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Remarks on the asymptotic behaviour of solutions to the compressible Navier–Stokes equations in the half-line

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We study the time-asymptotic behaviour of solutions to the Navier–Stokes equations for a one-dimensional viscous polytropic ideal gas in the half-line. Using a local representation for the specific volume, which is obtained by using a special cut-off function to localize the problem, and the weighted energy estimates, we prove that the specific volume is pointwise bounded from below and above for all x, t and that for all t the temperature is bounded from below and above locally in x. Moreover, global solutions are convergent as time goes to infinity. The large-time behaviour of solutions to the Cauchy problem is also examined.

1. Introduction

We study the large-time behaviour of solutions to the following initial boundary-value problem in the half-line,

$$u_t = v_x, \tag{1.1}$$

$$v_t = \sigma_x \qquad \left(\sigma := \mu\frac{v_x}{u} - R\frac{\theta}{u}\right), \tag{1.2}$$

$$c_V\theta_t = \left[\lambda\frac{\theta_x}{u}\right]_x + \sigma v_x, \tag{1.3}$$

together with the initial conditions

$$(u(x,0), v(x,0), \theta(x,0)) = (u_0(x), v_0(x), \theta_0(x)), \quad x \in \Omega, \tag{1.4}$$

and the boundary conditions

$$v|_{\partial\Omega} = 0, \qquad \theta_x|_{\partial\Omega} = 0, \tag{1.5}$$

where $\Omega = (0,\infty)$. The system (1.1)–(1.3) describes the motion of a one-dimensional viscous polytropic ideal gas in Ω in Lagrangian coordinates, where u, v and θ are the specific volume, the velocity and the absolute temperature, respectively, σ is the stress and μ, R, c_V, λ are positive constants.

Since the first work of Kazhikhov and Shelukhin [13] on the global existence for the equations of a one-dimensional viscous gas with large initial data, significant progress has been made on the mathematical aspect of the initial and initial

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boundary-value problems for (1.1)–(1.3). For initial boundary-value problems in *bounded* domains, the existence and uniqueness of global (generalized) solutions and the regularity have been proved. Furthermore, the global solution is asymptotically stable as time tends to infinity (see, for example, [1,2,4,14,16–18,21] and the references cited therein).

For the Cauchy problem (1.1)–(1.4) with $\Omega = \mathbb{R}$ and the initial boundary-value problem for (1.1)–(1.5) (in *unbounded* domains), Kazhikhov and Shelukhin [12,13] (also cf. [2,6,7,20]) proved that *if* $u_0 - 1, v_0, \theta_0 - 1 \in H^1$, $u_0, \theta_0 > 0$ *on* $\bar{\Omega}$, *and* u_0, v_0, θ_0 *are compatible with (1.5), then there exists a unique (generalized) solution of* (u, v, θ) *with* $u, \theta > 0$ *such that, for any* $T > 0$,

$$\left.\begin{aligned} u - 1, v, \theta - 1 &\in L^\infty((0,T), H^1), \\ u_t &\in L^\infty((0,T), L^2), \\ v_t, \theta_t, u_{xt}, v_{xx}, \theta_{xx} &\in L^2((0,T), L^2), \end{aligned}\right\} \tag{1.6}$$

and the regularity holds. The time-asymptotic behaviour as $t \to \infty$ of the solution has been studied under some smallness conditions on the initial data (see, for example, [5,8,10,11,15,19], among others). However, the large-time behaviour of the solution in the case of large data was not known until 1999. In 1999, the author [9] used a special cut-off function to derive a local representation for $u(x,t)$ and the new estimates for $\theta(x,t)$ and $\sigma(x,t)$ to obtain some partial results on the large-time behaviour of solutions.

The present paper is a continuation of the work [9]. In this paper we improve the results of [9] on the problem (1.1)–(1.5). The improvement is the threefold. We prove the pointwise boundedness of $u(x,t)$ from below and above for all (x,t) and the local in x pointwise boundedness of $\theta(x,t)$ for all t. The convergence of $v(x,t)$ in the H^1-norm is shown. The large-time behaviour of $\theta(x,t)$ is obtained. We point out that Feireisl and Petzeltová recently [3] studied the unconditional stability of stationary flows driven by large time-independent external forces for the system (1.1)–(1.3) with the heat conductivity λ tending to infinity with growing internal energy. They proved that the global solution converges to the stationary flow determined uniquely by the external forces as $t \to \infty$. Moreover, from [3], one can see how the temperature and external forces influence the asymptotic behaviour of the solution. Unfortunately, here we cannot deal with the case of external forces, since the boundedness of domains in [3] is essential in the arguments of [3].

To state the main result we define

$$u_1(t) := \frac{1}{k}\int_0^k u(x,t)\,\mathrm{d}x, \qquad \theta_1(t) := \frac{1}{k}\int_0^k \theta(x,t)\,\mathrm{d}x, \tag{1.7}$$

then the main result of this paper reads as follows.

