Math. Proc. Camb. Phil. Soc. (2022), **173**, 457–478
doi:10.1017/S0305004121000694
First published online 21 December 2021

Distal Actions of Automorphisms of Lie Groups G on Sub_G

BY RIDDHI SHAH
School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110067*, India,*
e-mail: riddhi.kausti@gmail.com, rshah@jnu.ac.in

ALOK KUMAR YADAV†
Department of Mathematical Sciences, Indian Institute of Science Education and Research, Mohali 140306*, India,*
e-mail: alokmath1729@gmail.com

(*Received* 16 *October* 2019*; revised* 24 *May* 2021*; accepted* 18 *October* 2021)

Abstract

For a locally compact metrisable group G, we study the action of $\mathrm{Aut}(G)$ on Sub_G, the set of closed subgroups of G endowed with the Chabauty topology. Given an automorphism T of G, we relate the distality of the T-action on Sub_G with that of the T-action on G under a certain condition. If G is a connected Lie group, we characterise the distality of the T-action on Sub_G in terms of compactness of the closed subgroup generated by T in $\mathrm{Aut}(G)$ under certain conditions on the center of G or on T as follows: G has no compact central subgroup of positive dimension or T is unipotent or T is contained in the connected component of the identity in $\mathrm{Aut}(G)$. Moreover, we also show that a connected Lie group G acts distally on Sub_G if and only if G is either compact or it is isomorphic to a direct product of a compact group and a vector group. All the results on the Lie groups mentioned above hold for the action on Sub^a_G, a subset of Sub_G consisting of closed abelian subgroups of G.

2020 Mathematics Subject Classification: 37B05 (Primary), 22D45, 22E25, 22E15 (Secondary)

1. *Introduction*

Let X be a (Hausdorff) topological space and let T be a homeomorphism of X. The map T is said to be distal if for any pair of distinct points $x, y \in X$, the closure of the double orbit $\{(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\}$ in $X \times X$ does not intersect the diagonal, i.e. for $x, y \in X$ with $x \neq y$, $\overline{\{(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\}} \cap \{(d, d) \mid d \in X\} = \emptyset$. The notion of distality was introduced by David Hilbert (see Ellis [12], Moore [21]) and studied by many in different contexts (see Abels [1, 2], Furstenberg [13], Raja–Shah [25, 26], Shah [28] and the references cited therein). For a semigroup Γ which acts on X by homeomorphisms, i.e. $\Gamma \to \mathrm{Homeo(X)}$ is a homomorphism, we say that Γ acts distally on X if for any two distinct points $x, y \in X$, the closure of $\{(\gamma(x), \gamma(y)) \mid \gamma \in \Gamma\}$ does not intersect the diagonal. Let G be a locally compact (Hausdorff) group with the identity e and let $T \in \mathrm{Aut}(G)$, the group of automorphisms

† Corresponding author: Alok Kumar Yadav

(homeomorphisms which are homomorphisms) of G. Then T is distal (on G) if and only if the closure of the T-orbit $\{T^n(x) \mid n \in \mathbb{Z}\}$ of x does not contain the identity e unless $x = e$. If a group Γ acts on G by automorphisms, then the Γ-action on G is distal if and only if $e \notin \overline{\Gamma(x)}$ unless $x = e$.

Let G be a locally compact (Hausdorff) group and let Sub_G be the set of all closed subgroups of G equipped with the Chabauty topology (see [9]). Note that Sub_G is compact and Hausdorff. Also, it is metrisable if G is second countable [7]. For various groups G, the space Sub_G has been identified and studied extensively, e.g. $\mathrm{Sub}_{\mathbb{R}^2}$ is homeomorphic to $\mathbb{S}^4$, the unit sphere in $\mathbb{R}^5$ [23], $\mathrm{Sub}_{\mathbb{R}}$ is homeomorphic to $[0, \infty]$ with a compact topology and $\mathrm{Sub}_{\mathbb{Z}}$ is homeomorphic to the subspace $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ of $[0,1]$ with the usual topology. The space of G-invariant measures on Sub_G and the subspace of lattices have also been studied extensively. We refer the reader to Bridson et al [8], Abert et al [4] and the references cited therein.

There is a natural group action of $\mathrm{Aut}(G)$ on Sub_G; namely, $(T, H) \mapsto T(H)$, $T \in \mathrm{Aut}(G)$, $H \in \mathrm{Sub}_G$. Since the image of $\mathrm{Aut}(G)$ under this action is a large subclass of homeomorphisms of Sub_G, it would be significant to study the dynamics of this special subclass. Here, we would like to study the action of an automorphism of G on Sub_G in terms of distality. We show that the distality of the T-action on Sub_G implies the distality of the T-action on G for a large class of locally compact groups G; namely the class of those locally compact Hausdorff first countable (metrisable) groups which do not admit nontrivial compact connected normal subgroups (more generally, see Theorem 3·6). Conversely, not all distal automorphisms of G act distally on Sub_G; see Example 3·2. In fact we know that any unipotent automorphism of a connected Lie group G is distal [1, 2], but it does not act distally on Sub_G unless it is trivial (see Theorem 4·3).

Ellis [12] characterised distal maps on compact spaces. If X is compact and T is a homeomorphism of X, then T is distal if and only if $E(T)$, the closure of $\{T^n \mid n \in \mathbb{N}\}$ in X^X is a group, where X^X is endowed with the product topology. Furstenberg [13] has a characterisation of minimal distal maps on compact metric spaces. Here we show that for a large class of connected Lie groups G, if an automorphism T acts distally on Sub_G, a stronger result holds; namely, T is contained in a compact subgroup of $\mathrm{Aut}(G)$. For a class of connected Lie groups G which do not admit any compact central subgroup of positive dimension, we characterise automorphisms T which act distally on Sub_G in terms of compactness of the closed group generated by T in $\mathrm{Aut}(G)$ (more generally see Theorem 4·1). For connected Lie groups, our results actually hold under a much weaker condition: e.g. if T acts distally on Sub^a_G, the space of closed abelian subgroups of G. Note that Sub^a_G is closed in Sub_G and hence compact, and it is invariant under the action of $\mathrm{Aut}(G)$. If $T \in \mathrm{Aut}(G)$ acts distally on Sub_G, then it acts distally on Sub^a_G. The converse holds for a large class of connected Lie groups as shown in Theorem 4·1.

We now discuss a class of automorphisms satisfying a certain weaker condition than those acting distally on Sub_G. For this, we introduce a subclass (NC) of $\mathrm{Aut}(G)$: An automorphism T of a locally compact group G is said to belong to (NC), if for any discrete (closed) nontrivial cyclic subgroup A of G, $T^{n_k}(A) \not\to \{e\}$ in Sub_G for any sequence $\{n_k\} \subset \mathbb{Z}$ such that $n_k \to \infty$ or $n_k \to -\infty$.

Since the cyclic groups are the most basic kind of groups and if the action of the group generated by an automorphism on Sub_G does not 'contract' any nontrivial closed cyclic group to the trivial group $\{e\}$, we call the class of such automorphisms (NC), where NC stands for 'non-contracting'. Calling such an automorphism T itself "non-contracting" could be misleading as it could imply that the contraction group of T is trivial.

https://doi.org/10.1017/S0305004121000694 Published online by Cambridge University Press

For $G = \mathbb{R}^n$ and $T = \alpha$ Id for any $\alpha \in \mathbb{R} \setminus \{0, 1, -1\}$, does not belong to (NC), as for any cyclic group A in $\mathbb{R}^n$, either $T^n(A) \to \{e\}$ or $T^{-n}(A) \to \{e\}$ as $n \to \infty$. All Lie groups and all connected locally compact non-compact groups admit nontrivial discrete cyclic subgroups. However, there are some totally disconnected groups which do not admit such subgroups, e.g. $\mathbb{Q}_p$, p a prime, has no nontrivial closed discrete cyclic subgroups.

