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GENERALIZED n-LIKE RINGS AND COMMUTATIVITY

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ABSTRACT. This note continues the investigation of those rings R with unity which also satisfy the polynomial identity $B(x, y) = (xy)^n - xy^n - x^n y + xy = 0$, for some integer $n > 1$. It is shown that when n is an even integer, or when $n = 3$, such rings are commutative. It is otherwise possible, as is shown by example, for such rings to fail to be commutative, although they are subdirect sums of local rings satisfying the polynomial identity. Each such ring has nilpotent commutator ideal.

In all that follows, R is an associative ring with unity 1 and satisfies the polynomial identity

$$B(x, y) = (xy)^n - xy^n - x^n y + xy = 0$$

for some integer $n > 1$. We begin with four lemmas whose proofs follow from results in [2] and [4].

LEMMA 1. (i) *$x^n - x$ is nilpotent for all $x \in R$.*
(*ii*) *If α is the characteristic of R then $x^{n\alpha - \alpha}$ is idempotent.*
(*iii*) *x nilpotent implies $x^2 = 0$.*

LEMMA 2. *Every idempotent element in R lies in the center of R.*

LEMMA 3. *The Jacobson radical J of R is precisely the set of all nilpotent elements of R.*

LEMMA 4. *If R is subdirectly irreducible then $x^{n\alpha - \alpha} = 0 = x^2$, or $x^{n\alpha - \alpha} = 1$.*

From these Lemmas, the fact that every ring is the subdirect sum of subdirectly irreducible rings, and the preservation of $B(x, y) = 0$ under homomorphic mappings we have.

THEOREM 1. *Let R be a ring with unity satisfying the polynomial identity $B(x, y) = 0$. Then R is a subdirect sum of (not necessarily commutative) local rings which satisfy the same identity.*

Proof. That each subdirect summand is a local ring [3] follows from lemmas 3 and 4. Each element of R is either a unit $x^{n\alpha - \alpha} = 1$; or else in J, $x^2 = x^{n\alpha - \alpha} = 0$. That these local rings need not be commutative is seen from the ring of Example 2.

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If R has a unity 1, $B(x, y)=0$ and α, the characteristic of R, divides n, then R is commutative. This was shown by Yaqub [4]. However, R need not be commutative if it fails to have a unity, as is seen from the ring of Example 1. If the characteristic α of R fails to divide n then R can also fail to be commutative as evidenced by the ring of Example 2. In this case, however, the commutator ideal C is nilpotent, $C^2=0$. (See [2].) For certain integers n, however, the identity $B(x, y)=0$ together with a unity is enough for R to be commutative.

THEOREM 2. *If R is a ring with unity and satisfies the polynomial identity $B(x, y)=(xy)^n-xy^n-x^ny+xy=0$ for an even positive integer $n=2k$, then R has characteristic 2 or 4 and is commutative.*

Proof. Consider the element $1+1=2=(-1)^n-(-1)$ for even n. Since for each x in R (x^n-x) is nilpotent, $[(-1)^n-(-1)]^2=0$, so $2^2=0$. Thus α, the characteristic of R, is 2 or 4.

Now suppose first that $\alpha=2$. Let $a\in R$ be nilpotent and $x\in R$ be arbitrary. Then from [4], $n(ax-xa)=ax-xa$ so

$$ax-xa=2k(ax-xa)=0.$$

Hence, every nilpotent element of R is in the center of R. Then, using the nilpotency of (x^n-x) and a theorem of Herstein [1], R is commutative.

If $\alpha=4$ and $n=4k$, the results are the same. In the case where $\alpha=4$ and $n=4k-2$, we have

$$ax-xa=(4k-2)(ax-xa)=2(ax-xa).$$

Therefore, $ax-xa=0$, and again the result follows.

Let us turn our attention to those rings with unity which satisfy the identity $B(x, y)=0$ for odd integers $n>1$. We have

THEOREM 3. *Let R be a ring with unity which satisfies $B_3(x, y)=(xy)^3-xy^3-x^3y+xy=0$. Then R is commutative.*

Proof. As we have already remarked prior to Theorem 1, it will be enough to consider R to be sub-directly irreducible. Then, by Theorem 1, R is a local ring. We shall show that J lies in the center of R, and, therefore, R is commutative by the same combination of Herstein's theorem and results of [2] used above.

