 \text{ on } \partial\Omega$$

is oblique, that is, it satisfies $G_p \cdot \gamma > 0$ where γ, is the outer unit normal.

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THEOREM 6.1. *Assume the conditions of Theorem 1.1. In addition, assume that $u, v \in C^2(\overline{\Omega})$ are generalised solutions of (1.1) subject to (1.2) and Ω^* is Y^*-convex with respect to each $x \in \Omega$ and an interval containing $u(\Omega) \cup v(\Omega)$. If there is $x_0 \in \partial\Omega$ with $u(x_0) = v(x_0)$ then $u \equiv v$ in $\overline{\Omega}$.*

PROOF. Using Theorem 1.1, it suffices to prove that there is $x \in \Omega$ with $u(x) = v(x)$. For a contradiction suppose that at some x_0 in $\partial\Omega$ we have $u(x_0) = v(x_0)$, yet $u > v$ in Ω. Hopf's lemma [4, Lemma 3.4] yields

$$D_\gamma(u - v)(x_0) < 0. \tag{6.1}$$

Here we used the fact that the linear elliptic inequality (4.2) is uniformly elliptic under the assumption $u \in C^2(\overline{\Omega})$ and that no sign condition is needed on the lowest-order coefficient in (4.2) since $u(x_0) - v(x_0) = 0$.

Consider the function $h(t) := G(x_0, u(x_0), tDv(x_0) + (1 - t)Du(x_0))$. A Taylor series yields

$$h(1) = h(0) + h'(0) + h''(\tau)/2,$$

for some $\tau \in [0, 1]$. Since $u(x_0) = v(x_0)$, we have $h(1), h(0) = 0$. Furthermore, convexity implies $h''(\tau) \geq 0$ and hence

$$0 \geq h'(0) = G_p \cdot D(v - u),$$

or equivalently $0 \leq G_p \cdot D(u - v)$. Combined with obliqueness, this gives

$$D_\gamma(u - v)(x_0) \geq 0,$$

which contradicts (6.1) and thus establishes the result. □

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