If T acts distally on Sub^a_G, then $T \in$ (NC). Some of the results, mainly those about the distal actions of T on Sub^a_G, for connected Lie groups G without compact central subgroups of positive dimension, are proven under a weaker assumption that $T \in$ (NC). All automorphisms contained in compact subgroups of Aut(G) belong to (NC). The converse also holds for a large class of connected Lie groups; one of the main results shows that if G is a connected Lie group without any compact central subgroup of positive dimension, then T belongs to (NC) if and only if T is contained in a compact subgroup of Aut(G) (see Theorem 4·1)

Note that the projective space $\mathbb{RP}^n$ can be identified with $\mathbb{L}^n$, the set of lines in $\mathbb{R}^{n+1}$, which is a closed subset of $\mathrm{Sub}^a_{\mathbb{R}^{n+1}}$ invariant under the action of $\mathrm{GL}(n+1, \mathbb{R})$. The action of $\mathrm{GL}(n+1, \mathbb{R})$ on $\mathbb{RP}^n$ has been studied extensively by many mathematicians. Shah and Yadav in [30] show that for $T \in \mathrm{SL}(n+1, \mathbb{R})$, if the homeomorphism of the unit n-sphere $\mathbb{S}^n$ corresponding to T is distal, then T is contained in a compact subgroup of $\mathrm{SL}(n+1, \mathbb{R})$ (more generally, see [30, Corollary 5]). This in turn implies that if the action of such T on $\mathbb{RP}^n$, and hence on $\mathbb{L}^n$ is distal, then T is contained in a compact subgroup of $\mathrm{SL}(n+1, \mathbb{R})$. It is easy to show that for $T \in \mathrm{GL}(n+1, \mathbb{R})$, if T belongs to (NC), then T acts distally on $\mathbb{L}^n \cong \mathbb{RP}^n$ (this also follows from Theorem 4·1). The converse is not true as $T = \alpha$ Id on $\mathbb{R}^{n+1}$, $(|\alpha| \neq 0, 1)$, acts trivially on $\mathbb{RP}^n \cong \mathbb{L}^n$ but it does not belong to (NC), as observed above. Therefore, in case of automorphisms T of $G = \mathbb{R}^{n+1}$, the statements in Theorem 4·1 are not equivalent to the statement that T acts distally on a smaller non-discrete compact T-invariant subset $\mathbb{L}^n \subset \mathrm{Sub}^a_G$.

In Theorem 4·1, the condition on the center of G is necessary as illustrated by the example in Remark 3·8. Moreover, Example 4·7 shows that for a certain group G, there are nontrivial automorphisms contained in a connected unipotent subgroup of $(\mathrm{Aut}(G))^0$ which belong to (NC), where $(\mathrm{Aut}(G))^0$ is the connected component of the identity in Aut(G) with respect to the compact-open topology. Therefore, the condition on the center in Theorem 4·1 is also necessary even if we restrict to the subclass of unipotent automorphisms or those belonging to $(\mathrm{Aut}(G))^0$. However for these subclasses, we show that a part of Theorem 4·1 can be generalised to any connected Lie group G, i.e. $T \in (\mathrm{Aut}(G))^0$ acts distally on Sub^a_G if and only if T is contained in a compact subgroup of Aut(G) (see Theorem 4·4). We also show that for a unipotent automorphism T of G, T acts distally on Sub^a_G if and only if $T = $ Id, the identity map of G (see Theorem 4·3).

The group G acts on Sub_G via Inn(G), the group of inner automorphisms of G. We study this action and characterise the class of connected Lie groups G which act distally on Sub^a_G. Namely, we show that G acts distally on Sub^a_G if and only if every inner automorphism of G acts distally on Sub^a_G, and that the latter statement is equivalent to the statement that G is either compact or $G = \mathbb{R}^n \times K$, where K is a compact group and $n \in \mathbb{N}$ (more generally, see Corollary 4·5).

Baik and Clavier in [6] show that for $G = \mathrm{PSL}(2, \mathbb{C})$, a specific subspace of Sub^a_G is homeomorphic to the one-point compactification of $\mathbb{S}^2 \times \mathbb{R}^4$. They also describe the space which is the closure of all cyclic subgroups of G, where $G = \mathrm{PSL}(2, \mathbb{R})$ or $G = \mathrm{PSL}(2, \mathbb{C})$ in [5, 6] respectively. Bridson, de la Harpe and Kleptsyn in [8] describe the structure of $\mathrm{Sub}^a_{\mathbb{H}}$ and

https://doi.org/10.1017/S0305004121000694 Published online by Cambridge University Press

various other subspaces of $\mathrm{Sub}_{\mathbb{H}}$, for the 3-dimensional Heisenberg group $\mathbb{H}$, and they also study and describe the action of $\mathrm{Aut}(G)$ on some of these spaces in detail. One can correlate the image of $\mathrm{Aut}(G)$ to a subclass of homeomorphisms of these spaces, and our results imply in particular that such a homeomorphism is distal if and only if it generates a compact group (as a closed group) in $\mathrm{Aut}(G)$, for a large class of connected Lie groups G which include those mentioned above from [5, 6, 8].

Throughout, we will assume that all groups are locally compact and Hausdorff. We will often assume that they are second countable. A locally compact Hausdorff group is second countable if and only if it is first countable (metrisable) and σ-compact. In particular, a locally compact Hausdorff first countable group is second countable if it is compactly generated or, more generally, its quotient modulo any open almost connected subgroup is countable. A closed subgroup of a connected Lie group is locally compact, Hausdorff and second countable. Let G^0 denote the connected component of the identity e in G. It is a closed (normal) characteristic subgroup of G.

2. *Actions of automorphisms of G on* Sub_G

In this section we discuss basic properties of the Chabauty topology on Sub_G, and discuss certain elementary results about convergence of sequences in Sub_G and the action of $\mathrm{Aut}(G)$ on Sub_G.

Let G be a locally compact (Hausdorff) group. A sub-basis of the Chabauty topology on Sub_G is given by the sets of the form $\mathcal{O}_1(K)=\{A\in\mathrm{Sub}_G \mid A\cap K=\emptyset\}$ and $\mathcal{O}_2(U)=\{A\in\mathrm{Sub}_G \mid A\cap U\neq\emptyset\}$, where K is a compact and U is an open subset of G. As observed earlier, Sub_G is compact and Hausdorff, and if G is second countable, then both G and Sub_G are metrisable. For more details on the Chabauty topology, see [8, 23].

Recall that $\mathrm{Aut}(G)$ is the group of all automorphisms of G. There is a natural group action of $\mathrm{Aut}(G)$ on Sub_G defined as follows:

$$\mathrm{Aut}(G)\times\mathrm{Sub}_G\longrightarrow\mathrm{Sub}_G,\ (T,H)\longmapsto T(H);\ T\in\mathrm{Aut}(G),\ H\in\mathrm{Sub}_G.$$

It is easy to see that the map $H\mapsto T(H)$ defines a homeomorphism of Sub_G for every $T\in\mathrm{Aut}(G)$ (see e.g. [14, Proposition 2.1]), and the corresponding map $\mathrm{Aut}(G)\to\mathrm{Homeo}(\mathrm{Sub}_G)$ is a group homomorphism.

All our groups G are locally compact (Hausdorff) and we now assume that they are second countable, which ensures that Sub_G is metrisable and also that G is metrisable.

