Ergod. Th. & Dynam. Sys. (2021), **41**, 1883–1920
doi:10.1017/etds.2020.23

Reducibility of a class of nonlinear quasi-periodic systems with Liouvillean basic frequencies

DONGFENG ZHANG and JUNXIANG XU

School of Mathematics, Southeast University, Nanjing 210096, PR China
(e-mail: zhdf@seu.edu.cn, xujun@seu.edu.cn)

(*Received* 10 *November* 2019 *and accepted in revised form* 3 *February* 2020)

Abstract. In this paper we consider the following nonlinear quasi-periodic system:

$$\dot{x} = (A + \epsilon P(t, \epsilon))x + \epsilon g(t, \epsilon) + h(x, t, \epsilon), \quad x \in \mathbb{R}^d,$$

where A is a $d \times d$ constant matrix of elliptic type, $\epsilon g(t, \epsilon)$ is a small perturbation with ϵ as a small parameter, $h(x, t, \epsilon) = O(x^2)$ as $x \to 0$, and P, g and h are all analytic quasi-periodic in t with basic frequencies $\omega = (1, \alpha)$, where α is irrational. It is proved that for most sufficiently small ϵ, the system is reducible to the following form:

$$\dot{x} = (A + B_*(t))x + h_*(x, t, \epsilon), \quad x \in \mathbb{R}^d,$$

where $h_*(x, t, \epsilon) = O(x^2)$ $(x \to 0)$ is a high-order term. Therefore, the system has a quasi-periodic solution with basic frequencies $\omega = (1, \alpha)$, such that it goes to zero when ϵ does.

Key words: reducibility, quasi-periodic, KAM iteration, Liouvillean frequencies
2010 Mathematics Subject Classification: 37J40 (Primary); 34C27 (Secondary)

1. *Introduction and main results*

Consider the following nonlinear quasi-periodic system:

$$\dot{x} = (A + \epsilon P(t, \epsilon))x + \epsilon g(t, \epsilon) + h(x, t, \epsilon), \quad x \in \mathbb{R}^d, \tag{1}$$

where A is a constant matrix of different purely imaginary and non-zero eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\epsilon g(t, \epsilon)$ is a small perturbation with ϵ as a small parameter, $h = O(x^2)$ as $x \to 0$, and P, g and h are all analytic quasi-periodic in t with basic frequencies $\omega = (\omega_1, \ldots, \omega_s)$.

In [**17**], Jorba and Simó proved that if the eigenvalues λ and basic frequencies ω satisfy the non-resonant conditions

$$|\langle k, \omega\rangle| \geq \frac{\gamma}{|k|^{\tau}}, \tag{2}$$
$$|\sqrt{-1}\langle k, \omega\rangle - \lambda_i| \geq \frac{\gamma}{|k|^{\tau}},$$
$$|\sqrt{-1}\langle k, \omega\rangle - \lambda_i + \lambda_j| \geq \frac{\gamma}{|k|^{\tau}} \quad \text{for all } i \neq j,$$

for all $k \in \mathbb{Z}^s\backslash\{0\}$, $1 \leq i, j \leq d$, where $\gamma > 0$, $\tau > s - 1$, then for most sufficiently small ϵ, system (1) can be reduced to a constant system with high-order terms:

$$\dot{x} = A_* x + h_*(x, t, \epsilon), \quad x \in \mathbb{R}^d,$$

where A_* is a constant matrix close to A, and $h_*(x, t, \epsilon) = O(x^2)$ $(x \to 0)$ is a high-order term close to h. Therefore, system (1) has a quasi-periodic solution with basic frequencies $\omega = (\omega_1, \ldots, \omega_s)$, such that it goes to zero when ϵ does.

Furthermore, a natural question, for Liouvillean basic frequencies $\omega = (1, \alpha)$, where α is irrational, is whether we can obtain the existence of a quasi-periodic solution for the nonlinear quasi-periodic system (1), which means that the Diophantine condition (2) can be eliminated. Moreover, we discuss the reducibility problems for the nonlinear quasi-periodic system (1) with Liouvillean basic frequencies.

Let us first recall some well-known results and development of reduction theory. Consider the system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^d, \tag{3}$$

where $A(t)$ is a $d \times d$ matrix which depends on time in a quasi-periodic way with basic frequencies $\omega = (\omega_1, \ldots, \omega_s)$.

Definition. (Reducibility) System (3) is said to be reducible, if there exists a non-singular quasi-periodic change of variables $x = \Phi(t)y$, such that $\Phi(t)$, $\Phi(t)^{-1}$ and $\dot{\Phi}(t)$ are quasi-periodic and bounded, and such that it transforms the system (3) into a constant system, that is, a linear system with constant coefficient.

Definition. (Rotations reducibility) System (3) is said to be rotations reducible, if there exists a quasi-periodic transformation $x = \Phi(t)y$ such that system (3) is transformed into a rotation system, that is, a linear system with so$(d, \mathbb{R})$-valued coefficients.

Definition. (Non-perturbative reducibility) Non-perturbative reducibility means that the smallness of the perturbation does not depend on the Diophantine constants (γ, τ) of ω in (2).

For $s = 1$ (the periodic case), the classical Floquet theory tells us that there exists a periodic change of variables such that the periodic system (3) can be reducible to a constant system.

For $s > 1$ (the quasi-periodic case), the system is not always reducible. The reducibility of quasi-periodic systems was initiated by Dinaburg and Sinai [**7**], who proved that the linear Schrödinger equation

$$-y'' + q(\omega t)y = Ey, \quad y \in \mathbb{R},$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

or equivalently the two-dimensional quasi-periodic system

$$\dot{y} = x, \quad \dot{x} = (q(\omega t) - E)y, \tag{4}$$

is reducible for most sufficiently large E, when the basic frequencies ω satisfy the Diophantine condition

$$|\langle k, \omega\rangle| \geq \frac{\gamma}{|k|^\tau} \quad \text{for all } 0 \neq k \in \mathbb{Z}^s,$$

where $\gamma > 0$, $\tau > s - 1$ are constants.

The reducibility of the system (4) implies the existence of an absolutely continuous spectrum of the Schrödinger operator $Ly = -d^2y/dt^2 + q(\omega t)y$. Due to its importance in dynamical systems and in the spectral theory of Schrödinger operators, the reducibility of quasi-periodic systems has been extensively investigated.

Liang and Xu [**19**] generalized the results of [**7**] to the high-dimensional case. Johnson and Sell [**15**] proved that if the quasi-periodic coefficients matrix $A(t)$ satisfies the full spectrum condition, then system (3) is reducible. Jorba and Simó [**16**] considered the linear quasi-periodic system

$$\dot{x} = (A + \epsilon P(t, \epsilon))x, \quad x \in \mathbb{R}^d, \tag{5}$$

where A is a constant matrix with different non-zero eigenvalues $\lambda_1, \ldots, \lambda_d$, $P(t)$ is a quasi-periodic matrix with frequencies $\omega = (\omega_1, \ldots, \omega_s)$, and ϵ is a small parameter. They proved that if

$$|\sqrt{-1}\langle k, \omega\rangle - \lambda_i + \lambda_j| \geq \frac{\gamma}{|k|^\tau} \quad \text{for all } 0 \neq k \in \mathbb{Z}^s, i, j = 1, \ldots, d,$$

and

$$\frac{d}{d\epsilon}(\lambda_i^0(\epsilon) - \lambda_j^0(\epsilon))\bigg|_{\epsilon=0} \neq 0, \quad i \neq j,$$

where $\gamma > 0$, $\tau > s - 1$, and $\lambda_i^0(\epsilon)$ $(i = 1, \ldots, d)$ are eigenvalues of $A + \epsilon[P(\epsilon)]$, with $[P(\epsilon)]$ being the average of $P(t, \epsilon)$ with respect to t, then for most sufficiently small parameters ϵ, the system (5) is reducible.

Later, Eliasson [**9**] proved that all quasi-periodic systems are almost reducible provided that the system satisfies the Diophantine condition and is close to constant. Eliasson [**8**] obtained a full measure reducibility result for the quasi-periodic Schrödinger equation. Krikorian [**18**] generalized the full measure reducibility result to linear systems with coefficients in the Lie algebra of the compact semi-simple Lie group. Her and You [**13**] and Chavaudret [**4**] established the full measure reducibility with coefficients in other groups. For the latest reducibility results of infinite-dimensional systems, we refer to [**2**, **3**, **10**] and the references therein.

In developing the reducibility of quasi-periodic systems and Kolmogorov–Arnold–Moser (KAM) theory, many scholars are dedicated to weakening the non-degeneracy condition and the non-resonant condition. Xu [**31**, **33**] obtained the reducibility of linear quasi-periodic system (5) in the case of multiple eigenvalues and more general non-degeneracy conditions, that is,

$$\frac{d^l}{d\epsilon^l}(\lambda_i^0(\epsilon) - \lambda_j^0(\epsilon))\bigg|_{\epsilon=0} \neq 0, \quad l \geq 1, i \neq j.$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

Zhao [**39**] proved the reducibility of nonlinear quasi-periodic system (1), when the eigenvalues of A are allowed to be multiple. Chavaudret [**6**] studied the reducibility of resonant cocycles. Moreover, the Diophantine condition (2) can also be weakened. Rüssmann [**25**] and Zhang and Liang [**34**] obtained the reducibility of the Schrödinger equation under Brjuno and Rüssmann's non-resonant condition:

$$|\langle k, \omega\rangle| \geq \frac{\gamma}{\triangle(|k|)} \quad \text{for all } 0 \neq k \in \mathbb{Z}^s,$$

where $\gamma > 0$, and $\triangle$ is a continuous, increasing, unbounded function $\triangle : [1, \infty) \to [1, \infty)$ such that $\triangle(1) = 1$ and

$$\int_1^\infty \frac{\ln \triangle(t)}{t^2}\, dt < \infty,$$

$\triangle$ is usually called the Brjuno–Rüssmann approximation function. For further KAM theory about Brjuno and Rüssmann's non-resonant condition, see [**5**, **21**, **22**, **24**, **35**–**37**].

In particular, for two-dimensional quasi-periodic systems, there have been some interesting results. Without imposing any non-degeneracy condition, the reducibility of two-dimensional quasi-periodic systems was obtained in [**28**, **32**]. These particular phenomena [**28**, **32**] are inherent in two-dimensional systems, but do not hold for high-dimensional systems. Recently, the reducibility of two-dimensional quasi-periodic systems with Liouvillean frequencies has been obtained, namely, $\omega = (1, \alpha)$, where α is irrational. Avila, Fayad and Krikorian [**1**] first introduced the CD bridge method and proved the rotations reducibility of SL(2, $\mathbb{R}$) cocycles with one frequency, irrespective of any Diophantine condition on the base dynamics. Hou and You [**14**] considered a quasi-periodic linear differential system with two frequencies in $sl(2, \mathbb{R})$,

$$\begin{cases} \dot{x} = A(\theta)x, \\ \dot{\theta} = \omega = (1, \alpha), \end{cases}$$

and obtained almost reducibility and rotations reducibility of the above system, provided that the coefficients are analytic and close to constant. Furthermore, if the rotation number of the system and the basic frequencies $\omega = (1, \alpha)$ satisfy the Diophantine condition, the system is reducible. Wang, You and Zhou [**30**] proved the existence of response solutions for quasi-periodically forced harmonic oscillators with forcing frequencies $\omega = (1, \alpha)$, where α is irrational. For other interesting results for two-dimensional systems, see to [**11**, **26**] and the references therein.

All the above results about Liouvillean frequencies are mainly concerned with two-dimensional or linear quasi-periodic systems. Naturally, in this paper we are mainly concerned with reducibility problems for the nonlinear and high -dimensional quasi-periodic system (1) with Liouvillean basic frequencies $\omega = (1, \alpha)$, where α is irrational.

Since the quasi-periodic systems in [**1**, **14**] are two-dimensional and linear, we can naturally introduce the rotation number and make good use of the rotation number property. But for the high-dimensional system (1) it is difficult to define the rotation number. The proof of [**1**] is based on the 'algebraic conjugacy' technique developed in [**12**]. In the proof of [**14**], the crucial observation is to analyze the structure of resonant terms, then to eliminate them by Floquet theory. Unfortunately, it is difficult to generalize the methods in [**1**, **14**] to nonlinear and high-dimensional problems. Comparing to [**30**], the former is a Hamiltonian system, while our system does not contain any structure, which may include Hamiltonian, reversible and dissipative systems. In fact, we can apply our results to quasi-periodically forced harmonic oscillators and obtain the existence of response solutions in [**30**].

For high-dimensional quasi-periodic systems, Zhou and Wang [**40**] used periodic approximation to study the reducibility of quasi-periodic $GL(d, \mathbb{R})$ cocycles with Liouvillean frequencies. Zhang, Xu and Xu [**38**] obtained the reducibility of a three-dimensional skew-symmetric linear system with Liouvillean basic frequencies. Compared to [**40**], the former is discrete and linear, while our system is continuous and nonlinear, so there are essential obstructions in applying the method of periodic approximation for the discrete case in [**40**] to the continuous case.

In our paper we mainly use the CD bridge method and improved KAM iteration with parameters. One iteration step will be completed by a family of sub-iterations, the sub-iteration steps will go to ∞, but we only need to delete the resonance parameters once in each iteration step (i.e. each family of sub-iterations). For nonlinear quasi-periodic systems, both zero-order and first-order non-resonant terms need to be eliminated, which may lead to more complicated iterations. It is pointed out in particular that in order to maintain the structure of the linear principal part, we need to eliminate all the non-diagonal terms in solving the homological equation, and we need to keep the system real in order to obtain real analytic quasi-periodic solutions. In Appendix A, we apply our theorem to the nonlinear Hill equation with quasi-periodic forcing terms, weakly forced oscillator and damped equation to study the existence of quasi-periodic solutions with Liouvillean frequencies. These three kinds of equations correspond to Hamiltonian systems, reversible systems and dissipative systems, respectively.

Before stating our results, we first give some notation and definitions. We usually denote by $\mathbb{Z}$ and $\mathbb{Z}_+$ the sets of integers and positive integers, respectively. A function $f(t)$ is called quasi-periodic with basic frequencies $\omega_1, \ldots, \omega_s$ if $f(t) = F(\theta) = F(\theta_1, \ldots, \theta_s)$, where F is 2π-periodic in all its arguments and $\theta_j = \omega_j t$ for $j = 1, \ldots, s$. Let $\omega = (\omega_1, \ldots, \omega_s)$. Thus, $f(t) = F(\omega t)$. Denote a strip domain in complex space $\mathbb{C}^s$ by

$$D(r) = \{\theta = (\theta_1, \ldots, \theta_s) \in \mathbb{C}^s/2\pi\mathbb{Z}^s : |\text{Im}\, \theta_j| \le r,\ j = 1, \ldots, s\}.$$

Furthermore, if $F(\theta)$ is analytic with respect to θ on $D(r)$, we say that $f(t)$ is analytic quasi-periodic on $D(r)$. Denote by

$$[f] = \frac{1}{(2\pi)^s} \int_{\mathbb{T}^s} F(\theta)\, d\theta$$

the average of f. Similarly, a function matrix $P(t) = (P_{ij})_{d\times d}$ is called analytic quasi-periodic on $D(r)$ if all $P_{ij}(t)$ are analytic quasi-periodic on it. Denote by $[P] = ([P_{ij}])_{d\times d}$ the average of P.

Let $\epsilon_0 > 0$ and denote $\Pi = (0, \epsilon_0)$. If $f(\epsilon)$ is differentiable with respect to parameters $\epsilon \in \Pi$ in the sense of Whitney, define the norm

$$|f|_\Pi = \sup_{\epsilon\in\Pi}(|f(\epsilon)| + |f'(\epsilon)|).$$

If a function

$$f(t,\epsilon)=\sum_{k\in\mathbb{Z}^s} f_k(\epsilon)e^{i\langle k,\omega\rangle t}$$

is analytic quasi-periodic in t on $D(r)$, and differentiable with respect to $\epsilon\in\Pi$, we define

$$\|f\|_{r,\Pi}=\sum_{k\in\mathbb{Z}^s}|f_k|_\Pi e^{|k|r},$$

where $|k|=|k_1|+\cdots+|k_s|$ for $k=(k_1,\ldots,k_s)$. Denote $\mathcal{A}_r(\Pi)=\{f:\|f\|_{r,\Pi}<+\infty\}$, which is a Banach algebra under norm $\|\cdot\|_{r,\Pi}$.

For any $K>0$, we define the truncating operators

$$\mathcal{T}_K f=\sum_{k\in\mathbb{Z}^s,|k|<K} f_k(\epsilon)e^{i\langle k,\omega\rangle t},$$

$$\mathcal{R}_K f=\sum_{k\in\mathbb{Z}^s,|k|\geq K} f_k(\epsilon)e^{i\langle k,\omega\rangle t}.$$

For a function matrix $P(t,\epsilon)=(P_{ij}(t,\epsilon))_{d\times d}$, similarly define a norm by

$$\|P\|_{r,\Pi}=\max_{1\leq i\leq d}\sum_{j=1}^{d}\|P_{ij}\|_{r,\Pi}.$$

We have $\|P_1P_2\|_{r,\Pi}\leq\|P_1\|_{r,\Pi}\|P_2\|_{r,\Pi}$.

Let $\alpha\in(0,1)$ be irrational and denote by p_n/q_n the nth convergence of α. Define

$$\beta(\alpha)=\limsup_{n\to\infty}\frac{\ln q_{n+1}}{q_n}. \tag{6}$$

Then $\beta(\alpha)$ measures how Liouvillean α is. Notice that $\beta(\alpha)$ has an equivalent definition:

$$\beta(\alpha)=\limsup_{|k|\to\infty}\frac{1}{|k|}\ln\frac{1}{|e^{2\pi ik\alpha}-1|}. \tag{7}$$

Assumption A. (Non-resonant conditions) Suppose that $\lambda=(\lambda_1,\ldots,\lambda_d)$ and $\omega=(1,\alpha)$ satisfy the conditions

$$|\sqrt{-1}\langle k,\omega\rangle-\lambda_i|\geq\frac{\gamma}{(|k|+1)^\tau},$$

$$|\sqrt{-1}\langle k,\omega\rangle-\lambda_i+\lambda_j|\geq\frac{\gamma}{(|k|+1)^\tau}\quad\text{for all } i\neq j,$$

for all $k\in\mathbb{Z}^2$, $1\leq i,j\leq d$, where $\gamma>0$, $\tau>2$.

Assumption B. (Non-degeneracy conditions) Let us denote by $\underline{x}(t,\epsilon)$ the unique analytic quasi-periodic solution of $\dot{x}=Ax+\epsilon g(t,\epsilon)$ (the existence of $\underline{x}(t,\epsilon)$ is shown by [**17**, Lemma 2.10]) and define $\hat{A}(\epsilon)=A+\epsilon\,[P(\epsilon)]+[D_xh(\underline{x}(t,\epsilon),t,\epsilon)]$. Let $\lambda_i^0(\epsilon)$ $(i=1,\ldots,d)$ be the eigenvalues of $\hat{A}$, such that $|(d\lambda_i^0(\epsilon))/d\epsilon|_{\epsilon=0}|\geq 2\delta>0$ and $|(d(\lambda_i^0(\epsilon)-\lambda_j^0(\epsilon)))/d\epsilon|_{\epsilon=0}|\geq 2\delta>0$.

THEOREM 1. *Consider the nonlinear system*

$$\dot{x}=(A+\epsilon P(t,\epsilon))x+\epsilon g(t,\epsilon)+h(x,t,\epsilon),\quad x\in\mathbb{R}^d,$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

where A is a $d \times d$ constant matrix of elliptic type, $\epsilon g(t, \epsilon)$ is a small perturbation with ϵ as a small parameter, $h(x, t, \epsilon)$ is analytic with respect to x on the ball $B_\kappa(0)$ such that $h(0, t, \epsilon) = 0$, $D_x h(0, t, \epsilon) = 0$, $\|D_{xx} h(x, t, \epsilon)\| \le G$, and P, g and h are all analytic quasi-periodic in t with basic frequencies $\omega = (1, \alpha)$, where α is irrational. Suppose that the Assumptions A and B are satisfied.

(I) *Then there exist a sufficiently small ϵ_0, which depends on r, γ and τ, but not on α, and a Cantor set $\Pi_* \subset (0, \epsilon_0)$ with positive Lebesgue measure such that for any $\epsilon \in \Pi_*$, there exists a real analytic quasi-periodic and near-identity transformation $x = e^{u(\omega t)} y + v(\omega t)$, which changes system (1) to*

$$\dot{y} = (A + B_*(t))y + h_*(y, t, \epsilon), \quad y \in \mathbb{R}^d,$$

where $h_*(y, t, \epsilon) = O(y^2)$ $(y \to 0)$ is a high-order term. Therefore, the system (1) has a quasi-periodic solution $x = v(\omega t)$ with basic frequencies $\omega = (1, \alpha)$, such that it goes to zero when ϵ does. Moreover, if ϵ_0 is small enough the relative measure of $(0, \epsilon_0)\backslash\Pi_*$ in $(0, \epsilon_0)$ is less than $c\epsilon_0^{13/15}$, where the constant c is independent of ϵ_0.

