MORE FIBONACCI VARIETIES

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The Fibonacci varieties introduced by Ann Chi Kim, *Bull. Austral.
Math. Soc.* **19** (1978), 191-196 (1979), are here generalised in an
obvious way suggested by the work of D.L. Johnson, J.W. Wamsley,
and D. Wright, *Proc. London Math. Soc.* (3) **29** (1974), 577-592.
The underlying groups of algebras in these varieties are studied,
and are shown to be abelian when the algebras are generated by a
single element, and in general are found to be extensions of
central subgroups by groups of finite exponent.

1. Introduction

We consider algebraic systems, or briefly *algebras*, that are groups
with an additional unary operation. We use conventional group notation:
$x \cdot y$, x^{-1}, e for the product, inverse, and unit element, respectively; the
additional unary operation is denoted by ϕ and written as a right-hand
operator. In addition to the group laws, which we assume but do not here
specify, we assume the laws

(1.1) $(x \cdot y)\phi = x\phi \cdot y\phi$, $x^{-1}\phi = (x\phi)^{-1}$, $e\phi = e$

(of which the first is well known to imply the other two);

(1.2) $x\phi^m = x\phi^{m-1} \cdot x\phi^{m-2} \cdot \ldots \cdot x\phi \cdot x$;

and

(1.3) $x\phi^n = x$.

Here m and n are integers, with $m \geq 2$ and $n \geq 0$; the case $n = 0$ means that the law (1.3) is trivially satisfied and can be omitted.

        The inspiration for this set-up is in the Fibonacci groups studied, *inter alia*, by Johnson, Wamsley, and Wright [3] (see also Chalk and Johnson [1] for further references). These authors define the Fibonacci groups

$F(m, n) = \mathrm{gp}\big(a_1, a_2, \ldots, a_n; a_{i+m} = a_{i+m-1} \cdot a_{i+m-2} \cdot \ldots \cdot a_i$,

$i = 1, 2, \ldots, n$, suffixes taken modulo $n\big)$,

together with the abelianised groups

$A(m, n) = F(m, n)/F(m, n)' = \mathrm{gp}\big(a_1, a_2, \ldots, a_n;$

$a_{i+m} = a_{i+m-1} \cdot a_{i+m-2} \cdot \ldots \cdot a_i, a_i \cdot a_j = a_j \cdot a_i,$

$i, j = 1, 2, \ldots, n$, suffixes taken modulo $n\big)$.

In order to allow also for the case $n = 0$, which makes not much sense in the Johnson *et al.* set-up, but is of interest in ours, we also define

$F(m, 0) = \mathrm{gp}\big(\ldots, a_{-1}, a_0, a_1, a_2, \ldots; a_{i+m} = a_{i+m-1} \cdot a_{i+m-2} \cdot \ldots \cdot a_i,$

for all $i\big)$,

$A(m, 0) = F(m, 0)/F(m, 0)' = \mathrm{gp}\big(\ldots, a_{-1}, a_0, a_1, a_2, \ldots;$

$a_{i+m} = a_{i+m-1} \cdot a_{i+m-2} \cdot \ldots \cdot a_i, a_i \cdot a_j = a_j \cdot a_i,$ for all $i, j\big)$.

It is not difficult to see that $F(m, 0)$ is a free group of rank m , and $A(m, 0)$ a free abelian group of rank m , both freely generated by $a_0, a_1, \ldots, a_{m-1}$, or by any other set of m consecutive a_i .

        These groups $F(m, n)$ and $A(m, n)$ have an obvious automorphism α , say, of order n , defined by

$a_i \alpha = a_{i+1}$ (suffixes modulo n).

Our operator ϕ imitates the action of α , but not only on a preferred set of generators of the group involved, but on all its elements. This imposes severe restrictions on the groups underlying our algebras; on the

other hand, our approach leads to some new phenomena: the Fibonacci groups
are clearly generated by a_1 and its images under powers of α , and thus
correspond to monogenic (that is one-generator) algebras in our context.
We shall, however, also look at algebras with more than one generator.

