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ASYMPTOTIC RUIN PROBABILITIES IN FINITE HORIZON WITH SUBEXPONENTIAL LOSSES AND ASSOCIATED DISCOUNT FACTORS

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Consider a discrete-time insurance risk model with risky investments. Under the assumption that the loss distribution belongs to a certain subclass of the subexponential class, Tang and Tsitsiashvili (Stochastic Processes and Their Applications 108(2): 299–325 (2003)) established a precise estimate for the finite time ruin probability. This article extends the result both to the whole subexponential class and to a nonstandard case with associated discount factors.

1. INTRODUCTION

Following the works of Nyrhinen [14,15] and Tang and Tsitsiashvili [17,19], we consider the finite time ruin probability of an insurer who invests his wealth into a risky asset. In this stochastic economic environment, the net loss during period n is denoted by a real-valued random variable X_n, $n = 1,2,\ldots$, and the discount factor from time n to time $n-1$ is denoted by another positive random variable Y_n, $n = 1,2,\ldots$.

Write $A_n = -X_n$ and $R_n = Y_n^{-1} - 1$, $n = 1,2,\ldots$. Then A_n denotes the total net income and R_n denotes the total stochastic return rate within period n. We tacitly assume that the income A_n or the loss X_n is calculated at time n. Let the initial surplus of the insurer be $S_0 = x \geq 0$. Then the surplus accumulated until time n, denoted by S_n, can be characterized by the recurrence equation

$$S_n = (1 + R_n)S_{n-1} + A_n, \qquad n = 1,2,\ldots. \tag{1.1}$$

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The probability of ruin within time n is defined as

$$\psi(x,n) = \Pr\left(\min_{0\le m\le n} S_m < 0 \,\middle|\, S_0 = x\right), \qquad n = 0,1,\ldots.$$

Iterating (1.1) and rewriting the resulting formulas in terms of $\{X_n : n = 1,2,\ldots\}$ and $\{Y_n : n = 1,2,\ldots\}$, we obtain that

$$S_0 = x, \qquad S_n = S_0 \prod_{i=1}^{n} Y_i^{-1} - \sum_{k=1}^{n} X_k \prod_{i=k+1}^{n} Y_i^{-1}, \qquad n = 1,2,\ldots.$$

It follows that

$$\psi(x,n) = \Pr\left(\max_{1\le m\le n} \sum_{k=1}^{m} X_k \prod_{i=1}^{k} Y_i > x\right) \qquad n = 1,2,\ldots. \tag{1.2}$$

By (1.2), we immediately see that the two-sided inequality

$$\Pr\left(\sum_{k=1}^{n} X_k \prod_{i=1}^{k} Y_i > x\right) \le \psi(x,n) \le \Pr\left(\sum_{k=1}^{n} X_k^{+} \prod_{i=1}^{k} Y_i > x\right) \tag{1.3}$$

holds for $n = 1,2,\ldots$, where $X^{+} = X1_{(X\ge 0)}$ denotes the positive part of a random variable X and 1_A denotes the indicator function of a set A. This inequality will be used below.

Tang and Tsitsiashvili [17,19] made the following standard assumptions:

P1: The net losses X_n, $n = 1,2,\ldots$, are independent and identically distributed (i.i.d.) with common distribution function F on the real line.

P2: The discount factors Y_n, $n = 1,2,\ldots$, are also i.i.d. with common distribution function G on the positive half-line.

P3: The two sequences $\{X_n : n = 1,2,\ldots\}$ and $\{Y_n : n = 1,2,\ldots\}$ are independent.

Under these assumptions, Tang and Tsitsiashvili [17] derived a precise asymptotic estimate for the finite time ruin probability for the case that the loss distribution F belongs to a certain subclass of the subexponential class; see also Tang and Tsitsiashvili [19] for a broader account.