(II) *Furthermore, if $\beta(\alpha) = 0$, then for the same ϵ_0 and $\epsilon \in \Pi_*$ in (I), system (1) is reducible to*

$$\dot{y} = A^* y + h^*(y, t, \epsilon), \quad y \in \mathbb{R}^d,$$

where A^* is a constant matrix close to A, and $h^*(y, t, \epsilon) = O(y^2)$ $(y \to 0)$.

If $0 < \beta(\alpha) < r$, then there exists a sufficiently small $\epsilon_0 = \epsilon_0(r, \gamma, \tau, \beta(\alpha))$ and a Cantor set $\Pi_* \subset (0, \epsilon_0)$ with positive Lebesgue measure, such that for any $\epsilon \in \Pi_*$ the system (1) is reducible to

$$\dot{y} = A^* y + h^*(y, t, \epsilon), \quad y \in \mathbb{R}^d,$$

where A^* is a constant matrix close to A, $h^*(y, t, \epsilon) = O(y^2)$ $(y \to 0)$.

Remark. From Theorem 1, for all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, system (1) has a quasi-periodic solution with basic frequencies $\omega = (1, \alpha)$, such that it goes to zero when ϵ does.

Remark. When $\beta(\alpha) = 0$, the smallness of ϵ_0 does not depend on α, therefore, we not only weaken the Diophantine condition (2) to Liouvillean frequencies, but also improve the results in [**17**] to be non-perturbative in the case of two-dimensional basic frequencies. In this sense, when $g = h \equiv 0$, our theorem generalizes partial conclusions of [**14**] to high dimensions.

2. *Outline of the proof*

We now give an outline of the proof of Theorem 1. The details are given in the next sections.

Since A is a constant matrix of elliptic type and $\det A \neq 0$, that is, A has different purely imaginary and non-zero eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_d)$, we suppose that A is in the real Jordan form (9). An essential idea of the proof is to construct a simplifying transformation, consisting of infinitely many successive iteration steps, to eliminate both zero-order and first-order terms, so that the transformed system has zero as an equilibrium. In the process of eliminating lower-order terms, the corresponding small divisors are $\sqrt{-1}\langle k, \omega\rangle - \lambda_i$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

and $\sqrt{-1}\langle k, \omega\rangle - \lambda_i + \lambda_j$. Since the basic frequencies are Liouvillean, the main difficulty is that we cannot kill the terms whose small divisor is $\langle k, \omega\rangle$, that is, the terms on the diagonal. Therefore, the terms on the diagonal will be retained in eliminating the first-order terms. We overcome this problem by putting the terms on the diagonal into the linear principal part. Thus the linear principal part in our work will be variable coefficients, which yields that the homological equation is an equation with variable coefficients. We overcome this problem by the diagonally dominant method to obtain an approximate solution to the homological equation. It is pointed out in particular that in order to maintain the structure of the linear principal part, we need to eliminate all the non-diagonal terms including $k = 0$ and $k \neq 0$.

Therefore, the proof of Theorem 1 is divided into two parts. First, we prove that system (1) can be reduced to a system with non-constant coefficients, which has a diagonal form:

$$\dot{y} = (A + B_*(t))y + h_*(y, t, \epsilon), \quad y \in \mathbb{R}^d,$$

where

$$B_*(t) = \operatorname{diag}\left(\begin{pmatrix} U_1^*(t) & V_1^*(t) \\ -V_1^*(t) & U_1^*(t) \end{pmatrix}, \ldots, \begin{pmatrix} U_{\tilde{d}}^*(t) & V_{\tilde{d}}^*(t) \\ -V_{\tilde{d}}^*(t) & U_{\tilde{d}}^*(t) \end{pmatrix}\right)$$

and $h_*(y, t, \epsilon) = O(y^2)$ $(y \to 0)$ is a high-order term. It follows that system (1) has a quasi-periodic solution with basic frequencies $\omega = (1, \alpha)$ such that it goes to zero when ϵ does. The first part of the proof can be achieved by infinite KAM iteration steps. Second, we eliminate the non-resonant terms containing t on the diagonal and transform the above system into a system with constant coefficients,

$$\dot{y} = A^* y + h^*(y, t, \epsilon), \quad y \in \mathbb{R}^d,$$

where A^* is a constant matrix close to A, and $h^*(y, t, \epsilon) = O(y^2)$ $(y \to 0)$. This process can be obtained by only one step if $0 \le \beta(\alpha) < r$.

Remark. If A has a general real Jordan form

$$A = \operatorname{diag}\left(\begin{pmatrix} 0 & \mu_1 \\ -\mu_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \mu_{\tilde{d}} \\ -\mu_{\tilde{d}} & 0 \end{pmatrix}, \mu_{\tilde{d}+1}, \ldots, \mu_{\tilde{d}+l}\right),$$

where $\mu_i \in \mathbb{R}$, $i = 1, \ldots, \tilde{d} + l$, $2\tilde{d} + l = d$, then $B_*(t)$ has the corresponding real Jordan form

$$B_*(t) = \operatorname{diag}\left(\begin{pmatrix} U_1^*(t) & V_1^*(t) \\ -V_1^*(t) & U_1^*(t) \end{pmatrix}, \ldots, \begin{pmatrix} U_{\tilde{d}}^*(t) & V_{\tilde{d}}^*(t) \\ -V_{\tilde{d}}^*(t) & U_{\tilde{d}}^*(t) \end{pmatrix}, U_{\tilde{d}+1}^*(t), \ldots, U_{\tilde{d}+l}^*(t)\right).$$

3. *Preliminary lemmas*

The aim of this section is to give the concept of the CD bridge, which first appeared in [**1**], and present some important lemmas, which are mainly used to solve the homological equation and estimate the measure of parameters.

3.1. *Continued fraction expansion.* Let $\alpha \in (0, 1)$ be irrational. Define $a_0 = 0$, $\alpha_0 = \alpha$, and inductively for $k \geq 1$,

$$a_k = [\alpha_{k-1}^{-1}], \quad \alpha_k = \alpha_{k-1}^{-1} - a_k = \left\{ \frac{1}{\alpha_{k-1}} \right\},$$

where $[\cdot]$ denotes the integer part and $\{\cdot\}$ denotes the fractional part.

We define $p_0 = 0$, $p_1 = 1$, $q_0 = 0$, $q_1 = a_1$, and inductively,

$$p_k = a_k p_{k-1} + p_{k-2},$$
$$q_k = a_k q_{k-1} + q_{k-2}.$$

Then the sequence (q_n) is the sequence of denominators of the best rational approximations for α, since it satisfies

$$\text{for all } 1 \leq k < q_n, \quad \|k\alpha\|_{\mathbb{T}} \geq \|q_{n-1}\alpha\|_{\mathbb{T}},$$

and

$$\frac{1}{q_n + q_{n+1}} < \|q_n \alpha\|_{\mathbb{T}} \leq \frac{1}{q_{n+1}},$$

where we use the norm

$$\|x\|_{\mathbb{T}} = \inf_{p \in \mathbb{Z}} |x - p|.$$

3.2. *CD bridge.* For any $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, we fix a particular subsequence (q_{n_k}) in the sequence of the denominators of α, which will be denoted by (Q_k) for simplicity. Denote the sequence (q_{n_k+1}) by $(\bar{Q}_k)$, and denote (p_{n_k}) by (P_k).

Definition. (CD bridge, [**1**]) Let $0 < \mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$. We say that the pair of denominators (q_l, q_n) forms a CD$(\mathcal{A}, \mathcal{B}, \mathcal{C})$ bridge, if:

(1) $q_{i+1} \leq q_i^{\mathcal{A}}$, for all $i = l, \ldots, n-1$;

(2) $q_l^{\mathcal{B}} \leq q_n \leq q_l^{\mathcal{C}}$.

LEMMA 2. [**1**] *For any $\mathcal{A} \geq 1$, there exists a subsequence (Q_k) such that $Q_0 = 1$ and for each $k \geq 0$, $Q_{k+1} \leq \bar{Q}_k^{\mathcal{A}^4}$, and either $\bar{Q}_k \geq Q_k^{\mathcal{A}}$, or the pairs $(\bar{Q}_{k-1}, Q_k)$ and the pairs (Q_k, Q_{k+1}) are both CD$(\mathcal{A}, \mathcal{A}, \mathcal{A}^3)$ bridges.*

LEMMA 3. [**1**] *For any $0 < r_* < r$, $\tau > 2$, $0 < \tilde{c} < 1$, there exists $C_1 = C_1(r_*, \tau, \tilde{c})$ such that if $f \in \mathcal{A}_r(\Pi)$, then the equation*

$$\partial_\omega g(t, \epsilon) = -\mathcal{T}_{Q_{n+1}} f(t, \epsilon) + [f] \tag{8}$$

has a solution with

$$\|g\|_{r(1-\eta), \Pi} \leq C_1(r_*, \tau, \tilde{c}) \|f - [f]\|_{r, \Pi} \left(\frac{\bar{Q}_n}{Q_n^{\mathcal{A}^4}} + \bar{Q}_n^{1/\mathcal{A}} \right).$$

LEMMA 4. [**30**] *For any $0 < \gamma < 1$, $\tau > 2$, there exists $c_2 = c_2(\tau)$ such that if $\Omega(\epsilon)$ satisfies $0 < |\Omega(\epsilon)|_\Pi < 2$, and the non-resonant condition*

$$|\sqrt{-1}\langle k, \omega\rangle - \Omega(\epsilon)| \geq \frac{\gamma}{(|k|+1)^{3\tau}} \quad \textit{for all } |k| < K,$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

where

$$K = \left[\frac{\gamma}{4\cdot 10^{\tau}} \max\left\{\frac{\bar{Q}_n}{Q_n^{\tau}}, \bar{Q}_n^{3/\mathcal{A}}\right\}\right],$$

then for any $|k| < K$, *we have*

$$|\sqrt{-1}\langle k, \omega\rangle - \Omega(\epsilon)| \geq c_2(\tau)\gamma^{\mathcal{A}\tau/2+1} Q_n^{-9\tau}.$$

3.3. *Homological equation.* Before describing the following lemmas, we first give some definitions and notation. Denote by diag$P$ the elements on the diagonal line or diagonal block of matrix P. Since A is a constant matrix of elliptic type and det $A \neq 0$, we suppose that A is in the real Jordan form

$$A = \operatorname{diag}\left(\begin{pmatrix} 0 & \mu_1 \\ -\mu_1 & 0\end{pmatrix}, \ldots, \begin{pmatrix} 0 & \mu_{\tilde{d}} \\ -\mu_{\tilde{d}} & 0\end{pmatrix}\right), \quad 2\tilde{d} = d, \tag{9}$$

and define

$$B(t) = \operatorname{diag}\left(\begin{pmatrix} U_1(t) & V_1(t) \\ -V_1(t) & U_1(t)\end{pmatrix}, \ldots, \begin{pmatrix} U_{\tilde{d}}(t) & V_{\tilde{d}}(t) \\ -V_{\tilde{d}}(t) & U_{\tilde{d}}(t)\end{pmatrix}\right) \tag{10}$$

and

$$b(t) = \operatorname{diag}\left(\begin{pmatrix} \hat{u}_1(t) & \hat{v}_1(t) \\ -\hat{v}_1(t) & \hat{u}_1(t)\end{pmatrix}, \ldots, \begin{pmatrix} \hat{u}_{\tilde{d}}(t) & \hat{v}_{\tilde{d}}(t) \\ -\hat{v}_{\tilde{d}}(t) & \hat{u}_{\tilde{d}}(t)\end{pmatrix}\right),$$

where $\mu_j \in \mathbb{R}$, $U_j(t)$, $V_j(t)$, $\hat{u}_j(t)$ and $\hat{v}_j(t)$ $(j = 1, \ldots, \tilde{d})$ are all real functions.

Let

$$S = \frac{1}{\sqrt{2}}\operatorname{diag}\left(\begin{pmatrix} 1 & 1 \\ -\sqrt{-1} & \sqrt{-1}\end{pmatrix}, \ldots, \begin{pmatrix} 1 & 1 \\ -\sqrt{-1} & \sqrt{-1}\end{pmatrix}\right).$$

Then

$$\tilde{A} = S^{-1}AS = \operatorname{diag}(-\sqrt{-1}\mu_1, \sqrt{-1}\mu_1, \ldots, -\sqrt{-1}\mu_{\tilde{d}}, \sqrt{-1}\mu_{\tilde{d}}),$$

$$\begin{aligned}\tilde{B}(t) &= S^{-1}B(t)S \\ &= \operatorname{diag}(U_1 - \sqrt{-1}V_1, U_1 + \sqrt{-1}V_1, \ldots, U_{\tilde{d}} - \sqrt{-1}V_{\tilde{d}}, U_{\tilde{d}} + \sqrt{-1}V_{\tilde{d}})\end{aligned}$$

and

$$\tilde{b}(t) = S^{-1}b(t)S = \operatorname{diag}(\hat{u}_1 - \sqrt{-1}\hat{v}_1, \hat{u}_1 + \sqrt{-1}\hat{v}_1, \ldots, \hat{u}_{\tilde{d}} - \sqrt{-1}\hat{v}_{\tilde{d}}, \hat{u}_{\tilde{d}} + \sqrt{-1}\hat{v}_{\tilde{d}}).$$

In what follows, for simplicity of notation we write $A = S\operatorname{diag}(\lambda_1, \ldots, \lambda_{2\tilde{d}})S^{-1}$, $B(t) = S\operatorname{diag}(\Xi_1(t), \ldots, \Xi_{2\tilde{d}}(t))S^{-1}$ and $b(t) = S\operatorname{diag}(\Delta_1(t), \ldots, \Delta_{2\tilde{d}}(t))S^{-1}$, where λ_{2j-1} and λ_{2j} are conjugate complex numbers, $\Xi_{2j-1}(t)$ and $\Xi_{2j}(t)$ are conjugate complex functions, and $\Delta_{2j-1}(t)$ and $\Delta_{2j}(t)$ are conjugate complex functions, $j = 1, \ldots, \tilde{d}$. The constant matrix S only affects the estimation constant, so we do not explicitly write this constant in the following estimates.

Remark. If A has a general real Jordan form, similar approaches can be applied. Noting that real eigenvalues do not produce the problem of small divisors, we only consider the case of purely imaginary eigenvalues here.

For $\tau > 2$, we define

$$\mathcal{A} = \tau + 3, \quad \mathcal{M} = \frac{\mathcal{A}^4}{2},$$

and let (Q_n) be the selected subsequence of α in Lemma 2 with this given $\mathcal{A}$. For $r, \gamma > 0$, we define

$$\eta = \frac{\tilde{c}}{Q_n^{1/2\mathcal{A}^4}}, \quad \mathcal{L} = e^{-c_0 \gamma r(\bar{Q}_n/Q_n^{\mathcal{M}} + \bar{Q}_n^{1/\mathcal{M}^{1/4}})},$$

where $0 < \tilde{c}, c_0 < 1$ are constants, $c_0 = (\tilde{c}/4^5 \cdot 10^\tau)$, and the definition of $\tilde{c}$ is given in (51).

LEMMA 5. *Let us consider the equation*

$$\dot{x} = (A + B(t) + b(t))x + \epsilon g(t), \quad x \in \mathbb{R}^d, \tag{11}$$

where A is a constant matrix of elliptic type and $\det A \neq 0$*, A, $B(t)$ and $b(t)$ have the concrete form as above, and $g(t) = (g_i(t))_{1 \le i \le d}$ with $g_i(t)$ being an analytic quasi-periodic function given by $g_i(t) = \sum_{k \in \mathbb{Z}^2} g_i^k e^{i\langle k, \omega\rangle t}$.*

Let $\tau > 2, 0 < r_* < r, 0 < \tilde{c} < 1$. There exist $C_3 = C_3(\tau)$ and $\varepsilon_1 = \varepsilon_1(\tau, r_*, \tilde{c})$ such that for $\sigma, \tilde{r}$ with $0 < \sigma < r_* < \tilde{r} \le r(1 - \eta)$, if $B(t) \in \mathcal{A}_r(\Pi)$ with $\mathcal{R}_{Q_{n+1}} B(t) = 0$, $b(t), g(t) \in \mathcal{A}_{\tilde{r}}(\Pi)$,

$$\|B\|_{r,\Pi} \cdot \left(\frac{\bar{Q}_n}{Q_n^{\mathcal{A}^4}} + \bar{Q}_n^{1/\mathcal{A}}\right) \le \varepsilon_1 \gamma \left(\frac{\bar{Q}_n}{Q_n^{\mathcal{M}}} + \bar{Q}_n^{1/\mathcal{M}^{1/4}}\right), \tag{12}$$

$$\|b\|_{\tilde{r},\Pi} < \frac{\gamma^{\mathcal{A}\tau+2}}{2C_3 Q_{n+1}^{18\tau}}, \tag{13}$$

and the eigenvalues of $A + [B]$ satisfy

$$|\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i])| \ge \frac{\gamma}{(|k|+1)^{3\tau}} \quad \text{for all } |k| < K,\ 1 \le i \le d,$$

then equation (11) has an approximate solution $x(t, \epsilon)$ with estimate

$$\|x\|_{\tilde{r},\Pi} \le \mathcal{L}^{-1/240} C_3(\tau) \gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau} \epsilon \|g\|_{\tilde{r},\Pi}.$$

Moreover, the error term $g_e = Sh_e$ with $h_e = (h_{ie})_{1 \le i \le d}$, $h_{ie} = e^{-\tilde{\Xi}i(t)} \mathcal{R}_K(e^{\tilde{\Xi}i(t)} (S^{-1}g(t))_i)$ and $d\tilde{\Xi}_i(t)/dt = -\Xi_i(t) + [\Xi_i]$, satisfies

$$\|g_e\|_{\tilde{r}-\sigma,\Pi} \le \mathcal{L}^{-1/240} e^{-K\sigma} \|g\|_{\tilde{r},\Pi}.$$

Proof. Making the change of variables $x = Sz$ and defining $h(t) = S^{-1}g(t)$, equation (11) becomes

$$\dot{z} = (\tilde{A} + \tilde{B}(t) + \tilde{b}(t))z + \epsilon h(t), \tag{14}$$

where $\tilde{A} = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2\tilde{d}})$, $\tilde{B}(t) = \mathrm{diag}(\Xi_1(t), \ldots, \Xi_{2\tilde{d}}(t))$ and $\tilde{b}(t) = \mathrm{diag}(\Delta_1(t), \ldots, \Delta_{2\tilde{d}}(t))$, where λ_{2j-1} and λ_{2j} are conjugate complex numbers, $\Xi_{2j-1}(t)$ and $\Xi_{2j}(t)$ are conjugate complex functions, and $\Delta_{2j-1}(t)$ and $\Delta_{2j}(t)$ are conjugate complex functions, $j = 1, \ldots, \tilde{d}$.

Since $z = (z_i)_{1 \le i \le d}$ and $h(t) = (h_i(t))_{1 \le i \le d}$, equation (14) can be written in the form of components:

$$\dot{z}_i = (\lambda_i + \Xi_i(t) + \Delta_i(t))z_i + \epsilon h_i(t). \tag{15}$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

Let $d\tilde{\Xi}_i(t)/dt = -\Xi_i(t) + [\Xi_i]$, $\tilde{z}_i(t) = e^{\tilde{\Xi}_i(t)} z_i(t)(y \to 0)$ and $\tilde{h}_i(t) = e^{\tilde{\Xi}_i(t)} h_i(t)$. Equation (15) is equivalent to

$$\dot{\tilde{z}}_i = (\lambda_i + [\Xi_i] + \Delta_i(t))\tilde{z}_i + \epsilon \tilde{h}_i(t). \tag{16}$$

Instead of solving equation (16), we first solve the truncation equation

$$\mathcal{T}_K \dot{\tilde{z}}_i = \mathcal{T}_K((\lambda_i + [\Xi_i] + \Delta_i(t))\tilde{z}_i + \epsilon \tilde{h}_i(t)). \tag{17}$$

If we write

$$\tilde{z}_i = \sum_{k \in \mathbb{Z}^2, |k|<K} \tilde{z}_i^k e^{i\langle k,\omega\rangle t}, \quad \tilde{h}_i = \sum_{k \in \mathbb{Z}^2, |k|<K} \tilde{h}_i^k e^{i\langle k,\omega\rangle t},$$

and compare the Fourier coefficients of equation (17), we have

$$(\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i]))\tilde{z}_i^k - \sum_{|k_1|<K} \Delta_i^{k-k_1} \tilde{z}_i^{k_1} = \epsilon \tilde{h}_i^k. \tag{18}$$

View (18) as a matrix equation

$$(D + F)\mathcal{W} = \mathcal{P},$$

where

$$D = \operatorname{diag}(\ldots, \sqrt{-1}\langle k, \omega\rangle - \Omega_i(\epsilon), \ldots)_{|k|<K},$$
$$\Omega_i(\epsilon) = (\lambda_i + [\Xi_i]),$$
$$F = (-\Delta_i^{k_1-k_2})_{|k_1|,|k_2|<K},$$
$$\mathcal{W} = (\tilde{z}_i^k)^T_{|k|<K}, \quad \mathcal{P} = \epsilon(\tilde{h}_i^k)^T_{|k|<K}.$$

If we denote $\Gamma_{\tilde{r}} = \operatorname{diag}(\ldots, e^{|k|\tilde{r}}, \ldots)_{|k|<K}$, then

$$(D + \Gamma_{\tilde{r}} F \Gamma_{\tilde{r}}^{-1})\Gamma_{\tilde{r}}\mathcal{W} = \Gamma_{\tilde{r}}\mathcal{P}.$$

Since

$$|\sqrt{-1}\langle k, \omega\rangle - \Omega_i(\epsilon)| \geq \frac{\gamma}{(|k|+1)^{3\tau}} \quad \text{for all } |k| < K,\ 1 \leq i \leq d,$$

then by Lemma 4, we have

$$\begin{aligned}\|D^{-1}\|_\Pi &= \max_{|k|<K} \sup_{\epsilon\in\Pi}\left(\frac{1}{|\sqrt{-1}\langle k, \omega\rangle - \Omega_i(\epsilon)|} + \frac{|\partial\Omega_i(\epsilon)/\partial\epsilon|}{|\sqrt{-1}\langle k, \omega\rangle - \Omega_i(\epsilon)|^2}\right)\\ &\leq C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau}/2,\end{aligned}$$

where

$$\|D\|_\Pi = \max_i \sup_{\epsilon\in\Pi} \sum_j \left(|D_{ij}(\epsilon)| + \left|\frac{\partial D_{ij}(\epsilon)}{\partial\epsilon}\right|\right),$$

and D_{ij} is the (i, j)th variable of the matrix D.