We denote the variety of algebras defined by the laws (1.1)-(1.3) by
$\underline{\underline{V}}(m, n)$. In [4], one of us studied the varieties $\underline{\underline{V}}(2, n)$ [there denoted
simply by $\underline{\underline{V}}_n$], and determined the free monogenic algebras of these
varieties, for all $n \geq 1$. In the present paper we deal with the cases
$m \geq 3$, and we specifically do not exclude the case $n = 0$, because this
involves the weakest assumptions and thus greatest generality. We also pay
special attention to the free algebras of $\underline{\underline{V}}(m, n)$ and to their underlying
groups. Denote the free algebra on d generators in $\underline{\underline{V}}(m, n)$ by
$A_d(m, n)$ and its underlying group by $G_d(m, n)$. Our results then are as
follows:

THEOREM 1. *The group $G_1(m, n)$ underlying the monogenic free
algebra of $\underline{\underline{V}}(m, n)$ is abelian, and isomorphic to $A(m, n)$* (in the
notation of Johnson, Wamsley, and Wright [3] - see above).

A proof that $A(m, n)$ is finite when $m \geq 2$, $n \geq 1$, and formulae
for the orders of these groups, are given by Johnson [2]. It has already
been remarked that $A(m, 0)$ is free abelian of rank m , hence infinite.

THEOREM 2. *The groups $G_d(3, n)$ underlying the d-generator free
algebras of $\underline{\underline{V}}(3, n)$ are abelian, for all $d \geq 1$ and all $n \geq 0$; in
fact the groups underlying arbitrary algebras in $\underline{\underline{V}}(3, n)$ are abelian.*

This carries commutativity of the underlying groups from $m = 2$ (see
the Lemma in [4]) over to $m = 3$; but there the commutativity stops:

THEOREM 3. *The groups G underlying algebras A in $\underline{\underline{V}}(m, n)$
satisfy "centre-by exponent $m - 1$ " law*

(1.4) $[x^{m-1}, y] = e$,

*where we use the customary commutator notation $[x, y] = x^{-1} \cdot y^{-1} \cdot x \cdot y$.
Thus such a group G has a subgroup Z in its centre such that G/Z lies
in the Burnside variety $\underline{\underline{B}}_{m-1}$ of groups of exponent $m - 1$. Conversely*

every group in $\underline{\underline{B}}_{m-1}$ *underlies an algebra in* $\underline{\underline{V}}(m, n)$ *, for arbitrary*
$n \geq 0$ *. Thus when* $m \geq 4$ *, there are non-abelian groups that underly*
algebras in $\underline{\underline{V}}(m, n)$ *.*

Note, however, that the law (1.4) is not sufficient to characterise
the groups underlying algebras in $\underline{\underline{V}}(m, n)$, especially when $n \geq 1$.

2. Preliminary results

It is clear that if the integer p divides the integer q , then the
law $x\phi^p = x$ implies the law $x\phi^q = x$. Hence we have the following fact:

LEMMA 1. *For all integers* p, q *such that* p *divides* q *, and all*
m *,*

$$\underline{\underline{V}}(m, 1) \subseteq \underline{\underline{V}}(m, p) \subseteq \underline{\underline{V}}(m, q) \subseteq \underline{\underline{V}}(m, 0) .$$

Next we show that ϕ is not only an endomorphism, but an automorphism
of the underlying group of every algebra in our varieties.

LEMMA 2. *Let* A *be an arbitrary algebra in* $\underline{\underline{V}}(m, n)$ *and* G *its*
underlying group; then ϕ *acts as an automorphism of* G *.*

Proof. The morphism property of ϕ is guaranteed by the laws (1.1),
so we need only show that ϕ is one-to-one and onto. If $n > 0$, this is
obvious, as then ϕ^n is the identity mapping of G , the *carrier* (or set
of elements) of G , which is also, of course, the carrier of A . For
arbitrary $n \geq 0$ we use the law (1.2), to write

$$x = (x\phi)^{-1} \cdot \left(x\phi^2\right)^{-1} \cdot \ldots \cdot \left(x\phi^{m-1}\right)^{-1} \cdot x\phi^m$$
$$= \left(x^{-1} \cdot x^{-1}\phi \cdot \ldots \cdot x^{-1}\phi^{m-2} \cdot x\phi^{m-1}\right)\phi .$$

This shows that every element x is an image under ϕ ; thus $\phi : G \to G$
is onto G . Also, if $x\phi = y\phi$, then also $x\phi^2 = y\phi^2$, and so on; thus

$$x = (x\phi)^{-1} \cdot \left(x\phi^2\right)^{-1} \cdot \ldots \cdot \left(x\phi^{m-1}\right)^{-1} \cdot x\phi^m$$
$$(y\phi)^{-1} \cdot \left(y\phi^2\right)^{-1} \cdot \ldots \cdot \left(y\phi^{m-1}\right)^{-1} \cdot y\phi^m = y .$$