THEOREM 1.1. *Let* $k > 0$ *be an arbitrary but fixed integer. Let* $u_0 - 1, \theta_0 - 1$, $v_0, (v_0)_x/u_0 \in H^1$ *and* $u_0, \theta_0 > 0$ *on* $\bar{\Omega}$. *Then, for the initial boundary-value problem (1.1)–(1.5), we have*

$$\begin{aligned} \alpha \leqslant u(x,t) \leqslant \beta \qquad &\textit{for all } (x,t) \in \bar{\Omega} \times [0,\infty), \\ C_1(k) \leqslant \theta(x,t) \leqslant C_2(k) \quad &\textit{for all } (x,t) \in [0,k] \times [0,\infty) \end{aligned}$$

and

$$\|(u(t)-u_1(t), v(t), \theta(t)-\theta_1(t))\|_{H^1(0,k)} \to 0 \quad \text{as } t\to\infty, \tag{1.8}$$

where α, β are positive constants independent of t and $C_1(k)$ and $C_2(k)$ are positive constants that depend on k but not on t.

Remark 1.2. Theorem 1.1 still holds for the Cauchy problem (1.1)–(1.4) with $\Omega=\mathbb{R}$, when the initial specific volume and initial temperature are even functions and the initial velocity is a odd function. This can easily be verified by using the reflection of solutions and the uniqueness.

In the proof of theorem 1.1, the derivation of pointwise boundedness of $u(x,t)$ is similar to that in [9]. The estimate of derivatives of v and θ, however, is different. In [9], the time-asymptotics of θ was not obtained due to the lack of spatial derivative estimates of θ. In this paper we control first v_t in $L^\infty([0,\infty), L^2_{\text{loc}})$ and θ_t in $L^2([0,\infty), L^2_{\text{loc}})$ by the $L^\infty([0,\infty), L^2_{\text{loc}})$-norm of θ_x, and then θ_x by v_t and θ_t by using the estimate deduced by the second law of thermodynamics and delicate weighted energy estimates. This process enables us to be able to close the estimate for v_t, θ_t, θ_x, but requires $(v_0)_x/u_0 \in H^1$, and consequently gives the uniform in t boundedness of the derivatives of (u,v,θ) by applying Gronwall's inequality. From the boundedness of the derivatives, theorem 1.1 follows. We shall prove theorem 1.1 in § 3. In § 2 we derive uniform pointwise bounds of $u(x,t)$.

Notation 1.3. Let G be a domain in $\mathbb{R}$. Let $m \geqslant 0$ be a non-negative integer and let $1\leqslant p\leqslant\infty$. By $W^{m,p}(G)$ we denote the usual Sobolev space defined over G with norm $\|\cdot\|_{W^{m,p}(G)}$, $W^{m,2}(G)\equiv H^m(G)$ with norm $\|\cdot\|_{H^m(G)}$, and $W^{0,p}(G)\equiv L^p(G)$ with norm $\|\cdot\|_{L^p(G)}$. For simplicity, we also use the following abbreviations:

$$L^p\equiv L^p(\Omega),\qquad H^m\equiv H^m(\Omega),\qquad \|\cdot\|_{L^p}\equiv\|\cdot\|_{L^p(\Omega)},\qquad \|\cdot\|_{H^m}\equiv\|\cdot\|_{H^m(\Omega)}.$$

$\|\cdot\|$ stands for the norm in $L^2(\Omega)$. $L^p(I,B)$ (respectively, $\|\cdot\|_{L^p(I,B)}$) denotes the space of all strongly measurable, pth-power integrable (essentially bounded if $p=\infty$) functions from I to B (respectively, its norm), where $I\subset\mathbb{R}$ is an interval, B a Banach space. For a vector-valued function $f=(f_1,\dots,f_m)$ and a normed space X with the norm $|||\cdot|||$, $f\in X$ means that each component of f is in X; we put $|||f|||:=|||f_1|||+\cdots+|||f_m|||$.

The same letter C (sometimes used as $C(k)$ to emphasize the dependence of C on k) will denote various positive constants that do not depend on the time t.

2. Pointwise estimates of u

In this section we prove the uniform upper and lower boundedness of $u(x,t)$. We begin with the following lemma (the proof of which can be found in [9]), which embodies the dissipative effects of viscosity and thermal diffusion.

Lemma 2.1.

(i) *There is a positive constant e_0, independent of t, such that*

$$\int_\Omega U(x,t)\,\mathrm{d}x + \int_0^t\int_\Omega\left(\lambda\frac{\theta_x^2}{u\theta^2}+\mu\frac{v_x^2}{u\theta}\right)\mathrm{d}x\mathrm{d}s \leqslant e_0 \quad \forall t\geqslant 0, \tag{2.1}$$

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where

$$U(x,t) := \{\tfrac{1}{2}v^2 + R(u - \log u - 1) + c_V(\theta - \log\theta - 1)\}(x,t). \tag{2.2}$$

