Proceedings of the Edinburgh Mathematical Society (2020) **63**, 761–779
doi:10.1017/S0013091520000152

A LOCAL STUDY NEAR THE WOLFF POINT ON THE BALL

FENGBAI LI[1] AND FENG RONG[2]

[1]*School of Mathematics, Shanghai University of Finance and Economics, 777 Guo Ding Road, Shanghai 200433, P.R. China*
(li.fengbai@mail.shufe.edu.cn)

[2]*School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dong Chuan Road, Shanghai 200240, P.R. China*
(frong@sjtu.edu.cn)

(Received 14 February 2019; first published online 17 June 2020)

Abstract Let f be a holomorphic self-map of the unit ball in dimension 2, which does not have an interior fixed point. Suppose that f has a Wolff point p with the boundary dilatation coefficient equal to 1 and the non-tangential differential $\mathrm{d}f_p = id$. The local behaviours of f near p are quite diverse, and we give a detailed study in many typical cases. As a byproduct, we give a dynamical interpretation of the Burns–Krantz rigidity theorem. Note also that similar results hold on two-dimensional contractible smoothly bounded strongly pseudoconvex domains.

Keywords: Wolff point; Siegel upper half-plane; holomorphic self-map

2010 *Mathematics Subject Classification:* Primary 32H50; 32T15

1. Introduction

Consider a holomorphic self-map f of $\mathbb{B}^2 := \{(z_1, z_2) \in \mathbb{C}^2 : \ |z_1|^2 + |z_2|^2 < 1\}$, which does not have an interior fixed point. It is well known that there exists a unique boundary point p, called the *Wolff point*, such that the iterates f^k converge uniformly on compact subsets to p (see e.g. [2]). We will always assume that the non-tangential limit of $\mathrm{d}f_z$ exists at p and denote it by $\mathrm{d}f_p$ (see e.g. [1, 3]). Let $\lambda \leq 1$ be the boundary dilatation coefficient of f at p (see e.g. [8]).

When $\lambda < 1$ or $\lambda = 1$ but $\mathrm{d}f_p \neq id$, the situation is much better understood (see e.g. [6] and the references therein). Therefore, we will focus on the case where $\lambda = 1$ and $\mathrm{d}f_p = id$. The main purpose of this paper is to give a detailed local analysis in many typical cases, which shows very diverse behaviours of f near p.

Suppose that f has $e_1 = (1, 0)$ as its Wolff point and $\mathrm{d}f_{e_1} = id$. We need the following notions of regularity of f at e_1 (cf. [6, 9]). (See also [4, 5], where universal holomorphic models and iteration properties are established with no regularity assumptions at the Wolff point.)

We say that f belongs to $\mathcal{C}^m(e_1)$, $m \geq 2$, if it can be written as

$$f(z_1, z_2) = \Bigg(z_1 + \sum_{2 \leq j+k \leq m} c_{jk} z_2^j (z_1 - 1)^k + \epsilon_1(z_1, z_2),$$
$$z_2 + \sum_{2 \leq j+k \leq m} d_{jk} z_2^j (z_1 - 1)^k + \epsilon_2(z_1, z_2) \Bigg), \quad \epsilon_i = o(\|(z_1 - 1, z_2)\|^m).$$

Similarly, we can define $\mathcal{C}^{m+\epsilon}(e_1)$, $0 < \epsilon < 1$, by assuming $\epsilon_i = O(\|(z_1 - 1, z_2)\|^{m+\epsilon})$.

The *order* of f is defined as

$$\min\{j + k : \; c_{jk} \neq 0 \text{ or } d_{jk} \neq 0\}.$$

Note that, by the Burns–Krantz rigidity theorem [**11**], the order of $f \neq id$ is either 2 or 3.

We say that f belongs to $\mathcal{D}^m(e_1)$, $m \geq 2$, if it can be written as

$$f(z_1, z_2) = \Bigg(z_1 + \sum_{j+k \geq 2, j+2k \leq m} c_{jk} z_2^j (z_1 - 1)^k + \epsilon_1(z_1, z_2),$$
$$z_2 + \sum_{j+k \geq 2, j+2k \leq m} d_{jk} z_2^j (z_1 - 1)^k + \epsilon_2(z_1, z_2) \Bigg), \quad \epsilon_i = o(|z_1 - 1|^{m/2}).$$

Similarly, we can define $\mathcal{D}^{m+\epsilon}(e_1)$, $0 < \epsilon < 1$, by assuming $\epsilon_i = O(|z_1 - 1|^{(m+\epsilon)/2})$. One can readily check that $\mathcal{C}^{2m}(e_1) \subset \mathcal{D}^{2m}(e_1) \subset \mathcal{C}^m(e_1)$.

Let σ be the Cayley transform which sends the unit ball to the Siegel upper half-plane $\mathbb{H}^2 := \{(z, w) \in \mathbb{C}^2 : \; \operatorname{Im} w > |z|^2\}$ and e_1 to the origin, i.e.

$$(z, w) = \sigma(z_1, z_2) = \left(\frac{z_2}{1 + z_1}, \; i \frac{1 - z_1}{1 + z_1} \right).$$

Set $F := \sigma \circ f \circ \sigma^{-1}$. Then, the regularity of f at e_1 naturally translates to the regularity of F at 0. And one can readily check that for $f \in \mathcal{C}^m(e_1)$ we have

$$F(z, w) = \Bigg(z + \sum_{2 \leq j+k \leq m} a_{jk} z^j w^k + \epsilon_1'(z, w), \; w + \sum_{2 \leq j+k \leq m} b_{jk} z^j w^k + \epsilon_2'(z, w) \Bigg), \quad (1.1)$$

where $\epsilon_i' = o(\|(z, w)\|^m)$.

For our purpose, it is more convenient to work in the right half-plane $\mathbb{H}_2 := \{(x, y) \in \mathbb{C}^2 : \; \operatorname{Re} x > |y|^2\}$. Set

$$(x, y) = \varphi(z, w) = (-iw, -z),$$

and $\tilde{F} := \varphi \circ F \circ \varphi^{-1}$. Then, from (1.1), we have

$$\tilde{F}(x, y) = \Bigg(x + \sum_{2 \leq j+k \leq m} (-1)^{j+1} i^{k+1} b_{jk} x^k y^j + \tilde{\epsilon}_1(x, y),$$
$$y + \sum_{2 \leq j+k \leq m} (-1)^{j+1} i^k a_{jk} x^k y^j + \tilde{\epsilon}_2(x, y) \Bigg), \quad (1.2)$$

where $\tilde{\epsilon}_i = o(\|(x, y)\|^m)$.

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

Consider the automorphism of $\mathbb{H}_2$ of the form

$$(u,v)=\psi(x,y)=\left(\frac{1}{x},\frac{y}{x}\right),$$

which sends the origin to the infinity. Set $G:=\psi\circ\tilde{F}\circ\psi^{-1}$. Then G has ∞ as an attracting fixed point. Denote $(u_n,v_n)=G^n(u,v)$.

Recall that the well-known Julia's lemma (see e.g. [16]) says that the holosphere

$$E(\alpha)=\left\{(z_1,z_2)\in\mathbb{B}^2:\ \frac{|1-z_1|^2}{1-|z_1|^2-|z_2|^2}<\alpha\right\},\quad 0<\alpha<1,$$

is invariant under f. One can readily check that this implies that G leaves $E(R)$ invariant, where

$$E(R)=\{(u,v)\in\mathbb{H}_2:\ \operatorname{Re}u>|v|^2+R\},\quad R=\frac{1}{\alpha}>1.$$

This fact will be used throughout the paper when estimating the higher-order terms.

In this paper, we give a detailed analysis of the asymptotic behaviour of f near the Wolff point. Our main results are as follows.