This shows that ϕ is also one-to-one.   //

3. Proof of Theorem 1

From now on we use the morphism property (1.1) of ϕ , and of powers of ϕ , without further mention. Apply ϕ to both sides of (1.2) to get

$$(3.1) \qquad x\phi^{m+1} = x\phi^m \cdot x\phi^{m-1} \cdot \ldots \cdot x\phi .$$

Then

$$(3.2) \qquad x\phi^{m+1} \cdot x = x\phi^m \cdot x\phi^{m-1} \cdot \ldots \cdot x\phi \cdot x$$
$$= x\phi^m \cdot x\phi^m .$$

Now we evaluate $(x\phi \cdot y)\phi^m$ by using (1.2) again, with $x\phi \cdot y$ and then $y \cdot x$ in place of x :

$$(x\phi \cdot y)\phi^m = (x\phi \cdot y)\phi^{m-1} \cdot (x\phi \cdot y)\phi^{m-2} \cdot \ldots \cdot x\phi \cdot y$$
$$= x\phi^m \cdot y\phi^{m-1} \cdot x\phi^{m-1} \cdot y\phi^{m-2} \cdot \ldots \cdot x\phi \cdot y$$
$$= x\phi^m \cdot (y \cdot x)\phi^{m-1} \cdot (y \cdot x)\phi^{m-2} \cdot \ldots \cdot (y \cdot x) \cdot x^{-1}$$
$$= x\phi^m \cdot (y \cdot x)\phi^m \cdot x^{-1}$$
$$= x\phi^m \cdot y\phi^m \cdot x\phi^m \cdot x^{-1} .$$

$$(3.3) \qquad x\phi^{m+1} \cdot y\phi^m = x\phi^m \cdot y\phi^m \cdot x\phi^m \cdot x^{-1} .$$

Put $y = x\phi$ to get

$$x\phi^{m+1} \cdot x\phi^{m+1} \cdot x = x\phi^m \cdot x\phi^{m+1} \cdot x\phi^m ,$$

and apply (3.2):

$$x\phi^{m+1} \cdot x\phi^m \cdot x\phi^m = x\phi^m \cdot x\phi^{m+1} \cdot x\phi^m .$$

This then gives

$$(3.4_1) \qquad x\phi^{m+1} \cdot x\phi^m = x\phi^m \cdot x\phi^{m+1} .$$

We now proceed by induction. Assume that we have, for some $j \geq 1$, the laws

$$(3.4_j) \qquad x\phi^{m+j} \cdot x\phi^m = x\phi^m \cdot x\phi^{m+j} ,$$

$$(3.5_j) \qquad x\phi^{m+j} \cdot x\phi^{m+1} = x\phi^{m+1} \cdot x\phi^{m+j} .$$

We note that $\left(3.4_1\right)$ has just been proved, and that $\left(3.5_1\right)$ is trivially satisfied. Applying ϕ to $\left(3.4_j\right)$ immediately gives $\left(3.5_{j+1}\right)$. To prove also $\left(3.4_{j+1}\right)$, we apply $\left(3.4_j\right)$ to $x\phi \cdot x$ in place of x :

$$(x\phi \cdot x)\phi^{m+j} \cdot (x\phi \cdot x)\phi^m = (x\phi \cdot x)\phi^m \cdot (x\phi \cdot x)\phi^{m+j} \ .$$

Thus

$$x\phi^{m+j+1} \cdot x\phi^{m+j} \cdot x\phi^{m+1} \cdot x\phi^m = x\phi^{m+1} \cdot x\phi^m \cdot x\phi^{m+j+1} \cdot x\phi^{m+j} \ .$$

Applying $\left(3.5_j\right)$ and $\left(3.5_{j+1}\right)$, we shift $x\phi^{m+1}$ on the left-hand side to the left, then cancel it:

$$x\phi^{m+j+1} \cdot x\phi^{m+j} \cdot x\phi^m = x\phi^m \cdot x\phi^{m+j+1} \cdot x\phi^{m+j} \ .$$

Here we use $\left(3.4_j\right)$ to shift $x\phi^{m+j}$ to the right on the left-hand side, then cancel it, giving finally

$\left(3.4_{j+1}\right)$ $$x\phi^{m+j+1} \cdot x\phi^m = x\phi^m \cdot x\phi^{m+j+1} \ .$$

By induction $\left(3.4_j\right)$ is then established for all positive j . Now given integers k, l with $k > l$, we apply ϕ^{l-m} to $\left(3.4_{k-l}\right)$ to get