We first state a criterion (cf. [7, p. 161]) for the convergence of a sequence in Sub_G in the following lemma, which will be used often.

LEMMA 2·1. *Let G be as above. A sequence $\{H_n\}\subset\mathrm{Sub}_G$ converges to $H\in\mathrm{Sub}_G$ if and only if the following hold*:

(i) *For $g\in G$, if there exists a subsequence $\{H_{n_k}\}$ of $\{H_n\}$ with $h_k\in H_{n_k}$, $k\in\mathbb{N}$, such that $h_k\to g$ in G, then $g\in H$.*

(ii) *For every $h\in H$, there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ such that $h_n\in H_n$, $n\in\mathbb{N}$, and $h_n\to h$.*

The following lemma lists some elementary facts about the convergence in Sub_G and also the interplay of the action of $\mathrm{Aut}(G)$ and the convergence.

LEMMA 2·2. *Let G be as above. The following statements hold*:

(i) *Let* $H_n, L_n \in \mathrm{Sub}_G$ *be such that* $H_n \subset L_n$, $n \in \mathbb{N}$, $H_n \to H$ *and* $L_n \to L$ *in* Sub_G. *Then* $H \subset L$.

(ii) *Let* $H, L_n \in \mathrm{Sub}_G$ *be such that* $H \subset L_n$ *for all* $n \in \mathbb{N}$, *and* $L_n \to L$ *in* Sub_G. *Then* $H \subset L$.

(iii) *Let* $H_0, L_0 \in \mathrm{Sub}_G$ *be such that* $H_0 \subset L_0$ *and* $T_n(H_0) \to H$ *and* $T_n(L_0) \to L$, *for some* $\{T_n\} \subset \mathrm{Aut}(G)$. *Then* $H \subset L$.

(iv) *Let* $T \in \mathrm{Aut}(G)$ *and let* $H, L \in \mathrm{Sub}_G$ *be such that* $T(H) = H$ *and* $H \subset L$. *If* L' *is any limit point of* $\{T^n(L)\}_{n\in\mathbb{Z}}$, *then* $H \subset L'$.

Proof. (i) Let $h \in H$. As $H_n \to H$, Lemma 2·1 (ii) implies that there exists a sequence $\{h_n\}$ such that $h_n \in H_n$ for all n, and $h_n \to h$. Since $H_n \subset L_n$, we have that $h_n \in L_n$, for all n. As $L_n \to L$, Lemma 2·1 (i) implies that $h \in L$. Hence $H \subset L$.

(i) $\Rightarrow$ (ii) as we can put $H_n = H$ in (1), for all n.
(i) $\Rightarrow$ (iii) as we can take $H_n = T_n(H_0)$ and $L_n = T_n(L_0)$ in (1), for all n.
(ii) $\Rightarrow$ (iv) is obvious.

The following elementary result about the relation between the convergence of a sequence of subgroups and that of the corresponding sequence in the quotient group will be useful.

LEMMA 2·3. *For G as above and a closed normal subgroup H of G, let* $\pi: G \to G/H$ *be the cannonical projection. Suppose* $\{L_n\} \subset \mathrm{Sub}_G$ *is such that* $H \subset \cap_{n\in\mathbb{N}} L_n$. *Then the following hold*:

(i) *If* $L_n \to L$ *in* Sub_G, *then* $\pi(L_n) \to \pi(L)$.

(ii) *If* $\pi(L_n) \to L'$, *then* $L_n \to \pi^{-1}(L')$.

Proof. (i) Suppose $L_n \to L$. Observe that $\{\pi(L_n)\}$ is relatively compact as $\mathrm{Sub}_{G/H}$ is compact. Let L' be a limit point of $\{\pi(L_n)\}$. As $\mathrm{Sub}_{G/H}$ is metrisable, there exists $\{n_k\} \subset \mathbb{N}$ such that $\pi(L_{n_k}) \to L'$. First we show that $\pi(L) \subset L'$. By Lemma 2·1 (ii), for any $x \in L$, there exists $\{x_n\}$ such that $x_n \in L_n$, $n \in \mathbb{N}$, and $x_n \to x$ in G. This implies that $\pi(x_n) \to \pi(x)$ (as π is continuous). Therefore, $\pi(x) \in L'$, and hence $\pi(L) \subset L'$.

Now we show that $L' \subset \pi(L)$. Suppose $x' \in L'$. Again by Lemma 2·1 (ii), there exists $\{x'_k\}$ such that $x'_k \in \pi(L_{n_k})$, $k \in \mathbb{N}$, and $x'_k \to x'$ in G/H. For $k \in \mathbb{N}$, let $x_k \in L_{n_k}$ be such that $\pi(x_k) = x'_k$. There exists a sequence $\{h_k\} \subset H$ such that $x_k h_k \to x$, for some $x \in G$. Now $x_k h_k \in L_{n_k}$ as $H \subset L_{n_k}$, for all k, and hence $x \in L$ (by Lemma 2·1 (i)). Moreover, $x' = \pi(x) \in \pi(L)$, and hence $\pi(L) = L'$. Since this holds for all limit points of $\{\pi(L_n)\}$, we have that $\pi(L_n) \to \pi(L)$.

(ii) Suppose $\pi(L_n) \to L'$. Here, $\{L_n\}$ is relatively compact as Sub_G is compact. Let $L \in \mathrm{Sub}_G$ be a limit point of $\{L_n\}$. We show that $L = \pi^{-1}(L')$. As Sub_G is metrisable, there exists a subsequence $\{L_{n_k}\}$ of $\{L_n\}$ such that $L_{n_k} \to L$. From (i), we get that $\pi(L) = L'$. As $H \subset L_{n_k}$, for all k, $H \subset L$ (from Lemma 2·2 (ii)), which in turn implies that $L = \pi^{-1}(L')$. Since this holds for all the limit points L of $\{L_n\}$, it follows that $L_n \to \pi^{-1}(L')$

Recall that for a locally compact (Hausdorff) group G, if $\mathrm{Aut}(G)$ is endowed with the compact-open topology, the map $\mathrm{Aut}(G) \times G \to G$, $(T, x) \mapsto T(x)$, $T \in \mathrm{Aut}(G)$, $x \in G$, is continuous. If $\mathrm{Aut}(G)$ is endowed with the modified compact-open topology (which is finer than the compact-open topology), then $\mathrm{Aut}(G)$ is a topological group [32, 9·17], and we

https://doi.org/10.1017/S0305004121000694 Published online by Cambridge University Press

show that the action of the group $\mathrm{Aut}(G)$ on Sub_G is continuous. Note that for a connected Lie group G with the Lie algebra $\mathcal{G}$, $\mathrm{Aut}(G)$ is a Lie group as a closed subgroup of $\mathrm{GL}(\mathcal{G})$ and, the topology on $\mathrm{Aut}(G)$ inherited from $\mathrm{GL}(\mathcal{G})$ coincides with the compact-open topology, as well as with the modified compact-open topology [3, 16].

LEMMA 2·4. *Let G be a locally compact Hausdorff group and let* $\mathrm{Aut}(G)$ *be endowed with the modified compact-open topology. Then the map* $\mathrm{Aut}(G) \times \mathrm{Sub}_G \to \mathrm{Sub}_G$, $(T, H) \mapsto T(H)$, $T \in \mathrm{Aut}(G)$, $H \in \mathrm{Sub}(H)$, *is continuous.*

Proof. For a compact set C and an open set U in G, let $\lfloor C, U \rfloor = \{\phi \in \mathrm{Aut}(G) \mid \phi(C) \subset U\}$ and $\lfloor C, U \rfloor^{-1} = \{\phi \in \mathrm{Aut}(G) \mid \phi^{-1} \in \lfloor C, U \rfloor\}$. They are open in $\mathrm{Aut}(G)$ and the collection of all such sets form a sub-basis for the modified compact-open topology on $\mathrm{Aut}(G)$ [32].