Meanwhile, since the (k_1, k_2)th variable of $\Gamma_{\tilde{r}} F \Gamma_{\tilde{r}}^{-1}$ is $-e^{(|k_1|-|k_2|)\tilde{r}}(\Delta_i^{k_1-k_2})$, we obtain that

$$\begin{aligned}\|\Gamma_{\tilde{r}} F \Gamma_{\tilde{r}}^{-1}\|_\Pi &\leq \max_{|k_1|<K} \sum_{|k_2|<K} |e^{(|k_1|-|k_2|)\tilde{r}} \Delta_i^{k_1-k_2}|_\Pi\\ &\leq \max_{|k_1|<K} \sum_{|k_2|<K} |\Delta_i^{k_1-k_2}|_\Pi e^{|k_1-k_2|\tilde{r}} \leq 2\|b\|_{\tilde{r},\Pi}.\end{aligned}$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

Thus if $\|b\|_{\tilde{r},\Pi} < \gamma^{\mathcal{A}\tau+2}/(2C_3(\tau)Q_{n+1}^{18\tau})$, then

$$\|D^{-1}\Gamma_{\tilde{r}}F\Gamma_{\tilde{r}}^{-1}\|_{\Pi} \le \|D^{-1}\|_{\Pi}\|\Gamma_{\tilde{r}}F\Gamma_{\tilde{r}}\|_{\Pi} < \tfrac{1}{2},$$

which implies $D + \Gamma_{\tilde{r}}F\Gamma_{\tilde{r}}^{-1}$ has a bounded inverse:

$$\begin{aligned}\|(D + \Gamma_{\tilde{r}}F\Gamma_{\tilde{r}}^{-1})^{-1}\|_{\Pi} &= \|(I + D^{-1}\Gamma_{\tilde{r}}F\Gamma_{\tilde{r}}^{-1})^{-1}D^{-1}\|_{\Pi}\\ &\le \|D^{-1}\|_{\Pi}\frac{1}{1-\|D^{-1}\Gamma_{\tilde{r}}F\Gamma_{\tilde{r}}^{-1}\|_{\Pi}}\\ &\le C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)}Q_{n+1}^{18\tau}.\end{aligned}$$

It follows that

$$\begin{aligned}\|\tilde{z}_i\|_{\tilde{r},\Pi} &= \sum_{|k|<K}|\tilde{z}_i^k|_{\Pi}e^{|k|\tilde{r}} = \|\Gamma_{\tilde{r}}\mathcal{W}\|_{\Pi}\\ &\le \|(D + \Gamma_{\tilde{r}}F\Gamma_{\tilde{r}}^{-1})^{-1}\|_{\Pi}\|\Gamma_{\tilde{r}}\mathcal{P}\|_{\Pi}\\ &\le C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)}Q_{n+1}^{18\tau}\epsilon\|\tilde{h}_i\|_{\tilde{r},\Pi}.\end{aligned} \tag{19}$$

Let $z_i(t) = e^{-\tilde{\Xi}_i(t)}\tilde{z}_i(t)$. By Lemma 3,

$$\|\tilde{\Xi}_i\|_{r(1-\eta),\Pi} \le C_1(r_*,\tau,\tilde{c})\|\Xi_i - [\Xi_i]\|_{r,\Pi}\bigg(\frac{\bar{Q}_n}{Q_n^{\mathcal{A}^4}} + \bar{Q}_n^{1/\mathcal{A}}\bigg).$$

Then, by assumption (12), if $\varepsilon_1 < (c_0(\tau,\tilde{c})r)/(960C_1(r_*,\tau,\tilde{c}))$, we get

$$\begin{aligned}\|e^{-\tilde{\Xi}_i}\|_{r(1-\eta),\Pi} &\le e^{\|\tilde{\Xi}_i\|_{r(1-\eta),\Pi}} \le e^{2C_1\|\tilde{\Xi}_i\|_{r,\Pi}(\bar{Q}_n/Q_n^{\mathcal{A}^4}+\bar{Q}_n^{1/\mathcal{A}})}\\ &\le e^{2C_1\varepsilon_1\gamma(\bar{Q}_n/Q_n^{\mathcal{M}}+\bar{Q}_n^{1/\mathcal{M}^{1/4}})} < \mathcal{L}^{-1/480}.\end{aligned}$$

Therefore, by estimate (19) and the definition of $z_i(t)$,

$$\|z_i\|_{\tilde{r},\Pi} \le \mathcal{L}^{-1/480}\|\tilde{z}_i\|_{\tilde{r},\Pi} \le \mathcal{L}^{-1/240}C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)}Q_{n+1}^{18\tau}\epsilon\|h_i\|_{\tilde{r},\Pi}.$$

Moreover, the error term $h_{ie} = e^{-\tilde{\Xi}i(t)}\mathcal{R}_K(e^{\tilde{\Xi}i(t)}h_i(t))$ satisfies

$$\|h_{ie}\|_{\tilde{r}-\sigma,\Pi} \le \mathcal{L}^{-1/240}e^{-K\sigma}\|h_i\|_{\tilde{r},\Pi}.$$

As $x = Sz$ and $g(t) = Sh(t)$, by the norm definition of vector function, the result follows. □

Remark. When solving equation (11), the concrete form of non-resonant conditions is

$$|\langle k,\omega\rangle \pm (\mu_i + [V_i])| \ge \frac{\gamma}{(|k|+1)^{3\tau}} \quad \text{for all } |k| < K,\ 1 \le i \le \tilde{d}.$$

LEMMA 6. *Consider the equation*

$$\dot{W} = (A + B(t) + b(t))W - W(A + B(t) + b(t)) + P^{(nre)}, \tag{20}$$

where A is a constant matrix of elliptic type and $\det A \ne 0$*, A, $B(t)$ and $b(t)$ have the concrete form as above, and $P^{(nre)}$ is an analytic quasi-periodic matrix and has the form*

$$\operatorname{diag} P^{(nre)} = \operatorname{diag}\left(\begin{pmatrix} u_1(t) & v_1(t)\\ v_1(t) & -u_1(t)\end{pmatrix},\ldots,\begin{pmatrix} u_{\tilde{d}}(t) & v_{\tilde{d}}(t)\\ v_{\tilde{d}}(t) & -u_{\tilde{d}}(t)\end{pmatrix}\right),$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

where $u_j(t)$, $v_j(t)$ $(j = 1, \ldots, \tilde{d})$ *are all real functions, that is,* $\mathrm{diag}(S^{-1}P^{(nre)}S) = 0$.

Let $\tau > 2$, $0 < r_* < r$, $0 < \tilde{c} < 1$. *There exist* $C_3 = C_3(\tau)$ *and* $\varepsilon_1 = \varepsilon_1(\tau, r_*, \tilde{c})$ *such that for any* σ, $\tilde{r}$ *with* $0 < \sigma < r_* < \tilde{r} \le r(1-\eta)$, *if* $B(t) \in \mathcal{A}_r(\Pi)$ *with* $\mathcal{R}_{Q_{n+1}}B(t) = 0$, $b(t)$, $P^{(nre)}(t) \in \mathcal{A}_{\tilde{r}}(\Pi)$,

$$\|b\|_{\tilde{r},\Pi} < \frac{\gamma^{\mathcal{A}\tau+2}}{2C_3 Q_{n+1}^{18\tau}}, \tag{21}$$

$$\|B\|_{r,\Pi} \cdot \left(\frac{\bar{Q}_n}{Q_n^{\mathcal{A}^4}} + \bar{Q}_n^{1/\mathcal{A}}\right) \le \varepsilon_1 \gamma \left(\frac{\bar{Q}_n}{Q_n^{\mathcal{M}}} + \bar{Q}_n^{1/\mathcal{M}^{1/4}}\right), \tag{22}$$

and the eigenvalues of $A + [B]$ *satisfy*

$$|\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i]) + (\lambda_j + [\Xi_j])| \ge \frac{\gamma}{(|k|+1)^{3\tau}} \quad \text{for all } |k| < K,\ i \ne j,$$

then the homological equation (20) has an approximate solution $W(t, \epsilon)$ *with the estimate*

$$\|W\|_{\tilde{r},\Pi} \le \mathcal{L}^{-1/240} C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau}\|P^{(nre)}\|_{\tilde{r},\Pi}.$$

Moreover, the error term $P_e^{(nre)}$ *satisfies*

$$\|P_e^{(nre)}\|_{\tilde{r}-\sigma,\Pi} \le \mathcal{L}^{-1/240} e^{-K\sigma}(\|P^{(nre)}\|_{\tilde{r},\Pi} + 2\|b\|_{\tilde{r},\Pi}\|W\|_{\tilde{r},\Pi}),$$

where $P_e^{(nre)} = SH_eS^{-1}$, *and the definition of* H_e *is given in (29).*

Proof. Making the change of variables of $W = SZS^{-1}$ and $H = S^{-1}P^{(nre)}S$, equation (20) becomes

$$\dot{Z} = (\tilde{A} + \tilde{B}(t) + \tilde{b}(t))Z - Z(\tilde{A} + \tilde{B}(t) + \tilde{b}(t)) + H, \tag{23}$$

where we continue to use the notation of Lemma 5, $\tilde{A} = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2\tilde{d}})$, $\tilde{B}(t) = \mathrm{diag}(\Xi_1, \ldots, \Xi_{2\tilde{d},})$ and $\tilde{b}(t) = \mathrm{diag}(\Delta_1, \ldots, \Delta_{2\tilde{d}})$, where λ_{2j-1} and λ_{2j} are conjugate complex numbers, Ξ_{2j-1} and Ξ_{2j} are conjugate complex functions, and Δ_{2j-1} and Δ_{2j} are conjugate complex functions, $j = 1, \ldots, \tilde{d}$.

Note that $\mathrm{diag} H = \mathrm{diag}(S^{-1}P^{(nre)}S) = 0$. Let $Z = (Z_{ij})_{1\le i,j\le d}$, $H = (H_{ij})_{1\le i,j\le d}$. Equation (23) can be written in the form of components:

$$\frac{dZ_{ij}}{dt} = ((\lambda_i + \Xi_i(t) + \Delta_i(t)) - (\lambda_j + \Xi_j(t) + \Delta_j(t)))Z_{ij} + H_{ij} \quad \text{for all } i \ne j. \tag{24}$$

Let

$$\frac{d\mathcal{B}_i(t)}{dt} = -\Xi_i(t) + [\Xi_i], \qquad \frac{d\mathcal{B}_j(t)}{dt} = -\Xi_j(t) + [\Xi_j]$$

and

$$\tilde{Z}_{ij}(t) = e^{(\mathcal{B}_i(t)-\mathcal{B}_j(t))}Z_{ij}(t), \qquad \tilde{H}_{ij}(t) = e^{(\mathcal{B}_i(t)-\mathcal{B}_j(t))}H_{ij}(t),$$

for $1 \le i, j \le d$. Equation (24) is transformed into

$$\frac{d\tilde{Z}_{ij}}{dt} - ((\lambda_i + [\Xi_i] + \Delta_i(t)) - (\lambda_j + [\Xi_j] + \Delta_j(t)))\tilde{Z}_{ij} = \tilde{H}_{ij} \quad \text{for all } i \ne j. \tag{25}$$

Instead of solving equation (25), we first solve the approximation equation

$$\mathcal{T}_K\left(\frac{d\tilde{Z}_{ij}}{dt} - ((\lambda_i + [\Xi_i] + \Delta_i(t)) - (\lambda_j + [\Xi_j] + \Delta_j(t)))\tilde{Z}_{ij}\right) = \mathcal{T}_k\tilde{H}_{ij} \quad \text{for all } i \ne j. \tag{26}$$

If we write

$$\tilde{Z}_{ij} = \sum_{k\in\mathbb{Z}^2,|k|<K} \tilde{Z}_{ij}^k e^{i\langle k,\theta\rangle}, \quad \tilde{H}_{ij} = \sum_{k\in\mathbb{Z}^2,|k|<K} \tilde{H}_{ij}^k e^{i\langle k,\theta\rangle},$$

and compare the Fourier coefficients of equation (26), then for $|k| < K$, we have

$$(\sqrt{-1}\langle k,\omega\rangle - (\lambda_i + [\Xi_i]) + (\lambda_j + [\Xi_j]))\tilde{Z}_{ij}^k - \sum_{|k_1|<K} (\Delta_i^{k-k_1} - \Delta_j^{k-k_1})\tilde{Z}_{ij}^{k_1} = \tilde{H}_{ij}^k. \tag{27}$$

View (27) as a matrix equation

$$(D+F)\mathcal{W} = \mathcal{P},$$

where

$$\begin{aligned} D &= \mathrm{diag}(\ldots, \sqrt{-1}\langle k,\omega\rangle - \Omega_{ij}(\epsilon), \ldots)_{|k|<K},\\ \Omega_{ij}(\epsilon) &= (\lambda_i + [\Xi_i]) - (\lambda_j + [\Xi_j]),\\ F &= (-(\Delta_i^{k_1-k_2} - \Delta_j^{k_1-k_2}))_{|k_1|,|k_2|<K},\\ \mathcal{W} &= (\tilde{Z}_{ij}^k)^T_{|k|<K}, \quad \mathcal{P} = (\tilde{H}_{ij}^k)^T_{|k|<K}. \end{aligned}$$

If we denote $\Gamma_{\tilde{r}} = \mathrm{diag}(\ldots, e^{|k|\tilde{r}}, \ldots)_{|k|<K}$, then

$$(D + \Gamma_{\tilde{r}} F \Gamma_{\tilde{r}}^{-1})\Gamma_{\tilde{r}}\mathcal{W} = \Gamma_{\tilde{r}}\mathcal{P}.$$

Since

$$|\sqrt{-1}\langle k,\omega\rangle - \Omega_{ij}(\epsilon)| \geq \frac{\gamma}{(|k|+1)^{3\tau}} \quad \text{for all } |k| < K,\ i \neq j,$$

then by Lemma 4, we have

$$\begin{aligned} \|D^{-1}\|_\Pi &= \max_{|k|<K} \sup_{\epsilon\in\Pi} \left(\frac{1}{|\sqrt{-1}\langle k,\omega\rangle - \Omega_{ij}(\epsilon)|} + \frac{|\partial\Omega_{ij}(\epsilon)/\partial\epsilon|}{|\sqrt{-1}\langle k,\omega\rangle - \Omega_{ij}(\epsilon)|^2} \right)\\ &\leq C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau}/2, \end{aligned}$$

where

$$\|D\|_\Pi = \max_i \sup_{\epsilon\in\Pi} \sum_j \left(|D_{ij}(\epsilon)| + \left| \frac{\partial D_{ij}(\epsilon)}{\partial\epsilon} \right| \right)$$

and D_{ij} is the (i, j)th variable of the matrix D.

Meanwhile, since the (k_1, k_2)th variable of $\Gamma_{\tilde{r}} F \Gamma_{\tilde{r}}^{-1}$ is $-e^{(|k_1|-|k_2|)\tilde{r}}(\Delta_i^{k_1-k_2} - \Delta_j^{k_1-k_2})$, we obtain that

$$\begin{aligned} \|\Gamma_{\tilde{r}} F \Gamma_{\tilde{r}}^{-1}\|_\Pi &\leq \max_{|k_1|<K} \sum_{|k_2|<K} |e^{(|k_1|-|k_2|)\tilde{r}}(\Delta_i^{k_1-k_2} - \Delta_j^{k_1-k_2})|_\Pi\\ &\leq \max_{|k_1|<K} \sum_{|k_2|<K} (|\Delta_i^{k_1-k_2}|_\Pi + |\Delta_j^{k_1-k_2}|_\Pi) e^{|k_1-k_2|\tilde{r}} \leq 2\|b\|_{\tilde{r},\Pi}. \end{aligned}$$

Thus if $\|b\|_{\tilde{r},\Pi} < \gamma^{\mathcal{A}\tau+2}/(2C_3(\tau)Q_{n+1}^{18\tau})$, then

$$\|D^{-1}\Gamma_{\tilde{r}} F \Gamma_{\tilde{r}}^{-1}\|_\Pi \leq \|D^{-1}\|_\Pi \|\Gamma_{\tilde{r}} F \Gamma_{\tilde{r}}\|_\Pi < \tfrac{1}{2},$$

which implies that $D + \Gamma_{\tilde r} F \Gamma_{\tilde r}^{-1}$ has a bounded inverse:

$$\begin{aligned}\|(D + \Gamma_{\tilde r} F \Gamma_{\tilde r}^{-1})^{-1}\|_\Pi &= \|(I + D^{-1}\Gamma_{\tilde r} F \Gamma_{\tilde r}^{-1})^{-1} D^{-1}\|_\Pi \\ &\le \|D^{-1}\|_\Pi \frac{1}{1 - \|D^{-1}\Gamma_{\tilde r} F \Gamma_{\tilde r}^{-1}\|_\Pi} \\ &\le C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau}.\end{aligned}$$

It follows that

$$\begin{aligned}\|\tilde Z_{ij}\|_{\tilde r,\Pi} &= \sum_{|k|<K} |\tilde Z_{ij}^k|_\Pi e^{|k|\tilde r} = \|\Gamma_{\tilde r}\mathcal{W}\|_\Pi \\ &\le \|(D + \Gamma_{\tilde r} F \Gamma_{\tilde r}^{-1})^{-1}\|_\Pi \|\Gamma_{\tilde r}\mathcal{P}\|_\Pi \\ &\le \|(D + \Gamma_{\tilde r} F \Gamma_{\tilde r}^{-1})^{-1}\|_\Pi \|\tilde H_{ij}\|_{\tilde r,\Pi}.\end{aligned} \tag{28}$$

Let $Z_{ij}(t) = e^{-(\mathcal{B}_i(t)-\mathcal{B}_j(t))}\tilde Z_{ij}(t)$. First, by Lemma 3, we have

$$\|\mathcal{B}_i\|_{r(1-\eta),\Pi} \le C_1(r_*, \tau, \tilde c)\|\Xi_i - [\Xi_i]\|_{r,\Pi}\bigg(\frac{\bar Q_n}{Q_n^{\mathcal{A}^4}} + \bar Q_n^{1/\mathcal{A}}\bigg).$$

Therefore, by assumption (22), if $\varepsilon_1 < (c_0(\tau, \tilde c)r)/(960 C_1(r_*, \tau, \tilde c))$, we get

$$\begin{aligned}\|e^{-(\mathcal{B}_i-\mathcal{B}_j)}\|_{r(1-\eta),\Pi} &\le e^{(\|\mathcal{B}_i\|_{r(1-\eta),\Pi}+\|\mathcal{B}_j\|_{r(1-\eta),\Pi})} \le e^{2C_1\|B\|_{r,\Pi}(\bar Q_n/Q_n^{\mathcal{A}^4}+\bar Q_n^{1/\mathcal{A}})} \\ &\le e^{2C_1\varepsilon_1\gamma(\bar Q_n/Q_n^{\mathcal{M}}+\bar Q_n^{1/\mathcal{M}^{1/4}})} < \mathcal{L}^{-1/480}.\end{aligned}$$

By estimate (28) and the definition of matrix norm, we have

$$\|Z_{ij}\|_{\tilde r,\Pi} \le \mathcal{L}^{-1/480}\|\tilde Z_{ij}\|_{\tilde r,\Pi} \le C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau}\mathcal{L}^{-1/240}\|H_{ij}\|_{\tilde r,\Pi}$$

and

$$\|Z\|_{\tilde r,\Pi} \le C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau}\mathcal{L}^{-1/240}\|H\|_{\tilde r,\Pi}.$$

Moreover, one can verify that the error term $H_e = (H_{ije})_{d\times d}$, with

$$H_{ije} = e^{-(\mathcal{B}_i(t)-\mathcal{B}_j(t))}\mathcal{R}_K(e^{(\mathcal{B}_i(t)-\mathcal{B}_j(t))}(H_{ij}(t) + (\Delta_i(t) - \Delta_j(t))Z_{ij})), \tag{29}$$

satisfies the estimate

$$\|H_e\|_{\tilde r-\sigma,\Pi} \le \mathcal{L}^{-1/240}e^{-K\sigma}(\|H\|_{\tilde r,\Pi} + 2\|b\|_{\tilde r,\Pi}\|Z\|_{\tilde r,\Pi}).$$

As $W = SZS^{-1}$ and $P_e^{(nre)} = SH_eS^{-1}$, the result follows. □

Remark. When solving equation (20), the concrete forms of non-resonant conditions are

$$|\langle k, \omega\rangle + (\mu_i + [V_i]) + (\mu_j + [V_j])| \ge \frac{\gamma}{(|k|+1)^{3\tau}} \quad \text{for all } |k| < K,$$

and

$$|\langle k, \omega\rangle + (\mu_i + [V_i]) - (\mu_j + [V_j])| \ge \frac{\gamma}{(|k|+1)^{3\tau}} \quad \text{for all } |k| < K, i \ne j.$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

3.4. *A lemma on measure estimates.* Finally, we give a lemma which will be used to estimate the measure of parameters.