(ii) *Let* α_1, α_2 *be two (positive) roots of the equation* $y - \log y - 1 = e_0/\min\{R, c_V\}$. *Then, for* $i = 0, 1, 2, \dots,$

$$\alpha_1 \leqslant \int_i^{i+1} u(x,t)\,\mathrm{d}x, \int_i^{i+1} \theta(x,t)\,\mathrm{d}x \leqslant \alpha_2, \quad t \geqslant 0, \tag{2.3}$$

and, for each $t \geqslant 0$, *there are points* $a_i(t), b_i(t) \in [i, i+1]$ *such that*

$$\alpha_1 \leqslant u(a_i(t), t), \quad \theta(b_i(t), t) \leqslant \alpha_2, \quad t \geqslant 0. \tag{2.4}$$

Next we derive a local representation of u by using a cut-off function. Let $\tilde{x} \in \bar{\Omega}$ be arbitrary but fixed. Let $\tilde{\varphi} \in W^{1,\infty}(\mathbb{R})$ be defined by

$$\tilde{\varphi}(x) := \begin{cases} 1, & x \leqslant [\tilde{x}] + 1, \\ [\tilde{x}] + 2 - x, & [\tilde{x}] + 1 \leqslant x \leqslant [\tilde{x}] + 2, \\ 0, & x \geqslant [\tilde{x}] + 2, \end{cases} \tag{2.5}$$

where $[x]$ denotes the largest integer that is less or equal to x. For simplicity, we denote $I := ([\tilde{x}] - 1, [\tilde{x}] + 1) \cap \Omega$.

We multiply (1.2) by $\tilde{\varphi}$ to obtain $[\tilde{\varphi}v]_t = [\sigma\tilde{\varphi}]_x - \tilde{\varphi}_x\sigma$. Integrating this over (x, ∞) ($x \in \bar{I}$) with respect to x, recalling (1.1) and the definition of $\tilde{\varphi}$ and σ, we arrive at

$$-\int_x^\infty [v\tilde{\varphi}]_t\,\mathrm{d}y = \sigma + \int_x^\infty \sigma\tilde{\varphi}_x\,\mathrm{d}y = \mu[\log u]_t - R\frac{\theta}{u} - \int_{[\tilde{x}]+1}^{[\tilde{x}]+2} \sigma(y,t)\,\mathrm{d}y, \quad x \in \bar{I}. \tag{2.6}$$

We integrate (2.6) over $(0,t)$ with respect to t and then take the exponential on both sides of the resulting equation to deduce that

$$\frac{1}{B(x,t)Y(t)} = \frac{1}{u(x,t)} \exp\biggl\{\frac{R}{\mu}\int_0^t \frac{\theta(x,s)}{u(x,s)}\,\mathrm{d}s\biggr\}, \quad x \in \bar{I}, \quad t \geqslant 0, \tag{2.7}$$

where

$$B(x,t) := u_0(x)\exp\biggl\{\frac{1}{\mu}\int_x^\infty (v_0(y) - v(y,t))\tilde{\varphi}(y)\,\mathrm{d}y\biggr\},$$

$$Y(t) := \exp\biggl\{\frac{1}{\mu}\int_0^t \int_{[\tilde{x}]+1}^{[\tilde{x}]+2} \sigma(y,s)\,\mathrm{d}y\mathrm{d}s\biggr\}.$$

Multiplying (2.7) by $R\theta(x,t)/\mu$ and integrating over $(0,t)$, we infer

$$\exp\biggl\{\frac{R}{\mu}\int_0^t \frac{\theta(x,s)}{u(x,s)}\,\mathrm{d}s\biggr\} = 1 + \frac{R}{\mu}\int_0^t \frac{\theta(x,s)}{B(x,s)Y(s)}\,\mathrm{d}s.$$

Substituting the above identity into (2.7), we obtain a local representation of $u(x,t)$ in a neighbourhood of $\tilde{x}$,

$$u(x,t) = B(x,t)Y(t) + \frac{R}{\mu}\int_0^t \frac{B(x,t)Y(t)}{B(x,s)Y(s)}\theta(x,s)\,\mathrm{d}s, \quad x \in \bar{I}, \quad t \geqslant 0. \tag{2.8}$$

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Now with the help of the local representation (2.8) and the new estimates for θ, σ obtained in [9], we are able to derive the uniform bounds on $u(x,t)$.

LEMMA 2.2. *There are positive constants C_1, C_2, independent of t, such that*

$$C_1 \leqslant u(x,t) \leqslant C_2 \quad \forall x \in \bar{\Omega}, \quad t \geqslant 0. \tag{2.9}$$

Proof. The idea of the proof is the same as that of lemma 2.4 in [9]. The only difference in the argument is that instead of deriving bounds of $u(x,t)$ for $x \in \bar{\Omega}_k$ in [9], we derive here upper and lower bounds of $u(x,t)$ for $x \in \bar{I}$, and the bounds of $u(x,t)$ are shown to be independent of $\tilde{x}$. Thus, if we use (2.1), (2.3), (2.4) and (2.8), replace the interval $[k+1,k+2]$ (respectively, $\bar{\Omega}_k$) in the proof of lemma 2.4 in [9] by $[[\tilde{x}]+1,[\tilde{x}]+2]$ (respectively, $\bar{I}$), then we obtain, by the completely same arguments as used for (2.14)–(2.24) in the proof of lemma 2.4 in [9], that

$$C_1 \leqslant u(x,t) \leqslant C_2 \quad \text{for all } x \in \bar{I}, \quad t \geqslant 0, \tag{2.10}$$

where C_1, C_2 are positive constants that do not depend on $\tilde{x}$. In fact, C_1, C_2 depend only on the measure of the interval I, i.e. $C_1 = C_1(|I|)$, $C_2 = C_2(|I|)$.