Theorem 1.1. *Let f be a holomorphic self-map of the ball $\mathbb{B}^2$ with Wolff point e_1 and $\mathrm{d}f_{e_1}=id$. Assume that $f\in\mathcal{D}^5(e_1)$, where f has order 2 and is non-degenerate at e_1. Let G be the associated self-map of the right half-plane $\mathbb{H}_2$ with ∞ as the Wolff point. Then, the following cases can occur:*

(1) $u_n\sim r_n+int,\ v_n\sim v,\ t\in\mathbb{R}\backslash\{0\},\ r_n=o(n),\ r_n\gtrsim|v|^2$;

(2) $u_n\sim r_n+int,\ v_n\sim v\log n,\ t\in\mathbb{R}\backslash\{0\},\ v\in\mathbb{C}\backslash\{0\},\ r_n=o(n),\ r_n\gtrsim\log^2 n$;

(3) $u_n\sim r_n+int,\ v_n\sim v\,\mathrm{e}^{\mathrm{i}s\log n},\ s,t\in\mathbb{R}\backslash\{0\},\ v\in\mathbb{C}\backslash\{0\},\ r_n=o(n),\ r_n\gtrsim|v|^2$;

(4) $u_n\sim n,\ v_n\sim s,\ s\geq 0$;

(5) $u_n\sim n,\ v_n\sim n^s,\ \frac{1}{2}\geq s>0$;

(6) $u_n\sim n,\ v_n\sim s\log n,\ s>0$;

(7) $u_n\sim mn,\ v_n\sim\sqrt{lmn},\ m>0,\ 1\geq l>0$.

Theorem 1.2. *Let f be a holomorphic self-map of the ball $\mathbb{B}^2$ with Wolff point e_1 and $\mathrm{d}f_{e_1}=id$. Assume that $f\in\mathcal{D}^7(e_1)$ and f has order 3. Let G be the associated self-map of the right half-plane $\mathbb{H}_2$ with ∞ as the Wolff point. Then, the following cases can occur:*

(1) $u_n\sim\sqrt{n},\ v_n\sim s,\ s\geq 0$;

(2) $u_n\sim\sqrt{n},\ v_n\sim n^s,\ \frac{1}{4}\geq s>0$;

(3) $u_n\sim\sqrt{n},\ v_n\sim s\log n,\ s>0$;

(4) $u_n\sim\sqrt{mn},\ v_n\sim\sqrt[4]{lmn},\ m>0,\ 1\geq l>0$.

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

Remark 1.3. Similar results hold for holomorphic self-maps of two-dimensional contractible smoothly bounded strongly pseudoconvex domains.

Remark 1.4. When f has order 2 and is degenerate at the Wolff point, the local dynamics are more complicated and we do not have a complete classification at this moment (cf. Remark 3.3).

In § 2, we study order 2 maps and prove Theorem 1.1. In § 3, we study order 3 maps and prove Theorem 1.2. In § 4, we study order 4 maps and give a dynamical interpretation of the Burns–Krantz rigidity theorem. In § 5, we give a brief discussion of strongly pseudoconvex domains.

2. Order 2 maps

Assigning weight 2 to w (respectively, x) and weight 1 to z (respectively, y), we say that the term $z^j w^k$ (respectively, $x^k y^j$) is of weighted order $2k + j$. Denote by $O_w(m)$ terms with weighted order at least m.

By [**10**, Theorem 3.1, **10**, Remark 3.2], we have the following.

Lemma 2.1. *Let F be a holomorphic self-map of the Siegel upper half-plane with the origin as its boundary fixed point and $\mathrm{d}F_0 = id$. Assume that $F \in \mathcal{D}^5(0)$ and the order of F is 2. Then*

$$F(z,w) = (z + a_{11}zw + a_{02}w^2 + a_{30}z^3 + O_w(4),\\ w + b_{02}w^2 + b_{21}z^2w + b_{40}z^4 + O_w(5)), \tag{2.1}$$

with $\operatorname{Im} b_{02} \geq 0$, $\operatorname{Im} a_{11} \geq 0$, and

$$\operatorname{Im} b_{02}(2\operatorname{Im} a_{11} - \operatorname{Im} b_{02}) \geq (\operatorname{Re} a_{11} - \operatorname{Re} b_{02})^2. \tag{2.2}$$

We say that F is *non-degenerate* at 0 (i.e. f is non-degenerate at e_1) if $b_{02} \neq 0$. Otherwise, we say that F is *degenerate* at 0. It will be clear that the non-degeneracy is preserved under the normalization below (Lemmas 2.2 and 2.3).

It is well known that the group Φ of automorphisms of $\mathbb{H}^2$ fixing the origin consists of the following two types of map (see e.g. [**17**]):

$$\phi_0(z,w) = (\lambda\, \mathrm{e}^{\mathrm{i}\theta} z, \lambda^2 w), \quad \lambda > 0, \ \ \theta \in \mathbb{R},$$

and

$$\phi_1(z,w) = \left(\frac{z + aw}{1 - 2i\bar{a}z - (r + i|a|^2)w}, \frac{w}{1 - 2i\bar{a}z - (r + i|a|^2)w}\right), \quad a \in \mathbb{C}, \ \ r \in \mathbb{R}.$$

We can use the group Φ to normalize $F(z,w)$ as follows.

Lemma 2.2. *Let F be as in Lemma 2.1, with $\operatorname{Im} b_{02} = 0$. Then, under ϕ_1, F can be normalized as*

$$F(z,w) = (z + (t + is)zw + \gamma w^2 + O(3), \ w + tw^2 + O(3)), \tag{2.3}$$

where $t \in \mathbb{R}$, $\gamma = 0$ if $s \neq 0$, and $\gamma = a_{02}$ if $s = 0$.

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

Proof. From (2.2) and $\operatorname{Im} b_{02} = 0$, we get that $\operatorname{Re} a_{11} = \operatorname{Re} b_{02}$. Set $a_{11} = t + is$. Set $(z', w') = \phi_1(z, w)$ and $F' = \phi_1 \circ F \circ \phi_1^{-1}$. Then one can readily check that $F'(z', w')$ takes the form

$$F'(z', w') = (z' + (t + is)z'w' + (a_{02} - isa)w'^2 + O(3),\ w' + tw'^2 + O(3)).$$

If $s \neq 0$, then setting $a = a_{02}/is$ we get $\gamma = 0$. If $s = 0$, then $\gamma = a_{02}$. □

Lemma 2.3. *Let F be as in Lemma 2.1, with $\operatorname{Im} b_{02} > 0$. Then, under ϕ_1, F can be normalized as*

$$F(z, w) = (z + a_{11}zw + \gamma w^2 + O(3),\ w + b_{02}w^2 + O(3)), \tag{2.4}$$

where $\gamma = 0$ if and only if $a_{02} = 0$ and $a_{11} = b_{02}$. Under ϕ_0, F can be further normalized as

$$F(z, w) = (z + (s + i\alpha)zw + \beta w^2 + O(3),\ w + (t + i)w^2 + O(3)), \tag{2.5}$$

where $s, t \in \mathbb{R}$, $\beta \geq 0$ with $\beta = 0$ if and only if $\gamma = 0$, and $\alpha \geq \frac{1}{2}$ with $\alpha = \frac{1}{2}$ only if $s = t$.

Proof. Set $(z', w') = \phi_1(z, w)$ and $F' = \phi_1 \circ F \circ \phi_1^{-1}$. Then one can readily check that $F'(z', w')$ takes the form

$$F'(z', w') = (z' + a_{11}z'w' + (a_{02} - a(a_{11} - b_{02}))w'^2 + O(3),\ w' + b_{02}w'^2 + O(3)).$$

It is obvious that $a_{02} - a(a_{11} - b_{02}) = 0$ for all a if and only if $a_{02} = 0$ and $a_{11} = b_{02}$.