LEMMA 7. *Let D be a $d \times d$ diagonal matrix with different non-zero eigenvalues $\lambda_1, \ldots, \lambda_d$. Let $P(\epsilon) = (P_{ij}(\epsilon))_{d\times d}$ be a matrix such that $P_{ij}(\epsilon) = O(\epsilon)$ and $P_{jj} = 0$ $(i, j = 1, \ldots, d)$. Let $\lambda_j^0(\epsilon)$ $(j = 1, \ldots, d)$ be the eigenvalues of $D + P(\epsilon)$. Then $\lambda_j^0(\epsilon) - \lambda_j = O(\epsilon^2)$.*

Proof. By [**16**, Lemma 7], there exists a regular matrix $I + S(\varepsilon)$ with $S_{jj} = 0$, $j = 1, \ldots, d$, such that

$$(I + S(\varepsilon))^{-1}(D + P(\epsilon))(I + S(\varepsilon)) = D + \hat{D}(\epsilon),$$

where $\hat{D}(\epsilon)$ is a diagonal matrix. The above equation is equivalent to

$$DS(\epsilon) - S(\epsilon)D = -P(\epsilon) - P(\epsilon)S(\epsilon) + \hat{D}(\epsilon) + S(\epsilon)\hat{D}(\epsilon).$$

Note that D and $\hat{D}$ are diagonal matrices, and $P_{jj} = 0$, $S_{jj} = 0$. We divide the matrix elements of the above equation into diagonal parts and non-diagonal parts. For $i \neq j$,

$$(\lambda_i - \lambda_j)S_{ij}(\epsilon) = -P_{ij}(\epsilon) - (P(\epsilon)S(\epsilon))_{ij} + (S(\epsilon)\hat{D}(\epsilon))_{ij}.$$

From the Gerschgorin circle theorem, $\lambda_j^0(\epsilon) - \lambda_j = O(\epsilon)$. In addition, $P_{ij}(\epsilon) = O(\epsilon)$ $(i \neq j)$ and $\lambda_i - \lambda_j \neq 0$, so $S_{ij}(\epsilon) = O(\epsilon)$. For $i = j$,

$$\lambda_j^0(\epsilon) - \lambda_j = (P(\epsilon)S(\epsilon))_{jj} - (S(\epsilon)\hat{D}(\epsilon))_{jj} = O(\epsilon^2). \qquad \square$$

4. *Proof of Theorem 1*

4.1. *KAM step.* Suppose that we are now in the nth step, and in what follows the quantities without subscripts refer to those of the nth step, while the quantities with subscripts '+' are those of the $(n + 1)$th step. Thus we consider the system

$$\dot{x} = (A + B(t, \epsilon) + \epsilon^\beta P(t, \epsilon))x + \epsilon^\beta g(t, \epsilon) + h(x, t, \epsilon), \quad x \in \mathbb{R}^d, \tag{30}$$

where A and $B(t)$ have real Jordan form as in (9) and (10), and P, g and h are all analytic quasi-periodic with respect to t on $D(r)$ and differentiable in the parameters $\epsilon \in \Pi$, satisfying

$$\max\{\|P\|_{r,\Pi}, \|g\|_{r,\Pi}\} \le M,$$

and with $h(x, t, \epsilon)$ analytic with respect to x on the ball $B_{\kappa_n}(0)$, where κ_n is a sequence with $\kappa_0 = \kappa$ and $\lim_{n\to\infty} \kappa_n > 0$.

For simplicity, we still write $A = S\mathrm{diag}(\lambda_1, \ldots, \lambda_{2\tilde{d}})S^{-1}$ and $B(t) = S\mathrm{diag}(\Xi_1, \ldots, \Xi_{2\tilde{d},})S^{-1}$, where λ_{2j-1} and λ_{2j} are conjugate complex numbers and Ξ_{2j-1} and Ξ_{2j} are conjugate complex functions, $j = 1, \ldots, \tilde{d}$.

For $\tau > 2$, we define

$$\mathcal{A} = \tau + 3, \quad \mathcal{M} = \frac{\mathcal{A}^4}{2}, \tag{31}$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

and let (Q_n) be the selected subsequence of α in Lemma 2 with this given $\mathcal{A}$. For $r, \gamma > 0$, we define

$$\eta = \frac{\tilde{c}}{Q_n^{1/2\mathcal{A}^4}}, \quad \mathcal{L} = e^{-c_0\gamma r(\bar{Q}_n/Q_n^{\mathcal{M}} + \bar{Q}_n^{1/\mathcal{M}^{1/4}})},$$
$$r_+ = r(1-\eta)^2, \quad \mathcal{L}_+ = e^{-c_0\gamma r_+(\bar{Q}_{n+1}/Q_{n+1}^{\mathcal{M}} + \bar{Q}_{n+1}^{1/\mathcal{M}^{1/4}})},$$
$$\mathcal{E}_+ = \mathcal{L}_+\mathcal{E}, \quad K = \left[\frac{\gamma}{4\cdot 10^\tau}\max\left\{\frac{\bar{Q}_{n+1}}{Q_{n+1}^\tau}, \bar{Q}_{n+1}^{3/\mathcal{A}}\right\}\right],$$

where $[\cdot]$ denotes the integer part, $0 < \tilde{c} < 1$ is a constant which will be defined in (51), and $c_0 = \tilde{c}/4^5 \cdot 10^\tau$ is a constant depending on $\tau, \tilde{c}$.

We summarize one KAM iteration step in the following lemma. The key point is to guarantee the non-resonant condition in the KAM iteration by adjusting some parameters [**16**, **17**, **19**, **29**, **31**].

LEMMA 8. (Step lemma) *We consider the nonlinear quasi-periodic system (30). Let* $\tau > 2$, $0 < r_* < r$, $0 < \tilde{c} < 1$. *There exist* $\varepsilon_0 = \varepsilon_0(\tau, r_*, \tilde{c})$, $\varepsilon_1 = \varepsilon_1(\tau, r_*, \tilde{c})$, $J = J(\tau)$ *and* $T_0 = T_0(\tau, r_*, \tilde{c})$ *such that for* $B(t) \in \mathcal{A}_r(\Pi)$ *with* $\mathcal{R}_{Q_{n+1}}B(t) = 0$, *and* $P(t) \in \mathcal{A}_r(\Pi)$, *if*

$$\bar{Q}_{n+1} \geq T_0 \cdot \gamma^{-\mathcal{A}/2},$$
$$\|B\|_{r,\Pi}\left(\frac{\bar{Q}_n}{Q_n^{\mathcal{A}^4}} + \bar{Q}_n^{1/\mathcal{A}}\right) \leq \varepsilon_1\gamma\left(\frac{\bar{Q}_n}{Q_n^{\mathcal{M}}} + \bar{Q}_n^{1/\mathcal{M}^{1/4}}\right), \tag{32}$$
$$\max\{\|\epsilon^\beta P\|_{r,\Pi}, \|\epsilon^\beta g\|_{r,\Pi}\} \leq \epsilon^\beta M = \mathcal{E} \leq \varepsilon_0\gamma^J\mathcal{L},$$

and the eigenvalues of $A + [B]$ *satisfy*

$$|\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i])| \geq \frac{\gamma}{(|k|+1)^{3\tau}},$$
$$|\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i]) + (\lambda_j + [\Xi_j])| \geq \frac{\gamma}{(|k|+1)^{3\tau}} \quad \textit{for all } i \neq j,$$

for any $|k| < K$, $1 \leq i, j \leq d$, *then there exists a quasi-periodic transformation* Φ, *such that* Φ *reduces the system (30) to*

$$\dot{x}_+ = (A + B_+(t,\epsilon) + \epsilon^{\beta_+}P_+(t,\epsilon))x_+ + \epsilon^{\beta_+}g_+(t,\epsilon) + h_+(x_+, t, \epsilon), \quad x_+ \in \mathbb{R}^d,$$

where

$$B_+(t) = \operatorname{diag}\left(\begin{pmatrix} U_1^+(t) & V_1^+(t) \\ -V_1^+(t) & U_1^+(t)\end{pmatrix}, \ldots, \begin{pmatrix} U_{\tilde{d}}^+(t) & V_{\tilde{d}}^+(t) \\ -V_{\tilde{d}}^+(t) & U_{\tilde{d}}^+(t)\end{pmatrix}\right),$$

which is defined in $D(r_+) \times \Pi_+$, *satisfies* $\mathcal{R}_{Q_{n+2}}B_+(t) = 0$, $\beta_+ = (\frac{16}{15})^L\beta$ *with* L *being the steps of KAM iteration, and*

$$\max\{\|\epsilon^{\beta_+}P_+\|_{r_+,\Pi_+}, \|\epsilon^{\beta_+}g_+\|_{r_+,\Pi_+}\} \leq \mathcal{E}_+, \tag{33}$$
$$\|B_+ - B\|_{r_+,\Pi_+} \leq 2\mathcal{E}^{14/15}. \tag{34}$$

Moreover, the following estimate holds:

$$\|\Phi - \mathrm{id}\|_{r_+,\Pi_+} \leq \frac{\mathcal{E}^{14/15}}{4}. \tag{35}$$

In the case where $B(t,\epsilon)=0$ in (30), the same conclusions are true if one replaces (32) by

$$\max\{\|\epsilon^{\beta}P\|_{r,\Pi},\ \|\epsilon^{\beta}g\|_{r,\Pi}\}\le\mathcal{E}\le\min\bigg\{\varepsilon_0\gamma^{J},\ \frac{1}{Q_{n_0}^{120\tau}}\bigg\}.$$

Remark. From the following proof, the quasi-periodic transformation Φ is the finite combination of these transformations: $\phi_i : x_i = e^{\epsilon^{\beta_i}W_i}x_{i+1}+\underline{x_i}$, where $\underline{x_i}$ is the approximation solution of $\dot{x}_i(t)=(A+B(t)+b_i(t))x_i(t)+\epsilon^{\beta_i}g_i(t)$ and $W_i(t,\epsilon)$ is an analytic quasi-periodic matrix.

Remark. We can also write $B_+(t)=S\mathrm{diag}(\Xi_1^+(t),\ldots,\Xi_{2\tilde{d}}^+(t))S^{-1}$, where $\Xi_{2j-1}^+(t)$ and $\Xi_{2j}^+(t)$ are conjugate complex functions, $j=1,\ldots,\tilde{d}$.

Proof. Before giving the proof, we first collect some useful estimates. Let

$$\tilde{\mathcal{E}}_0=\mathcal{E},\qquad \tilde{r}_0=r(1-\eta),$$

and define inductively the sequences

$$\tilde{\mathcal{E}}_m=\tilde{\mathcal{E}}_{m-1}^{16/15}=\tilde{\mathcal{E}}_0^{(16/15)^m}.$$

Let $m_0=\min\{m\in\mathbb{Z}_+ : K\tilde{\mathcal{E}}_m^{1/15}<1\}$, and define

$$\sigma_m=\begin{cases}\dfrac{\eta}{2^{m+3}}, & m<m_0,\\[2mm] -\dfrac{2\ln\tilde{\mathcal{E}}_m}{15K\tilde{r}_0}, & m\ge m_0,\end{cases}$$

and $\hat{r}_{m-1}=\tilde{r}_{m-1}-\tilde{r}_0\sigma_{m-1}$, $\tilde{r}_m=\tilde{r}_{m-1}-3\tilde{r}_0\sigma_{m-1}$.

Once we have these parameters, there exist $J=J(\tau)$, $\varepsilon_0=\varepsilon_0(\tau,r_*,\tilde{c})$ and $T_0=T_0(\tau,r_*,\tilde{c})$ such that if

$$\tilde{\mathcal{E}}_0\le\varepsilon_0\gamma^{J(\tau)}\mathcal{L},\qquad \bar{Q}_{n+1}\ge T_0\gamma^{-\mathcal{A}/2},$$

then we have the useful estimates

$$\tilde{\mathcal{E}}_0\le\min\bigg\{\frac{1}{Q_{n+1}^{120\tau}},\ \bigg(\frac{\gamma^{\mathcal{A}\tau+2}}{16C_3(\tau)}\bigg)^{30}\bigg\},\tag{36}$$

$$e^{-K\tilde{r}_0\sigma_m}\le\tilde{\mathcal{E}}_m^{2/15},\qquad \tilde{\mathcal{E}}_m\le(\tilde{r}_0\sigma_m)^{15},\tag{37}$$

where $C_3(\tau)$ is the constant in Lemmas 5 and 6.

In what follows we first check the above estimates. For estimate (36), let $J(\tau)=[120\tau\mathcal{A}^5]$. By $\mathcal{L}=e^{-c_0\gamma r(\bar{Q}_n/Q_n^{\mathcal{M}}+\bar{Q}_n^{1/\mathcal{M}^{1/4}})}$ and the choice of $(Q_n): Q_{n+1}\le\bar{Q}_n^{\mathcal{A}^4}$, if ε_0 is sufficiently small, we have

$$\tilde{\mathcal{E}}_0\le\varepsilon_0\gamma^{J(\tau)}\mathcal{L}\le\frac{\varepsilon_0\gamma^{J(\tau)}J!}{(c_0\gamma r\bar{Q}_n^{1/\mathcal{M}^{1/4}})^{J(\tau)}}\le\frac{\varepsilon_0 J!}{(c_0r_*)^{J(\tau)}}\frac{1}{Q_{n+1}^{120\tau}}\le\frac{1}{Q_{n+1}^{120\tau}}$$

and

$$\tilde{\mathcal{E}}_0\le\varepsilon_0\gamma^{J(\tau)}\mathcal{L}\le\varepsilon_0\gamma^{30(\mathcal{A}\tau+2)}\le\bigg(\frac{\gamma^{\mathcal{A}\tau+2}}{16C_3(\tau)}\bigg)^{30}.$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

For the first estimate of (37), if $m \geq m_0$, by the definition of σ_m, the estimate apparently holds. If $m < m_0$, by the definition of $m_0 K$ and σ_m, we have

$$K\sigma_m\tilde{r}_0 \geq \left(\frac{1}{\tilde{\mathcal{E}}_m}\right)^{1/30}\left(\frac{\gamma}{4\cdot 10^\tau}\right)^{1/2}\bar{Q}_{n+1}^{3/2\mathcal{A}}\frac{\tilde{c}}{Q_n^{1/2\mathcal{A}^4}}\frac{r_*}{2^{m+3}}$$
$$\geq \left(\frac{1}{\tilde{\mathcal{E}}_m}\right)^{1/30}\left(\frac{\gamma}{4\cdot 10^\tau}\right)^{1/2}\frac{r_*\tilde{c}}{2^{m+3}} \geq \frac{2}{15}\ln\left(\frac{1}{\tilde{\mathcal{E}}_m}\right)$$

for sufficiently small ε_0. For the second estimate of (37), if $m \geq m_0$, then

$$\tilde{r}_0\sigma_m = \frac{2}{15}\ln\frac{1}{\tilde{\mathcal{E}}_m}\frac{1}{K} \geq \frac{2}{15}\ln\frac{1}{\tilde{\mathcal{E}}_m}\tilde{\mathcal{E}}_m^{1/15} \geq \tilde{\mathcal{E}}_m^{1/15}.$$

If $m < m_0$, then

$$\tilde{\mathcal{E}}_m^{1/15} = \tilde{\mathcal{E}}_0^{1/15(16/15)^m} \leq \tilde{\mathcal{E}}_0^{1/18(16/15)^m}\left(\frac{1}{Q_{n+1}}\right)^{4\tau/3(16/15)^m}$$
$$\leq \frac{\tilde{r}_0\tilde{c}}{2^{m+3}}\frac{1}{Q_n^{1/2\mathcal{A}^4}} = \tilde{r}_0\sigma_m$$

for sufficiently small ε_0.

We now prove Lemma 8 by induction. For simplicity, we omit the dependence on ϵ. Consider

$$\frac{dx}{dt} = (A + B(t) + \epsilon^\beta P(t))x + \epsilon^\beta g(t) + h(x,t), \quad x \in \mathbb{R}^d,$$

where A and $B(t)$ have real Jordan form as in (9) and (10), $B(t) \in \mathcal{A}_r(\Pi)$ with $\mathcal{R}_{Q_{n+1}}B(t) = 0$, and $P(t)$, $g(t) \in \mathcal{A}_r(\Pi)$ with

$$\max\{\|\epsilon^\beta P\|_{r,\Pi}, \|\epsilon^\beta g\|_{r,\Pi}\} \leq \tilde{\mathcal{E}}_0.$$

Assume that for $j = 1, \ldots, \nu$ we can find the transformation $\phi_{j-1} : x_{j-1} = e^{\epsilon^{\beta_{j-1}}W_{j-1}}x_j + \underline{x_{j-1}}$, where $\underline{x_{j-1}}$ is the approximation solution of $\dot{x}_{j-1} = (A + B(t) + b_{j-1}(t))x_{j-1} + \epsilon^{\beta_{j-1}}g_{j-1}(t)$, and W_{j-1} is an analytic quasi-periodic matrix, such that the system

$$\frac{dx_{j-1}}{dt} = (A + B(t) + b_{j-1}(t) + \epsilon^{\beta_{j-1}}P_{j-1}(t))x_{j-1} + \epsilon^{\beta_{j-1}}g_{j-1}(t) + h_{j-1}(x_{j-1},t),$$

can be transformed into

$$\frac{dx_j}{dt} = (A + B(t) + b_j(t) + \epsilon^{\beta_j}P_j(t))x_j + \epsilon^{\beta_j}g_j(t) + h_j(x_j,t),$$

satisfying $\mathcal{R}_{Q_{n+2}}b_j = 0$, and

$$\begin{gathered}\max\{\|\epsilon^{\beta_j}g_j\|_{\tilde{r}_j,\Pi}, \|\epsilon^{\beta_j}P_j\|_{\tilde{r}_j,\Pi}\} \leq \tilde{\mathcal{E}}_j,\\ \|b_j - b_{j-1}\|_{\tilde{r}_j,\Pi} \leq \tilde{\mathcal{E}}_{j-1}^{14/15}, \quad \|D_{x_jx_j}h_j\|_{\tilde{r}_j} \leq G_j,\\ \|\underline{x_{j-1}}\|_{\tilde{r}_{j-1},\Pi} \leq 2C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)}Q_{n+1}^{18\tau}\mathcal{L}^{-1/240}\tilde{\mathcal{E}}_{j-1},\\ \|\epsilon^{\beta_{j-1}}W_{j-1}\|_{\hat{r}_{j-1},\Pi} \leq 2G_{j-1}(C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)}Q_{n+1}^{18\tau}\mathcal{L}^{-1/240})^2\tilde{\mathcal{E}}_{j-1}.\end{gathered} \tag{38}$$

For $j = \nu + 1$, we want to find a transformation ϕ_ν such that it can reduce the system

$$\frac{dx_\nu}{dt} = (A + B(t) + b_\nu(t) + \epsilon^{\beta_\nu} P_\nu(t))x_\nu + \epsilon^{\beta_\nu} g_\nu(t) + h_\nu(x_\nu, t), \tag{39}$$

to the desired form

$$\frac{dx_{\nu+1}}{dt} = (A + B(t) + b_{\nu+1}(t) + \epsilon^{\beta_{\nu+1}} P_{\nu+1}(t))x_{\nu+1} + \epsilon^{\beta_{\nu+1}} g_{\nu+1}(t) + h_{\nu+1}(x_{\nu+1}, t),$$

and satisfies the corresponding estimates.