In particular, the estimate (2.10) gives $C_1 \leqslant u(\tilde{x},t) \leqslant C_2$ for all $t \geqslant 0$. Because $\tilde{x} \in \bar{\Omega}$ is arbitrary and C_1, C_2 are independent of $\tilde{x}$, we conclude that (2.9) holds. The proof is complete. □

REMARK 2.3. From the proof of lemma 2.2, we easily see that if we consider the Cauchy problem (1.1)–(1.4) with $\Omega = \mathbb{R}$, then (2.9) still remains valid.

3. Proof of theorem 1.1

We have proved the uniform lower and upper boundedness of u in theorem 1.1 (i.e. lemma 2.2) in § 2. In this section we apply the results obtained in § 2 and the weighted energy method to derive bounds for derivatives of (u,v,θ). As mentioned in § 1, we first estimate v_t, θ_t by θ_x and then θ_x by v_t and θ_t.

Let $\psi \in W^{1,\infty}(\mathbb{R})$ be defined by

$$\psi(x) := \begin{cases} 1, & x \leqslant k, \\ k+1-x, & k \leqslant x \leqslant k+1, \\ 0, & x \geqslant k+1. \end{cases}$$

Using lemma 2.1 and (2.9), we can obtain the following lemma, the proof of which is completely the same as that of lemmas 3.1 and 3.2 in [9], and therefore will be omitted here.

LEMMA 3.1.

$$\int_0^t \max_{[0,k+1]} v^2(\cdot,s)\,\mathrm{d}s, \ \int_0^t \max_{[0,k+1]} [\theta(\cdot,s) - \bar{\theta}(s)]^2\,\mathrm{d}s \leqslant C \quad \textit{for all } t \geqslant 0,$$

$$\int_\Omega u_x^2(x,t)\psi(x)\,\mathrm{d}x + \int_0^t \int_\Omega u_x^2 \psi\,\mathrm{d}x\mathrm{d}s \leqslant C \quad \textit{for all } t \geqslant 0,$$

where $\bar{\theta}(t) := \theta(b_0(t),t)$ and $b_0(t)$ is the same as in lemma 2.1.

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Now we multiply (1.2) by ψv and integrate the resulting equation over Ω, integrate by parts with respect to x and use the boundary conditions (1.5) to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^2\psi\,\mathrm{d}x = -\int_\Omega\left(\mu\frac{v_x^2\psi}{u} + \mu\frac{v_x\psi_x v}{u} + R\frac{\theta_x}{u}\psi v - R\frac{\theta u_x}{u^2}\psi v\right)\mathrm{d}x.$$

If we integrate the above equation over $(0,t)$, use (2.9), lemmas 2.1 and 3.1, we obtain

$$\begin{aligned}
&\int_\Omega v^2\psi\,\mathrm{d}x + \int_0^t\int_\Omega v_x^2\psi\,\mathrm{d}x\mathrm{d}s \\
&\quad\leqslant C\int_0^t\int_0^{k+1}|v_x v|\,\mathrm{d}x\mathrm{d}s + C\int_0^t\int_\Omega(|\theta_x| + \theta|u_x|)|v|\psi\,\mathrm{d}x\mathrm{d}s \\
&\quad\leqslant C + C\int_0^t\int_0^{k+1}\left(\frac{v_x^2}{u\theta} + v^2\theta\right)\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega\left(\frac{\theta_x^2}{u\theta^2} + u_x^2\psi + \theta^2 v^2\psi\right)\mathrm{d}x\mathrm{d}s \\
&\quad\leqslant C + C\int_0^t\max_{[0,k+1]} v^2(\cdot,s)\,\mathrm{d}s + C\int_0^t\max_{[0,k+1]}(\theta-\bar{\theta})^2\int_0^{k+1}v^2\,\mathrm{d}x\mathrm{d}s \\
&\quad\leqslant C \quad \text{for all } t\geqslant 0,
\end{aligned}\tag{3.1}$$

where $\bar{\theta}\equiv\bar{\theta}(t)$ is the same as in lemma 2.1. Noting that k is arbitrary, we replace k by $k+1$ in the definition of ψ and use (3.1) to obtain

$$\int_0^t\int_0^{k+1}v_x^2\,\mathrm{d}x\mathrm{d}s\leqslant C \quad \forall t\geqslant 0. \tag{3.2}$$

In the sequel we derive bounds of v_t and θ_t by θ_x.