Now assume that $\operatorname{Im} b_{02} > 0$. Set $(z'', w'') = \phi_0(z', w')$ and $F'' = \phi_1 \circ F' \circ \phi_1^{-1}$. Then one can readily check that $F''(z'', w'')$ takes the form

$$F''(z'', w'') = (z'' + \lambda^{-2}a_{11}z''w'' + \gamma\lambda^{-3}\,\mathrm{e}^{\mathrm{i}\theta}w''^2 + O(3),\ w'' + \lambda^{-2}b_{02}w''^2 + O(3)).$$

Thus, taking $\lambda = (\operatorname{Im} b_{02})^{1/2}$ and $\theta = -\operatorname{Arg}\gamma$ when $\gamma \neq 0$, and $\theta = 0$ when $\gamma = 0$, one gets the desired normal form. Note that $\alpha = \operatorname{Im} a_{11}/\operatorname{Im} b_{02}$, and thus by Lemma 2.1 we have $\alpha \geq 1/2$ with $\alpha = 1/2$ implying $\operatorname{Re} a_{11} = \operatorname{Re} b_{02}$, i.e. $s = t$. □

By Lemma 2.1 and (1.2), we can write $\tilde{F}$ as

$$\begin{aligned}\tilde{F}(x, y) = (&x + ib_{02}x^2 + b_{21}xy^2 - ib_{40}y^4 + O_w(5),\\ &y + ia_{11}xy + a_{02}x^2 + a_{30}y^3 + O_w(4)).\end{aligned} \tag{2.6}$$

Then G takes the form

$$\begin{aligned}G(u, v) = \Bigg(&u\left(1 - ib_{02}\frac{1}{u} - b_{21}\frac{v^2}{u^2} + ib_{40}\frac{v^4}{u^3} + \mu(u, v)\right),\\ &v\left(1 + i(a_{11} - b_{02})\frac{1}{u} + a_{02}\frac{1}{vu} + (a_{30} - b_{21})\frac{v^2}{u^2} + ib_{40}\frac{v^4}{u^3}\right) + \nu(u, v)\Bigg),\end{aligned}$$

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

where

$$\mu(u,v) = O\bigg(\frac{v}{u^2}, \frac{1}{u^2}, \frac{v^3}{u^3}, \frac{v^5}{u^4}\bigg),$$

and

$$\nu(u,v) = O\bigg(\frac{v^2}{u^2}, \frac{v}{u^2}, \frac{1}{u^2}, \frac{v^4}{u^3}, \frac{v^6}{u^4}\bigg).$$

For $(u,v) \in E(R)$ with R large, one can readily check that

$$\mu(u,v) = o\bigg(\frac{1}{u}\bigg), \quad \nu(u,v) = o\bigg(\frac{1}{u}\bigg) + o\bigg(\frac{1}{vu}\bigg). \tag{2.7}$$

In this section, we always assume that F is non-degenerate at 0. And, for simplicity, we first assume that $a_{30} = b_{21} = b_{40} = 0$.

First, consider the case $\operatorname{Im} b_{02} = 0$. Then, by Lemma 2.2, we can write $\tilde{F}$ as

$$\tilde{F}(x,y) = (x + itx^2 + O_w(5),\ y + (it - s)xy + \gamma x^2 + O_w(4)), \quad t \neq 0.$$

Thus, G takes the form

$$G(u,v) = \bigg(u\bigg(1 - it\frac{1}{u} + \mu(u,v)\bigg),\ v\bigg(1 - s\frac{1}{u} + \gamma\frac{1}{uv}\bigg) + \nu(u,v)\bigg). \tag{2.8}$$

Since for $|u|$ large, $\mu(u,v) = o(1/u)$, we have

$$\operatorname{Im} u_n \sim -nt, \quad \operatorname{Re} u_n = o(n). \tag{2.9}$$

If $s = 0$, then G takes the form

$$G(u,v) = \bigg(u\bigg(1 - it\frac{1}{u} + \mu(u,v)\bigg),\ v\bigg(1 + a_{02}\frac{1}{uv}\bigg) + \nu(u,v)\bigg).$$

If $a_{02} = 0$, then from (2.7), we have

$$v_n \sim v. \quad \text{[Theorem 1.1(1)]} \tag{2.10}$$

If $a_{02} \neq 0$, then from (2.7) and (2.9), we have

$$v_n \sim \frac{ia_{02}}{t}\log n. \quad \text{[Theorem 1.1(2)]} \tag{2.11}$$

If $s \neq 0$, then G takes the form

$$G(u,v) = \bigg(u\bigg(1 - it\frac{1}{u} + \mu(u,v)\bigg), v\bigg(1 - s\frac{1}{u}\bigg) + \nu(u,v)\bigg).$$

From (2.7) and (2.9), one can readily check that

$$\log v_n \sim \log v + i\bigg(-\frac{s}{t}\log n\bigg). \quad \text{[Theorem 1.1(3)]} \tag{2.12}$$

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

Remark 2.4. Consider the following holomorphic automorphism of $\mathbb{H}^2$:

$$\tau_t(z,w) = (z, w+t), \quad t \in \mathbb{R}\backslash\{0\}.$$

Then, conjugating with the Cayley transform which sends $(1,0)$ on $\partial\mathbb{B}^2$ to $(0,0)$, τ_t induces a holomorphic automorphism f_t of $\mathbb{B}^2$ with $(-1,0)$ as its Wolff point. Conjugating f_t with the Cayley transform which sends $(-1,0)$ on $\partial\mathbb{B}^2$ to $(0,0)$, we get the following holomorphic automorphism of $\mathbb{H}^2$ with $(0,0)$ as its Wolff point:

$$F_t(z,w) = \left(\frac{z}{1-tw}, \frac{w}{1-tw}\right).$$

The corresponding $G_t(u,v)$ is of the form

$$G_t(u,v) = (u - it, v).$$

From the structure of the isotropy group Φ, we know that $F_t(z,w)$ are the only automorphisms of $\mathbb{H}^2$ tangent to the identity at the origin. Therefore, we will say that holomorphic self-maps F of $\mathbb{H}^2$, whose associated map G has asymptotic behaviour as in Theorem 1.1(1), are of **automorphic type**. All other holomorphic self-maps of $\mathbb{H}^2$ are of **non-automorphic type**. This dichotomy is similar to the one-dimensional case (cf. [**7**]), where the notion of a hyperbolic step is used. For instance, for a typical orbit of automorphic type of the form $(u_n, v_n) \sim (1 + int, 0)$, one can readily check that the limit of the Kobayashi distance between (u_n, v_n) and (u_{n+1}, v_{n+1}) is

$$\lim_{n\to\infty} d_{\kappa_{\mathbb{H}_2}}((u_n,v_n),(u_{n+1},v_{n+1})) = \frac{1}{2}\ln\frac{\sqrt{4+t^2}+t}{\sqrt{4+t^2}-t},$$

which is positive for $t \in \mathbb{R}\backslash\{0\}$.

Next, consider the case $\operatorname{Im} b_{02} > 0$. Then, by Lemma 2.3, we can write $\tilde{F}$ as

$$\tilde{F}(x,y) = (x + (-1+it)x^2 + O_w(5),\ y + (-\alpha + is)xy + \beta x^2 + O_w(4)), \quad \beta \geq 0,\ \ \alpha \geq \tfrac{1}{2}.$$

Thus, G takes the form

$$G(u,v) = \left(u\left(1 + a\frac{1}{u} + \mu(u,v)\right),\ v\left(1 + b\frac{1}{u} + \beta\frac{1}{uv}\right) + \nu(u,v)\right), \tag{2.13}$$

where $\operatorname{Re} a = 1$ and $\operatorname{Re} b = 1 - \alpha \leq \frac{1}{2}$.

Remark 2.5. In [**6**], Bayart gave three examples to illustrate the diverse behaviours of G near infinity: [**6**, Example 5.13] corresponds to taking $a = 1$, $\beta = \frac{1}{10}$ and $b = 0$ in (2.13); and [**6**, Example 5.14] corresponds to taking $a = 1$, $b = \lambda$ and $\beta = 0$ in (2.13). Note, however, that [**6**, Example 5.12] is not a self-map of $\mathbb{H}_2$, and it can not be modified by only changing the coefficients to exhibit the desired behaviour. See the discussion at the end of this section.

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

Assume that a and b are real. Then $a = 1$, $b \leq \frac{1}{2}$, and we have

$$(u_1, v_1) = \left(u\left(1 + \frac{1}{u} + \mu(u,v)\right), v\left(1 + b\frac{1}{u} + \beta\frac{1}{uv}\right) + \nu(u,v)\right). \tag{2.14}$$

Since for $|u|$ large, $\mu(u,v) = o(1/u)$, we have

$$u_n \sim n. \tag{2.15}$$

For the estimate of $|v_n|$, we consider the following three typical cases.