First, we apply the change of variables $x_\nu(t) = y_\nu(t) + \underline{x_\nu}(t)$ to system (39), obtaining

$$\begin{aligned}\frac{dy_\nu}{dt} &= (A + B(t) + b_\nu(t) + \epsilon^{\beta_\nu} P_\nu(t))y_\nu + \epsilon^{\beta_\nu} P_\nu(t)\underline{x_\nu}(t) + h_\nu(y_\nu(t) + \underline{x_\nu}(t), t)\\ &\quad + (A + B(t) + b_\nu(t))\underline{x_\nu}(t) + \epsilon^{\beta_\nu} g_\nu(t) - \frac{d\underline{x_\nu}(t)}{dt}.\end{aligned}$$

We wish to find $\underline{x_\nu}(t)$ from the homological equation

$$\frac{d\underline{x_\nu}(t)}{dt} = (A + B(t) + b_\nu(t))\underline{x_\nu}(t) + \epsilon^{\beta_\nu} g_\nu(t). \tag{40}$$

By the inductive hypothesis (38), it follows that $\mathcal{R}_{Q_{n+2}} b_\nu = 0$ and

$$\|b_\nu\|_{\tilde{r}_\nu,\Pi} \le \|b_\nu - b_{\nu-1}\|_{\tilde{r}_\nu,\Pi} + \sum_{j=1}^{\nu-1} \|b_j - b_{j-1}\|_{\tilde{r}_j,\Pi} \le 2\tilde{\mathcal{E}}_0^{14/15}.$$

Then, by (36), we have

$$\|b_\nu\|_{\tilde{r}_\nu,\Pi} \le 2\tilde{\mathcal{E}}_0^{14/15} < \frac{\gamma^{\mathcal{A}\tau+2}}{2C_3(\tau)Q_{n+1}^{18\tau}},$$

which implies that condition (13) is satisfied.

On the other hand, by the assumptions of Lemma 8, $B(t)$ satisfies (12), $\mathcal{R}_{Q_{n+1}} B(t) = 0$, and

$$|\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i])| \ge \frac{\gamma}{(|k|+1)^{3\tau}} \quad \text{for all } |k| < K.$$

Then for any $|k| < K$, we can apply Lemma 5 to get an approximate solution $\underline{x_\nu}$ of (40) with the error term $\epsilon^{\beta_\nu}(g_\nu(t))_e$. That is to say, instead of solving (40), we first solve the approximation equation

$$\frac{d\underline{x_\nu}(t)}{dt} = (A + B(t) + b_\nu(t))\underline{x_\nu}(t) + \epsilon^{\beta_\nu}(g_\nu(t) - (g_\nu(t))_e).$$

By Lemma 5, we have

$$\|\underline{x_\nu}\|_{\tilde{r}_\nu,\Pi} \le C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau} \mathcal{L}^{-1/240} \tilde{\mathcal{E}}_\nu \le \frac{\tilde{\mathcal{E}}_\nu^{14/15}}{32}. \tag{41}$$

Moreover, the error term $(g_\nu(t))_e$ satisfies

$$\|(g_\nu)_e\|_{\tilde{r}_\nu - \tilde{r}_0\sigma_\nu,\Pi} \le \mathcal{L}^{-1/240} e^{-K\tilde{r}_0\sigma_\nu} \|g_\nu\|_{\tilde{r}_\nu,\Pi}.$$

Thus system (39) becomes

$$\frac{dy_\nu}{dt} = (A + B(t) + b_\nu(t) + \epsilon^{\beta_\nu} \hat{P}_\nu(t))y_\nu + \epsilon^{\beta_{\nu+1}} \hat{g}_\nu(t) + \hat{h}_\nu(y_\nu, t), \tag{42}$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

where

$$\hat{P}_\nu(t) = P_\nu(t) + \epsilon^{-\beta_\nu} D_x h_\nu(\underline{x_\nu}(t), t), \quad \beta_{\nu+1} = \tfrac{16}{15}\beta_\nu,$$
$$\hat{g}_\nu(t) = \epsilon^{-1/15\beta_\nu} P_\nu(t)\underline{x_\nu}(t) + \epsilon^{-1/15\beta_\nu}(g_\nu(t))_e + \epsilon^{-\beta_{\nu+1}} h_\nu(\underline{x_\nu}(t), t),$$
$$\hat{h}_\nu(y_n, t) = h_\nu(y_\nu + \underline{x_\nu}(t), t)) - h_\nu(\underline{x_\nu}(t), t) - D_x h_\nu(\underline{x_\nu}(t), t) y_\nu,$$

and the analyticity strip has been reduced to $\hat{r}_\nu = \tilde{r}_\nu - \tilde{r}_0\sigma_\nu$.

The terms of this equation must be bounded. Using Lemma 5, we have

$$\begin{aligned}\|\epsilon^{\beta_\nu}\hat{P}_\nu\|_{\hat{r}_\nu} &\le \|\epsilon^{\beta_\nu} P_\nu\|_{\tilde{r}_\nu} + G_\nu\|\underline{x_\nu}\|_{\tilde{r}_\nu}\\ &\le \epsilon^{\beta_\nu}\|P_\nu\|_{\tilde{r}_\nu} + G_\nu \mathcal{L}^{-1/240} C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau}\epsilon^{\beta_\nu}\|g_\nu\|_{\tilde{r}_\nu}\\ &\le 2G_\nu\mathcal{L}^{-1/240}C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)}Q_{n+1}^{18\tau}\tilde{\mathcal{E}}_\nu \le \frac{\tilde{\mathcal{E}}_\nu^{14/15}}{32}.\end{aligned} \tag{43}$$

Then let us bound $\|\epsilon^{\beta_{\nu+1}}\hat{g}_\nu\|_{\hat{r}_\nu}$, again by means of Lemma 5:

$$\begin{aligned}\|\epsilon^{\beta_{\nu+1}}\hat{g}_\nu\|_{\hat{r}_\nu} &\le \epsilon^{\beta_\nu}\|P_\nu\|_{\tilde{r}_\nu}\|\underline{x_\nu}\|_{\tilde{r}_\nu} + \epsilon^{\beta_\nu}\|(g_\nu)_e\|_{\hat{r}_\nu} + \frac{G_\nu}{2}(\|\underline{x_\nu}\|_{\tilde{r}_\nu})^2\\ &\le \mathcal{L}^{-1/240}C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)}Q_{n+1}^{18\tau}\tilde{\mathcal{E}}_\nu^2 + \mathcal{L}^{-1/240}e^{-K\tilde{r}_0\sigma_\nu}\tilde{\mathcal{E}}_\nu\\ &\quad + \frac{G_\nu}{2}(\mathcal{L}^{-1/240}C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)}Q_{n+1}^{18\tau}\tilde{\mathcal{E}}_\nu)^2 \le \frac{\tilde{\mathcal{E}}_\nu^{16/15}}{2} = \frac{\tilde{\mathcal{E}}_{\nu+1}}{2}.\end{aligned}$$

Finally, we bound $D_{y_\nu y_\nu}\hat{h}_\nu(y_\nu, t)$:

$$\|D_{y_\nu y_\nu}\hat{h}_\nu\|_{\hat{r}_\nu} \le \|D_{x_\nu x_\nu} h_\nu(y_\nu + \underline{x_\nu}(t), t))\|_{\tilde{r}_\nu} \le G_\nu,$$

where $y_\nu \in B_{\hat{\kappa}_\nu}(0)$, $\hat{\kappa}_\nu = \kappa_\nu - \|\underline{x_\nu}\|_{\tilde{r}_\nu}$.

For system (42), we first truncate $\hat{P}_\nu(t)$ as the form

$$\begin{aligned}\hat{P}_\nu &= \mathcal{T}_K\hat{P}_\nu + \mathcal{R}_K\hat{P}_\nu\\ &= (\mathcal{T}_K\hat{P}_{\nu,ij})_{d\times d} + (\mathcal{R}_K\hat{P}_{\nu,ij})_{d\times d},\end{aligned}$$

where

$$\mathcal{T}_K\hat{P}_{\nu,ij} = \sum_{|k|<K}\hat{P}_{\nu,ij}^k e^{i\langle k,\omega\rangle t}, \quad \mathcal{R}_K\hat{P}_{\nu,ij} = \sum_{|k|\ge K}\hat{P}_{\nu,ij}^k e^{i\langle k,\omega\rangle t}.$$

Then we further divide $\epsilon^{\beta_\nu}\hat{P}_\nu(t)$ into three parts:

$$\epsilon^{\beta_\nu}\hat{P}_\nu = \epsilon^{\beta_\nu}(\mathcal{T}_K^1\hat{P}_\nu + \mathcal{T}_K^2\hat{P}_\nu + \mathcal{R}_K\hat{P}_\nu),$$

where

$$\epsilon^{\beta_\nu}\mathcal{T}_K^1\hat{P}_\nu = \operatorname{diag}\left(\begin{pmatrix}\hat{u}_1^\nu(t) & \hat{v}_1^\nu(t)\\ -\hat{v}_1^\nu(t) & \hat{u}_1^\nu(t)\end{pmatrix}, \ldots, \begin{pmatrix}\hat{u}_{\tilde{d}}^\nu(t) & \hat{v}_{\tilde{d}}^\nu(t)\\ -\hat{v}_{\tilde{d}}^\nu(t) & \hat{u}_{\tilde{d}}^\nu(t)\end{pmatrix}\right)_{|k|<K}, \tag{44}$$

with

$$\hat{u}_1^\nu(t) = \frac{\epsilon^{\beta_\nu}}{2}(\hat{P}_{\nu 11} + \hat{P}_{\nu 22}), \quad \hat{v}_1^\nu(t) = \frac{\epsilon^{\beta_\nu}}{2}(\hat{P}_{\nu 12} - \hat{P}_{\nu 21}),$$
$$\ldots, \quad \ldots,$$
$$\hat{u}_{\tilde{d}}^\nu(t) = \frac{\epsilon^{\beta_\nu}}{2}(\hat{P}_{\nu 2\tilde{d}-1,2\tilde{d}-1} + \hat{P}_{\nu 2\tilde{d},2\tilde{d}}), \quad \hat{v}_{\tilde{d}}^\nu(t) = \frac{\epsilon^{\beta_\nu}}{2}(\hat{P}_{\nu 2\tilde{d}-1,2\tilde{d}} - \hat{P}_{\nu 2\tilde{d},2\tilde{d}-1}),$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

and $\mathcal{T}_K^2\hat{P}_\nu=\mathcal{T}_K\hat{P}_\nu-\mathcal{T}_K^1\hat{P}_\nu$, that is,

$$\mathrm{diag}\mathcal{T}_K^2\hat{P}_\nu=\mathrm{diag}\left(\begin{pmatrix}u_1^\nu(t) & v_1^\nu(t)\\ v_1^\nu(t) & -u_1^\nu(t)\end{pmatrix},\ldots,\begin{pmatrix}u_{\tilde{d}}^\nu(t) & v_{\tilde{d}}^\nu(t)\\ v_{\tilde{d}}^\nu(t) & -u_{\tilde{d}}^\nu(t)\end{pmatrix}\right),$$

with

$$u_1^\nu(t)=\tfrac{1}{2}(\hat{P}_{\nu 11}-\hat{P}_{\nu 22}),\quad v_1^\nu(t)=\tfrac{1}{2}(\hat{P}_{\nu 12}+\hat{P}_{\nu 21}),$$
$$\ldots,\quad \ldots,$$
$$u_{\tilde{d}}^\nu(t)=\tfrac{1}{2}(\hat{P}_{\nu 2\tilde{d}-1,2\tilde{d}-1}-\hat{P}_{\nu 2\tilde{d},2\tilde{d}}),\quad v_{\tilde{d}}^\nu(t)=\tfrac{1}{2}(\hat{P}_{\nu 2\tilde{d}-1,2\tilde{d}}+\hat{P}_{\nu 2\tilde{d},2\tilde{d}-1}).$$

Moreover, these truncation functions satisfy the estimates

$$\|\mathcal{T}_K^1\hat{P}_\nu\|_{\hat{r}_\nu,\Pi}+\|\mathcal{T}_K^2\hat{P}_\nu\|_{\hat{r}_\nu,\Pi}\le 2(1+G_\nu\mathcal{L}^{-1/240}C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)}Q_{n+1}^{18\tau})M_\nu$$

and

$$\|\mathcal{R}_K\hat{P}_\nu\|_{\tilde{r}_{\nu+1},\Pi}=\sum_{|k|\ge K}|\hat{P}_\nu|_\Pi e^{|k|(\hat{r}_\nu-2\tilde{r}_0\sigma_\nu)}$$
$$\le e^{-2K\tilde{r}_0\sigma_\nu}\|\hat{P}_\nu\|_{\hat{r}_\nu,\Pi}\le\tilde{\mathcal{E}}_\nu^{4/15}\|\hat{P}_\nu\|_{\hat{r}_\nu,\Pi}.$$

For simplicity, we define $\hat{b}_\nu(t)=\epsilon^{\beta_\nu}\mathcal{T}_K^1\hat{P}_\nu$ and $\hat{P}_\nu^{(nre)}=\mathcal{T}_K^2\hat{P}_\nu$, with

$$\mathrm{diag}\hat{P}_\nu^{(nre)}=\mathrm{diag}\left(\begin{pmatrix}u_1^\nu(t) & v_1^\nu(t)\\ v_1^\nu(t) & -u_1^\nu(t)\end{pmatrix},\ldots,\begin{pmatrix}u_{\tilde{d}}^\nu(t) & v_{\tilde{d}}^\nu(t)\\ v_{\tilde{d}}^\nu(t) & -u_{\tilde{d}}^\nu(t)\end{pmatrix}\right).$$

Let

$$S=\frac{1}{\sqrt{2}}\mathrm{diag}\left(\begin{pmatrix}1 & 1\\ -\sqrt{-1} & \sqrt{-1}\end{pmatrix},\ldots,\begin{pmatrix}1 & 1\\ -\sqrt{-1} & \sqrt{-1}\end{pmatrix}\right).$$

Then $S^{-1}\hat{b}_\nu(t)S=\mathrm{diag}(\hat{\Delta}_1^\nu,\hat{\Delta}_2^\nu,\ldots,\hat{\Delta}_{2\tilde{d}-1}^\nu,\hat{\Delta}_{2\tilde{d}}^\nu)$ and $\mathrm{diag}(S^{-1}\hat{P}_\nu^{(nre)}S)=0$, where

$$\hat{\Delta}_1^\nu=\hat{u}_1^\nu(t)-\sqrt{-1}\hat{v}_1^\nu(t),\quad \hat{\Delta}_2^\nu=\hat{u}_1^\nu(t)+\sqrt{-1}\hat{v}_1^\nu(t),$$
$$\ldots,\quad\ldots,$$
$$\hat{\Delta}_{2\tilde{d}-1}^\nu=\hat{u}_{\tilde{d}}^\nu(t)-\sqrt{-1}\hat{v}_{\tilde{d}}^\nu(t),\quad \hat{\Delta}_{2\tilde{d}}^\nu=\hat{u}_{\tilde{d}}^\nu(t)+\sqrt{-1}\hat{v}_{\tilde{d}}^\nu(t).$$

We rewrite system (42) as

$$\frac{dy_\nu}{dt}=(A+B(t)+b_{\nu+1}(t)+\epsilon^{\beta_\nu}(\mathcal{T}_K^2\hat{P}_\nu+\mathcal{R}_K\hat{P}_\nu))y_\nu+\epsilon^{\beta_{\nu+1}}\hat{g}_\nu(t)+\hat{h}_\nu(y_\nu,t),\quad (45)$$

where $b_{\nu+1}(t)=b_\nu(t)+\hat{b}_\nu(t)$.

Let $y_\nu(t)=e^{\epsilon^{\beta_\nu}W_\nu(t)}x_{\nu+1}$, where $W_\nu=W_\nu(\omega t,\epsilon)$ is an analytic quasi-periodic matrix. Denote $\hat{A}_\nu(t)=A+B(t)+b_{\nu+1}(t)$. Then we apply the change of variables $y_\nu(t)=e^{\epsilon^{\beta_\nu}W_\nu(t)}x_{\nu+1}$ to system (45). We have

$$\frac{dx_{\nu+1}}{dt}=\left(\hat{A}_\nu+\epsilon^{\beta_\nu}\left(\hat{A}_\nu W_\nu-W_\nu\hat{A}_\nu+\mathcal{T}_K^2\hat{P}_\nu-\frac{dW_\nu}{dt}\right)\right)x_{\nu+1}+\hat{P}'_{\nu+1}x_{\nu+1},$$
$$+\epsilon^{\beta_{\nu+1}}e^{-\epsilon^{\beta_\nu}W_\nu}\hat{g}_\nu(t)+e^{-\epsilon^{\beta_\nu}W_\nu}\hat{h}_\nu(e^{\epsilon^{\beta_\nu}W_\nu}x_{\nu+1},t),$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

where

$$\hat{P}'_{\nu+1} = (e^{-\epsilon^{\beta_\nu} W_\nu} - I + \epsilon^{\beta_\nu} W_\nu)\hat{A}_\nu e^{\epsilon^{\beta_\nu} W_\nu} + \hat{A}_\nu (e^{\epsilon^{\beta_\nu} W_\nu} - I - \epsilon^{\beta_\nu} W_\nu)$$
$$+ \epsilon^{\beta_\nu}(\mathcal{T}_K^2 \hat{P}_\nu - W_\nu \hat{A}_\nu)(e^{\epsilon^{\beta_\nu} W_\nu} - I) + \epsilon^{\beta_\nu}(e^{-\epsilon^{\beta_\nu} W_\nu} - I)\mathcal{T}_K^2 \hat{P}_\nu e^{\epsilon^{\beta_\nu} W_\nu}$$
$$+ e^{\epsilon^{\beta_\nu} W_\nu} \epsilon^{\beta_\nu}(\mathcal{R}_K \hat{P}_\nu) e^{-\epsilon^{\beta_\nu} W_\nu} + \frac{d(e^{-\epsilon^{\beta_\nu} W_\nu} - I + \epsilon^{\beta_\nu} W_\nu)}{dt}$$
$$+ \frac{de^{-\epsilon^{\beta_\nu} W_\nu}}{dt}(e^{\epsilon^{\beta_\nu} W_\nu} - I). \tag{46}$$

The point is to find W_ν such that

$$\frac{dW_\nu}{dt} - \hat{A}_\nu W_\nu + W_\nu \hat{A}_\nu = \mathcal{T}_K^2 \hat{P}_\nu, \tag{47}$$

where $\mathrm{diag}(S^{-1}\mathcal{T}_K^2 \hat{P}_\nu S) = 0$.

Thanks to the definition of $b_{\nu+1}$ and (43), we get $\|b_{\nu+1} - b_\nu\|_{\hat{r}_\nu,\Pi} \le (\epsilon^{\beta_\nu} M_\nu)^{14/15} = \tilde{\mathcal{E}}_\nu^{14/15}$. From (38), it follows that

$$\|b_{\nu+1}\|_{\hat{r}_\nu,\Pi} \le \|b_{\nu+1} - b_\nu\|_{\hat{r}_\nu,\Pi} + \sum_{j=1}^{\nu} \|b_j - b_{j-1}\|_{\tilde{r}_j,\Pi} \le 2\tilde{\mathcal{E}}_0^{14/15}. \tag{48}$$

Then, by (36), we have

$$\|b_{\nu+1}\|_{\hat{r}_\nu,\Pi} \le 2\tilde{\mathcal{E}}_0^{14/15} < \frac{\gamma^{\mathcal{A}\tau+2}}{2C_3(\tau) Q_{n+1}^{18\tau}},$$

which implies that condition (21) is satisfied. What is more, since $K < Q_{n+2}$, it is obvious that $\mathcal{R}_{Q_{n+2}}(b_{\nu+1} - b_\nu) = 0$, which implies $\mathcal{R}_{Q_{n+2}} b_{\nu+1} = 0$ by $\mathcal{R}_{Q_{n+2}} b_\nu = 0$.

On the other hand, by the assumptions of Lemma 8, $B(t)$ satisfies condition (22), $\mathcal{R}_{Q_{n+1}} B = 0$, and

$$|\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i]) + (\lambda_j + [\Xi_j])| \ge \frac{\gamma}{(|k| + 1)^{3\tau}} \quad \text{for all } |k| < K,\ i \ne j.$$

Then for any $|k| < K$, $i \ne j$, we can apply Lemma 6 to get an approximate solution W_ν of (47) with the error term $(\mathcal{T}_K^2 \hat{P}_\nu)_e$. That is to say, instead of solving (47), we first solve the approximation equation

$$\frac{dW_\nu}{dt} - \hat{A}_\nu W_\nu + W_\nu \hat{A}_\nu = \mathcal{T}_K^2 \hat{P}_\nu - (\mathcal{T}_K^2 \hat{P}_\nu)_e.$$

By Lemma 6, we have

$$\|W_\nu\|_{\hat{r}_\nu,\Pi} \le 2C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau} \mathcal{L}^{-1/240} \|\hat{P}_\nu\|_{\hat{r},\Pi}.$$

Moreover, the error term $(\mathcal{T}_K^2 \hat{P}_\nu)_e$ satisfies

$$\|(\mathcal{T}_K^2 \hat{P}_\nu)_e\|_{\hat{r}_\nu - \tilde{r}_0\sigma_\nu,\Pi} \le 2\mathcal{L}^{-1/240} e^{-K\tilde{r}_0\sigma_\nu}(\|\hat{P}_\nu\|_{\hat{r}_\nu} + 2\|b_{\nu+1}\|_{\hat{r}_\nu} \cdot \|W\|_{\hat{r}_\nu}).$$

Then system (42) becomes

$$\frac{dx_{\nu+1}}{dt} = (A + B(t) + b_{\nu+1}(t) + \epsilon^{16/15\beta_\nu} P_{\nu+1}(t))x_{\nu+1} + \epsilon^{16/15\beta_\nu} e^{-\epsilon^{\beta_\nu} W_\nu} \hat{g}_\nu(t)$$
$$+ e^{-\epsilon^{\beta_\nu} W_\nu} \hat{h}_\nu(e^{\epsilon^{\beta_\nu} W_\nu} x_{\nu+1}, t), \tag{49}$$

where $P_{\nu+1} = \epsilon^{-16/15\beta_\nu}(\hat{P}'_{\nu+1} + \epsilon^{\beta_\nu}(\mathcal{T}_K^2 \hat{P}_\nu)_e)$. Thus system (49) can be rewritten as

$$\frac{dx_{\nu+1}}{dt} = (A + B(t) + b_{\nu+1}(t) + \epsilon^{\beta_{\nu+1}} P_{\nu+1}(t))x_{\nu+1} + \epsilon^{\beta_{\nu+1}} g_{\nu+1}(t) + h_{\nu+1}(x_{\nu+1}, t),$$

where

$$\begin{gathered} b_{\nu+1}(t) = b_\nu(t) + \hat{b}_\nu(t), \quad \hat{b}_\nu(t) = \epsilon^{\beta_\nu} \mathcal{T}_K^1 \hat{P}_\nu, \\ P_{\nu+1} = \epsilon^{-16/15\beta_\nu}(\hat{P}'_{\nu+1} + \epsilon^{\beta_\nu}(\mathcal{T}_K^2 \hat{P}_\nu)_e), \\ g_{\nu+1}(t) = e^{-\epsilon^{\beta_\nu} W_\nu} \hat{g}_\nu(t), \quad \beta_{\nu+1} = \tfrac{16}{15}\beta_\nu, \\ h_{\nu+1}(x_{\nu+1}, t) = e^{-\epsilon^{\beta_\nu} W_\nu} \hat{h}_\nu(e^{\epsilon^{\beta_\nu} W_\nu} x_{\nu+1}, t), \end{gathered}$$

and the analyticity strip has been reduced to $\tilde{r}_{\nu+1}$.