Multiplying (1.2) by $uv_t\psi^3$ and integrating over $(0,t)\times\Omega$, we integrate by parts and use (2.9) to deduce that

$$\begin{aligned}
&\int_0^t\int_\Omega uv_t^2\psi^3\,\mathrm{d}x\mathrm{d}s + \tfrac{1}{2}\mu\int_\Omega v_x^2\psi^3\,\mathrm{d}x \\
&\qquad\leqslant C + C\int_0^t\int_\Omega\{(|v_xu_xv_t| + |\theta_x v_t| + |\theta u_x v_t|)\psi^3 + |v_xv_t\psi_x|\psi^2\}\,\mathrm{d}x\mathrm{d}s,
\end{aligned}$$

which, by the Cauchy–Schwarz inequality and (3.1), as well as lemma 3.1, yields

$$\begin{aligned}
&\int_0^t\int_\Omega v_t^2\psi^3\,\mathrm{d}x\mathrm{d}s + \int_\Omega v_x^2\psi^3\,\mathrm{d}x \\
&\quad\leqslant C + C\int_0^t\int_\Omega\{(v_x^2u_x^2 + \theta_x^2 + \theta^2u_x^2)\psi^3 + v_x^2\psi\}\,\mathrm{d}x\mathrm{d}s \\
&\quad\leqslant C + C\int_0^t\int_\Omega(v_x^2u_x^2 + \theta_x^2)\psi^3\,\mathrm{d}x\mathrm{d}s + C\int_0^t\max_{[0,k+1]}((\theta-\bar{\theta})^2+1)\int_\Omega u_x^2\psi^3\,\mathrm{d}x\mathrm{d}s \\
&\quad\leqslant C + C\int_0^t\max_{\bar{\Omega}} v_x^2\psi^2\int_\Omega u_x^2\psi\,\mathrm{d}x\mathrm{d}s + C\int_0^t\int_\Omega\theta_x^2\psi^3\,\mathrm{d}x\mathrm{d}s.
\end{aligned}\tag{3.3}$$

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Here, the second term on the right-hand side of (3.3) can be bounded as follows, using lemma 3.1, Sobolev's imbedding theorem ($W^{1,1} \hookrightarrow L^\infty$), and (3.1) and (1.2):

$$\begin{aligned}
\int_0^t \max_{\bar\Omega} v_x^2\psi^2 \int_\Omega u_x^2\psi \,\mathrm{d}x\mathrm{d}s
&\leqslant C\int_0^t \max_{\bar\Omega}\{(\sigma^2+(\theta-\bar\theta)^2+1)\psi^2\}\int_\Omega u_x^2\psi\,\mathrm{d}x\mathrm{d}s\\
&\leqslant C + C\int_0^t\int_\Omega (\sigma^2\psi^2+|\sigma\sigma_x|\psi^2+\sigma^2\psi|\psi_x|)\,\mathrm{d}x\int_\Omega u_x^2\psi\,\mathrm{d}x\mathrm{d}s\\
&\leqslant C + C\int_0^t\int_\Omega (\epsilon^{-1}\sigma^2\psi+\epsilon\sigma_x^2\psi^3)\,\mathrm{d}x\int_\Omega u_x^2\psi\,\mathrm{d}x\mathrm{d}s\\
&\leqslant C + \frac{C}{\epsilon}\int_0^t\int_\Omega (v_x^2+(\theta-\bar\theta)^2+1)\psi\,\mathrm{d}x\int_\Omega u_x^2\psi\,\mathrm{d}x\mathrm{d}s\\
&\qquad + C\epsilon\int_0^t\int_\Omega \sigma_x^2\psi^3\,\mathrm{d}x\int_\Omega u_x^2\psi\,\mathrm{d}x\mathrm{d}s\\
&\leqslant \frac{C}{\epsilon} + C\epsilon\int_0^t\int_\Omega v_t^2\psi^3\,\mathrm{d}x\mathrm{d}s. \qquad (3.4)
\end{aligned}$$

Inserting (3.4) into (3.3) and taking ϵ suitably small, one gets

$$\int_0^t\int_\Omega v_t^2\psi^3\,\mathrm{d}x\mathrm{d}s + \int_\Omega v_x^2\psi^3\,\mathrm{d}x \leqslant C + C\int_0^t\int_\Omega \theta_x^2\psi^3\,\mathrm{d}x\mathrm{d}s, \quad t\geqslant 0. \qquad (3.5)$$