Let $T \in \mathrm{Aut}(G)$ and $H \in \mathrm{Sub}_G$. Then $T(H)$ belongs to either $O_1(K)$ or $O_2(U)$ for some compact set K or an open set U in G, where $O_1(K)$ and $O_2(U)$ are open sets in the sub-basis of the topology on Sub_G. Suppose $T(H) \in O_1(K) = \{L \in \mathrm{Sub}_G \mid L \cap K = \emptyset\}$. Then $H \cap T^{-1}(K) = \emptyset$. As H is closed, $T^{-1}(K)$ is compact and since G is locally compact, there exists an open relatively compact set W containing $T^{-1}(K)$ such that $H \cap \overline{W} = \emptyset$. Then $\lfloor K, W \rfloor^{-1} \times O_1(\overline{W})$ is open in $\mathrm{Aut}(G) \times \mathrm{Sub}_G$, it contains (T,H) and if $\phi \in \lfloor K, W \rfloor^{-1}$ and $L \in O_1(\overline{W})$, then $\phi(L) \in O_1(K)$.

Now suppose $T(H) \in O_2(U) = \{L \in \mathrm{Sub}_G \mid L \cap U \neq \emptyset\}$. Since $T(H) \cap U \neq \emptyset$, there exists $x \in H$ such that $T(x) \in U$. Since G is locally compact and U is open, there exists an open relatively compact set W_1 such that $T(x) \in W_1 \subset \overline{W}_1 \subset U$. Let $V = T^{-1}(W_1)$. Then V is relatively compact. As $x \in V$ and $T(\overline{V}) \subset U$, we get that $(T, H) \in \lfloor \overline{V}, U \rfloor \times O_2(V)$ which is open in $\mathrm{Aut}(G) \times \mathrm{Sub}_G$. Moreover, if $\phi \in \lfloor \overline{V}, U \rfloor$ and $L \in O_2(V)$, then $\phi(L) \cap U \neq \emptyset$. Therefore, the map $\mathrm{Aut}(G) \times \mathrm{Sub}_G \to \mathrm{Sub}_G$ as above is continuous.

3. *Distality of automorphisms on G and* Sub_G

In this section, for $T \in \mathrm{Aut}(G)$, we show that the distality of the T-action on Sub_G implies the distality of the corresponding action on $\mathrm{Sub}_{G/H}$ for any closed normal T-invariant subgroup H of G. We also compare the distality of the T-action on Sub_G and that of T on G. [28, theorem 1·3] shows that an automorphism T of G is distal if and only if for any closed normal T-invariant subgroup H, $T|_H$ is distal and T acts distally on G/H. Lemma 3·1 and Example 3·2 illustrate that only a partial analogue of this theorem holds for the action of automorphisms on Sub_G. We prove some results about the distality of automorphisms on a connected Lie group which belong to (NC). We also prove a result on the structure of a nilpotent group admitting a unipotent automorphism which is a generalisation of Kolchin's Theorem for vector spaces; this will be useful to prove some of the main results. We end the section with a useful lemma about a criterion for the behaviour of an automorphism which implies that it does not belong to (NC).

LEMMA 3·1. *Let G be a locally compact Hausdorff group, $T \in$* $\mathrm{Aut}(G)$ *and let H be a closed normal T-invariant subgroup of G. Let $\overline{T} \in$* $\mathrm{Aut}(G/H)$ *be the corresponding map defined as $\overline{T}(gH) = T(g)H$, for all $g \in G$. If T acts distally on* Sub_G, *then T acts distally on* Sub_H *and $\overline{T}$ acts distally on* $\mathrm{Sub}_{G/H}$.

Proof. Suppose T acts distally on Sub_G. As Sub_H is closed and T-invariant, it is easy to see that T acts distally on Sub_H. Now we show that $\overline{T}$ acts distally on $\mathrm{Sub}_{G/H}$. Suppose G is second countable. For $i=1,2$, suppose $H_i \in \mathrm{Sub}_{G/H}$ and a sequence $\{n_k\} \subset \mathbb{Z}$ are such that $\overline{T}^{n_k}(H_i) \to L$ in $\mathrm{Sub}_{G/H}$. Then $T(H)=H \subset \pi^{-1}(H_i)$. By Lemma 2·3 (ii), $T^{n_k}(\pi^{-1}(H_i)) \to \pi^{-1}(L)$, for $i=1,2$. As T is distal on Sub_G, $\pi^{-1}(H_1)=\pi^{-1}(H_2)$, and hence $H_1=H_2$. Thus, $\overline{T}$ acts distally on $\mathrm{Sub}_{G/H}$ in this case.

Now suppose G is not second countable. Let $S_H := \{L \in \mathrm{Sub}_G \mid H \subset L\}$. As $T(H)=H$, it follows that S_H is a closed T-invariant subspace of Sub_G. Let $\bar{\pi}: S_H \to \mathrm{Sub}_{G/H}$, $\bar{\pi}(L)=\pi(L)=L/H$, $L \in S_H$. Then $\bar{\pi}$ is a homeomorphism such that $\overline{T} \circ \bar{\pi} = \bar{\pi} \circ T$. As T acts distally on Sub_G, it acts distally on S_H. Now using the homeomorphism $\bar{\pi}$, it is easy to see that $\overline{T}$ acts distally on $\mathrm{Sub}_{G/H}$.

The converse of the lemma does not hold as illustrated by the following.

Example 3·2. Let $T=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ in $\mathrm{GL}(2,\mathbb{R})$. It is clear that the T-action on $\mathbb{R}^2$ is distal. Also, for $H=\mathbb{R}\times\{0\}$, T acts trivially on H and on $\mathbb{R}^2/H$, hence it acts trivially on both Sub_H and $\mathrm{Sub}_{\mathbb{R}^2/H}$, but the T-action on $\mathrm{Sub}_{\mathbb{R}^2}$ is not distal, as $T(H)=H$ and for $H_1=\{0\}\times\mathbb{R}$, $T^n(H_1)\to H$ in $\mathrm{Sub}_{\mathbb{R}^2}$ as $n\to\pm\infty$.

Recall that $T\in\mathrm{Aut}(G)$ belongs to (NC) if for any discrete (closed) nontrivial cyclic subgroup A of G, $T^{n_k}(A)\not\to\{e\}$ in Sub_G for any sequence $\{n_k\}\subset\mathbb{Z}$ such that $n_k\to\infty$ or $n_k\to-\infty$. Note that for T as in Example 3·2 and $G=\mathbb{R}^2$, $T\notin$ (NC). This is because one can choose a discrete cyclic subgroup $A=\{0\}\times\mathbb{Z}$, for which $T^n(A)=\{(nz,z)\mid z\in\mathbb{Z}\}$, $n\in\mathbb{Z}$, and $T^n(A)\to\{(0,0)\}$ as $n\to\pm\infty$.

A topological group is said to be monothetic if it has a dense cyclic subgroup. Monothetic groups are abelian. In a locally compact group, a closed monothetic subgroup is either compact or a discrete infinite cyclic group. The following lemma will be useful.