Now we estimate the new perturbation terms $\epsilon^{\beta_{\nu+1}} P_{\nu+1}$ and $\epsilon^{\beta_{\nu+1}} g_{\nu+1}$. Suppose that $\|\epsilon^{\beta_\nu} P_\nu\|_{\tilde{r}_\nu,\Pi} \le \epsilon^{\beta_\nu} M_\nu = \tilde{\mathcal{E}}_\nu$, and $\tilde{\mathcal{E}}_0$ is sufficiently small such that

$$\|\epsilon^{\beta_\nu} W_\nu\|_{\hat{r}_\nu,\Pi} \le 2C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau} \mathcal{L}^{-1/240} \epsilon^{\beta_\nu} \|\hat{P}_\nu\|_{\hat{r}_\nu,\Pi} \le \frac{\tilde{\mathcal{E}}_\nu^{14/15}}{32} \le \frac{1}{2}. \tag{50}$$

Then by $e^W = I + W + W^2/2! + \cdots + W^n/n! + \cdots$, we have

$$\|e^{\pm\epsilon^{\beta_\nu} W_\nu}\|_{\hat{r}_\nu,\Pi} \le \frac{1}{1 - \|\epsilon^{\beta_\nu} W_\nu\|_{\hat{r}_\nu,\Pi}} \le 2.$$

Similarly, we have

$$\|e^{\pm\epsilon^{\beta_\nu} W_\nu} - I\|_{\hat{r}_\nu,\Pi} \le 4C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau} \mathcal{L}^{-1/240} \tilde{\mathcal{E}}_\nu \le \frac{\tilde{\mathcal{E}}_\nu^{14/15}}{16},$$

$$\|e^{\pm\epsilon^{\beta_\nu} W_\nu} - I \mp \epsilon^{\beta_\nu} W_\nu\|_{\hat{r}_\nu,\Pi} \le 8C_3^2(\tau)\gamma^{-2(\mathcal{A}\tau+2)} Q_{n+1}^{36\tau} \mathcal{L}^{-1/120} \tilde{\mathcal{E}}_\nu^2 \le \frac{\tilde{\mathcal{E}}_\nu^{17/15}}{16}.$$

By Cauchy's estimates and in view of

$$\hat{r}_\nu - \tilde{r}_{\nu+1} = 2\tilde{r}_0\sigma_\nu,$$

it follows that

$$\left\|\frac{d(\epsilon^{\beta_\nu} W_\nu)}{dt}\right\|_{\tilde{r}_{\nu+1},\Pi} \le \frac{\|\epsilon^{\beta_\nu} W_\nu\|_{\hat{r}_\nu}}{2\tilde{r}_0\sigma_\nu} \le \frac{2C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau} \mathcal{L}^{-1/240} \tilde{\mathcal{E}}_\nu}{2\tilde{r}_0\sigma_\nu} \le \frac{\tilde{\mathcal{E}}_\nu^{13/15}}{16},$$

$$\left\|\frac{de^{-\epsilon^{\beta_\nu} W_\nu}}{dt}\right\|_{\tilde{r}_{\nu+1},\Pi} \le \frac{4C_3(\tau)\gamma^{-(\mathcal{A}\tau+2)} Q_{n+1}^{18\tau} \mathcal{L}^{-1/240} \tilde{\mathcal{E}}_\nu}{2\tilde{r}_0\sigma_\nu} \le \frac{\tilde{\epsilon}_\nu^{13/15}}{16},$$

and

$$\left\|\frac{d(e^{-\epsilon^{\beta_\nu} W_\nu} - I + \epsilon^{\beta_\nu} W_\nu)}{dt}\right\|_{\tilde{r}_{\nu+1},\Pi} \le \frac{8C_3^2(\tau)\gamma^{-2(\mathcal{A}\tau+2)} Q_{n+1}^{36\tau} \mathcal{L}^{-1/120} \tilde{\mathcal{E}}_\nu^2}{2\tilde{r}_0\sigma_\nu} \le \frac{\tilde{\mathcal{E}}_\nu^{16/15}}{8}.$$

Altogether, combining the definition (46) of $\hat{P}'_{\nu+1}$, the estimate of $(\mathcal{T}_K^2 \hat{P}_\nu)_e$ and all the above estimates, we obtain

$$\begin{aligned} \|P_{\nu+1}\|_{\tilde{r}_{\nu+1},\Pi} &\le \epsilon^{-16/15\beta_\nu}(\|\hat{P}'_{\nu+1}\|_{\tilde{r}_{\nu+1},\Pi} + \|\epsilon^{\beta_\nu}(\mathcal{T}_K^2 \hat{P}_\nu)_e\|_{\tilde{r}_{\nu+1},\Pi}) \\ &\le \epsilon^{-16/15\beta_\nu} \tilde{\mathcal{E}}_\nu^{16/15} = M_\nu^{16/15}. \end{aligned}$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

Then

$$\|\epsilon^{\beta_{\nu+1}} P_{\nu+1}\|_{\tilde{r}_{\nu+1},\Pi} \le \epsilon^{16/15\beta_\nu} M_\nu^{16/15} = \tilde{\mathcal{E}}_\nu^{16/15} = \tilde{\mathcal{E}}_{\nu+1}.$$

Similarly,

$$\|\epsilon^{\beta_{\nu+1}} g_{\nu+1}\|_{\tilde{r}_{\nu+1},\Pi} \le 2\epsilon^{\beta_{\nu+1}} \|\hat{g}_\nu\|_{\hat{r}_\nu,\Pi} \le \tilde{\mathcal{E}}_\nu^{16/15} = \tilde{\mathcal{E}}_{\nu+1}$$

and

$$\begin{aligned}\|D_{x_{\nu+1}x_{\nu+1}} h_{\nu+1}\|_{\tilde{r}_{\nu+1}} &\le \frac{G_\nu}{1-\epsilon^{\beta_\nu}\|W_\nu\|_{\tilde{r}_{\nu+1}}} (\|e^{\epsilon_\nu W_\nu}\|_{\tilde{r}_{\nu+1}})^2 \\ &\le G_\nu \frac{(1+\epsilon^{\beta_\nu}\|W_\nu\|_{\tilde{r}_{\nu+1}})^2}{1-\epsilon^{\beta_\nu}\|W_\nu\|_{\tilde{r}_{\nu+1}}} = G_{\nu+1},\end{aligned}$$

where $x_{\nu+1} \in B_{\kappa_{\nu+1}}(0)$, $\kappa_{\nu+1} = \hat{\kappa}_\nu/(1+2\epsilon^{\beta_\nu}\|W_\nu\|_{\tilde{r}_{\nu+1}})$ and $\hat{\kappa}_\nu = \kappa_\nu - \|x_\nu\|_{\tilde{r}_\nu}$. The convergence of the sequences of analytic radius $\{\kappa_\nu\}$ and second-order derivatives, $\{G_\nu\}$ is given in §4.2.

Notice that there exists $L \ge m_0$ such that $\tilde{\mathcal{E}}_L \le \mathcal{L}_+ \tilde{\mathcal{E}}_0 = \mathcal{E}_+$ and $\tilde{\mathcal{E}}_{L-1} > \mathcal{E}_+$. Once we reach the Lth step, we stop the above iteration. Let $\Phi = \phi_0 \circ \cdots \circ \phi_{L-1}$, $x_+ = x_L$, $B_+(t) = B(t) + b_L(t)$, where $b_L(t) = \sum_{j=0}^{L-1} \epsilon^{\beta_j} \mathcal{T}_k^1 \hat{P}_j(t)$ has the form as in (44), $\beta_+ = (\frac{16}{15})^L \beta$, $P_+(t) = P_L(t)$, $g_+(t) = g_L(t)$, $h_+(t) = h_L(t)$. Then the transformation $x = \Phi x_+$ reduces the system (30) to

$$\frac{dx_+}{dt} = (A + B_+(t) + \epsilon^{\beta_+} P_+(t))x_+ + \epsilon^{\beta_+} g_+(t) + h_+(x_+, t).$$

Moreover, since $K < Q_{n+2}$, it follows that $\mathcal{R}_{Q_{n+2}} B_+(t) = 0$.

Since $\Phi = \phi_0 \circ \cdots \circ \phi_{L-1}$, where $\phi_i : x_i = e^{\epsilon^{\beta_i} W_i} x_{i+1} + \underline{x_i}$ $(i = 0, \ldots, L-1)$, we obtain $x_0 = \Phi x_L$, that is,

$$\begin{aligned} x_0 &= e^{\epsilon^{\beta_0} W_0} x_1 + \underline{x_0} \\ &= e^{\epsilon^{\beta_0} W_0} e^{\epsilon^{\beta_1} W_1} x_2 + e^{\epsilon^{\beta_0} W_0} \underline{x_1} + \underline{x_0} \\ &= e^{\epsilon^{\beta_0} W_0} e^{\epsilon^{\beta_1} W_1} \cdots e^{\epsilon^{\beta_{L-1}} W_{L-1}} x_L + \sum_{i=1}^{L-1} e^{\epsilon^{\beta_0} W_0} \cdots e^{\epsilon^{\beta_{i-1}} W_{i-1}} \underline{x_i} + \underline{x_0}. \end{aligned}$$

Then estimates (41) and (50) yield that

$$\begin{aligned} \|\Phi - \mathrm{id}\|_{r_+} &\le \left\| \prod_{i=0}^{L-1} e^{\epsilon^{\beta_i} W_i} - \mathrm{Id} \right\| + \sum_{i=1}^{L-1} \|e^{\epsilon^{\beta_0} W_0} \cdots e^{\epsilon^{\beta_{i-1}} W_{i-1}} \underline{x_i}\| + \|\underline{x_0}\| \\ &\le \|e^{\sum_{i=0}^{L-1} \epsilon^{\beta_i} W_i} - \mathrm{Id}\| + \sum_{i=1}^{L-1} \|e^{\sum_{j=0}^{i-1} \epsilon^{\beta_j} W_j} \underline{x_i}\| + \|\underline{x_0}\| \\ &\le 2 \sum_{i=0}^{L-1} \|\epsilon^{\beta_i} W_i\| + 4\|\underline{x_0}\| \le \frac{\tilde{\mathcal{E}}_0^{14/15}}{4}. \end{aligned}$$

In addition, it follows from (48) that estimate (34) holds. Thus, we get all the estimates (33)–(35).

Recall that

$$r_+ = \tilde{r}_0(1-\eta).$$

By the definition of σ_m, we know $\sigma_{m+1} = \frac{16}{15}\sigma_m$ if $m \geq m_0$. Then, by the selection of $L \in \mathbb{Z}_+$ (i.e. $\tilde{\mathcal{E}}_L \leq \mathcal{E}_+, \tilde{\mathcal{E}}_{L-1} > \mathcal{E}_+ = \mathcal{L}_+\tilde{\mathcal{E}}_0$), we have

$$\begin{aligned}\sum_{j=0}^{L-1}\sigma_j &= \sum_{j=0}^{m_0-1}\sigma_j + \sum_{j=m_0}^{L-1}\sigma_j \\ &\leq \frac{\eta}{4} - \frac{2\ln\tilde{\mathcal{E}}_0}{K\tilde{r}_0}\bigg(\frac{16}{15}\bigg(\frac{16}{15}\bigg)^{L-1} - \bigg(\frac{16}{15}\bigg)^{m_0}\bigg) \\ &\leq \frac{\eta}{4} - \frac{32}{15}\frac{\ln\mathcal{L}_+}{K\tilde{r}_0} - \frac{32}{15}\frac{\ln\tilde{\mathcal{E}}_0}{K\tilde{r}_0} + \frac{2\ln\tilde{\mathcal{E}}_0}{K\tilde{r}_0}\bigg(\frac{16}{15}\bigg)^{m_0}.\end{aligned}$$

If $m_0 \geq 1$, we have that

$$\sum_{j=0}^{L-1}\sigma_j \leq \frac{\eta}{4} - \frac{32}{15}\frac{\ln\mathcal{L}_+}{K\tilde{r}_0} \leq \frac{\eta}{4} + \frac{\tilde{c}}{8}\bigg(\frac{1}{Q_{n+1}^{\mathcal{M}-\tau}} + \frac{1}{\bar{Q}_{n+1}^{2/\mathcal{A}}}\bigg) \leq \frac{\eta}{3}.$$

If $m_0 = 0$, that is to say $K\tilde{\mathcal{E}}_0^{16/15} < 1$, we have

$$\begin{aligned}\sum_{j=0}^{L-1}\sigma_j &\leq -\frac{32}{15}\frac{\ln\mathcal{L}_+}{K\tilde{r}_0} - \frac{32}{15}\frac{\ln\tilde{\mathcal{E}}_0}{K\tilde{r}_0} + \frac{2\ln\tilde{\mathcal{E}}_0}{K\tilde{r}_0} \\ &= -\frac{32}{15}\frac{\ln\mathcal{L}_+}{K\tilde{r}_0} - \frac{2}{15}\frac{\ln\tilde{\mathcal{E}}_0}{K\tilde{r}_0}.\end{aligned}$$

Suppose that $\tilde{\mathcal{E}}_0 \geq \mathcal{L}_+$. Then

$$\sum_{j=0}^{L-1}\sigma_j \leq -\frac{34}{15}\frac{\ln\mathcal{L}_+}{K\tilde{r}_0} \leq \frac{\eta}{3},$$

which implies $\tilde{r}_L = \tilde{r}_0 - 3\tilde{r}_0\sum_{j=0}^{L-1}\sigma_j \geq r_+$.

Remark. If $\tilde{\mathcal{E}}_0 < \mathcal{L}_+$, which means that $\tilde{\mathcal{E}}_0$ is small enough, we do not need to do the above iteration and let $\Phi = \mathrm{id}$, where id is an identity mapping. Since $\tilde{\mathcal{E}}_0 < \mathcal{L}_+ = e^{-c_0\gamma r_+(\bar{Q}_{n_0+1}/Q_{n_0+1}^{\mathcal{M}} + \bar{Q}_{n_0+1}^{1/\mathcal{M}^{1/4}})}$ and $\mathcal{L}_{n+1} = e^{-c_0\gamma_n r_{n+1}(\bar{Q}_{n_0+n+1}/Q_{n+1}^{\mathcal{M}} + \bar{Q}_{n_0+n+1}^{1/\mathcal{M}^{1/4}})}$ is a series of monotonically decreasing numbers, there exists $\mathcal{L}_{k+1} (k \geq 1)$ such that $\cdots < \mathcal{L}_{k+2} < \mathcal{L}_{k+1} < \tilde{\mathcal{E}}_0 < \mathcal{L}_k$. Then we let $\Phi_0 = \Phi_1 = \cdots = \Phi_{k-1} = \mathrm{id}$, and apply the above iteration to find Φ_k from the $(k+1)$th step, since $\tilde{\mathcal{E}}_0 > \mathcal{L}_{k+1}$. Therefore, without loss of generality we suppose that $\tilde{\mathcal{E}}_0 \geq \mathcal{L}_+$. □

4.2. *Iteration and convergence.* Let $0 < r_* < r$, $\tau > 2$, $\gamma > 0$, and $\mathcal{A}$ and $\mathcal{M}$ be defined as in (31). Since $r/r_* > 1$, there exists $\tilde{c}_0 = \frac{1}{2}(r/r_* + 1)$ such that $r > \tilde{c}_0 r_*$. Let

$$\begin{aligned}\tilde{c} = \frac{1}{12}\bigg(1 - \frac{1}{\tilde{c}_0}\bigg) < 1, \quad T_1 = \bigg(\frac{4\cdot 10^\tau J(\tau)\ln 2}{\tilde{c}r_*}\bigg)^{\mathcal{A}}, \\ T = \max\bigg\{T_0\bigg(\frac{\gamma}{2}\bigg)^{-\mathcal{A}/2}, T_1\gamma^{-\mathcal{A}}, 4^{\mathcal{A}^4}\bigg\}.\end{aligned} \tag{51}$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

These constants $\varepsilon_0(\tau, r_*, \tilde{c})$, $\varepsilon_1(\tau, r_*, \tilde{c})$, $J(\tau)$, $T_0(\tau, r_*, \tilde{c})$ are defined as in Lemma 8.

Due to the choice of Lemma 2, there exists some $n_0 \in \mathbb{Z}_+$ such that $Q_{n_0} \le T^{\mathcal{A}^4}$, whereas $\bar{Q}_{n_0} \ge T$. Since $Q_{n_0} \le T^{\mathcal{A}^4}$, we can take ϵ sufficiently small, depending on r, r_*, γ and τ, but not on α, such that

$$\mathcal{E} < \min\left\{\varepsilon_0\left(\frac{\gamma}{2}\right)^J, \left(\frac{\varepsilon_1\gamma}{2}\right)^2, \frac{1}{T^{120\tau\mathcal{A}^4}}\right\} \le \min\left\{\varepsilon_0\left(\frac{\gamma}{2}\right)^J, \left(\frac{\varepsilon_1\gamma}{2}\right)^2, \frac{1}{Q_{n_0}^{120\tau}}\right\}. \quad (52)$$

For any given $\mathcal{E} > 0$ satisfying (52), we define some sequences inductively:

$$\mathcal{E}_0 = \mathcal{E}, \quad r_0 = r, \quad \gamma_0 = \gamma,$$

$$\gamma_n = \frac{\gamma}{2^n}, \quad \mathcal{L}_{n+1} = e^{-c_0\gamma_n r_{n+1}(\bar{Q}_{n_0+n+1}/Q^{\mathcal{M}}_{n_0+n+1} + \bar{Q}^{1/\mathcal{M}^{1/4}}_{n_0+n+1})}, \quad \mathcal{E}_{n+1} = \mathcal{L}_{n+1}\mathcal{E}_n,$$

$$\eta_n = \tilde{c}Q_{n_0+n}^{-1/2\mathcal{A}^4}, \quad r_{n+1} = r_n(1-\eta_n)^2,$$

$$K_n = \left[\frac{\gamma_n}{4\cdot 10^\tau}\max\left\{\frac{\bar{Q}_{n_0+n}}{Q^\tau_{n_0+n}}, \bar{Q}^{3/\mathcal{A}}_{n_0+n}\right\}\right].$$

Suppose that on Π_n the non-resonant conditions (53) for $|k| < K_n$ hold. Define

$$R^1_{n+1}(\gamma_{n+1}) = \left\{\epsilon \in \Pi_n : |\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i^{n+1}])| < \frac{\gamma_{n+1}}{(|k|+1)^{3\tau}}, \forall |k| < K_{n+1}\right\},$$

$$R^2_{n+1}(\gamma_{n+1}) = \left\{\epsilon \in \Pi_n : |\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i^{n+1}]) + (\lambda_j + [\Xi_j^{n+1}])| < \frac{\gamma_{n+1}}{(|k|+1)^{3\tau}},\right.$$
$$\left.\forall |k| < K_{n+1}, i \ne j\right\},$$

and

$$R_{n+1}(\gamma_{n+1}) = R^1_{n+1}(\gamma_{n+1}) \bigcup R^2_{n+1}(\gamma_{n+1}).$$

Then for any $\epsilon \in \Pi_{n+1} = \Pi_n \backslash R_{n+1}(\gamma_{n+1})$, the corresponding non-resonant conditions of the $(n+1)$th step hold.