On the other hand, with the help of lemma 2.1 and (2.9), one has

$$\begin{aligned}
\int_0^t\int_\Omega \theta_x^2\psi^3\,\mathrm{d}x\mathrm{d}s &\leqslant C\int_0^t\int_\Omega \theta_x^2\biggl(\frac{1}{u\theta^2}+\theta^2\psi^6\biggr)\mathrm{d}x\mathrm{d}s\\
&\leqslant C + C\int_0^t \max_{[0,k+1]}(\theta-\bar\theta)^2\int_\Omega \theta_x^2\psi^6\,\mathrm{d}x\mathrm{d}s + C\int_0^t\int_\Omega \theta_x^2\psi^6\,\mathrm{d}x\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t\int_\Omega \theta_x^2\psi^6\,\mathrm{d}x\mathrm{d}s
&\leqslant C\int_0^t\int_\Omega \frac{\theta_x^2\psi^6}{u\theta^2}(\theta-\bar\theta)^2\,\mathrm{d}x\mathrm{d}s + C\int_0^t\int_\Omega \frac{\theta_x^2\psi^6}{u\theta^2}\,\mathrm{d}x\mathrm{d}s\\
&\leqslant C\int_0^t\biggl\{\int_{\{x|\theta\leqslant 1\}} + \int_{\{x|\theta>1\}}\biggr\}\frac{\theta_x^2\psi^6}{u\theta^2}(\theta-\bar\theta)^2\,\mathrm{d}x\mathrm{d}s + C\\
&\leqslant C\int_0^t\int_{\{x|\theta\leqslant 1\}}\frac{\theta_x^2}{u\theta^2}\,\mathrm{d}x\mathrm{d}s + C\int_0^t\int_{\{x|\theta>1\}}\theta_x^2\psi^6(\theta-\bar\theta)^2\,\mathrm{d}x\mathrm{d}s + C\\
&\leqslant C + C\int_0^t \max_{[0,k+1]}(\theta-\bar\theta)^2\int_\Omega \theta_x^2\psi^6\,\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\int_0^t\int_\Omega \theta_x^2\psi^3\,\mathrm{d}x\mathrm{d}s \leqslant C + C\int_0^t \max_{[0,k+1]}(\theta-\bar\theta)^2\int_\Omega \theta_x^2\psi^6\,\mathrm{d}x\mathrm{d}s.$$

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Multiplying the above inequality by $2C$ and adding the resulting inequality to (3.5), we find that

$$\int_0^t\int_\Omega (v_t^2+\theta_x^2)\psi^3\,\mathrm{d}x\mathrm{d}s+\int_\Omega v_x^2\psi^3\,\mathrm{d}x \leqslant C+C\int_0^t \max_{[0,k+1]}(\theta-\bar\theta)^2\int_\Omega \theta_x^2\psi^6\,\mathrm{d}x\mathrm{d}s \quad \forall t\geqslant 0. \tag{3.6}$$

Next we estimate v_t in $L^\infty((0,\infty),L^2_{\mathrm{loc}})$. We differentiate (1.2) with respect to t and then multiply the resulting equation by $v_t\psi^6$ in $L^2((0,t)\times\Omega)$. If we integrate by parts and make use of (2.9), we deduce

$$\begin{aligned}\frac12\int_\Omega v_t^2\psi^6\,\mathrm{d}x \leqslant C-\mu\int_0^t\int_\Omega \frac{v_{tx}^2}{u}\psi^6\,\mathrm{d}x\mathrm{d}s &+C\int_0^t\int_\Omega\{v_x^2+|\theta_t|+\theta|v_x|\}|v_{tx}|\psi^6\,\mathrm{d}x\mathrm{d}s\\ &+C\int_0^t\int_\Omega\{|v_{tx}|+v_x^2+|\theta_t|+\theta|v_x|\}|v_t|\psi^5\,\mathrm{d}x\mathrm{d}s,\end{aligned}$$

which, by (3.1), yields

$$\begin{aligned}\int_\Omega v_t^2\psi^6\,\mathrm{d}x+\int_0^t\int_\Omega v_{tx}^2\psi^6\,\mathrm{d}x\mathrm{d}s &\leqslant C+C\int_0^t\int_\Omega\{(v_x^4+\theta_t^2+\theta^2v_x^2)\psi^6+v_t^2\psi^4\}\,\mathrm{d}x\mathrm{d}s\\ &\leqslant C+C\int_0^t\int_\Omega\{(\theta_t^2+v_x^4)\psi^6+v_t^2\psi^3\}\,\mathrm{d}x\mathrm{d}s+C\int_0^t\max_{[0,k+1]}(\theta-\bar\theta)^2\int_\Omega v_x^2\psi^3\,\mathrm{d}x\mathrm{d}s\end{aligned} \tag{3.7}$$

for all $t\geqslant 0$. The term $\int_0^t\int_\Omega v_x^4\psi^6\,\mathrm{d}x\mathrm{d}s$ in (3.7) can be bounded similarly to (3.4) as follows, using (3.1), $W^{1,1}\hookrightarrow L^\infty$ and (1.2):