LEMMA 3·3. *Let G be a locally compact Hausdorff second countable group and let* $T\in\mathrm{Aut}(G)$. *Let Z be a closed central T-invariant subgroup of G and let* $\overline{T}$ *be the corresponding automorphism on G/Z. Suppose T acts distally on* Sub^a_G. *Then* $\overline{T}$ *satisfies the following: for any two closed monothetic subgroups* A_1 *and* A_2 *in G/Z, if for a sequence* $\{n_k\}\subset\mathbb{Z}$, $\overline{T}^{n_k}(A_i)\to B$, $i=1,2$, *for some closed subgroup B in G/Z, then* $A_1=A_2$. *In particular,* $\overline{T}\in$ (NC) *in* $\mathrm{Aut}(G/Z)$.

Proof. Let $\pi: G\to G/Z$ be the natural projection. Since each A_i is monothetic and Z is a central subgroup of G, it is easy to see that each $\pi^{-1}(A_i)$ is abelian. By Lemma 2·3 (ii), $T^{n_k}(\pi^{-1}(A_i))\to\pi^{-1}(B)$, $i=1,2$. Therefore, $\pi^{-1}(B)\in\mathrm{Sub}^a_G$. As T acts distally on Sub_G, we get that $\pi^{-1}(A_1)=\pi^{-1}(A_2)$ and hence that $A_1=A_2$. The second assertion follows from the first.

The following useful lemma can be proved easily using Lemma 2·4. The lemma will apply in particular to the case of a connected Lie group G (with the Lie algebra $\mathcal{G}$) as $\mathrm{Aut}(G)$ is a Lie group whose topology (inherited from $\mathrm{GL}(\mathcal{G})$) coincides with the compact-open topology, as well as with the modified compact-open topology.

LEMMA 3·4. *Let G be a locally compact Hausdorff group and let* $T\in\mathrm{Aut}(G)$ *be such that* $T=S\phi=\phi S$ *for some* $\phi, S\in\mathrm{Aut}(G)$, *where* ϕ *is contained in a compact subgroup of* $\mathrm{Aut}(G)$ *with respect to the modified compact-open topology. Then* $T\in$ (NC) *if and only if* $S\in$ (NC).

https://doi.org/10.1017/S0305004121000694 Published online by Cambridge University Press

For a locally compact group G, $T \in \mathrm{Aut}(G)$ and a compact T-invariant subgroup K of G, $C_K(T) = \{x \in G \mid T^n(x)K \to K \text{ in } G/K \text{ as } n \to \infty\}$ is known as the K-contraction group of T. The group $C(T) = C_{\{e\}}(T)$ is known as the contraction group of T. It has been shown in [26] that T is distal if and only if both $C(T)$ and $C(T^{-1})$ are trivial. For a connected Lie group G, it is well–known that $C_K(T) = C(T)K$ [15]; see [26] for a more general result on this decomposition.

For an almost connected locally compact group G, let K be the largest compact normal subgroup. Then K is characteristic in G and G/K is a Lie group. As observed in [26], every inner automorphism of G acts distally on K. If any $T \in \mathrm{Aut}(G)$ acts distally on K, then $C(T)$ is closed [26, proposition 4·3]. The following useful lemma gives a more general result in the case of connected Lie groups. Note that the largest compact connected central subgroup of G is characteristic in G.

LEMMA 3·5. *Let G be a connected Lie group and let C be the largest compact connected central subgroup of G. Let $T \in \mathrm{Aut}(G)$. Then the following hold*:

(i) $\overline{C(T)} \subset C(T)C$.

(ii) If T acts distally on C, then C(T) is closed. In particular, if G does not have any compact central subgroup of positive dimension, then C(T) is closed.

Proof. (i) Let $D = \{d_n\}$ be a countable subgroup in $C(T)$ which is dense in $\overline{C(T)}$ and let $\mu = \sum_{n\in\mathbb{N}} 2^{-n}\delta_{d_n}$ be a probability measure on G. It is easy to see that $T^i(\mu) \to \delta_e$ as $i \to \infty$. Now we can apply [11, theorem 1·1] and get that $\overline{C(T)} = \mathrm{supp}\mu \subset C_C(T)$. As G is a connected Lie group, we have that $C_C(T) = C(T)C$ [15, theorem 2·4]. Therefore, we get that $\overline{C(T)} \subset C(T)C$ and (1) follows. Moreover, $C(T)C = \overline{C(T)}C$ and hence $C(T)C$ is closed.

(ii) Now suppose T acts distally on C. Then $C(T) \cap C = \{e\}$. Let $G' = C(T)C = \overline{C(T)}C$. Then G' is a closed T-invariant subgroup of G and $G' = C_C(\tau)$, where $\tau = T|_{G'}$. As $C(T) = C(\tau)$, $C(\tau) \cap C = \{e\}$. By [15, corollary 2·7], we get that $C(\tau)$ is closed in G'. As G' is closed in G, we get that $C(T)$ is closed. In particular, if G does not have any compact central subgroup of positive dimension, i.e. if C is trivial, then $C(T)$ is closed.

For $T \in \mathrm{Aut}(G)$, if the T-action on G is distal, it does not imply that the T-action on Sub_G is distal; see Example 3·2. Conversely, the following theorem shows that for a large class of locally compact groups G, the distality of the T-action on Sub_G implies the distality of the T-action on G.

THEOREM 3·6. *Let G be a locally compact first countable (metrisable) group, $T \in$ $\mathrm{Aut}(G)$ and let K be the largest compact normal subgroup of G^0. Suppose T acts distally on* Sub_G. *Then the T-action on G/K^0 is distal. Moreover, if T acts distally on K^0, then T acts distally on G.*

Proof. Let G be as above and let $T \in \mathrm{Aut}(G)$. Suppose T acts distally on Sub_G. Let K be as in the hypothesis. Note that K^0 is characteristic in G. In particular, it is T-invariant. We want to first show that the T-action on G/K^0 is distal.

Since T acts distally on Sub_G, by Lemma 3·1, T acts distally on Sub_{G/K^0}. Hence, without loss of any generality, we may assume that K as above is totally disconnected and show that T is distal. By [26, theorem 4·1], T is distal if and only if both $C(T)$ and $C(T^{-1})$ are trivial.

https://doi.org/10.1017/S0305004121000694 Published online by Cambridge University Press

Step 1. We first assume that G is second countable. Suppose G is totally disconnected. Then $K = \{e\}$. If possible, suppose that $C(T)$ is nontrivial. Since G is totally disconnected, there exists a neighbourhood basis of open compact subgroups $\{C_m\}_{m\in\mathbb{N}}$ at the identity e in G. Choose m such that $C(T) \not\subset C_m$. Let $H = C_m \cap \overline{C(T)}$. As Sub_G is compact, there exists a sequence $\{n_k\} \subset \mathbb{N}$ such that $T^{-n_k}(H) \to L$ for some $L \in \mathrm{Sub}_G$. As $\overline{C(T)}$ is T-invariant and $H \subset \overline{C(T)}$, by Lemma 2·2 (iii) we get that $L \subset \overline{C(T)}$. Let $x \in C(T)$. Since C_m is an open subgroup in G and $C(T)$ is T-invariant, we get that $T^n(x) \in C_m \cap C(T) \subset H$ for all large n, and hence, that $x = T^{-n}(T^n(x)) \in T^{-n}(H)$ for all large n. Therefore, $x \in L$ and hence, $C(T) \subset L$. As L is closed, $\overline{C(T)} \subset L$. Thus, we have that $\overline{C(T)} = L$, i.e. $T^{-n_k}(H) \to \overline{C(T)}$. As $\overline{C(T)}$ is T-invariant and $H \neq \overline{C(T)}$, we get that T does not act distally on Sub_G, a contradiction. Therefore, $C(T)$ is trivial. Interchanging T and T^{-1} and arguing as above, we can show that $C(T^{-1})$ is trivial.