In the present article we aim at extensions of the result of Tang and Tsitsiashvili [17]. In the rest of this article, after a brief review on heavy-tailed distributions in Section 2, we give in Section 3 the first main result in which the loss distribution F ranges over the whole subexponential class, and we give in Section 4 the second main result in which the discount factors $\{Y_n : n = 1,2,\ldots\}$ or, equivalently, the return rates $\{R_n : n = 1,2,\ldots\}$ are associated.

2. HEAVY-TAILED DISTRIBUTIONS

The most important class of heavy-tailed distributions is the subexponential class. By definition, a distribution $F = 1 - \bar{F}$ on $[0,\infty)$ or its corresponding random variable is said to be subexponential, denoted by $F \in \mathcal{S}$, if the relation

$$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\bar{F}(x)} = n \tag{2.1}$$

holds for some (or, equivalently, for all) $n = 2,3,\ldots$, where F^{*n} denotes the n-fold convolution of F. More generally, a distribution F on $(-\infty,\infty)$ is still said to be subexponential if the distribution $F^+(x) = F(x)1_{(x\geq 0)}$ is subexponential. By Lemma 2.1 and the last inclusion of (2.3) below, it is easy to verify that (2.1) remains valid for the latter general case. The class $\mathcal{S}$ contains the Pareto-like, the lognormal-like, and the Weibull-like distributions.

Closely related are the class $\mathcal{L}$ of long-tailed distributions and the class $\mathcal{D}$ of distributions with dominatedly varying tails. A distribution F on $(-\infty,\infty)$ belongs to the class $\mathcal{L}$ if the relation

$$\lim_{x\to\infty} \frac{\bar{F}(x+y)}{\bar{F}(x)} = 1$$

holds for some (or, equivalently, for all) $y > 0$; F belongs to the class $\mathcal{D}$ if the relation

$$\limsup_{x\to\infty} \frac{\bar{F}(xy)}{\bar{F}(x)} < \infty \tag{2.2}$$

holds for some (or, equivalently, for all) $0 < y < 1$.

It is well known that

$$\mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L}. \tag{2.3}$$

For more details of heavy-tailed distributions, we refer the reader to Embrechts, Klüppelberg, and Mikosch [8] and references therein.

In what follows, all limiting relationships are for $x \to \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) = O(b(x))$ if lim sup $a(x)/b(x) < \infty$, $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$, $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$, and $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leq 1$.

The following lemma is well known; see Embrechts and Goldie [7], Cline [5, Cor. 1], and Tang and Tsitsiashvili [17, Lemma 3.2].

Lemma 2.1: *Let F be the convolution of two distributions F_1 and F_2. If $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{L}$, and $\bar{F}_2(x) = O(\bar{F}_1(x))$, then $F \in \mathcal{S}$ and*

$$\bar{F}(x) \sim \bar{F}_1(x) + \bar{F}_2(x).$$

The following lemma is from Cline and Samorodnitsky [6, Thm. 2.1].

Lemma 2.2: *Let X and Y be two independent random variables with distributions F and G, respectively, satisfying $F \in \mathcal{S}$ and $G(0) = 0$. The distribution H of the product XY is subexponential if there is a positive function $a(x) = o(x)$ such that $\bar{F}(x - a(x)) \sim \bar{F}(x)$ and $\bar{G}(a(x)) = o(\bar{H}(x))$.*

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3. FOR THE STANDARD CASE

Let us go back to the model introduced in Section 1. Hereafter, denote the generic random variable of $\{X_n : n = 1,2,\ldots\}$ (under assumption P1) by X, the generic random variable of $\{Y_n : n = 1,2,\ldots\}$ (under assumption P2) by Y, and the distribution of XY (under assumptions P1, P2, and P3) by $H = F \otimes G$.

The main result of Tang and Tsitsiashvili [17] is that, under assumptions P1, P2, and P3, the relation

$$\psi(x,n) \sim \sum_{k=1}^{n} \Pr\left(X \prod_{i=1}^{k} Y_i > x \right) \tag{3.1}$$

holds for each $n = 1,2,\ldots$ if $F \in \mathcal{L} \cap \mathcal{D}$ and $\mathrm{E}Y^p < \infty$ for some large $p > 0$ (more precisely, for some p larger than the upper Matuszewska index of the distribution F). The estimate given by (3.1) enables us to recursively calculate the ruin probability $\psi(x,n)$. However, an obvious drawback is that the condition $F \in \mathcal{L} \cap \mathcal{D}$ excludes many popular distributions such as the lognormal-like and the Weibull-like distributions; recall (2.2).