Case 1. $b < 0$. Since for $|u|$ large, $\nu(u,v) = o(1/u, 1/vu)$, we have

$$v_1 = v\left(1 + \left(\frac{\beta}{v} + b\right)\frac{1}{u} + o\left(\frac{1}{u}, \frac{1}{vu}\right)\right). \tag{2.16}$$

From (2.16), we have

$$\mathrm{Re}\, v_1 = \mathrm{Re}\, v + \mathrm{Re}\left(\frac{\beta + bv}{u}\right) + o\left(\frac{v}{u}, \frac{1}{u}\right). \tag{2.17}$$

Since $u_n \sim n$, we have $\mathrm{Re}\, v_1 > \mathrm{Re}\, v$ if $\beta + b\mathrm{Re}\, v > 0$, i.e. $\mathrm{Re}\, v < -\beta/b$, and $\mathrm{Re}\, v_1 < \mathrm{Re}\, v$ if $\beta + b\mathrm{Re}\, v > 0$, i.e. $\mathrm{Re}\, v > -\beta/b$. Therefore, we get

$$\lim_{n\to\infty} \mathrm{Re}\, v_n = -\frac{\beta}{b}. \tag{2.18}$$

From (2.16), we also have

$$|v_1|^2 = |v|^2 + \frac{2\mathrm{Re}\,(u(\beta v + b|v|^2))}{|u|^2} + o\left(\frac{v^2}{u}, \frac{v}{u}\right). \tag{2.19}$$

Since $u_n \sim n$, we have $|v_1| > |v|$ if $\beta\mathrm{Re}\, v + b|v|^2 > 0$, i.e. $|v|^2 < -(\beta/b)\mathrm{Re}\, v$, and $|v_1| < |v|$ if $\beta\mathrm{Re}\, v + b|v|^2 < 0$, i.e. $|v|^2 > -(\beta/b)\mathrm{Re}\, v$. Therefore, we get

$$\lim_{n\to\infty} |v_n|^2 + \frac{\beta}{b}\mathrm{Re}\, v_n = 0. \tag{2.20}$$

Combining (2.18) and (2.20), we get

$$\lim_{n\to\infty} v_n = -\frac{\beta}{b}. \quad \text{[Theorem 1.1(4)]} \tag{2.21}$$

Remark 2.6. Under the conditions $a = 1$ and $b \neq 0$, $\tilde{F}$ takes the form

$$\tilde{F}(x,y) = (x - x^2 + O(3), y + (b-1)xy + \beta x^2 + O(3)).$$

One can readily check that $[1 : -\beta/b]$ is a non-degenerate characteristic direction for $\tilde{F}$ at $(0,0)$ with director equal to $-b$ (see e.g. [**15**]).

Case 2. $b > 0$. In this case, (2.17) and (2.19) still hold.

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

From (2.17), we have $\operatorname{Re} v_1 > \operatorname{Re} v$ if $\beta + b\operatorname{Re} v > 0$, i.e. $\operatorname{Re} v > -\beta/b$, and $\operatorname{Re} v_1 < \operatorname{Re} v$ if $\beta + b\operatorname{Re} v > 0$, i.e. $\operatorname{Re} v < -\beta/b$. Therefore, we get $\lim_{n\to\infty} |\operatorname{Re} v_n| = \infty$ and

$$\operatorname{Re} v_1 = \operatorname{Re} v\left(1 + b\frac{1}{u} + o\left(\frac{1}{u}\right)\right).$$

Hence, we have

$$|\operatorname{Re} v_n| \sim n^b. \tag{2.22}$$

From (2.19), we have $|v_1| > |v|$ if $\beta\operatorname{Re} v + b|v|^2 > 0$, i.e. $|v|^2 > -(\beta/b)\operatorname{Re} v$, and $|v_1| < |v|$ if $\beta\operatorname{Re} v + b|v|^2 < 0$, i.e. $|v|^2 < -(\beta/b)\operatorname{Re} v$. Therefore, we get $\lim_{n\to\infty} |v_n| = \infty$ and

$$|v_1| = |v|\left(1 + b\frac{1}{u} + o\left(\frac{1}{u}\right)\right).$$

Hence, we have

$$|v_n| \sim n^b. \tag{2.23}$$

Combining (2.22) and (2.23), we get

$$v_n \sim n^b. \quad \text{[Theorem 1.1(5)]} \tag{2.24}$$

Case 3. $b = 0$ and $\beta > 0$. In this case, we have

$$v_1 = v + \frac{\beta}{u} + o\left(\frac{1}{u}\right). \tag{2.25}$$

Since $u_n \sim n$, we get

$$v_n \sim \beta \log n. \quad \text{[Theorem 1.1(6)]} \tag{2.26}$$

Note that if $\lim_{n\to\infty}(|v_n|^2/|u_n|) = 0$, then $v^2/u^2 = o(1/u)$ and $v^4/u^3 = o(1/u)$, hence the above discussion is still valid without assuming $c = d = e = 0$.

We next consider the case where $\lim_{n\to\infty}(|v_n|^2/|u_n|) > 0$. Note that this implies that $\lim_{n\to\infty}|v_n| = \infty$ and $1/uv = o(1/u)$. We only consider the case where c, d and e are real. Then, from (2.13) and (2.7), we have

$$\begin{aligned}(u_1, v_1) = \Bigg(&u\left(1 + \frac{1}{u} + c\frac{v^2}{u^2} + d\frac{v^4}{u^3} + o\left(\frac{1}{u}\right)\right),\\ &v\left(1 + b\frac{1}{u} + e\frac{v^2}{u^2} + d\frac{v^4}{u^3} + o\left(\frac{1}{u}\right)\right)\Bigg).\end{aligned} \tag{2.27}$$

Thus,

$$\frac{v_1^2}{u_1} = \frac{v^2}{u}\left(1 + \left((2b-1) + (2e-c)\frac{v^2}{u} + d\frac{v^4}{u^2}\right)\frac{1}{u} + o\left(\frac{1}{u}\right)\right). \tag{2.28}$$

Set $\gamma = 2e - c$, $\delta = 2b - 1$, $w = v^2/u$ and $L(w) = dw^2 + \gamma w + \delta$. Anticipating $\lim_{n\to\infty}(v_n^2/u_n) = l$ with $0 < l \leq 1$, we need

$$L(l) = 0, \quad L(w) > 0 \quad \text{for } 0 < w < l, \qquad L(w) < 0 \quad \text{for } w > l.$$

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

Thus, we get

$$\delta < 0, \quad d < 0, \quad \gamma^2 > 4d\delta, \quad \frac{-\gamma - \sqrt{\gamma^2 - 4d\delta}}{2d} = l \leq 1. \tag{2.29}$$

Note also that $\lim_{n\to\infty}(v_n^2/u_n) = l$ implies that $u_n \sim (1 + cl + dl^2)n$. Thus, we need

$$m := 1 + cl + dl^2 > 0. \tag{2.30}$$

Therefore, for b, c, d and e satisfying (2.29) and (2.30), we have the estimates

$$u_n \sim mn, \quad v_n \sim \sqrt{lmn}. \quad \text{[Theorem 1.1(7)]} \tag{2.31}$$

Example 2.7. Set $b = \frac{1}{4}$, $c = 4$, $d = -8$ and $e = \frac{9}{2}$. Then $\gamma = 5$, $\delta = -\frac{1}{2}$ and $L(w) = -8w^2 + 5w - \frac{1}{2}$ has a positive root $l = \frac{1}{2}$. Thus, (2.27) takes the form

$$\begin{aligned}(u_1, v_1) = \Bigg(&u\left(1 + \frac{1}{u} + 4\frac{v^2}{u^2} - 8\frac{v^4}{u^3} + o\left(\frac{1}{u}\right)\right),\\ &v\left(1 + \frac{1}{4}\frac{1}{u} + \frac{9}{2}\frac{v^2}{u^2} - 8\frac{v^4}{u^3} + o\left(\frac{1}{u}\right)\right)\Bigg),\end{aligned}$$

and we have the estimates

$$u_n \sim n, \quad v_n \sim \sqrt{\frac{n}{2}}.$$

3. Order 3 maps

By [10, Lemma 3.3], we have the following.