Step 2. Suppose G is not totally disconnected. Note that G^0 is T-invariant and G/G^0 is totally disconnected. Let $\overline{T}: G/G^0 \to G/G^0$ be the automorphism corresponding to T. Then from Lemma 3·1, $\overline{T}$ acts distally on Sub_{G/G^0}. Hence, from the assertion in Step 1, $C(\overline{T})$ and $C(\overline{T}^{-1})$ are trivial. Therefore, $C(T) \cup C(T^{-1}) \subset G^0$. As K is totally disconnected and normal in G^0, by [29, lemma 2·2], it is central in G^0 (this also follows from [18, theorem 1′]). Now $\mathrm{Sub}_K = \mathrm{Sub}^a_K$. From Step 1, if T acts distally on Sub^a_K, we get that T acts distally on K. Now it is enough to show that T acts distally on G/K. As T acts distally on $\mathrm{Sub}_{G/K}$, replacing G by G/K and T by the corresponding automorphism of G/K, we may assume that G is a connected Lie group without any nontrivial compact normal subgroup. We have also assumed that T acts distally on Sub_G. We want to show that T is distal.

Step 3. We prove a more general statement: if G is any connected Lie group, $T \in \mathrm{Aut}(G)$ acts distally on the largest compact connected central subgroup of G and if $T \in (\mathrm{NC})$, then T is distal.

Assume that G and T are as above. By Lemma 3·5 (ii), both $C(T)$ and $C(T^{-1})$ are closed, and hence, simply connected and nilpotent.

If possible, suppose $C(T)$ is nontrivial. Let $V = Z(C(T))$, the center of $C(T)$. Then $V \cong \mathbb{R}^m$, for some $m \in \mathbb{N}$. Let $T_1 = T|_V$ be the restriction of T to V. Then $T_1 \in \mathrm{GL}(V)$ and $C(T_1) = V$, which is turn implies that all the eigenvalues of T_1 have absolute value less than 1. Suppose T_1 has a real eigenvalue λ. Then $0 < |\lambda| < 1$ and there exists a subspace $W \cong \mathbb{R}$ in V such that $T_1(x) = \lambda x$ for all $x \in W$. For the discrete subgroup $\mathbb{Z}$ in $\mathbb{R} \cong W$, it is easy to see that $T_1^{-n}(\mathbb{Z}) = \lambda^{-n}\mathbb{Z} \to \{0\}$ as $n \to \infty$ in $\mathrm{Sub}_{\mathbb{R}}$ [5]. As $T|_W = T_1|_W$, this leads to a contradiction as $T \in (\mathrm{NC})$.

Now suppose all the eigenvalues of T_1 are complex. There exists a T_1-invariant subspace $W' \cong \mathbb{R}^2$ in V such that $T_1|_{W'}$ has a complex eigenvalue of the form $r(\cos\theta + i \sin\theta)$, where $r \in \mathbb{R}$ and $0 < r < 1$. Let $T_2 \in \mathrm{GL}(2,\mathbb{R})$ be such that $T|_{W'} = T_1|_{W'} = T_2$ (under the isomorphism of $\mathbb{R}^2$ with W'). Then $T_2 = rA_\theta$, where $A_\theta = AR_\theta A^{-1}$ for some $A \in \mathrm{GL}(2,\mathbb{R})$, and R_θ is the rotation by the angle θ on $\mathbb{R}^2$. Here, A_θ generates a relatively compact group in $\mathrm{GL}(2,\mathbb{R})$. Let $Z = \{(m,0) \mid m \in \mathbb{Z}\}$. Then $Z \in \mathrm{Sub}_{\mathbb{R}^2}$. Since $0 < r < 1$, we have $r^{-n}Z \to \{(0,0)\}$ as $n \to \infty$ [5], and that $r\,\mathrm{Id} \notin (\mathrm{NC})$. Since $T_2 = (r\,\mathrm{Id})A_\theta = A_\theta(r\,\mathrm{Id})$, and A_θ generates a relatively compact group, by Lemma 3·4, we get that $T_2 \notin (\mathrm{NC})$. As $T_2 = T|_{W'}$, this leads to a contradiction. Hence $C(T)$ is trivial. Replacing T by T^{-1} and arguing as above, we get that $C(T^{-1})$ is also trivial. Therefore, T is distal.

We have proved the first assertion in case G is second countable. Now suppose G is not second countable. As noted before step 1, K can be assumed to be totally disconnected and T acts distally on Sub_G. We need to show that T is distal. Suppose $x \in G$ is such that $T^{n_k}(x) \to e$ in G for sone sequence $\{n_k\} \subset \mathbb{Z}$. Let L_x be the subgroup generated by $O_x = \{T^n(x) \mid n \in \mathbb{Z}\}$ in G and let $H = \overline{L_x G^0}$. Then H is a closed T-invariant subgroup of G. As O_x, and hence, L_x is countable, we have that H/H' is countable for some open almost connected subgroup H' of H. Since H' is compactly generated and first countable, and hence second countable, we get that H is second countable. As $H^0 = G^0$, K is the largest compact normal subgroup of H^0 and K is totally disconnected. Since T acts distally on Sub_G and since Sub_H is closed in Sub_G, we have that $T|_H$ acts distally on Sub_H. As H is second countable, we get from above that $T|_H$ is distal. Therefore, $x = e$ and T is distal.

We have proved that if T acts distally on Sub_G, then T acts distally on G/K^0, where K is the largest compact normal subgroup of G^0. Moreover, if T also acts distally on K^0, then it is easy to show that T acts distally on G.

We get a stronger result in case of Lie groups as follows.

COROLLARY 3·7. *Let G be a Lie group with not necessarily finitely many connected components and let $T \in \mathrm{Aut}(G)$ and let C be the largest compact connected central subgroup of G^0. Then the following hold*:

(a) *If T acts distally on C and $T \in$ (NC), then T acts distally on G.*

(b) *If T acts distally on Sub^a_G, then T acts distally on G/C.*

Proof. As G/G^0 is discrete and C as above is characteristic in G, T acts distally on G (resp. G/C) if and only if T acts distally on G^0 (resp. G^0/C). Moreover, $\mathrm{Sub}^a_{G^0}$ is closed in Sub^a_G and, if $T \in$ (NC) (resp. T acts distally on Sub^a_G), then $T|_{G^0} \in$ (NC) (resp. $T|_{G^0}$ acts distally on $\mathrm{Sub}^a_{G^0}$). Thus, to prove (*a*) and (*b*), we may assume that G is connected and C is central in G. Now (*a*) follows from the general statement in step 3 of the proof of Theorem 3·6. To prove (*b*), suppose T acts distally on Sub^a_G, where G is connected as assumed above. As C is central in G, by Lemma 3·3 we get that the automorphism of G/C corresponding to T belongs to (NC). As G/C does not have a compact central subgroup of positive dimension, (*a*) implies that T acts distally on G/C.

Remark 3·8. Note that the first condition in (a) of Corollary 3·7 is necessary as every automorphism of the compact connected abelian Lie group $G = \mathbb{T}^n$, $(n \geq 2)$, belongs to (NC), but G admits automorphisms which are not distal. Here, $\mathrm{Aut}(G)$ is isomorphic to $\mathrm{GL}(n, \mathbb{Z})$. Any closed cyclic group A of G is finite and it is contained in a finite characteristic subgroup A_m, the set of all m-th roots of unity, for some $m \in \mathbb{N}$ which depends on A. Now for any $T \in \mathrm{Aut}(G)$, $T|_{A_m}$ has finite order, and hence $T \in$ (NC). However, if we take any hyperbolic map in $\mathrm{GL}(2, \mathbb{Z})$, its eigenvalues are of absolute value other than 1, and hence it is not distal on $\mathbb{T}^2$. For $n \geq 3$, we can take a hyperbolic map on $\mathbb{T}^2$ and extend it to $\mathbb{T}^n \cong \mathbb{T}^2 \times \mathbb{T}^{n-2}$ naturally by putting the identity map on $\mathbb{T}^{n-2}$.