The following is our first main result, which extends the scope of the loss distribution to the whole subexponential class $\mathcal{S}$:

THEOREM 3.1: *Assume P1, P2, and P3. If $F \in \mathcal{S}$ and there is some $0 < \tau < 1$ such that $\bar{F}(x - x^\tau) \sim \bar{F}(x)$ and $\bar{G}(x^\tau) = o(\bar{H}(x))$, then (3.1) holds for each $n = 1,2,\ldots$.*

Two concrete cases of Theorem 3.1 are listed below without proof.

COROLLARY 3.1: *Assume P1, P2, and P3. Relation (3.1) holds for each $n = 1,2,\ldots$ if one of the following groups of conditions is valid:*

(A) F is lognormal-like with a tail satisfying $\bar{F}(x) \sim c\bar{F}_1(x)$ for some $c > 0$, where the distribution F_1 has a density function

$$f_1(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left\{ \frac{-(\ln x - \mu)^2}{2\sigma^2} \right\}, \qquad x > 0,$$

with $-\infty < \mu < \infty$ and $\sigma > 0$, and $\bar{G}(x^\tau) = o(\bar{H}(x))$ for some $0 < \tau < 1$.

(B) F is Weibull-like with a tail satisfying $\bar{F}(x) \sim c\exp\{-dx^\nu\}$ for some c, $d > 0$, $0 < \nu < 1$, and $\bar{G}(x^\tau) = o(\bar{H}(x))$ for some $0 < \tau < 1 - \nu$.

Clearly, both in Theorem 3.1 and Corollary 3.1, the condition $\bar{G}(x^\tau) = o(\bar{H}(x))$ is implied by $\bar{G}(x^\tau) = o(\bar{F}(x))$. More concretely, in Corollary 3.1(A) the G can be every Weibull distribution or every lognormal distribution with a density function

$$g(x) = \frac{1}{\sqrt{2\pi}\tilde{\sigma} x} \exp\left\{ \frac{-(\ln x - \tilde{\mu})^2}{2\tilde{\sigma}^2} \right\}, \qquad x > 0,$$

as long as $-\infty < \tilde{\mu} < \infty$ and $0 < \tilde{\sigma} < \sigma$, and in Corollary 3.1(B) the G can be every Weibull-like distribution with a tail $\bar{G}(x) \sim \tilde{c}\exp\{-\tilde{d}x^{\tilde{v}}\}$ as long as $\tilde{c}, \tilde{d} > 0$ and $\tilde{v} > v/(1-v)$.

In the proof of Theorem 3.1 we will need the following lemma.

LEMMA 3.1: *Under the conditions of Theorem 3.1, it holds for each* $k = 1, 2, \ldots$ *that*

$$\Pr\left(X\prod_{i=1}^{k} Y_i > x - x^{\tau}\right) \sim \Pr\left(X\prod_{i=1}^{k} Y_i > x\right).$$

PROOF: We only prove the result for $k = 1$ since the general case extends by induction. Trivially, the condition $\bar{F}(x - x^{\tau}) \sim \bar{F}(x)$ implies that $\bar{F}(x - Cx^{\tau}) \sim \bar{F}(x)$ holds for every constant $C > 0$. Choose some $0 < \varepsilon < 1$ such that $\bar{G}(\varepsilon) > 0$. Then it holds for all large $x > 0$ and $t \in (\varepsilon, x^{\tau}]$ that

$$\frac{x^{\tau}}{t} \leq \frac{1}{\varepsilon}\left(\frac{x}{t}\right)^{\tau}.$$