(3.6) $$x\phi^k \cdot x\phi^l = x\phi^l \cdot x\phi^k \ .$$

As this is symmetric in k, l and the case $k = l$ is trivial, we have proved the first statement in the following lemma:

   LEMMA 3. *In every variety $\underline{\underline{V}}(m, n)$ the law (3.6) is valid for arbitrary integers k, l . If A is a monogenic algebra in $\underline{\underline{V}}(m, n)$, then its underlying group G is abelian.*

   The second statement follows from the fact that if A is generated, *qua* algebra, by the single element a , then its underlying group G is generated, *qua* group, by the set $\left\{a\phi^j \mid j \text{ an integer}\right\}$, and even, because of the law (1.2), by the set $\left\{a\phi^j \mid 0 \leq j \leq m-1\right\}$.   //

   Now the proof of Theorem 1 is the same, *mutatis mutandis*, as that of the Theorem in [4], and we omit it.   //

4. Proof of Theorem 2

We now turn to the special case $m = 3$, and prove commutativity of
the underlying groups in this case. The law (1.2) becomes

$$(4.1) \qquad\qquad x\phi^3 = x\phi^2 \cdot x\phi \cdot x \ ;$$

and we shall proceed without the law (1.3), that is to say, we work in the
variety $\underline{V}(3, 0)$, of which all $\underline{V}(3, n)$ are, as we know, subvarieties.

We begin by applying (4.1) to a product $x \cdot z$:

$$(x \cdot z)\phi^3 = (x \cdot z)\phi^2 \cdot (x \cdot z)\phi \cdot x \cdot z \ ,$$

or

$$(4.21) \qquad\qquad x\phi^3 \cdot z\phi^3 = x\phi^2 \cdot z\phi^2 \cdot x\phi \cdot z\phi \cdot x \cdot z \ .$$

Applying (4.1) to x and z separately, and multiplying, we also get

$$(4.22) \qquad\qquad x\phi^3 \cdot z\phi^3 = x\phi^2 \cdot x\phi \cdot x \cdot z\phi^2 \cdot z\phi \cdot z \ .$$

Comparison of (4.21) and (4.22) then gives, after cancelling $x\phi^2$ and z ,

$$z\phi^2 \cdot x\phi \cdot z\phi \cdot x = x\phi \cdot x \cdot z\phi^2 \cdot z\phi \ .$$

Here we put $z\phi = y$, and note that, as a consequence of Lemma 2, y
ranges with z over the whole carrier of our algebra; thus we get the law

$$y\phi \cdot x\phi \cdot y \cdot x = x\phi \cdot x \cdot y\phi \cdot y \ ,$$

or

$$(4.3) \qquad\qquad (y \cdot x)\phi \cdot y \cdot x = x\phi \cdot x \cdot y\phi \cdot y \ .$$

Now we define a new operation ψ by

$$x\psi = x\phi \cdot x \ .$$

Then (4.3) says that

$$(y \cdot x)\psi = x\psi \cdot y\psi \ ;$$

that is to say, ψ is an *anti*morphism: it reverses the order of the
factors in a product. Then ψ^2 must restore that order, and be a
morphism. Now $x\psi^2$ is, in terms of ϕ ,

$$x\psi^2 = (x\phi \cdot x)\phi \cdot x\phi \cdot x$$

$$= x\phi^2 \cdot x\phi \cdot x\phi \cdot x$$

$$= x\phi^2 \cdot x\phi \cdot x \cdot x\phi \ ,$$

where we have used the law (3.6). Applying (4.1) we get

(4.4) $x\psi^2 = x\phi^3 \cdot x\phi$.

Now as ψ^2 is a morphism, so is $\psi^2\phi^{-1}$, which, using (4.4), is given by

$$x\psi^2\phi^{-1} = x\phi^2 \cdot x$$.

Hence, applying this to the product $x \cdot z$, we have

$$(x \cdot z)\phi^2 \cdot x \cdot z = x\phi^2 \cdot x \cdot z\phi^2 \cdot z \ ,$$

or

$$x\phi^2 \cdot z\phi^2 \cdot x \cdot z = x\phi^2 \cdot x \cdot z\phi^2 \cdot z$$.

Cancelling $x\phi^2$ on the left, z on the right, and putting $z\phi^2 = y$ (which again ranges with z over the whole carrier of our algebra), we finally have the commutative law

$$y \cdot x = x \cdot y \ ,$$

and Theorem 2 follows.   //

5. Proof of Theorem 3

We return to the case of general m , and again dispense with the law (1.3), that is to say, we work in $\underline{V}(m, 0)$. We deduce from (3.2) the law

(5.1) $\left(x^{-1} \cdot x\phi\right)\phi^m = \left(x\phi^m\right)^{-1} \cdot x\phi^{m+1} = x\phi^m \cdot x^{-1}$.