A locally compact group G is said to be pointwise distal if every inner automorphism is distal on G. The group G is said to be distal if the conjugacy action of G on G is distal, i.e. for $x \in G$, the closure of $\{gxg^{-1} \mid g \in G\}$ does not contain the identity e unless $x = e$. We get the following for inner automorphisms of almost connected groups.

COROLLARY 3·9. *Let G be an almost connected locally compact Hausdorff group. If every inner automorphism of G belongs to* (NC)*, then G is distal. In particular, if every inner automorphism of G acts distally on* Sub^a_G*, then G is distal.*

Proof. Let K be the largest compact normal subgroup of G. Let T be an inner automorphism of G. It is easy to see that T acts distally on K (see the last part of the proof of [26, theorem 5·2]). Hence $C(T)$ is closed and a simply connected nilpotent Lie group [26, proposition 4·3]. In particular, it has no nontrivial compact subgroups. Moreover, if $T \in$ (NC), then $T|_{C(T)}$ also belongs to (NC), and from Corollary 3·7 (a), it follows that $T|_{C(T)}$ is distal i.e. $C(T) = \{e\}$. Since this is true for all inner automorphisms of G, by [26, theorem 4·1], we have that G is pointwise distal, and hence it is distal [27, theorem 9]. The second assertion follows from the first.

Recall that an automorphism T of a connected Lie group is unipotent if the corresponding map dT on the Lie algebra is a unipotent linear transformation, i.e. all the eigenvalues of dT are equal to 1. The following proposition will be very useful. In the special case of vector spaces, it is well known as Kolchin's Theorem. It is also easy to deduce for compact connected abelian Lie groups by considering the corresponding map on the Lie algebra (see also [2, lemma 2·5]).

PROPOSITION 3·10. *Let G be a connected nilpotent Lie group and let $T \in \mathrm{Aut}(G)$ be unipotent. Then either $T = \mathrm{Id}$ or G has an increasing sequence of closed connected normal T-invariant subgroups $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$, $n \geq 2$, such that T acts trivially on G_i/G_{i-1}, $1 \leq i \leq n$, and T does not act trivially on G_{i+1}/G_{i-1}, $1 \leq i \leq n-1$.*

Proof. Suppose $T \neq \mathrm{Id}$. As G is nilpotent, G/K is simply connected, where K is the largest compact connected central subgroup in G. Then G has a sequence of closed connected normal T-invariant subgroups $\{e\} = Z_0 \subset Z_1 = K \subset \cdots \subset Z_m = G$, such that $Z_1/Z_0 = K$ and if G is not compact, Z_i/Z_{i-1} is a vector group and it is the center of G/Z_{i-1}, for all $i = 2, \ldots, m$. The automorphism on each Z_i/Z_{i-1} corresponding to T is unipotent. From Kolchin's Theorem for vector spaces and [2, lemma 2·5] for compact connected abelian Lie groups, it follows that there exists a sequence of closed connected normal subgroups in each Z_i/Z_{i-1} such that T-acts trivially on each successive quotient. Taking the pre-images of these subgroups in Z_i, we see that they are closed and T-invariant. Also since Z_i/Z_{i-1} is central in G/Z_{i-1}, any subgroup of Z_i which contains Z_{i-1} is normal in G. Now we have a finite sequence of closed connected normal T-invariant subgroups $\{e\} = H_0 \subset \cdots \subset H_l = G$, such that T acts trivially on H_i/H_{i-1}, $i = 1, \ldots l$. As $T \neq \mathrm{Id}$ and T acts trivially on H_1, there exists i_1 such that $1 \leq i_1 < l$, T acts trivially on H_{i_1} and T does not act trivially on H_{i_1+1}. Let $G_1 = H_{i_1}$. Having chosen $G_k = H_{i_k}$ for $k \geq 1$, if $G_k \neq G$ (equivalently, $i_k \neq l$), we choose $G_{k+1} = H_j$, where $i_k < j \leq l$ and j is the largest such natural number such that T acts trivially on H_j/G_k. As l is finite, there exists n such that $G_n = H_l = G$.

The following is known but we give a proof for the sake of completeness. It does not hold for general compact abelian groups as there are some compact totally disconnected abelian groups which are torsion-free, e.g. $\mathbb{Z}_p$, the ring of p-adic integers in $\mathbb{Q}_p$, p a prime.

LEMMA 3·11. *Let G be a nontrivial connected compact abelian Lie group. Let $B_n = \{x \in G \mid x^n = e\}$, $n \in \mathbb{N}$. Then the following holds: if $\{a_n\} \subset \mathbb{N}$ is an unbounded sequence, then $\cup_n B_{a_n}$ is dense in G. In particular, the torsion group $B = \cup_n B_n$ is a dense subgroup in G.*

https://doi.org/10.1017/S0305004121000694 Published online by Cambridge University Press

Proof. Note that each B_n is a finite subgroup and as G is abelian, $B_m B_n \subset B_{mn}$, $m, n \in \mathbb{N}$. Hence B is a group. It follows from the first statement that B is dense in G.

Let $B' = \cup_n B_{a_n}$. We may assume that $\{a_n\}$ is strictly increasing and that $a_n \to \infty$. As $G = \mathbb{T}^k = (\mathbb{S}^1)^k$, where $\mathbb{S}^1$ is the unit circle and $k = \dim G$, it is enough to prove the statement for $G = \mathbb{S}^1$. There exists a continuous real one-parameter subgroup $\{x(t)\}_{t\in\mathbb{R}}$ such that $x(0) = x(1) = e$, $x(t) \neq e$ for all $t \in]0, 1[$ and $G = \{x(t) \mid t \in [0, 1[\}$. Now $x(1/a_n) \in B_{a_n}$ and $x(1/a_n) \to x(0) = e$. Let $t \in]0, 1[$. As $1/a_n \to 0$, there exist $m_n \in \mathbb{N}$, $n \in \mathbb{N}$, such that $m_n < a_n$ for all large n, and $m_n/a_n \to t$. Therefore, $x(m_n/a_n) \to x(t)$. As $x(m_n/a_n) \in B_{a_n}$, we get that B' is dense in G.

Note that in a locally compact group G, the subgroup G_x generated by an element x is either closed and hence a discrete cyclic group, or its closure $\overline{G_x}$, being a compactly generated (locally compact) abelian group, is compact. If G is a Lie group, then either G_x is discrete or $\overline{G_x}$ is a compact Lie group with finitely many connected components.

We now state and prove a lemma which will play a crucial role in the proofs of main results in Section 4. Since we need to use Lemma 3·11 for the proof, we state it only for Lie groups even though it may hold for some locally compact (non-discrete) groups.

For $T \in \mathrm{Aut}(G)$, let $S_T = \{x \in G \mid T(x) = x\}$ be the stabiliser of T. It is a closed subgroup of G.