For all large $x > 0$, we derive

$$\begin{aligned}
\bar{H}(x - x^{\tau}) &= \left(\int_0^{\varepsilon} + \int_{\varepsilon}^{x^{\tau}} + \int_{x^{\tau}}^{\infty}\right)\bar{F}\left(\frac{x - x^{\tau}}{t}\right)G(dt) \\
&\leq \Pr(\varepsilon X > x - x^{\tau})G(\varepsilon) + \int_{\varepsilon}^{x^{\tau}} \bar{F}\left(\frac{x}{t} - \frac{1}{\varepsilon}\left(\frac{x}{t}\right)^{\tau}\right)G(dt) + \bar{G}(x^{\tau}) \\
&= \Pr(\varepsilon X > x - x^{\tau}, Y > \varepsilon)\,\frac{G(\varepsilon)}{\bar{G}(\varepsilon)} \\
&\quad + (1 + o(1))\int_{\varepsilon}^{x^{\tau}} \bar{F}\left(\frac{x}{t}\right)G(dt) + \bar{G}(x^{\tau}) \\
&\leq \Pr(XY > x - x^{\tau})\,\frac{G(\varepsilon)}{\bar{G}(\varepsilon)} + (1 + o(1))\bar{H}(x) + \bar{G}(x^{\tau}) \\
&= \bar{H}(x - x^{\tau})\,\frac{G(\varepsilon)}{\bar{G}(\varepsilon)} + (1 + o(1))\bar{H}(x).
\end{aligned}$$

It follows that

$$\frac{\bar{H}(x - x^{\tau})}{\bar{H}(x)} \leq (1 + o(1))\left(1 - \frac{G(\varepsilon)}{\bar{G}(\varepsilon)}\right)^{-1}.$$

Since $G(0) = 0$ and $\varepsilon > 0$ can be arbitrarily small, we obtain

$$\limsup_{x\to\infty} \frac{\bar{H}(x - x^{\tau})}{\bar{H}(x)} \leq 1,$$

which actually amounts to $\bar{H}(x - x^{\tau}) \sim \bar{H}(x)$. ■

Proof of Theorem 3.1: Recall the two-sided inequality (1.3). If we can prove the relation

$$\Pr\left(\sum_{k=1}^{n} X_k \prod_{i=1}^{k} Y_i > x\right) \sim \sum_{k=1}^{n} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x\right), \qquad n = 1,2,\ldots,$$

without using $F(0-) > 0$, then the same proof should also be valid for the relation

$$\Pr\left(\sum_{k=1}^{n} X_k^+ \prod_{i=1}^{k} Y_i > x\right) \sim \sum_{k=1}^{n} \Pr\left(X_k^+ \prod_{i=1}^{k} Y_i > x\right), \qquad n = 1,2,\ldots,$$

and we immediately obtain (3.1). Write

$$V_n = \sum_{k=1}^{n} X_k \prod_{i=k}^{n} Y_i, \qquad n = 1,2,\ldots.$$

Under assumptions P1, P2, and P3, it is clear that

$$V_n =^d \sum_{k=1}^{n} X_k \prod_{i=1}^{k} Y_i, \qquad n = 1,2,\ldots,$$

where $=^d$ denotes "equal in distribution." Based on this analysis, it suffices to prove the relation

$$\Pr(V_n > x) \sim \sum_{k=1}^{n} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x\right), \qquad n = 1,2,\ldots. \tag{3.2}$$

In view of Theorem 4.1 below, we only need to consider the case that Y is unbounded. We prove the asymptotic relation (3.2) by the inductive method.