Lemma 3.1. *Let F be a holomorphic self-map of the Siegel upper half-plane with the origin as its boundary fixed point and $\mathrm{d}F_0 = id$. Assume that $F \in \mathcal{D}^7(0)$ and the order of F is 3. Then*

$$\begin{aligned}F(z, w) = (&z + a_{12}zw^2 + a_{03}w^3 + a_{31}z^3w + a_{50}z^5 + O_w(6),\\ &w + b_{03}w^3 + b_{22}z^2w^2 + a_{41}z^4w + b_{60}z^6 + O_w(7)),\end{aligned} \tag{3.1}$$

with $\mathrm{Im}\, b_{03} = 0$, and

$$\tfrac{1}{2}b_{03} \leq \mathrm{Re}\, a_{12} \leq \tfrac{3}{2}b_{03}, \quad (3b_{03} - 2\mathrm{Re}\, a_{12})(2\mathrm{Re}\, a_{12} - b_{03}) \geq 16(\mathrm{Im}\, a_{12})^2.$$

We say that F is *non-degenerate* at 0 (i.e. f is non-degenerate at e_1) if $b_{03} > 0$. If $b_{03} = 0$, then we say that F is *degenerate* at 0. Unlike the order 2 case, we can actually show that F must be non-degenerate if F is not the identity map.

Lemma 3.2. *Let F be as in Lemma 3.1 with 0 being its Wolff point. Then F is non-degenerate at 0.*

Proof. Write $F(z, w) = (F_1(z, w), F_2(z, w))$ and $g(w) := F_2(0, w)$. Then $F(0, w) = (F_1(0, w), g(w))$ with $\mathrm{Im}\, g(w) > |F_1(0, w)|^2 \geq 0$. Thus, g is a holomorphic self-map of $\{\mathrm{Im}\, w > 0\}$ with $w = 0$ as its boundary fixed point.

Suppose that F is degenerate at 0. Then we have $g(w) = w + O_w(7) = w + o(3)$ near $w = 0$. By the Burns–Krantz rigidity theorem, we must have $g(w) \equiv w$.

Write $f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2)) = \sigma^{-1} \circ F(z, w) \circ \sigma$ and $h(z_1) := f_2(z_1, 0)$. Then one can readily check that $g(w) \equiv w$ gives $f_1(z_1, 0) \equiv z_1$. Since f is a holomorphic self-map of the unit ball, this implies that $\limsup_{|z_1|\to 1} |h(z_1)| = 0$. By the maximum modulus principle, we get $h(z_1) \equiv 0$. But this means that all points with $z_2 = 0$ are fixed by f, a contradiction. □

Remark 3.3. A similar argument to that used in Lemma 3.2 shows that in the order 2 case b_{02} and b_{03} can not both be zero in (2.1).

As in the previous section, we can normalize $F(z, w)$ using Φ as follows.

Lemma 3.4. *Let F be as in Lemma 3.1 with 0 being its Wolff point. Then, under ϕ_1, F can be normalized as*

$$F(z, w) = (z + a_{12}zw^2 + \gamma w^3 + O(4),\ w + b_{03}w^3 + O(4)), \tag{3.2}$$

where $\gamma = 0$ if and only if $a_{03} = 0$ and $a_{12} = b_{03}$. Under ϕ_0, F can be further normalized as

$$F(z, w) = (z + (\alpha + is)zw^2 - i\beta w^3 + O(4),\ w + \tfrac{1}{2}w^3 + O(4)), \tag{3.3}$$

where $s \in \mathbb{R}$, $\beta \geq 0$ with $\beta = 0$ if and only if $\gamma = 0$, and $\frac{1}{4} \leq \alpha \leq \frac{3}{4}$ with $\alpha = \frac{1}{4}$ or $\frac{3}{4}$ only if $s = 0$.

Proof. Set $(z', w') = \phi_1(z, w)$ and $F' = \phi_1 \circ F \circ \phi_1^{-1}$. Then one can readily check that $F'(z', w')$ takes the form

$$F'(z', w') = (z' + a_{12}z'w'^2 + (a_{03} - a(a_{12} - b_{03}))w'^3 + O(4),\ w' + b_{03}w'^3 + O(4)).$$

It is obvious that $a_{03} - a(a_{12} - b_{03}) = 0$ for all a if and only if $a_{03} = 0$ and $a_{12} = b_{03}$.

Note that $b_{03} > 0$ by Lemma 3.2. Set $(z'', w'') = \phi_0(z', w')$ and $F'' = \phi_1 \circ F' \circ \phi_1^{-1}$. Then one can readily check that $F''(z'', w'')$ takes the form

$$F''(z'', w'') = (z'' + \lambda^{-4}a_{12}z''w''^2 + \gamma\lambda^{-5}\,\mathrm{e}^{\mathrm{i}\theta}w''^3 + O(4),\ w'' + \lambda^{-4}b_{03}w''^3 + O(4)).$$

Thus, taking $\lambda = (2b_{03})^{1/4}$ and $\theta = -\mathrm{Arg}\gamma - \pi/2$ when $\gamma \neq 0$ or $\theta = 0$ when $\gamma = 0$, one gets the desired normal form. Note that $\alpha = \mathrm{Re}\, a_{12}/2b_{03}$, and thus by Lemma 3.1 we have $\frac{1}{4} \leq \alpha \leq \frac{3}{4}$ with $\alpha = \frac{1}{4}$ or $\frac{3}{4}$ implying $\mathrm{Im}\, a_{12} = 0$, i.e. $s = 0$. □

By Lemma 3.1 and (1.2), we can write $\tilde{F}$ as

$$\begin{aligned}\tilde{F}(x, y) = (&x - b_{03}x^3 + ib_{22}x^2y^2 + b_{41}xy^4 - ib_{60}y^6 + O_w(7),\\ &y - a_{12}x^2y + ia_{03}x^3 + ia_{31}xy^3 + a_{50}y^5 + O_w(6)).\end{aligned} \tag{3.4}$$

By Lemma 3.4, $\tilde{F}$ takes the form

$$\begin{aligned}\tilde{F}(x, y) = (&x - \tfrac{1}{2}x^3 + \tilde{b}_{22}x^2y^2 + \tilde{b}_{41}xy^4 + \tilde{b}_{60}y^6 + O_w(7),\\ &y - (\alpha + is)x^2y + \beta x^3 + \tilde{a}_{31}xy^3 + \tilde{a}_{50}y^5 + O_w(6)),\end{aligned} \tag{3.5}$$

where $\frac{1}{4} \leq \alpha \leq \frac{3}{4}$.

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

Suppose $s = 0$. Then $G(u,v)$ takes the form

$$G(u,v) = \left(u\left(1 + \frac{1}{2u^2} + a_1\frac{v^2}{u^3} + a_2\frac{v^4}{u^4} + a_3\frac{v^6}{u^5} + \mu(u,v)\right),\right.$$
$$\left. v\left(1 + \left(\frac{1}{2} - \alpha\right)\frac{1}{u^2} + \frac{\beta}{vu^2} + b_1\frac{v^2}{u^3} + b_2\frac{v^4}{u^4} + a_3\frac{v^6}{u^5}\right) + \nu(u,v)\right), \tag{3.6}$$

where

$$\mu(u,v) = O\left(\frac{v}{u^3}, \frac{1}{u^3}, \frac{v^3}{u^4}, \frac{v^5}{u^5}, \frac{v^7}{u^6}\right),$$

and

$$\nu(u,v) = O\left(\frac{v^2}{u^3}, \frac{v}{u^3}, \frac{1}{u^3}, \frac{v^4}{u^4}, \frac{v^6}{u^5}, \frac{v^8}{u^6}\right).$$

Set $b = \frac{1}{2} - \alpha$. Then $-\frac{1}{4} \leq b \leq \frac{1}{4}$. Consider first the case $a_1 = a_2 = a_3 = b_1 = b_2 = 0$. Then $G(u,v)$ takes the form

$$G(u,v) = \left(u\left(1 + \frac{1}{2}\frac{1}{u^2} + \mu(u,v)\right), v\left(1 + b\frac{1}{u^2} + \beta\frac{1}{vu^2}\right) + \nu(u,v)\right).$$

Set $z = u^2$. Then $G(u,v)$ induces $\tilde{G}(z,v)$ of the form

$$\tilde{G}(z,v) = \left(z + 1 + \tilde{\mu}(z,v), v + b\frac{v}{z} + \beta\frac{1}{z} + \tilde{\nu}(z,v)\right), \tag{3.7}$$

where

$$\tilde{\mu}(z,v) = O\left(\frac{v}{z^{1/2}}, \frac{1}{z^{1/2}}, \frac{v^3}{z}, \frac{v^5}{z^{3/2}}, \frac{v^7}{z^2}\right),$$

and

$$\tilde{\nu}(z,v) = O\left(\frac{v^2}{z^{3/2}}, \frac{v}{z^{3/2}}, \frac{1}{z^{3/2}}, \frac{v^4}{z^2}, \frac{v^6}{z^{5/2}}, \frac{v^8}{z^3}\right).$$

Therefore, by the discussion in the previous section, we have the following cases.