Since as shown in [4] nilpotent elements already commute with each other, let a be nilpotent and let x be not in J. Hence, x is a unit. As before, $(x^3-x)^2=x^2(x^2-1)^2=0$. Since x is a unit, $(x^2-1)^2=(x-1)^2(x+1)^2=0$. If $(x-1)$ is a unit, then $x+1$ lies in J and so $0=a(x+1)=(x+1)a$ or $ax+a=xa+a$ and $ax=xa$. If $x-1$ is not a unit, it is in J. Then $0=a(x-1)=(x-1)a$ or $ax=xa$. In any case, nilpotent elements commute not only with themselves but with the units as well. Therefore, R is commutative.

When the characteristic of R is 2, R need not be commutative even when R is a local ring. Consider the ring of Example 2.

We turn now to noncommutative rings satisfying $B(x, y)=0$. By Theorem 1 they are subdirect sums of certain local rings.

THEOREM 4. *If R is a subdirectly irreducible ring with unity and satisfies $B(x, y)=0$, then R is a local ring of characteristic p or p^2, where p is a prime divisor of 2^n-2. If R is not commutative, then $n=1+kp^i$ where k and p are relatively prime.*

Proof. R is local by Theorem 1. Let S be the intersection of all non-zero ideals of R. $S\neq(0)$ because R is subdirectly irreducible. Let α be the characteristic of R and let p be any prime divisor of α. Set $R_p=\{x\in R \mid px=0\}$. Clearly, $R_p\neq(0)$ and is an ideal of R so $S\subseteq R_p$. Suppose for some other prime $q\neq p$, $R_q\neq(0)$. Then $S\subseteq R_p\cap R_q=(0)$—a contradiction. Therefore, α is a power of the single prime p, say $\alpha=p^i$. Now $(p.1)^i=p^i\cdot 1=0$, so the element $p=p\cdot 1$ of R is nilpotent. But then $p^2=0$. Hence $\alpha=p$ or $\alpha=p^2$. Furthermore, since commutators and p are both nilpotent as was shown in [2, Theorem 2], we have $p(xy-yx)=0$. (See [4].) Thus, if R is not-commutative, since $(n-1)(xy-yx)=0$ whenever y is nilpotent and x is arbitrary, p divides $n-1$. In this case n is odd. So $n=1+kp^i$, where k is even and k and p are relatively prime.

Let us conclude with two examples of rings which satisfy $B(x, y)=0$ but are not-commutative.

EXAMPLE 1. Let R be the sub-ring of the ring of 2×2 matrices over $GF(2)$ consisting of those matrices of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$. It is readily verified that R satisfies $B(x, y)=(xy)^2-xy^2-x^2y+xy$ and $2x=0$, but fails to be commutative. R fails to have a unity, but the matrices $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ are both left multiplicative identities. Therefore, the requirement of a one-sided multiplicative identity is not enough in conjunction with *both* $B(x, y)=0$ and $nx=0$ (same n) to assure commutativity.

EXAMPLE 2. Let R be the subring of the ring of all 3×3 matrices over $GF(4)$ which consists of all matrices of the form $\begin{bmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix}$. It is readily verified that R is a non-commutative local ring with unity and characteristic 2. One can calculate that for each $x\in R$, $x^7=x$ or $x^7=x^2=0$. Therefore, R satisfies

$$B(x, y)=(xy)^7-xy^7-x^7y+xy=0$$

and $2x = 0$. The presence of a unity and the polynomial identity $B(x, y) = 0$ are not sufficient for commutativity.

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References

1. I. N. Herstein, "*A Generalization of a Theorem of Jacobson,*" Amer. I. Math. 75 (1953) 105–111.

2. H. G. Moore and Adil Yaqub, "*On the Commutator Ideal of Certain Rings*" Port. Math. 29 (1970) 119–124.

3. H. G. Moore and Adil Yaqub "*Equational Definability of Addition in Certain Rings*" Pac. J. of Math. 74 (1978) 407–417.

4. A. Yaqub, "*A Generalization of Certain Rings of A. L. Foster,*" Canad. Math. Bull. 6 (1963) 55–60.

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