Trivially, (3.2) holds for $n = 1$. Applying Lemma 2.2, we also know that V_1 is subexponential. Now we assume by induction that (3.2) holds for $n = m \ge 1$ and that V_m is subexponential. Clearly, $\bar{F}(x) = O(\Pr(V_m > x))$ since $\bar{G}(1) > 0$. From Lemmas 2.1 and 3.1 and the inductive hypothesis, it follows that the sum $X_{m+1} + V_m$ is subexponential and that

$$\begin{aligned}
\Pr(X_{m+1} + V_m > x - x^\tau) &\sim \Pr(X_{m+1} > x - x^\tau) + \Pr(V_m > x - x^\tau) \\
&\sim \Pr(X_{m+1} > x - x^\tau) + \sum_{k=1}^{m} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x - x^\tau\right) \\
&\sim \Pr(X_{m+1} > x) + \sum_{k=1}^{m} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x\right) \\
&\sim \Pr(X_{m+1} + V_m > x).
\end{aligned}$$

Hence, by Lemma 2.2, the random variable V_{m+1} is subexponential. By Lemma 2.1 and the inductive hypothesis, we derive that

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$$
\begin{aligned}
&\Pr(V_{m+1} > x) \\
&\quad = \left(\int_0^{x^\tau} + \int_{x^\tau}^{\infty}\right) \Pr(X_{m+1} + V_m > x/t) G(dt) \\
&\quad = (1 + o(1)) \int_0^{x^\tau} (\Pr(X_{m+1} > x/t) + \Pr(V_m > x/t)) G(dt) + O(\bar{G}(x^\tau)) \\
&\quad = (1 + o(1)) \int_0^{x^\tau} \left(\Pr(X_{m+1} > x/t) + \sum_{k=1}^{m} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x/t\right)\right) G(dt) \\
&\qquad + O(\bar{G}(x^\tau)) \\
&\quad = (1 + o(1)) \sum_{k=1}^{m+1} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x\right) + O(\bar{G}(x^\tau)) \\
&\quad = (1 + o(1)) \sum_{k=1}^{m+1} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x\right).
\end{aligned}
$$

This proves that (3.2) holds for $n = m + 1$. By the mathematical inductive method, we conclude that (3.2) holds for each $n = 1,2,\ldots$. ■

4. FOR THE CASE OF ASSOCIATED DISCOUNT FACTORS

Recently, the study on ruin probabilities of nonstandard models has become an important part of risk theory. We refer the reader to Cai [1,2] and Cai and Dickson [4], among many others.

Now we propose a general (positively) dependence structure for the discount factors. We say that a sequence of random variables $\{Y_n : n = 1,2,\ldots\}$ is (positively) associated if the inequality

$$\mathrm{E}f_1(Y_1,\ldots,Y_n)f_2(Y_1,\ldots,Y_n) \geq \mathrm{E}f_1(Y_1,\ldots,Y_n)\mathrm{E}f_2(Y_1,\ldots,Y_n) \tag{4.1}$$

holds for all $n = 1,2,\ldots$ and all coordinatewise (not necessarily strictly) increasing functions f_1 and f_2 for which the moments involved exist. Since it was introduced by Esary, Proschan, and Walkup [9], this dependence structure has been extensively studied and applied by many researchers in statistics, applied probability, insurance, and finance. Trivially, if in the above definition f_1 is coordinatewise increasing but f_2 is coordinatewise decreasing, then (4.1) is changed to

$$\mathrm{E}f_1(Y_1,\ldots,Y_n)f_2(Y_1,\ldots,Y_n) \leq \mathrm{E}f_1(Y_1,\ldots,Y_n)\mathrm{E}f_2(Y_1,\ldots,Y_n). \tag{4.2}$$

The following is our second main result, which partially extends Theorem 3.1 to the proposed nonstandard case:

Theorem 4.1: *Assume P1, P3, and*

P2′: The discount factors $\{Y_n : n = 1,2,\ldots\}$ *constitute a sequence of bounded, associated, and positive random variables.*

If $F \in \mathcal{S}$, *then (3.1) holds for each* $n = 1,2,\ldots$.

Theorem 4.1 indicates that the association of the bounded discount factors does not influence the asymptotic relation (3.1). Moreover, if we restrict the discussion to the case of Pareto-like loss distributions, then under assumptions P1 and P3, using a result of Resnick and Willekens [16], it is not difficult to prove that (3.1) even holds for arbitrarily dependent discount factors $\{Y_n : n = 1,2,\ldots\}$ as long as they satisfy suitable summability conditions.