First, we claim that $r_n \ge r_*$ for all n. In fact, by our selection $Q_{n_0+1} \ge \bar{Q}_{n_0} \ge T \ge 4^{\mathcal{A}^4}$, we have

$$\prod_{k=1}^{\infty}(1-2\eta_k) \ge 1 - 4\sum_{k=1}^{\infty}\eta_k \ge 1 - 8\tilde{c}Q_{n_0+1}^{-1/2\mathcal{A}^4} > 1 - 8\tilde{c},$$

which implies that for any $n \ge 0$,

$$r_n = r\prod_{k=0}^{n}(1-\eta_k)^2 > r(1-2\eta_0)\prod_{k=1}^{\infty}(1-2\eta_k) > r(1-12\tilde{c}) > r_*.$$

Second, by the choice of parameters we can verify that B_n, $\bar{Q}_{n_0+n}$, P_n satisfy that

$$\bar{Q}_{n_0+n} \ge T_0\gamma_n^{-\mathcal{A}/2},$$

$$\|B_n\|_{r_n,\Pi_n}\left(\frac{\bar{Q}_{n_0+n-1}}{Q^{\mathcal{A}^4}_{n_0+n-1}} + \bar{Q}^{1/\mathcal{A}}_{n_0+n-1}\right) \le \varepsilon_1\gamma_n\left(\frac{\bar{Q}_{n_0+n-1}}{Q^{\mathcal{M}}_{n_0+n-1}} + \bar{Q}^{1/\mathcal{M}^{1/4}}_{n_0+n-1}\right),$$

$$\max\{\|\epsilon^{\beta_n}P_n\|_{r_n,\Pi_n}, \|\epsilon^{\beta_n}g_n\|_{r_n,\Pi_n}\} \le \epsilon^{\beta_n}M_n = \mathcal{E}_n \le \varepsilon_0\gamma_n^J\mathcal{L}_n.$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

In what follows we check the above estimates by induction. By $\bar{Q}_{n+1} \geq \bar{Q}_n^{\mathcal{A}}$ and $\bar{Q}_{n_0} \geq T_0\gamma_0^{-\mathcal{A}/2}$, we have

$$\bar{Q}_{n_0+n} \geq \bar{Q}_{n_0}^{\mathcal{A}^n} \geq (T_0\gamma_0^{-\mathcal{A}/2})^{\mathcal{A}^n} \geq T_0\gamma_n^{-\mathcal{A}/2}.$$

By $Q_{n+1} \geq Q_n^{\mathcal{A}}$, $\bar{Q}_{n+1} \geq \bar{Q}_n^{\mathcal{A}}$ and $\mathcal{E}_0 \leq (\varepsilon_1\gamma/2)^2$, it follows that

$$\begin{aligned}\|B_n\|_{r_n,\Pi_n}\bigg(\frac{\bar{Q}_{n_0+n-1}}{Q_{n_0+n-1}^{\mathcal{A}^4}} + \bar{Q}_{n_0+n-1}^{1/\mathcal{A}}\bigg) &\leq \varepsilon_1\gamma\bigg(\frac{\bar{Q}_{n_0+n-1}}{Q_{n_0+n-1}^{\mathcal{A}^4}} + \bar{Q}_{n_0+n-1}^{1/\mathcal{A}}\bigg)\\ &\leq \varepsilon_1\gamma_n\bigg(\frac{\bar{Q}_{n_0+n-1}}{Q_{n_0+n-1}^{\mathcal{M}}} + \bar{Q}_{n_0+n-1}^{1/\mathcal{M}^{1/4}}\bigg).\end{aligned}$$

Since $Q_{n_0} \geq T_1\gamma^{-\mathcal{A}}$, one can check that $\mathcal{L}_1 \leq 1/2^{J(\tau)}$. Then the definition $\mathcal{L}_n$ and the estimate $\mathcal{E}_0 \leq \varepsilon_0(\gamma/2)^J$ yield that

$$\begin{aligned}\mathcal{E}_n &= \mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_1\mathcal{E}_0 \leq \mathcal{L}_n\frac{1}{2^{(n-1)J}}\mathcal{E}_0\\ &\leq \mathcal{L}_n\frac{1}{2^{(n-1)J}}\varepsilon_0\bigg(\frac{\gamma}{2}\bigg)^J = \varepsilon_0\gamma_n^J\mathcal{L}_n.\end{aligned}$$

After setting the parameters, we will perform infinite KAM iteration and prove its convergence. For the first step, let $B_0 = 0$, $\beta_0 = 1$, $P_0 = P$, $g_0 = g$ and $h_0 = h$. By our selection $\bar{Q}_{n_0} \geq T \geq T_0\gamma^{-\mathcal{A}/2}$, and (52),

$$\max\{\epsilon_0\|P_0\|_{r_0,\Pi_0}, \epsilon_0\|g_0\|_{r_0,\Pi_0}\} \leq \epsilon_0 M_0 = \mathcal{E}_0 \leq \min\bigg\{\varepsilon_0\bigg(\frac{\gamma}{2}\bigg)^J, \frac{1}{Q_{n_0}^{120\tau}}\bigg\}.$$

Meanwhile, we can check that (36) and (37) hold. Thus we apply Lemma 8 and get a transformation $\Phi_0 : D(r_1) \times \Pi_1 \to D(r_0) \times \Pi_0$ such that system (1) is transformed into

$$\frac{dx_1}{dt} = (A + B_1(t) + \epsilon^{\beta_1}P_1(t))x_1 + \epsilon^{\beta_1}g_1(t) + h_1(x_1, t),$$

satisfying

$$\|B_1\|_{r_1,\Pi_1} \leq 2\mathcal{E}_0^{14/15} \leq \varepsilon_1\gamma,$$

and

$$\max\{\|\epsilon^{\beta_1}P_1\|_{r_1,\Pi_1}, \|\epsilon^{\beta_1}g_1\|_{r_1,\Pi_1}\} \leq \mathcal{E}_0\mathcal{L} \leq \varepsilon_0\gamma^J\mathcal{L}.$$

Note that the above estimates mean that all the conditions of Lemma 8 for the next step hold.

Inductively, there exists a subset $\Pi_n \subset \Pi$ with

$$\begin{aligned}\Pi_n = \bigg\{\epsilon \in \Pi : |\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i^n]) + (\lambda_j + [\Xi_j^n])| \geq \frac{\gamma_n}{(|k|+1)^{3\tau}},\\ |\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i^n])| \geq \frac{\gamma_n}{(|k|+1)^{3\tau}}, \forall |k| < K_n, i \neq j\bigg\},\end{aligned} \tag{53}$$

and for $\epsilon \in \Pi_n$ there exists a transformation Φ_{n-1} such that it reduces the system of the previous step to a desired form and satisfies

$$\|\Phi_{n-1} - \mathrm{id}\|_{r_n,\Pi_n} \leq \frac{\mathcal{E}_{n-1}^{14/15}}{4}.$$

Let $\Phi^n = \Phi_0 \circ \cdots \circ \Phi_{n-1}$. Then, for any $\epsilon \in \Pi_n$, the transformation $x = \Phi^n x_n$ reduces system (1) to the form

$$\frac{dx_n}{dt} = (A + B_n(t) + \epsilon^{\beta_n} P_n(t))x_n + \epsilon^{\beta_n} g_n(t) + h_n(x_n, t),$$

where A has the real Jordan form (9), and $B_n(t)$ has the form

$$B_n(t) = \mathrm{diag}\left(\begin{pmatrix} U_1^n(t) & V_1^n(t) \\ -V_1^n(t) & U_1^n(t) \end{pmatrix}, \ldots, \begin{pmatrix} U_{\tilde{d}}^n(t) & V_{\tilde{d}}^n(t) \\ -V_{\tilde{d}}^n(t) & U_{\tilde{d}}^n(t) \end{pmatrix}\right),$$

with $U_j^n(t)$, $V_j^n(t)$ $(j = 1, \ldots, \tilde{d})$ being real functions,

$$\max\{\|\epsilon^{\beta_n} P_n\|_{r_n,\Pi_n}, \|\epsilon^{\beta_n} g_n\|_{r_n,\Pi_n}\} \le \epsilon^{\beta_n} M_n = \mathcal{E}_n, \quad \|B_n - B_{n-1}\|_{r_n,\Pi_n} \le 2\mathcal{E}_{n-1}^{14/15},$$

and $h_n(x_n, t)$ is analytic with respect to x_n on the ball $B_{\kappa_n}(0)$, satisfying

$$\|D_{x_n x_n} h_n\|_{r_n} \le G_n = G_{n-1} \frac{(1 + \epsilon^{\beta_{n-1}} \|W_{n-1}\|_{r_n})^2}{1 - \epsilon^{\beta_{n-1}} \|W_{n-1}\|_{r_n}}.$$

Consider now the convergence of Φ^n. By Lemma 8, we observe that

$$\|\Phi_n - \mathrm{id}\|_{r_{n+1},\Pi_{n+1}} \le \frac{\mathcal{E}_n^{14/15}}{4}$$

and

$$\|(D\Phi_n - \mathrm{Id})\|_{r_{n+1},\Pi_{n+1}} \le \frac{\mathcal{E}_n^{14/15}}{16},$$

where D denotes the Jacobian with respect to x. By induction we have $D\Phi^n = D\Phi_0 \cdots D\Phi_{n-1}$, with the Jacobians evaluated at different points, and

$$\|D\Phi^n\| = \|D\Phi_0 \cdots D\Phi_{n-1}\| \le \prod_{k\ge 0}\left(1 + \frac{\mathcal{E}_k^{14/15}}{16}\right) < \infty,$$

which is uniformly bounded on $D_n \times \Pi_n$.

Then

$$\begin{aligned} \|\Phi^{n+1} - \Phi^n\|_{r_{n+1},\Pi_{n+1}} &= \|\Phi^n \circ \Phi_n - \Phi^n\|_{r_{n+1},\Pi_{n+1}} \\ &\le \|D\Phi^n\|_{r_n,\Pi_n} \|\Phi_n - \mathrm{id}\|_{r_{n+1},\Pi_{n+1}} \\ &\le \frac{\mathcal{E}_n^{14/15}}{2} \end{aligned}$$

and

$$\|\Phi^{n+1} - \mathrm{id}\|_{r_{n+1},\Pi_{n+1}} \le \sum_{k\ge 0} \frac{\mathcal{E}_k^{14/15}}{2} \le \mathcal{E}_0^{14/15}.$$

By the methods in [**17**], we can prove the convergence of the radius κ_n of the ball where h_n is analytic with respect to x_n, and the second-order derivative G_n of h_n with respect to x_n.

It has been shown that

$$\kappa_{n+1} = \frac{\hat{\kappa}_n}{1 + 2\epsilon^{\beta_n}\|W_n\|_{r_{n+1}}} = \frac{\kappa_n}{1 + 2\epsilon^{\beta_n}\|W_n\|_{r_{n+1}}} - \frac{\|\underline{x_n}\|_{r_n}}{1 + 2\epsilon^{\beta_n}\|W_n\|_{r_{n+1}}}.$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

We now define

$$a_n = \frac{1}{1+2\epsilon^{\beta_n}\|W_n\|_{r_{n+1}}}, \quad b_n = \frac{\|\underline{x_n}\|_{r_n}}{1+2\epsilon^{\beta_n}\|W_n\|_{r_{n+1}}},$$

where $\prod_{n=0}^{\infty} a_n$ converges,

$$\left|\ln \prod_{n=0}^{\infty} a_n\right| \le \sum_{n=0}^{\infty} |\ln a_n| \le 2\sum_{n=0}^{\infty} \epsilon^{\beta_n}\|W_n\|_{r_{n+1}} < \infty,$$

and $\sum_{n=0}^{\infty} b_n$ converges.

Suppose that $\prod_{n=0}^{\infty} a_n = a \in (0, 1)$ and $\sum_{n=0}^{\infty} b_n = b$. By induction

$$\begin{aligned}\kappa_{n+1} &= \left(\prod_{i=0}^{n} a_i\right)\kappa_0 - \sum_{i=0}^{n-1}\left[\left(\prod_{j=i+1}^{n} a_j\right)b_i\right] - b_n \\ &\ge a\kappa_0 - b.\end{aligned}$$

Therefore, the sequence $\{\kappa_n\}$ is monotonically decreasing and has a lower bound, so it is convergent, and $\kappa_\infty \ge a\kappa_0 - b$, which is positive if ϵ is taken small enough.

Consider now the value G_n. From Lemma 8, we know that

$$G_{n+1} = G_n \frac{(1+\epsilon^{\beta_n}\|W_n\|_{r_{n+1}})^2}{1-\epsilon^{\beta_n}\|W_n\|_{r_{n+1}}},$$

and, by means of the inequality $1/(1-x) \le 1+2x$, if $0 \le x \le \frac{1}{2}$, we get

$$\begin{aligned}G_{n+1} &\le (1+2\epsilon^{\beta_n}\|W_n\|_{r_{n+1}})^3 G_n \\ &\le \prod_{j=0}^{n}(1+2\epsilon^{\beta_j}\|W_j\|_{r_{j+1}})^3 G_0 \\ &\le (1+4\epsilon^{\beta_0}\|W_0\|_{r_1})^3 G_0 \le 2G_0.\end{aligned}$$

Therefore, the sequence $\{G_n\}$ is monotonically increasing and has an upper bound, so it is convergent.

Let $\Pi_* = \bigcap_{n\ge 1}\Pi_n$, $\Phi^* = \lim_{n\to\infty}\Phi^n$, $B_* = \lim_{n\to\infty} B_n$ and $h_* = \lim_{n\to\infty} h_n$. Because $\lim_{n\to\infty}\epsilon^{\beta_n}P_n = 0$ and $\lim_{n\to\infty}\epsilon^{\beta_n}g_n = 0$, the transformation $x = \Phi^* y$ reduces system (1) into

$$\frac{dy}{dt} = (A + B_*(t))y + h_*(y, t, \epsilon), \qquad y \in \mathbb{R}^d,$$

where A has the real Jordan form as in (9), $B_*(t)$ has the form

$$B_*(t) = \operatorname{diag}\left(\begin{pmatrix} U_1^*(t) & V_1^*(t) \\ -V_1^*(t) & U_1^*(t)\end{pmatrix}, \ldots, \begin{pmatrix} U_{\tilde d}^*(t) & V_{\tilde d}^*(t) \\ -V_{\tilde d}^*(t) & U_{\tilde d}^*(t)\end{pmatrix}\right), \tag{54}$$

$U_j^*(t)$, $V_j^*(t)$ $(j = 1, \ldots, \tilde d)$ are all real functions, and $h_*(y, t, \epsilon) = O(y^2)$ as $y \to 0$.

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

4.3. *Measure estimates.* We now estimate the Lebesgue measure of the set $\Pi\backslash\Pi_*$. Recall that $\Pi_* = \bigcap_{n=0}^{\infty}\Pi_n$, where $\Pi \supset \Pi_0 \supset \Pi_1 \supset \cdots$ is a decreasing sequence of closed sets defined inductively during the iteration process by

$$\Pi_n = \Pi_{n-1}\backslash \bigcup_{n=0}^{\infty} R_n(\gamma_n), \quad \Pi_{-1} = \Pi,\ n = 0, 1, \ldots,$$

where

$$R_n(\gamma_n) = R_n^1(\gamma_n) \bigcup R_n^2(\gamma_n),$$

$$R_n^1(\gamma_n) = \left\{\epsilon \in (0, \epsilon_0) : |\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i^n])| < \frac{\gamma_n}{(|k|+1)^{3\tau}}, \forall |k| < K_n\right\},$$

and

$$R_n^2(\gamma_n) = \left\{\epsilon \in (0, \epsilon_0) : |\sqrt{-1}\langle k, \omega\rangle - (\lambda_i + [\Xi_i^n]) + (\lambda_j + [\Xi_j^n])| < \frac{\gamma_n}{(|k|+1)^{3\tau}}, \forall |k| < K_n, i \neq j\right\}.$$

Let $\varphi_i^n = \lambda_i + [\Xi_i^n]$, $\varphi_{ij}^n = (\lambda_i + [\Xi_i^n]) - (\lambda_j + [\Xi_j^n])$, $i \neq j$. By Lemma 7 and KAM iteration, $\lambda_i + [\Xi_i^0] - \lambda_i^0(\epsilon) = O(\epsilon^2)$. Then it follows from non-degeneracy condition B and $[\Xi_i^n] - [\Xi_i^0] = O(\epsilon^2)$ that

$$\left|\frac{d\varphi_i^n}{d\epsilon}|_{\epsilon=0}\right| = \left|\frac{d(\lambda_i^0(\epsilon) + (\lambda_i + [\Xi_i^0] - \lambda_i^0(\epsilon)) + ([\Xi_i^n] - [\Xi_i^0]))}{d\epsilon}|_{\epsilon=0}\right| \geq 2\delta > 0.$$

Similarly,

$$\left|\frac{d\varphi_{ij}^n}{d\epsilon}|_{\epsilon=0}\right| \geq 2\delta > 0 \quad \text{for all } i \neq j.$$

Let $f^1(\epsilon) = \sqrt{-1}\langle k, \omega\rangle - \varphi_i^n$ and $f^2(\epsilon) = \sqrt{-1}\langle k, \omega\rangle - \varphi_{ij}^n$, $i \neq j$. Then there exists a sufficiently small ϵ_0 such that for $\epsilon \in (0, \epsilon_0)$, the above iteration is convergent, and

$$\left|\frac{df^1}{d\epsilon}\right| = \left|\frac{d\varphi_i^n}{d\epsilon}\right| \geq \delta, \quad \left|\frac{df^2}{d\epsilon}\right| = \left|\frac{d\varphi_{ij}^n}{d\epsilon}\right| \geq \delta. \tag{55}$$

For $\epsilon \in (0, \epsilon_0)$, by (34), we have

$$\begin{aligned}\|B_n - B_0\| &\leq \|B_n - B_{n-1}\| + \cdots + \|B_1 - B_0\| \\ &\leq 2\mathcal{E}_{n-1}^{14/15} + \cdots + 2\mathcal{E}_0^{14/15} \\ &\leq 4\mathcal{E}_0^{14/15} = \tfrac{1}{2}\hat{M}\epsilon_0^{14/15},\end{aligned}$$

where $B_0 = 0$, $\hat{M} = 8\|P_0\|^{14/15}$. Hence,

$$\begin{aligned}|f^2(\epsilon)| &\geq |\sqrt{-1}\langle k, \omega\rangle - \lambda_i + \lambda_j| - |[\Xi_i^n]| - |[\Xi_j^n]| \\ &\geq \frac{\gamma_0}{(|k|+1)^\tau} - 2\|B_n - B_0\| \\ &\geq \frac{\gamma_0}{(|k|+1)^\tau} - \hat{M}\epsilon_0^{14/15}.\end{aligned}$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

If $1/(|k|+1)^\tau \geq 2\hat{M}\epsilon_0^{14/15}/\gamma_0$, then $|f^2(\epsilon)| \geq \gamma_0/2(|k|+1)^\tau > \gamma_n/(|k|+1)^{3\tau}$, that is, $R_n^2(\gamma_n) = \emptyset$. Suppose that $1/(|k|+1)^\tau < 2\hat{M}\epsilon_0^{14/15}/\gamma_0$. By (55), it follows that

$$\begin{aligned}\operatorname{meas}(R_n^2(\gamma_n)) &\leq \frac{d^2\gamma_n}{\delta} \sum_{1/(|k|+1)^\tau < 2\hat{M}\epsilon_0^{14/15}/\gamma_0} \frac{1}{(|k|+1)^{3\tau}} \\ &\leq \frac{d^2\gamma_n}{\delta} \cdot \frac{4\hat{M}^2\epsilon_0^{28/15}}{\gamma_0^2} \sum_{k\in\mathbb{Z}^2} \frac{1}{(|k|+1)^\tau} \\ &\leq \frac{c\epsilon_0^{28/15}}{2^n},\end{aligned}$$

where c is a constant depending on γ_0. Similarly, $\operatorname{meas}(R_n^1(\gamma_n)) \leq c\epsilon_0^{28/15}/2^n$.

Noting that

$$(0, \epsilon_0)\backslash\Pi_* = (0, \epsilon_0)\backslash\bigg(\bigcap_{n=0}^{\infty}\Pi_n\bigg) = \bigcup_{n=0}^{\infty}((0, \epsilon_0)\backslash\Pi_n),$$

we have

$$\operatorname{meas}((0, \epsilon_0)\backslash\Pi_*) \leq \sum_{n=0}^{\infty}\operatorname{meas}((0, \epsilon_0)\backslash\Pi_n) \leq \sum_{n=0}^{\infty}\operatorname{meas}(R_n(\gamma_n)) \leq c\epsilon_0^{28/15}$$

and

$$\lim_{\epsilon_0\to 0}\frac{\operatorname{meas}((0, \epsilon_0)\backslash\Pi_*)}{\epsilon_0} = 0.$$

Therefore, Π_* is a non-empty subset of $(0, \epsilon_0)$.