Case 1. $-\frac{1}{4} \leq b < 0$.

$$z_n \sim n, \quad v_n \sim -\frac{\beta}{b}. \quad \text{[Theorem 1.2(1)]}$$

Case 2. $0 < b \leq \frac{1}{4}$.

$$z_n \sim n, \quad v_n \sim n^b. \quad \text{[Theorem 1.2(2)]}$$

Case 3. $b = 0$ and $\beta > 0$.

$$z_n \sim n, \quad v_n \sim \beta \log n. \quad \text{[Theorem 1.2(3)]}$$

Note that if $\lim_{n\to\infty}(|v_n|^2/|u_n|) = 0$, then $v^2/u^3 = o(1/u^2)$, $v^4/u^4 = o(1/u^2)$ and $v^6/u^5 = o(1/u^2)$; hence, the above discussion is still valid without assuming $a_1 = a_2 = a_3 = b_1 = b_2 = 0$.

Let us next consider the case where $\lim_{n\to\infty}(|v_n|^2/|u_n|) > 0$. Note that this implies that $\lim_{n\to\infty}|v_n| = \infty$ and $1/vu^2 = o(1/u^2)$. We only consider the case where a_1, a_2, a_3, b_1 and b_2 are all real. Then, from (3.6), we have

$$G(u,v) = \left(u\left(1 + \frac{1}{2}\frac{1}{u^2} + a_1\frac{v^2}{u^3} + a_2\frac{v^4}{u^4} + a_3\frac{v^6}{u^5} + o\left(\frac{1}{u^2}\right)\right),\right.$$
$$\left. v\left(1 + b\frac{1}{u^2} + b_1\frac{v^2}{u^3} + b_2\frac{v^4}{u^4} + a_3\frac{v^6}{u^5} + o\left(\frac{1}{u^2}\right)\right)\right).$$

Thus,

$$\frac{v_1^2}{u_1} = \frac{v^2}{u}\left(1 + \left(\left(2b - \frac{1}{2}\right) + (2b_1 - a_1)\frac{v^2}{u} + (2b_2 - a_2)\frac{v^4}{u^2} + a_3\frac{v^6}{u^3}\right)\frac{1}{u^2} + o\left(\frac{1}{u^2}\right)\right).$$

Set $c_0 = 2b - \frac{1}{2}$, $c_1 = 2b_1 - a_1$, $c_2 = 2b_2 - a_2$, $w = v^2/u$ and $L(w) = a_3w^3 + c_2w^2 + c_1w + c_0$. Then, for a positive root l of $L(w)$ with $L'(l) < 0$, we have $\lim_{n\to\infty}(v_n^2/u_n) = l$. Note that $c_0 < 0$. Then, it is easy to see that such an l exists only if $a_3 < 0$ or $a_3 = 0$ and $c_2 < 0$ and it is unique. For such an l, we get the following estimates:

$$z_n \sim mn, \quad m = 1 + 2(a_1 + a_2 + a_3)l > 0,$$

$$v_n \sim \sqrt{l\sqrt{mn}}, \quad 0 < l \le 1. \quad \text{[Theorem 1.2(4)]}$$

Remark 3.5. In [**13**, Example 4], Huang gave a family of holomorphic self-maps of $\mathbb{B}^2$ with $(1, 0)$ as the Wolff point as follows:

$$f(z_1, z_2) = \left(\frac{z_1 + a(1 - z_1)^2}{1 + a(1 - z_1)^2}, \frac{z_2}{1 + a(1 - z_1)^2}\right), \quad a > 0.$$

Then one can readily check that $f(z_1, z_2)$ induces $F(z, w)$ of the form

$$F(z, w) = (z(1 + 4aw^2 + O(3)),\ w(1 + 4aw^2 + O(3))).$$

4. Order 4 maps

By the well-known Burns–Krantz rigidity theorem (see [**11**]), we know that the only holomorphic self-map of the unit ball tangent to the identity at the Wolff point of order greater or equal to 4 is the identity. In this section, we give a dynamical interpretation of this rigidity phenomenon.

For this purpose, suppose that there exist holomorphic self-maps of the unit ball tangent to the identity at the Wolff point of order equal to 4 which is not the identity. Then, similar to Lemmas 2.1 and 3.1, we have the following lemma, whose proof we defer to the appendix.

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

Lemma 4.1. *Let F be a holomorphic self-map of the Siegel upper half-plane with the origin as its boundary fixed point and* $\mathrm{d}F_0 = id$. *Assume that F is of order* 4. *Then*

$$F(z,w) = (z + a_{13}zw^3 + a_{04}w^4 + O(5),\ w + b_{04}w^4 + O(5)), \tag{4.1}$$

with $\operatorname{Im} b_{04} \geq 0$, $\operatorname{Im} b_{04} \geq 2\operatorname{Im} a_{13}$, $a_{50} = a_{41} = a_{60} = 0$ *and* $b_{50} = b_{41} = b_{32} = b_{60} = b_{51} = b_{70}$.

By Lemma 4.1 and (1.2), we can write $\tilde{F}$ as

$$\begin{aligned}\tilde{F}(x,y) = (&x - ib_{04}x^4 - b_{23}x^3y^2 + ib_{42}x^2y^4 + b_{61}xy^6 - ib_{80}y^8 + O_w(9),\\ &y - ia_{13}x^2y - a_{04}x^4 - a_{32}x^2y^3 + ia_{51}xy^5 + a_{70}y^7 + O_w(8)).\end{aligned} \tag{4.2}$$

In this section, we only consider the case $\operatorname{Im} b_{04} > 0$. For simplicity, we assume that b_{04} and a_{13} are purely imaginary and $b_{23} = b_{42} = b_{61} = b_{80} = a_{32} = a_{51} = a_{70} = 0$.

Since $\operatorname{Im} b_{04} \geq 2\operatorname{Im} a_{13}$, by a scaling of the form $(x,y) \mapsto (ax, \sqrt{a}\,\mathrm{e}^{\mathrm{i}\theta}y)$ with $a > 0$, $\tilde{F}$ takes the form

$$\tilde{F}(x,y) = (x + \tfrac{1}{3}x^4 + O_w(9), y + \alpha x^3 y - \beta x^4 + O_w(8)), \tag{4.3}$$

where $\alpha \leq \frac{1}{6}$ and $\beta \geq 0$ with $\beta = 0$ if and only if $a_{04} = 0$.

Then $G(u,v)$ takes the form

$$G(u,v) = \left(u\left(1 - \frac{1}{3u^3} + o\left(\frac{1}{u^3}\right)\right), v\left(1 - \left(\frac{1}{3} - \alpha\right)\frac{1}{u^3} - \frac{\beta}{vu^3} + o\left(\frac{1}{u^3}, \frac{1}{vu^3}\right)\right)\right). \tag{4.4}$$

Set $b = \frac{1}{3} - \alpha$. Then $b \geq \frac{1}{6}$. Set $z = -u^3$. Then $G(u,v)$ induces $\tilde{G}(z,v)$ of the form

$$\tilde{G}(z,v) = \left(z\left(1 + \frac{1}{z} + o\left(\frac{1}{z}\right)\right), v + b\frac{v}{z} + \beta\frac{1}{z} + o\left(\frac{1}{z}, \frac{1}{vz}\right)\right). \tag{4.5}$$

Therefore, we have the estimates $z_n \sim n$ and $v_n \sim n^b$. Thus, we get $u_n \sim n^{1/3}\,\mathrm{e}^{\mathrm{i}\pi/3}$ or $u_n \sim n^{1/3}\,\mathrm{e}^{-\mathrm{i}\pi/3}$.