We also remark that the boundedness condition of Theorem 4.1 is not so restrictive for application. For example, it allows for a realistic case below (see also Example 4.1 of Tang and Tsitsiashvili [19]).

Suppose that an insurer invests his wealth not only in a risk-free asset (a bank) but also in a risky asset (a stock market). At time $n-1$, the insurer has wealth S_{n-1}, and he keeps a nonrandom fraction, say $0 < a_n \le 1$, of his wealth in the bank and invests the remaining part in the stock market. Then, at time n, the first part becomes $a_n(1 + r_n)S_{n-1}$ with some deterministic interest rate $r_n \ge 0$ and the second part becomes $(1 - a_n)(1 + R_n)S_{n-1}$ with some stochastic return rate $R_n \in [-1,\infty)$. Consequently, the discount factors equal

$$Y_n = \frac{1}{a_n(1 + r_n) + (1 - a_n)(1 + R_n)}, \qquad n = 1,2,\ldots, \tag{4.3}$$

which are obviously bounded from above by positive constants.

For related discussions in continuous-time settings, see Hipp and Plum [12], Gaier and Grandits [10], Gaier, Grandits, and Schachermayer [11], Cai [3], Liu and Yang [13], among others.

Additionally, from (4.3) we see that if $0 < a_n < 1$ for $n = 1,2,\ldots$, the association of $\{Y_n : n = 1,2,\ldots\}$ is equivalent to that of $\{R_n : n = 1,2,\ldots\}$.

In the proof of Theorem 4.1, we will need the following lemma, which is a restatement of Proposition 5.1 of Tang and Tsitsiashvili [18].

Lemma 4.1: *Let* $\{X_1,\ldots,X_n\}$ *be n i.i.d. real-valued random variables with common distribution* $F \in \mathcal{S}$. *Then, for arbitrarily fixed* $0 < a \le b < \infty$, *the relation*

$$\Pr\left(\sum_{k=1}^{n} c_k X_k > x\right) \sim \sum_{k=1}^{n} \bar{F}(x/c_k) \tag{4.4}$$

holds uniformly for $(c_1,\ldots,c_n) \in [a,b] \times \cdots \times [a,b]$; *that is,*

$$\lim_{x\to\infty} \sup_{(c_1,\cdots,c_n)\in[a,b]\times\cdots\times[a,b]} \left| \frac{\Pr\left(\sum_{k=1}^{n} c_k X_k > x\right)}{\sum_{k=1}^{n} \bar{F}(x/c_k)} - 1 \right| = 0.$$

PROOF OF THEOREM 4.1: Choose some constant $d = d_n > 1$ as a common upper bound of the random variables $\{Y_1, \dots, Y_n\}$. First, we derive an asymptotic upper bound for $\psi(x, n)$. For an arbitrarily fixed $0 < \varepsilon < 1$ such that $\Pr(\varepsilon < Y_j \le d) > 0$ for each $j = 1, \dots, n$, we split the probability on the right-hand side of (1.3) into two parts as

$$\begin{aligned}
\Pr\left(\sum_{k=1}^{n} X_k^+ \prod_{i=1}^{k} Y_i > x\right)
&= \Pr\left(\sum_{k=1}^{n} X_k^+ \prod_{i=1}^{k} Y_i > x, \bigcup_{j=1}^{n} (Y_j \le \varepsilon)\right) \\
&\quad + \Pr\left(\sum_{k=1}^{n} X_k^+ \prod_{i=1}^{k} Y_i > x, \bigcap_{j=1}^{n} (\varepsilon < Y_j \le d)\right) \\
&= J_1(x, \varepsilon) + J_2(x, \varepsilon). \qquad \textbf{(4.5)}
\end{aligned}$$