$$\begin{aligned}\int_0^t\int_\Omega v_x^4\psi^6\,\mathrm{d}x\mathrm{d}s &\leqslant C\int_0^t\max_{x\in\bar\Omega}\{\sigma^2+(\theta-\bar\theta)^2+1\}\psi^3\int_\Omega v_x^2\psi^3\,\mathrm{d}x\mathrm{d}s\\ &\leqslant C+C\int_0^t\max_{[0,k+1]}(\theta-\bar\theta)^2\int_\Omega v_x^2\psi^3\,\mathrm{d}x\mathrm{d}s\\ &\quad+C\int_0^t\int_\Omega\{\sigma^2\psi^3+|\sigma\sigma_x|\psi^3+\sigma^2\psi^2|\psi_x|\}\int_\Omega v_x^2\psi^3\,\mathrm{d}x\mathrm{d}s\\ &\leqslant C+C\int_0^t\max_{[0,k+1]}(\theta-\bar\theta)^2\int_\Omega v_x^2\psi^3\,\mathrm{d}x\mathrm{d}s\\ &\quad+C\int_0^t\left\{\int_\Omega v_t^2\psi^6\,\mathrm{d}x+\int_0^{k+1}\sigma^2\,\mathrm{d}x\right\}\int_\Omega v_x^2\psi^3\,\mathrm{d}x\mathrm{d}s\\ &\leqslant C+C\int_0^t\left\{\max_{[0,k+1]}(\theta-\bar\theta)^2+\int_0^{k+1}v_x^2\,\mathrm{d}x\right\}\int_\Omega(v_x^2\psi^3+v_t^2\psi^6)\,\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{3.8}$$

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Inserting (3.8) into (3.7), we thus conclude that

$$\begin{aligned}\int_\Omega v_t^2\psi^6\,\mathrm{d}x + \int_0^t\int_\Omega v_{tx}^2\psi^6\,\mathrm{d}x\mathrm{d}s &\\ \leqslant C + C\int_0^t\int_\Omega \{\theta_t^2\psi^6 + v_t^2\psi^3\}\,\mathrm{d}x\mathrm{d}s &\\ + C\int_0^t\biggl\{\max_{[0,k+1]}(\theta-\bar\theta)^2 + \int_0^{k+1} v_x^2\,\mathrm{d}x\biggr\}\int_\Omega (v_x^2\psi^3 + v_t^2\psi^6)\,\mathrm{d}x\mathrm{d}s.&\end{aligned} \tag{3.9}$$

Now, multiplying (1.3) by $u\theta_t\psi^6$ in $L^2((0,t)\times\Omega)$, integrating by parts and applying (2.9), one finds that

$$\begin{aligned}C\int_0^t\int_\Omega \theta_t^2\psi^6\,\mathrm{d}x\mathrm{d}s + \tfrac{1}{2}\kappa\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta_x^2\psi^6\,\mathrm{d}x\mathrm{d}s &\\ \leqslant \int_0^t\int_\Omega \biggl\{\sigma v_x u\psi^6 - \kappa\frac{\theta_x}{u}u_x\psi^6 - 6\kappa\theta_x\psi^5\psi_x\biggr\}\theta_t\,\mathrm{d}x\mathrm{d}s,&\end{aligned}$$

whence

$$\int_0^t\int_\Omega \theta_t^2\psi^6\,\mathrm{d}x\mathrm{d}s + \int_\Omega \theta_x^2\psi^6\,\mathrm{d}x \leqslant C + C\int_0^t\int_\Omega \{(\sigma^2 v_x^2 + \theta_x^2 u_x^2)\psi^6 + \theta_x^2\psi^3\}\,\mathrm{d}x\mathrm{d}s. \tag{3.10}$$

Using (3.8) and (3.1), we easily obtain

$$\begin{aligned}\int_0^t\int_\Omega \sigma^2 v_x^2\psi^6\,\mathrm{d}x\mathrm{d}s &\\ \leqslant C\int_0^t\int_\Omega v_x^4\psi^6\,\mathrm{d}x\mathrm{d}s + C\int_0^t \{\max_{\bar\Omega}(\theta-\bar\theta)^2 + 1\}\psi^3\int_\Omega v_x^2\psi^3\,\mathrm{d}x\mathrm{d}s &\\ \leqslant C + C\int_0^t\biggl\{\max_{[0,k+1]}(\theta-\bar\theta)^2 + \int_0^{k+1} v_x^2\,\mathrm{d}x\biggr\}\int_\Omega (v_x^2\psi^3 + v_t^2\psi^6)\,\mathrm{d}x\mathrm{d}s,&\end{aligned} \tag{3.11}$$

while, by lemma 3.1 and (1.3) and (3.11),

$$\begin{aligned}\int_0^t\int_\Omega \theta_x^2 u_x^2\psi^6 &\leqslant C\int \max_{\bar\Omega}\biggl\{\biggl(\frac{\theta_x}{u}\biggr)^2\psi^5\biggr\}\,\mathrm{d}s\\ &\leqslant C\int_0^t\int_\Omega \theta_x^2\psi^4\,\mathrm{d}x\mathrm{d}s + C\int_0^t \biggl|\frac{\theta_x}{u}\biggr|\biggl|\biggl[\frac{\theta_x}{u}\biggr]_x\biggr|\psi^5\,\mathrm{d}x\mathrm{d}s\\ &\leqslant \frac{C}{\epsilon}\int_0^t\int_\Omega \theta_x^2\psi^4\,\mathrm{d}x\mathrm{d}s + \epsilon\int_0^t\int_\Omega \biggl[\frac{\theta_x}{u}\biggr]_x^2\psi^6\,\mathrm{d}x\mathrm{d}s\\ &\leqslant \frac{C}{\epsilon}\int_0^t\int_\Omega \theta_x^2\psi^3\,\mathrm{d}x\mathrm{d}s + C\epsilon\int_0^t\int_\Omega \theta_t^2\psi^6\,\mathrm{d}x\mathrm{d}s\\ &\quad + C\int_0^t\biggl\{\max_{[0,k+1]}(\theta-\bar\theta)^2 + \int_0^{k+1} v_x^2\,\mathrm{d}x\biggr\}\int_\Omega (v_x^2\psi^3 + v_t^2\psi^6)\,\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10) and taking ϵ appropriately small, we infer that