We digress by noting that this implies that for any element, say g , of the carrier of an algebra in $\underline{V}(m, n)$,

$$g\phi^m = g$$

is equivalent with

$$g\phi = g :$$

   LEMMA 4. *An element g of the carrier of an algebra in $\underline{\underline{V}}(m, n)$ is*

fixed by ϕ^m *if, and only if,* g *is fixed by* ϕ .

   COROLLARY. *The varieties* $\underline{V}(m, m)$ *and* $\underline{V}(m, 1)$ *coincide; and they coincide with all varieties* $\underline{\underline{V}}(m, m')$ *, where* m' *divides* m .

   The last statement follows from Lemma 1. Now we re-write (3.3) in the form

$$\left(x\phi^m\right)^{-1}\cdot x\phi^{m+1}\cdot y\phi^m = y\phi^m\cdot x\phi^m\cdot x^{-1} \ .$$

Using (5.1), this gives

$$\left(x^{-1}\cdot x\phi\right)\phi^m\cdot y\phi^m = y\phi^m\cdot\left(x^{-1}\cdot x\phi\right)\phi^m \ ,$$

and applying ϕ^{-m} , we finally get

(5.2) $x^{-1}\cdot x\phi\cdot y = y\cdot x^{-1}\cdot x\phi$.

This law says that $x^{-1}\cdot x\phi$ commutes with all elements, in other words: is central in the underlying group G of the algebra A under consideration.

   The set $\{g^{-1}\cdot g\phi \mid g \in G\}$, where G is the carrier of G (and of A), generates a subgroup Z of G , and we have proved, by (5.2):

   LEMMA 5. *The subgroup* Z *of* G *lies in the centre of* G .

   It is not difficult to verify that the carrier Z of Z is, in fact, the set $\{g^{-1}\cdot g\phi \mid g \in G\}$ we have used to generate Z ; and that Z is the underlying group of a subalgebra T of A . In fact T is the kernel of the natural epimorphism of A onto an algebra A_1 , with underlying group $G_1 = G/Z$, where ϕ acts on G_1 as the identity automorphism. The algebra A_1 is obtained from A by introducing the law

(5.3) $x\phi = x$,

or strengthening the law (1.3) to (5.3). With this law, (1.2) becomes

$$x = x^m \ ,$$

or

(5.4) $x^{m-1} = e$,

the "Burnside" law with exponent $m - 1$. Thus G_1 lies in the Burnside

variety $\underline{\underline{B}}_{m-1}$ of groups of exponent $m - 1$. Conversely, if G_1 is an arbitrary group of exponent $m - 1$, and if we define the operator ϕ as the identity automorphism of G_1, then all the laws (1.1), (1.2), (1.3) are trivially satisfied, so that we get an algebra A_1 with G_1 as its underlying group, and A_1 belongs to $\underline{\underline{V}}(m, 1)$ (and thus to all $\underline{\underline{V}}(m, n)$, with $n \geq 0$). Theorem 3 follows.   //

If we take G_1 in particular as the free d-generator group $B_{d,m-1}$ of $\underline{\underline{B}}_{m-1}$, we thus clearly obtain the group $G_d(m, 1)$ underlying the free d-generator algebra $A_d(m, 1)$ of $\underline{\underline{V}}(m, 1)$. As the variety $\underline{\underline{V}}(m, 1)$ is obtained from $\underline{\underline{V}}(m, n)$ by strengthening the law (1.3) to (5.3), the free d-generator algebra $A_d(m, n)$ of $\underline{\underline{V}}(m, n)$ maps epimorphically on $A_d(m, 1)$, and thus its underlying group $G_d(m, n)$ maps epimorphically on this Burnside group $B_{d,m-1} = G_d(m, 1)$. The kernel is, as we know, a central subgroup $Z_d(m, n)$ of $G_d(m, n)$, generated by the set $\left\{ g^{-1} \cdot g\phi \mid g \in G_d(m, n) \right\}$. This depends on the values of the parameters m, n, d. For example, we know that for $n = 1$, or indeed for any n dividing m, this subgroup is trivial (see the Corollary to Lemma 4). We state without proof that at the other extreme, namely $n = 0$, this kernel $Z_d(m, 0)$ is free abelian of rank $d \cdot m$, and that for $n > 0$ and finite d it is a finite abelian group.

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