Since the limiting behaviour of u_n is not unique, there is more than one attracting basin at the Wolff point, which is impossible since the whole $\mathbb{B}^2$ is the attracting basin.

Remark 4.2. For order 5 or higher maps, a similar analysis will provide the same contradiction.

5. Strongly pseudoconvex domains

In this section, let D be a two-dimensional contractible smoothly bounded strongly pseudoconvex domain and f a holomorphic self-map of D without any interior fixed point.

First, it is known that the Wolff–Denjoy theorem holds on D (see e.g. [**14**]). Thus, there exists a unique Wolff point p on the boundary of D for f. Moreover, a version of Julia's lemma in terms of small and large horospheres holds on D at p (see e.g. [**1**, **14**]).

Second, it is also known that the Julia–Wolff–Carathéodory theorem holds on D (see e.g. [**3**]). Thus, the non-tangential differential of f at p, $\mathrm{d}f_p$, exists. We assume that $\mathrm{d}f_p = id$.

Third, by [**12**], there exists a holomorphic embedding ρ of $\bar{D}$ into $\overline{\mathbb{B}^2}$ such that $\rho(\bar{D}) \cap \overline{\mathbb{B}^2} = e_1$ and $\rho(D)$ is tangent to $\mathbb{B}^2$ at e_1 with $\rho(p) = e_1$. Set $\Omega = \rho(D)$.

Finally, [**10**, Theorem 3.1, **10**, Lemma 3.3] are stated for holomorphic maps between two strongly pseudoconvex domains, using the Chern–Moser normal forms. However, for holomorphic self-maps of a strongly pseudoconvex domain, the Chern–Moser components cancel each other and thus Lemmas 2.1 and 3.1 also hold on Ω.

Therefore, a similar local analysis can be carried out on Ω, yielding a version of Theorems 1.1 and 1.2 for two-dimensional contractible smoothly bounded strongly pseudoconvex domains. Since the proof of Lemma 3.2 depends on $D = \mathbb{B}^2$, we need to add the assumption of non-degeneracy in Theorem 1.2 in the generalized version.

Appendix A

In this appendix, we prove Lemma 4.1.

First, we recall two lemmas from [**10**], adapted to our setting.

Lemma A.1. *Let $p(x_1, x_2)$ be a weighted homogeneous polynomial of degree d in $(x_1, x_2) \in \mathbb{R}^2$ with weight (ν_1, ν_2), i.e. $p(t^{\nu_1}x_1, t^{\nu_2}x_2) = t^d p(x_1, x_2)$. Let r be a real function satisfying*

$$r(x_1, x_2) = o((|x_1|^{1/\nu_1} + |x_2|^{1/\nu_2})^d), \quad (x_1, x_2) \to (0, 0).$$

Suppose that $p(x) + r(x) \geq 0$ for $x = (x_1, x_2)$ in a neighbourhood of 0. Then $p(x) \geq 0$. Furthermore, if $p_0(x_1, x_2)$ is the non-trivial bihomogeneous component of p of minimal degree in x_1 (or in x_2), then also $p_0(x) \geq 0$.

Lemma A.2. *Let $p(z, \bar{z}) = \sum_k p_k z^k \bar{z}^{d-k}$ be a homogeneous real-valued polynomial of degree d for $z \in \mathbb{C}$. Assume that $p(z, \bar{z}) \geq 0$ in a neighbourhood of 0. Then,*

(1) *if d is odd, then $p \equiv 0$;*

(2) *if d is even, then $p_s \geq 0$ for $s = d/2$;*

(3) *if $d = 2s$ and $p_s = 0$, then $p \equiv 0$.*

Now, we prove Lemma 4.1. Write

$$F(z, w) = (f_1(z, w), f_2(z, w)) = \left(z + \sum_{j \geq 4} p_j(z, w),\ w + \sum_{j \geq 4} q_j(z, w)\right),$$

where $p_j(z, w)$ and $q_j(z, w)$ are homogeneous of degree j. Write $w = u + iv$.

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

Since F maps $\mathbb{H}^2$ into itself, we must have

$$\operatorname{Im} f_2(z,w) \geq |f_1(z,w)|^2 \quad \text{when } \operatorname{Im} w \geq |z|^2.$$

This implies that

$$\sum_{j\geq 4} \operatorname{Im} q_j(z, u+i|z|^2) \geq 2\sum_{j\geq 4} \operatorname{Re}\,(z\bar{p}_j(z, u+i|z|^2)) + \left|\sum_{j\geq 4} p_j(z, u+i|z|^2)\right|^2. \tag{A.1}$$

Considering weighted order 4 terms in (A.1) and applying Lemma A.1, we have

$$\operatorname{Im}\,(b_{40}z^4) \geq 0,$$

which clearly implies that

$$b_{40} = 0. \tag{A.2}$$

Considering weighted order 5 terms in (A.1) and applying Lemma A.1, we have

$$\operatorname{Im}\,(b_{50}z^5) + \operatorname{Im}\,(b_{31}(u+i|z|^2)) \geq 2\operatorname{Re}\,(\bar{a}_{40}z\bar{z}^4).$$

Applying Lemma A.1 again, we have

$$\operatorname{Im}\,(b_{31}z^3u) \geq 0, \tag{A.3}$$

and

$$\operatorname{Im}\,(b_{50}z^5) + \operatorname{Im}\,(ib_{31}z^3|z|^2) \geq 2\operatorname{Re}\,(\bar{a}_{40}z\bar{z}^4). \tag{A.4}$$

From (A.3), we have

$$b_{31} = 0. \tag{A.5}$$

Combining (A.4) and (A.5) and applying Lemma A.2, we have

$$b_{50} = 0, \quad a_{40} = 0. \tag{A.6}$$

Considering weighted order 6 terms in (A.1) and applying Lemma A.1, we have

$$\begin{aligned} &\operatorname{Im}\,(b_{60}z^6) + \operatorname{Im}\,(b_{41}z^4(u+i|z|^2)) + \operatorname{Im}\,(b_{22}z^2(u+i|z|^2)^2) \\ &\geq 2\operatorname{Re}\,(\bar{a}_{50}z\bar{z}^5) + 2\operatorname{Re}\,(\bar{a}_{31}z\bar{z}^3(u-i|z|^2)). \end{aligned} \tag{A.7}$$

Applying Lemma A.1 again, we have

$$\operatorname{Im}\,(b_{22}z^2u^2) \geq 0, \tag{A.8}$$

and

$$\operatorname{Im}\,(b_{60}z^6) + \operatorname{Im}\,(ib_{41}z^4|z|^2) \geq 2\operatorname{Re}\,(\bar{a}_{50}|z|^2\bar{z}^4) - 2\operatorname{Re}\,(i\bar{a}_{31}|z|^4\bar{z}^2). \tag{A.9}$$

From (A.8), we have

$$b_{22} = 0. \tag{A.10}$$

Applying Lemma A.2 to (A.9), we have

$$\mathrm{Im}\,(b_{60}z^6) = 0, \tag{A.11}$$

$$\mathrm{Re}\,(i\bar{a}_{31}z^2\bar{z}^4) = 0, \tag{A.12}$$

and

$$\mathrm{Im}\,(ib_{41}z^5\bar{z}) = 2\mathrm{Re}\,(\bar{a}_{50}z\bar{z}^5). \tag{A.13}$$