Since the random variables $\{Y_1, \dots, Y_n\}$ are associated and are independent of the nonnegative random variables $\{X_1^+, \dots, X_n^+\}$, by (4.2) it holds that

$$\begin{aligned}
J_1(x, \varepsilon) &= \mathrm{E}\left[1_{\left(\sum_{k=1}^{n} X_k^+ \prod_{i=1}^{k} Y_i > x\right)} 1_{\left(\bigcup_{j=1}^{n} (Y_j \le \varepsilon)\right)}\right] \\
&\le \mathrm{E}\left[1_{\left(\sum_{k=1}^{n} X_k^+ \prod_{i=1}^{k} Y_i > x\right)}\right] \mathrm{E}\left[1_{\left(\bigcup_{j=1}^{n} (Y_j \le \varepsilon)\right)}\right] \\
&= \Pr\left(\sum_{k=1}^{n} X_k^+ \prod_{i=1}^{k} Y_i > x\right) \Pr\left(\bigcup_{j=1}^{n} (Y_j \le \varepsilon)\right). \qquad \textbf{(4.6)}
\end{aligned}$$

Substituting (4.6) into (4.5) and rearranging the resulting inequality, we have

$$\Pr\left(\sum_{k=1}^{n} X_k^+ \prod_{i=1}^{k} Y_i > x\right) \le \frac{J_2(x, \varepsilon)}{1 - \Pr\left(\bigcup_{j=1}^{n} (Y_j \le \varepsilon)\right)}. \qquad \textbf{(4.7)}$$

For $J_2(x, \varepsilon)$, on the event $\bigcap_{j=1}^{n} (\varepsilon < Y_j \le d)$, we have

$$\left(\prod_{i=1}^{1} Y_i, \dots, \prod_{i=1}^{n} Y_i\right) \in [\varepsilon^n, d^n] \times \cdots \times [\varepsilon^n, d^n].$$

Hence, by Lemma 4.1 and the independence between $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_n\}$,

$$J_2(x, \varepsilon) \sim \sum_{k=1}^{n} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x, \bigcap_{j=1}^{n} (\varepsilon < Y_j \le d)\right) \le \sum_{k=1}^{n} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x\right). \qquad \textbf{(4.8)}$$

Substituting (4.8) into (4.7) yields

$$\Pr\left(\sum_{k=1}^{n} X_k^+ \prod_{i=1}^{k} Y_i > x\right) \lesssim \frac{1}{1-\Pr\left(\bigcup_{j=1}^{n}(Y_j \le \varepsilon)\right)} \sum_{k=1}^{n} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x\right).$$

Since the random variables $\{Y_1,\ldots,Y_n\}$ are positive and $\varepsilon > 0$ can be arbitrarily small, we prove that

$$\psi(x,n) \le \Pr\left(\sum_{k=1}^{n} X_k^+ \prod_{i=1}^{k} Y_i > x\right) \lesssim \sum_{k=1}^{n} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x\right). \tag{4.9}$$

Now we aim at an asymptotic lower bound. From (1.3), Lemma 4.1, and the association of the random variables $\{Y_1,\ldots,Y_n\}$, we derive

$$\begin{aligned}
\psi(x,n) &\ge \Pr\left(\sum_{k=1}^{n} X_k \prod_{i=1}^{k} Y_i > x, \bigcap_{j=1}^{n}(\varepsilon < Y_j \le d)\right) \\
&\sim \sum_{k=1}^{n} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x, \bigcap_{j=1}^{n}(\varepsilon < Y_j \le d)\right) \\
&= \sum_{k=1}^{n} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x\right) - \sum_{k=1}^{n} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x, \bigcup_{j=1}^{n}(Y_j \le \varepsilon)\right) \\
&\ge \sum_{k=1}^{n} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x\right)\left(1-\Pr\left(\bigcup_{j=1}^{n}(Y_j \le \varepsilon)\right)\right).
\end{aligned}$$

As earlier, by letting $\varepsilon \searrow 0$, we conclude that

$$\psi(x,n) \gtrsim \sum_{k=1}^{n} \Pr\left(X_k \prod_{i=1}^{k} Y_i > x\right). \tag{4.10}$$

Combining (4.9) and (4.10) leads to the announced result (3.1). ■

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