$$\begin{aligned}\int_0^t\int_\Omega \theta_t^2\psi^6\,\mathrm{d}x\mathrm{d}s+\int_\Omega \theta_x^2\psi^6\,\mathrm{d}x &\leqslant C+C\int_0^t\int_\Omega \theta_x^2\psi^3\,\mathrm{d}x\mathrm{d}s\\ &\quad+C\int_0^t\Big\{\max_{[0,k+1]}(\theta-\bar\theta)^2+\int_0^{k+1}v_x^2\,\mathrm{d}x\Big\}\int_\Omega(v_x^2\psi^3+v_t^2\psi^6)\,\mathrm{d}x\mathrm{d}s. \end{aligned}\tag{3.13}$$

Therefore, (3.6) + (3.9) $\times\,\epsilon$ + (3.13) $\times\sqrt{\epsilon}$, with ϵ appropriately small, yields

$$\begin{aligned}\int_0^t\int_\Omega\{v_t^2\psi^3+\theta_x^2\psi^3+v_{tx}^2\psi^6+\theta_t^2\psi^6\}\,\mathrm{d}x\mathrm{d}s+\int_\Omega\{v_x^2\psi^3+v_t^2\psi^6+\theta_x^2\psi^6\}(x,t)\,\mathrm{d}x\\ \leqslant C+C\int_0^t\Big\{\max_{[0,k+1]}(\theta-\bar\theta)^2+\int_0^{k+1}v_x^2\,\mathrm{d}x\Big\}\int_\Omega\{v_x^2\psi^3+v_t^2\psi^6+\theta_x^2\psi^6\}\,\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{3.14}$$

Applying Gronwall's inequality to (3.14) and using lemma 3.1 and (3.2), we obtain

$$\int_0^t\int_\Omega\{(v_t^2+\theta_x^2)\psi^3+(v_{tx}^2+\theta_t^2)\psi^6\}\,\mathrm{d}x\mathrm{d}s+\int_\Omega\{v_x^2\psi^3+(v_t^2+\theta_x^2)\psi^6\}(x,t)\,\mathrm{d}x\leqslant C\tag{3.15}$$

for all $t\geqslant 0$. Hence, from (1.2) and (1.3), (3.15), (3.4), lemma 3.1, and (3.11) and (3.12), it follows that

$$\begin{aligned}\int_0^t\int_\Omega(v_{xx}^2+\theta_{xx}^2)\psi^6\,\mathrm{d}x\mathrm{d}s &\leqslant C\int_0^t\int_\Omega\{v_t^2+v_x^2u_x^2+(\theta-\bar\theta)^2u_x^2+u_x^2+\theta_x^2+\theta_t^2+\sigma^2v_x^2+\theta_x^2u_x^2\}\psi^6\,\mathrm{d}x\mathrm{d}s\\ &\leqslant C\quad\forall t\geqslant 0.\end{aligned}\tag{3.16}$$

By lemma 3.1, (3.15), (3.16) and the identity

$$\int_\Omega\theta_x\theta_{xt}\psi^8\,\mathrm{d}x=-\int_\Omega\theta_{xx}\theta_t\psi^8\,\mathrm{d}x-8\int_\Omega\theta_x\theta_t\psi^7\psi_x\,\mathrm{d}x,$$

we see that

$$\int_0^\infty\left|\frac{\mathrm{d}}{\mathrm{d}t}\|(u_x\psi^4,v_x\psi^4,\theta_x\psi^4)\|^2\right|\mathrm{d}t\leqslant C,$$

which gives

$$\|(u_x(t),v_x(t),\theta_x(t))\|_{L^2(0,k)}\to 0\quad\text{as } t\to\infty.\tag{3.17}$$

We apply Poincaré's inequality and (3.17) to obtain (1.8). To complete the proof of theorem 1.1, it remains to prove the upper and lower boundedness of θ. By (2.3), we see that $\alpha_1\leqslant\theta_1(t)\leqslant\alpha_2$ for any $t\geqslant 0$. Hence, from (1.8) and Sobolev's imbedding theorem, we get

$$C^{-1}\leqslant\theta(x,t)\leqslant C\quad\text{for all } x\in[0,k],\quad t\geqslant T_0,\tag{3.18}$$

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where T_0 is a (large) constant. On the other hand, from the proof in [2,13], we have $1/(Ce^{Ct}) \leqslant \theta(x,t) \leqslant Ce^{Ct}$ for all $(x,t) \in \bar{\Omega} \times [0,\infty)$, which, combined with (3.18), implies the local lower and upper bounds of θ. This complete the proof of theorem 1.1.

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