4.4. *Elimination of the non-resonant terms on the diagonal.* By infinite KAM iteration steps, system (1) can be reduced to the following system with non-constant coefficients:

$$\dot{x} = (A + B_*(t))x + h_*(x, t, \epsilon), \quad x \in \mathbb{R}^d, \tag{56}$$

where A and $B_*(t)$ have real Jordan form as in (9) and (54), and $h_*(x, t, \epsilon) = O(x^2)$ as $x \to 0$. For simplicity, we continue to write $A = S\operatorname{diag}(\lambda_1, \ldots, \lambda_{2\tilde{d}})S^{-1}$, $B_*(t) = S\operatorname{diag}(\Xi_1^*(t), \ldots, \Xi_{2\tilde{d}}^*(t))S^{-1}$, where λ_{2j-1} and λ_{2j} are conjugate complex numbers, and $\Xi_{2j-1}^*(t)$ and $\Xi_{2j}^*(t)$ are conjugate complex functions, $j = 1, \ldots, \tilde{d}$.

We will now eliminate all the non-resonant terms containing t on the diagonal. First, making the change of variables $x = Sz$, system (56) becomes

$$\dot{z} = (\tilde{A} + \tilde{B}_*(t))z + \tilde{h}_*(z, t, \epsilon), \tag{57}$$

where $\tilde{A} = \operatorname{diag}(\lambda_1, \ldots, \lambda_{2\tilde{d}})$, $\tilde{B}_*(t) = \operatorname{diag}(\Xi_1^*(t), \ldots, \Xi_{2\tilde{d}}^*(t))$ and $\tilde{h}_*(z, t, \epsilon) = S^{-1}h_*(Sz, t, \epsilon)$.

Second, notice that $\beta(\alpha)$ has an equivalent definition (6) and (7); this implies that the equation

$$\frac{dH(t)}{dt} = \tilde{B}_*(t) - [\tilde{B}_*]$$

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

has an analytic solution if $0 \le \beta(\alpha) < r_*$. Let $z = e^{H(t)} z_*$, where

$$H(t) = \begin{pmatrix} \sum_{0 \neq k \in \mathbb{Z}^2} \frac{\Xi_1^*(k) e^{i\langle k, \omega\rangle t}}{i\langle k, \omega\rangle} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \sum_{0 \neq k \in \mathbb{Z}^2} \frac{\Xi_{2\tilde{d}}^*(k) e^{i\langle k, \omega\rangle t}}{i\langle k, \omega\rangle} \end{pmatrix}.$$

Then the above system (57) becomes

$$e^{H(t)} \frac{dH(t)}{dt} z_* + e^{H(t)} \frac{dz_*}{dt} = (\tilde{A} + \tilde{B}_*(t)) e^{H(t)} z_* + \tilde{h}_*(e^{H(t)} z_*, t, \epsilon),$$

which can be reduced to

$$\frac{dz_*}{dt} = (\tilde{A} + [\tilde{B}_*]) z_* + e^{-H(t)} \tilde{h}_*(e^{H(t)} z_*, t, \epsilon). \tag{58}$$

Third, making the change of variables $z_* = S^{-1} x^*$, system (58) becomes

$$\frac{dx^*}{dt} = (A + B^*) x^* + h^*(x^*, t, \epsilon), \quad x^* \in \mathbb{R}^d,$$

where $B^* = S[\tilde{B}_*]S^{-1}$ is a real constant matrix, and $h^*(x^*, t, \epsilon) = Se^{-H(t)} S^{-1} h_*(Se^{H(t)} S^{-1} x^*, t, \epsilon)$ is a real and high-order term. Thus, Theorem 1 is proved in full.

5. *Applications*

By way of applications of Theorem 1, we apply our theorem to the nonlinear Hill equation with quasi-periodic forcing terms, weakly forced oscillator and damped equation to study the existence of a quasi-periodic solution with basic frequencies as its frequencies. This kind of solution is also known as a response solution. These three kinds of equations correspond to Hamiltonian systems, reversible systems and dissipative systems, respectively.

Example 1. Consider the nonlinear Hill equation with quasi-periodic forcing terms

$$\ddot{x} + x^{2n+1} + (a_1 + \epsilon a(t)) x = \epsilon p(t), \quad n \ge 1, \tag{59}$$

where a_1 is a positive constant, and $a(t)$ and $p(t)$ are real analytic and quasi-periodic in t with basic frequencies $\omega = (1, \alpha)$, where α is irrational.

When $a(t)$ and $p(t)$ are continuous and 1-periodic, Liu [**20**] considered the nonlinear Hill equation with periodic forcing terms

$$\ddot{x} + x^{2n+1} + (a_1 + \epsilon a(t)) x = p(t), \quad n \ge 1, \tag{60}$$

and used Moser's twist theorem to obtain the following results: there exist a large constant $\alpha_* > 0$ and a small constant $\epsilon_0 > 0$ such that for every irrational number $\alpha > \alpha_*$ satisfying

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{2|q|^{2+\delta}}$$

for all integers p and $q \neq 0$ with some $\delta > 0$, there is a quasi-periodic solution of (60) having frequencies $(1, \alpha)$ when $0 < \epsilon < \epsilon_0$.

https://doi.org/10.1017/etds.2020.23 Published online by Cambridge University Press

The natural question is whether there are quasi-periodic response solutions for nonlinear Hill equation (59). Let $\dot{x} = \sqrt{a_1}\,y$. Then equation (59) can be rewritten in the form

$$\dot{z} = (A + \epsilon P(t, \epsilon))z + \epsilon g(t, \epsilon) + h(z, t, \epsilon), \quad z \in \mathbb{R}^2,$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \sqrt{a_1} \\ -\sqrt{a_1} & 0 \end{pmatrix}, \quad P(t) = \begin{pmatrix} 0 & 0 \\ -\dfrac{1}{\sqrt{a_1}}a(t) & 0 \end{pmatrix},$$

$$g(t) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{a_1}}p(t) \end{pmatrix}, \quad h(z, t) = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{a_1}}x^{2n+1} \end{pmatrix}.$$

By Theorem 1, for most sufficiently small ϵ, for all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, system (59) has a quasi-periodic solution with basic frequencies $\omega = (1, \alpha)$, such that it goes to zero when ϵ does.

Example 2. (Moser [**23**]) Consider a weakly forced oscillator

$$\ddot{x} + \lambda^2 x = \epsilon f(x, \dot{x}, t), \tag{61}$$

where $f(-t, x, -\dot{x}) = f(t, x, \dot{x})$ is quasi-periodic in t of basic frequencies $\omega = (\omega_1, \ldots, \omega_s)$, and real analytic with respect to x, $\dot{x}$ and t.

Moser [**23**] has obtained some well-known results: if ω satisfies

$$|\langle k, \omega\rangle + j_0| \geq \frac{\gamma}{|k|^\tau} \quad \text{for } k \in \mathbb{Z}^s\backslash\{0\},\ j_0 = 0, 1, 2,$$

there exists an analytic function $a(\epsilon)$ such that for $a = a(\epsilon)$ the above forced oscillator possesses a quasi-periodic solution of frequencies $\omega = (\omega_1, \ldots, \omega_s)$.

Our aim is to find a quasi-periodic solution of Liouvillean frequencies $\omega = (1, \alpha)$ for small ϵ. Let $\dot{x} = \lambda y$ and $f(x, \dot{x}, t) = f(0, 0, t) + (\partial f/\partial x)(0, 0, t)x + (\partial f/\partial \dot{x})(0, 0, t)\dot{x} + O(x^2) + O(\dot{x}^2)$. Then equation (61) can be rewritten in the form

$$\dot{z} = (A + \epsilon P(t, \epsilon))z + \epsilon g(t, \epsilon) + h(z, t, \epsilon), \quad z \in \mathbb{R}^2,$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}, \quad P(t) = \begin{pmatrix} 0 & 0 \\ \dfrac{1}{\lambda}\dfrac{\partial f}{\partial x}(0, 0, t) & \dfrac{1}{\lambda}\dfrac{\partial f}{\partial y}(0, 0, t) \end{pmatrix},$$

$$g(t) = \begin{pmatrix} 0 \\ \dfrac{1}{\lambda}f(0, 0, t) \end{pmatrix}, \quad h(z, t) = \begin{pmatrix} 0 \\ O(z^2) \end{pmatrix}.$$

Then by Theorem 1, for most sufficiently small ϵ, system (61) has a quasi-periodic solution with Liouvillean frequencies $\omega = (1, \alpha)$, such that it goes to zero when ϵ does.

Example 3. Consider the differential equation

$$\ddot{x} + a_1\dot{x} + a_2 x + x^2 = \epsilon f(x, \dot{x}, t), \tag{62}$$

where $f(x, \dot{x}, t)$ is real analytic in all variables and quasi-periodic in t with basic frequencies $\omega = (\omega_1, \ldots \omega_s)$. The usual choice of $a_1 > 0$ corresponds to a damped equation. In these equations the damping dominates the forcing term.

For equation (62), Stoker [27] has proved some well-known results: if the basic frequencies ω satisfy the Diophantine condition (2), $a_1 \neq 0$ and ϵ/a_1 is sufficiently small, then there exist quasi-periodic solutions possessing the same basic frequencies.

We are mainly concerned with the existence of quasi-periodic solutions with Liouvillean frequencies $\omega = (1, \alpha)$. Let $\dot{x} = y$. Then equation (62) can be rewritten in the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -x^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \epsilon f(x, y, t) \end{pmatrix}.$$

By Theorem 1, if $a_1^2 - 4a_2 \neq 0$ and $a_2 \neq 0$ (different and non-zero eigenvalues), for most sufficiently small ϵ (elliptic case) or for all sufficiently small ϵ (hyperbolic case), system (62) has quasi-periodic solutions with Liouvillean frequencies.

Acknowledgements. We thank the referees for their careful reading and valuable suggestions, which have helped improve the presentation of this paper. The first author is supported by NSFC (grant nos 11001048, 11571072) and the Fundamental Research Funds for the Central Universities (grant no. 2242017k41046), The second author is supported by NSFC (grant no. 11871146).

A. *Appendix*

The proof of Lemma 3 first appeared in [1], and we include the full proof for the convenience of readers.

Proof. The idea is to select an appropriate subsequence according to the definition of the CD bridge for which we will have a unified and nice control on the solution of homological equation (8).

Recall that $\mathcal{A} = \tau + 3$, (Q_n) is the selected subsequence of α in Lemma 2 with this given $\mathcal{A}$, and $\eta = \tilde{c}/Q_n^{1/2\mathcal{A}^4}$, where $0 < \tilde{c} < 1$. The solution of equation (8) can be written as

$$g(t, \epsilon) = \sum_{0<|k|<Q_{n+1}} \frac{i f_k(\epsilon)}{\langle k, \omega\rangle} e^{i\langle k,\omega\rangle t}.$$

Now we consider two cases to estimate the norm of $g(t, \epsilon)$.

In the case of $\bar{Q}_n < Q_n^{\mathcal{A}}$, $(\bar{Q}_{n-1}, Q_n)$, (Q_n, Q_{n+1}) are both $CD(\mathcal{A}, \mathcal{A}, \mathcal{A}^3)$ bridges. Let $q_u = \bar{Q}_{n-1}$, $q_{v+1} = Q_{n+1}$. Then

$$\begin{aligned}
\|g\|_{r(1-\eta),\Pi} &= \left(\sum_{0<|k|<q_u} + \sum_{q_u \le |k| < q_{u+1}} + \cdots + \sum_{q_v \le |k| < q_{v+1}} \right) \left| \frac{f_k(\epsilon)}{\langle k, \omega\rangle} \right| e^{|k| r(1-\eta)} \\
&\le 2q_u \sum_{0<|k|<q_u} |f_k(\epsilon)| e^{|k|r(1-\eta)} + 2q_{u+1} \sum_{q_u \le |k| < q_{u+1}} |f_k(\epsilon)| e^{|k|r(1-\eta)} \\
&\quad + \cdots + 2q_{v+1} \sum_{q_v \le |k| < q_{v+1}} |f_k(\epsilon)| e^{|k|r(1-\eta)} \\
&\le \left(2q_u + 2\sum_{j=u}^{v} q_{j+1} e^{-r\eta q_j} \right) \|f - [f]\|_{r,\Pi}.
\end{aligned}$$

By the definition of CD bridge, we have $Q_n \le \bar{Q}_{n-1}^{A^3} \le q_j^{A^3}$ for $j \in [u, v]$, which implies

$$q_{j+1}e^{-r\eta q_j} \le c(\tau)q_{j+1}\bigg(\frac{Q_n^{1/2\mathcal{A}^4}}{\tilde{c}rq_j}\bigg)^{2\mathcal{A}^5} \le c(r_*, \tau, \tilde{c})\frac{q_{j+1}}{q_j^{\mathcal{A}^4}}.$$

Moreover, again by the definition of CD bridge, we know that for any $j \in [u, v]$, $q_{j+1} \le q_j^{\mathcal{A}}$ and $q_u^{\mathcal{A}} \le Q_n$. Then we have

$$\begin{aligned}\|g\|_{r(1-\eta),\Pi} &\le c(r_*, \tau, \tilde{c})\left(q_u + \sum_{j=u}^{v}\frac{q_{j+1}}{q_j^{\mathcal{A}^4}}\right)\|f - [f]\|_{r(1-\eta),\Pi}\\ &\le C_1(r_*, \tau, \tilde{c})Q_n^{1/\mathcal{A}}\|f - [f]\|_{r(1-\eta),\Pi}.\end{aligned}$$

In the case of $\bar{Q}_n \ge Q_n^{\mathcal{A}}$, let $q_u = Q_n$ and $q_{v+1} = Q_{n+1}$. By the construction of the sequences (Q_k), we know that $\bar{Q}_n \ge Q_n^{\mathcal{A}}$ and for any $j \in [u+1, v]$, $q_{j+1} \le q_j^{\mathcal{A}}$. Then by the similar argument as in case 1, we have

$$\begin{aligned}\|g\|_{r(1-\eta),\Pi} &\le c(r_*, \tau, \tilde{c})\bigg(q_u + \sum_{j=u}^{v}\frac{q_{j+1}}{q_j^{\mathcal{A}^4}}\bigg)\|f - [f]\|_{r(1-\eta),\Pi}\\ &\le C_1(r_*, \tau, \tilde{c})\bigg(\bar{Q}_n^{1/\mathcal{A}} + \frac{\bar{Q}_n}{Q_n^{\mathcal{A}^4}}\bigg)\|f - [f]\|_{r(1-\eta),\Pi}.\end{aligned}$$

□

REFERENCES

[1] A. Avila, B. Fayad and R. Krikorian. A KAM scheme for SL(2, R) cocycles with Liouvillean frequencies. *Geom. Funct. Anal.* **21** (2011), 1001–1019.

[2] D. Bambusi. Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations, II. *Comm. Math. Phys.* **353**(1) (2017), 353–378.

[3] D. Bambusi. Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations, I. *Trans. Amer. Math. Soc.* **370**(3) (2018), 1823–1865.

[4] C. Chavaudret. Strong almost reducibility for analytic and Gevrey quasi-periodic cocycles. *Bull. Soc. Math. France* **141** (2013), 47–106.

[5] C. Chavaudret and S. Marmi. Reducibility of quasiperiodic cocycles under a Brjuno–Rüssmann arithmetical condition. *J. Mod. Dyn.* **6**(1) (2012), 59–78.

[6] C. Chavaudret and L. Stolovitch. Analytic reducibility of resonant cocycles to a normal form. *J. Inst. Math. Jussieu* **15**(1) (2016), 203–223.

[7] E. I. Dinaburg and Y. G. Sinai. The one dimensional Schrödinger equation with quasi-perioidc potential. *Funct. Anal. Appl.* **9** (1975), 8–21.

[8] L. H. Eliasson. Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation. *Comm. Math. Phys.* **146** (1992), 447–482.

[9] L. H. Eliasson. Almost reducibility of linear quasi-periodic systems. *Smooth Ergodic Theory and its Applications (Seattle, WA, 1999) (Proceedings of Symposia in Pure Mathematics, 69)*. American Mathematical Society, Providence, RI, 2001, pp. 679–705.

[10] L. H. Eliasson and S. B. Kuksin. On reducibility of Schrödinger equations with quasi-periodic in time potentials. *Comm. Math. Phys.* **286** (2009), 125–135.

[11] B. Fayad and R. Krikorian. Herman's last geometric theorem. *Ann. Sci. Éc. Norm. Supér. (4)* **42** (2009), 193–219.

[12] B. Fayad and R. Krikorian. Rigidity results for quasiperiodic SL(2, R)-cocycles. *J. Mod. Dyn.* **3**(4) (2009), 497–510.

[**13**] H. Her and J. You. Full measure reducibility for generic one-parameter family of quasi-periodic linear systems. *J. Dynam. Differential Equations* **20** (2008), 831–866.
[**14**] X. Hou and J. You. Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems. *Invent. Math.* **190** (2012), 209–260.
[**15**] R. A. Johnson and G. R. Sell. Smoothness of spectral subbundles and reducibility of quasi-periodic linear differential systems. *J. Differential Equations* **41** (1981), 262–288.
[**16**] À. Jorba and C. Simó. On the reducibility of linear differential equation with quasi-perioidc coefficients. *J. Differential Equations* **98**(1) (1992), 111–124.
[**17**] À. Jorba and C. Simó. On quasi-periodic perturbations of elliptic equilibrium points. *SIAM J. Math. Anal.* **27**(6) (1996), 1704–1737.
[**18**] R. Krikorian. Global density of reducible quasi-periodic cocycles on $T^1 \times \mathrm{SU}(2)$. *Ann. of Math. (2)* **154** (2001), 269–326.
[**19**] J. Liang and J. Xu. A note on the extension of Dinaburg–Sinai theorem to higher dimension. *Ergod. Th. & Dynam. Sys.* **25** (2005), 1539–1549.
[**20**] B. Liu. Boundedness for solutions of nonlinear Hill's equations with periodic forcing terms via Moser's twist theorem. *J. Differential Equations* **79** (1989), 304–315.
[**21**] J. Lopes Dias. A normal form theorem for Brjuno skew systems through renormalization. *J. Differential Equations* **230**(1) (2006), 1–23.
[**22**] J. Lopes Dias. Brjuno condition and renormalization for Poincaré flows. *Discrete Contin. Dyn. Syst.* **15**(2) (2006), 641–656.
[**23**] J. Moser. Combination tones for Duffing's equation. *Comm. Pure Appl. Math.* **18** (1965), 167–181.
[**24**] J. Pöschel. KAM à la R. *Regul. Chaotic Dyn.* **16** (2011), 17–23.
[**25**] H. Rüssmann. On the one dimensional Schrödinger equation with a quasi-periodic potential. *Ann. N. Y. Acad. Sci.* **357** (1980), 90–107.
[**26**] H. Rüssmann. Convergent transformations into a normal form in analytic Hamiltonian systems with two degrees of freedom on the zero energy surface near degenerate elliptic singularities. *Ergod. Th. & Dynam. Sys.* **24** (2004), 1787–1832.
[**27**] J. J. Stoker. *Nonlinear Vibrations*. Interscience, New York, 1950, esp. pp. 235–239.
[**28**] X. Wang and J. Xu. On the reducibility of a class of nonlinear quasi-periodic system with small perturbation parameter near zero equilibrium point. *Nonlinear Anal.* **69** (2008), 2318–2329.
[**29**] X. Wang, J. Xu and D. Zhang. On the persistence of degenerate lower-dimensional tori in reversible systems. *Ergod. Th. & Dynam. Sys.* **35** (2015), 2311–2333.
[**30**] J. Wang, J. You and Q. Zhou. Response solutions for quasi-periodically forced harmonic oscillators. *Trans. Amer. Math. Soc.* **369** (2017), 4251–4274.
[**31**] J. Xu. On the reducibility of a class of linear differential equation with quasi-periodic coefficients. *Mathematika* **46** (1999), 443–451.
[**32**] J. Xu and X. Lu. On the reducibility of two-dimensional linear quasi-periodic systems with small parameter. *Ergod. Th. & Dynam. Sys.* **35** (2015), 2334–2352.
[**33**] J. Xu and Q. Zheng. On the reducibility of linear differential equations with quasiperiodic coefficients which are degenerate. *Proc. Amer. Math. Soc.* **126** (1998), 1445–1451.
[**34**] D. Zhang and J. Liang. On high dimensional Schrödinger equation with quasi-periodic potentials. *J. Dyn. Control Syst.* **23** (2017), 655–666.
[**35**] D. Zhang and J. Xu. Invariant curves of analytic reversible mappings under Brjuno–Rüssmann's non-resonant condition. *J. Dynam. Differential Equations* **26** (2014), 989–1005.
[**36**] D. Zhang and J. Xu. On invariant tori of vector field under weaker non-degeneracy condition. *NoDEA Nonlinear Differential Equations Appl.* **22** (2015), 1381–1394.
[**37**] D. Zhang, J. Xu and H. Wu. On invariant tori with prescribed frequency in Hamiltonian systems. *Adv. Nonlinear Stud.* **16**(4) (2016), 719–737.
[**38**] D. Zhang, J. Xu and X. Xu. Reducibility of three dimensional skew symmetric system with Liouvillean basic frequencies. *Discrete Contin. Dyn. Syst.* **38**(6) (2018), 2851–2877.
[**39**] H. Zhao. A note on quasi-periodic perturbations of elliptic equilibrium points. *Bull. Korean Math. Soc.* **49**(6) (2012), 1223–1240.
[**40**] Q. Zhou and J. Wang. Reducibility results for quasiperiodic cocycles with Liouvillean frequency. *J. Dynam. Differential Equations* **24** (2012), 61–83.