From (A.11) and (A.12), we have

$$b_{60} = 0, \quad a_{31} = 0. \tag{A.14}$$

Putting (A.10) into (A.7) and applying Lemma A.1, we have

$$\mathrm{Im}\,(b_{41}z^4u) \geq 0. \tag{A.15}$$

From (A.15), we have

$$b_{41} = 0. \tag{A.16}$$

Combining (A.13) and (A.16), we have

$$a_{50} = 0. \tag{A.17}$$

Considering weighted order 7 terms in (A.1) and applying Lemma A.1, we have

$$\begin{aligned} &\mathrm{Im}\,(b_{70}z^7) + \mathrm{Im}\,(b_{51}z^5(u+i|z|^2)) + \mathrm{Im}\,(b_{32}z^3(u+i|z|^2)^2) + \mathrm{Im}\,(b_{13}z(u+i|z|^2)^3) \\ &\quad \geq 2\mathrm{Re}\,(\bar{a}_{60}z\bar{z}^6) + 2\mathrm{Re}\,(\bar{a}_{41}z\bar{z}^4(u-i|z|^2)) + 2\mathrm{Re}\,(\bar{a}_{22}z\bar{z}^2(u-i|z|^2)^2). \end{aligned} \tag{A.18}$$

Applying Lemma A.1 again, we have

$$\mathrm{Im}\,(b_{13}zu^3) \geq 0,$$

which implies that

$$b_{13} = 0. \tag{A.19}$$

Putting (A.19) into (A.18) and applying Lemma A.1, we have

$$\mathrm{Im}\,(b_{32}z^3u^2) \geq 2\mathrm{Re}\,(\bar{a}_{22}z\bar{z}^2u^2),$$

which, by Lemma A.2, implies that

$$b_{32} = 0, \quad a_{22} = 0. \tag{A.20}$$

Putting (A.20) into (A.18) and applying Lemma A.1, we have

$$\mathrm{Im}\,(b_{51}z^5u) \geq 2\mathrm{Re}\,(\bar{a}_{41}z\bar{z}^4u),$$

which, by Lemma A.2, implies that

$$b_{51} = 0, \quad a_{41} = 0. \tag{A.21}$$

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

Putting (A.21) into (A.18) and applying Lemma A.1, we have

$$\operatorname{Im}(b_{70}z^7) \geq 2\operatorname{Re}(\bar{a}_{60}z\bar{z}^6),$$

which, by Lemma A.2, implies that

$$b_{70} = 0, \quad a_{60} = 0. \tag{A.22}$$

Considering weighted order 8 terms in (A.1) and applying Lemma A.1, we have

$$\begin{aligned} &\operatorname{Im}(b_{80}z^8) + \operatorname{Im}(b_{61}z^6(u+i|z|^2)) + \operatorname{Im}(b_{42}z^4(u+i|z|^2)^2) \\ &\quad + \operatorname{Im}(b_{23}z^2(u+i|z|^2)^3) + \operatorname{Im}(b_{04}(u+i|z|^2)^4) \\ &\geq 2\operatorname{Re}(\bar{a}_{70}z\bar{z}^7) + 2\operatorname{Re}(\bar{a}_{51}z\bar{z}^5(u-i|z|^2)) \\ &\quad + 2\operatorname{Re}(\bar{a}_{32}z\bar{z}^3(u-i|z|^2)^2) + 2\operatorname{Re}(\bar{a}_{13}z\bar{z}(u-i|z|^2)^3). \end{aligned} \tag{A.23}$$

Applying Lemma A.1 again, we have

$$\operatorname{Im}(b_{04}u^4) \geq 0,$$

which implies that

$$\operatorname{Im} b_{04} \geq 0. \tag{A.24}$$

Putting $u = t|z|^2$ into (A.23) and applying Lemma A.2, we have

$$\operatorname{Im}(b_{04}(t+i)^4) \geq 2\operatorname{Re}(\bar{a}_{13}(t-i)^3). \tag{A.25}$$

Writing out (A.25) into a polynomial of t, we have

$$\begin{aligned} &\operatorname{Im} b_{04}t^4 + (4\operatorname{Re} b_{04} - 2\operatorname{Re} a_{13})t^3 - (6\operatorname{Im} b_{04} - 6\operatorname{Im} a_{13})t^2 \\ &\quad - (4\operatorname{Re} b_{04} - 6\operatorname{Re} a_{13})t + (\operatorname{Im} b_{04} - 2\operatorname{Im} a_{13}) \geq 0. \end{aligned} \tag{A.26}$$

From (A.24) and (A.26), we have

$$\operatorname{Im} b_{04} \geq 2\operatorname{Im} a_{13}. \tag{A.27}$$

Combining (A.2), (A.5), (A.6), (A.10), (A.14), (A.16), (A.17), (A.19), (A.20), (A.21), (A.22), (A.24) and (A.27), Lemma 4.1 is proven.

Acknowledgements. We would like to thank John Erik Fornæss, Xiaojun Huang and Kang-Tae Kim for some stimulating discussions. Part of this work was done while the second author was visiting the Department of Mathematical Sciences at the Norwegian University of Science and Technology. He would like to thank the Department and, in particular, John Erik Fornæss and Berit Stensønes for the generous hospitality and support. We also thank the referee for many useful comments, especially for pointing out Lemma 3.2.

The first author is partially supported by the National Natural Science Foundation of China (grant nos. 11971290 and 11501347). The second author is partially supported by the National Natural Science Foundation of China (grant no. 11871333). Part of this work was done while the second author was supported by a Simons Visiting Professorship at the Mathematisches Forschungsinstitut Oberwolfach.

https://doi.org/10.1017/S0013091520000152 Published online by Cambridge University Press

References

1. M. Abate, Horospheres and iterates of holomorphic maps, *Math. Z.* **198** (1988), 225–238.
2. M. Abate, *Iteration theory of holomorphic maps on taut manifolds* (Mediterranean Press, Rende, Cosenza, 1989).
3. M. Abate, Angular derivatives in strongly pseudoconvex domains, in *Proceedings of Symposia in Pure Mathematics*, Volume 52 (2), pp. 23–40 (American Mathematical Society, Providence, RI, 1991).
4. L. Arosio, Canonical models for the forward and backward iteration of holomorphic maps, *J. Geom. Anal.* **27** (2017), 1178–1210.
5. L. Arosio and F. Bracci, Canonical models for holomorphic iteration, *Trans. Amer. Math. Soc.* **368** (2016), 3305–3339.
6. F. Bayart, The linear fractional model on the ball, *Rev. Mat. Iberoam.* **24** (2008), 765–824.
7. P. S. Bourdon and J. H. Shapiro, *Cyclic phenomena for composition operators*, Memoirs of the American Mathematical Society, Volume 125, no. 596 (American Mathematical Society, Providence, RI, 1997).
8. F. Bracci, Dilatation and order of contact for holomorphic self-maps of strongly convex domains, *Proc. Lond. Math. Soc.* **86** (2003), 131–152.
9. F. Bracci and G. Gentili, Solving the Schröder equation at the boundary in several variables, *Michigan Math. J.* **53** (2005), 337–356.
10. F. Bracci and D. Zaitsev, On biholomorphisms between bounded quasi-Reinhardt domains, *J. Funct. Anal.* **254** (2008), 1449–1466.
11. D. Burns and S. G. Krantz, Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary, *J. Amer. Math. Soc.* **7** (1994), 661–676.
12. K. Diederich, J. E. Fornæss and E. F. Wold, Exposing points on the boundary of a strictly pseudoconvex or a locally convexifiable domain of finite 1-type, *J. Geom. Anal.* **24** (2014), 2124–2134.
13. X. Huang, A preservation principle of extremal mappings near a strongly pseudoconvex point and its applications, *Illinois J. Math.* **38** (1994), 283–302.
14. D. Ma, On iterates of holomorphic maps, *Math. Z.* **207** (1991), 417–428.
15. F. Rong, A brief survey on local holomorphic dynamics in higher dimensions, in *Complex analysis and geometry*, Springer Proceedings in Mathematics and Statistics, Volume 144, pp. 295–307 (Springer, Tokyo, 2015).
16. W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Grundlehren der mathematischen Wissenschaften, Volume 241 (Springer-Verlag, New York, 1980).
17. A. G. Vitushkin, Holomorphic mappings and the geometry of hypersurfaces, in *Introduction to complex analysis*, pp. 159–214 (Springer-Verlag, Berlin Heidelberg, 1997).