LEMMA 3·12. *Let G be a Lie group with not necessarily finitely many connected components and let $T \in \mathrm{Aut}(G)$. Let H be a closed normal T-invariant subgroup of G such that H does not contain any nontrivial compact subgroup and $T|_H = \mathrm{Id}$. Suppose there exists $x \in G$ such that $T(x) = xy$ for some $y \in H$, $y \neq e$. Let G_x be the cyclic subgroup generated by x. If $G_x \cap S_T = \{e\}$, then $T \notin$ (NC).*

Proof. Let G, T, H, x and y be as above. Suppose $G_x \cap S_T = \{e\}$. Let $G(x,y)$ be the subgroup of G generated by x and y. Then $G(x,y)$ is countable and T-invariant, and its closure $\overline{G(x,y)}$ is a T-invariant Lie subgroup, which is separable, and hence second countable. In particular, $\overline{G(x,y)}$ has countably many connected components. Replacing G by $\overline{G(x,y)}$, T by its restriction to $\overline{G(x,y)}$, and H by $\overline{G(x,y)} \cap H$, we may assume that G is second countable and G/H is an abelian Lie group which is either discrete or compact. Now Sub_G is metrisable. We show that $T \notin$ (NC). Since H is T-invariant and normal in G, we get for every $n \in \mathbb{Z}$ that $T(x^n) = x^n y_n$ for some $y_n \in H$, where $y_n \neq e$ unless $x^n = e$, as $G_x \cap S_T = \{e\}$.

Step 1. Suppose G/H is discrete, i.e. $G = G_x H$. We first show that G_x is discrete. Observe that if $G_x H/H$ is finite, then for some $n \in \mathbb{N}$, $x^n \in G_x \cap H \subset G_x \cap S_T = \{e\}$, and hence G_x is finite. If possible, suppose G_x is not discrete. Then G_x is infinite and relatively compact. This implies that $G/H = G_x H/H$ is compact, and hence it is finite. Then we get as above that G_x is finite, and it leads to a contradiction. Therefore, G_x is discrete, and hence, closed.

There exists an unbounded sequence $\{n_k\} \subset \mathbb{N}$ such that $T^{n_k}(G_x) \to L$ (say). We show that $L = \{e\}$. If possible, suppose $a \in L$ is such that $a \neq e$. Replacing a by a^{-1} if necessary, we get that there exists $\{m_k\} \subset \mathbb{N}$ such that $T^{n_k}(x^{m_k}) \to a$. If G_x is finite, then passing to a subsequence if necessary, we can choose $m_k = m_1$ for all k. Suppose G_x is infinite. As observed above, G/H is also infinite. Let $\pi : G \to G/H$ be the natural projection. Since $T(x) \in xH$, we have that $\pi(x^{m_k}) \to \pi(a)$. As G/H is infinite and discrete, we get that $\{m_k\}$ is eventually constant. In either case, we have that $T^{n_k}(x^m) \to a$ for some $m \in \mathbb{N}$. Then $T^{n_k}(x^m) = x^m y_m^{n_k} \to a$. Also, since $a \neq e$, we get that $x^m \neq e$ and hence $y_m \neq e$. Therefore,

https://doi.org/10.1017/S0305004121000694 Published online by Cambridge University Press

$\{y_m^{n_k}\}_{k\in\mathbb{N}}$ is convergent, and hence y_m is contained in a nontrivial compact subgroup in H. This leads to a contradiction. Therefore, $L=\{e\}$ and $T\notin$ (NC).

Step 2. Now suppose G/H is not discrete. As G_xH is dense in G, we have that G/H is compact and G_x is infinite. Since G/H is abelian and $H\subset S_T$, we have that S_T is normal in G. Moreover, G/S_T is infinite, as G_x is infinite and $G_x\cap S_T=\{e\}$. As G is a Lie group, G^0, and hence, G^0H is an open subgroup in G. Therefore, G/G^0H, being compact, is finite. We may replace x by x^n for some $n\in\mathbb{N}$ and assume that $G=G^0H$. Here, $G^0/(G^0\cap H)$ is connected, and it is also compact as it is isomorphic to G/H. Since G/S_T is infinite and $H\subset S_T$, we get that $G/S_T=(G^0H)/S_T$, and hence $G^0/(G^0\cap S_T)$ is compact and infinite.

Note that G^0 is T-invariant, T acts trivially on $H\cap G^0$ and on $G^0/(H\cap G^0)$. Also, $xh\in G^0$ for some $h\in H$, $T(xh)\in xH\cap G^0=xh(H\cap G^0)$ and $G_{xh}\cap S_T=\{e\}$. Moreover $G_{xh}(H\cap G^0)$ is dense in G^0 and G_{xh} is infinite, as $G_{xh}H=G_xH$. We may replace x by xh and assume that $x\in G^0$, and we may also replace G, H by G^0, $H\cap G^0$ respectively, and assume G is a connected Lie group, G/H is a compact connected abelian Lie group and G/S_T is infinite and compact. Note that $T(g)\in gH$ for all $g\in G$. Let $Q=\{g^{-1}T(g)\mid g\in G\}$. Then $e\in Q\subset H$ and, Q is connected as G is so. Therefore, $Q\subset H^0$, and hence $T(g)\in gH^0$ for all $g\in G$. Here, H/H^0, being a discrete normal subgroup of G/H^0, is central in G/H^0. Hence, G/H^0 is a covering group of G/H, and the later is abelian. Therefore, G/H^0 is abelian and, being connected, it is isomorphic to $\mathbb{R}^n\times\mathbb{T}^m$, for $n,m\geq 0$ and $m+n\in\mathbb{N}$. Let $\pi_0{:}G\to G/H^0$ be the natural projection.

Suppose $m\in\mathbb{N}$. Let $M=\pi_0^{-1}(\mathbb{T}^m)$. Then M is a closed connected T-invariant Lie subgroup and $H^0\subset M$. By Theorem 3·7 in Ch. XV of [17], $M=CH^0$, where C is a maximal compact connected subgroup of G. Now $M=C\ltimes H^0$, since H^0 does not have any nontrivial compact subgroup. Observe that C is isomorphic to $\pi_0(C)$, and hence it is nontrivial and abelian. Suppose $M\not\subset S_T$, i.e. $C\cap S_T\subsetneq C$. Note that $C\cap S_T$ is a closed (abelian) subgroup of C.

As C is a compact connected abelian Lie group and the set of primes is an unbounded set in $\mathbb{N}$, by Lemma 3·11, the set of elements of prime orders in C is dense in C. As $C\setminus(C\cap S_T)$ is open in C, there exists $b\in C\setminus(C\cap S_T)$ such that $b^p=e$ for some prime p. Since $C/(C\cap S_T)$ is also a compact connected abelian Lie group, we have that $b^n\in C\setminus(C\cap S_T)$ for all $n<p$ and $T(b^n)=b^nh_n$ for some $h_n\in H$ and $h_n\neq e$, $1\leq n<p$. Let G_b be the cyclic subgroup generated by b. Then $G_b\cap S_T=\{e\}$. Note that $\pi_0(G_b)$ is finite and discrete in M/H^0, and hence in G/H^0. Now arguing as in Step 1 for b, H^0 instead of x, H respectively, we get that $T\notin$ (NC).

Now suppose $m=0$ or $C\subset S_T$. Then $G/(CH^0)=\mathbb{R}^n$, where $n\in\mathbb{N}$ and, either $C=\{e\}$ or $C\subset S_T$. As $H\subset S_T\subsetneq G$ and G/S_T, being a compact connected abelian Lie group, is monothetic, there exists $x_1\in G\setminus S_T$ such that its image in G/S_T generates a (dense) infinite group. Hence $G_{x_1}\cap S_T=\{e\}$, where G_{x_1} is the group generated by x_1 in G. Moreover $\pi_0(x_1)$ generates a discrete infinite group in G/CH^0, and hence in G/H^0. Arguing as in Step 1, for x_1, H^0 instead of x, H respectively, we get that $T\notin$ (NC).

4. *Characterisation of automorphisms of Lie groups G which act distally on* Sub_G

In this section, we state and prove some of the main results characterising the class of automorphisms which belong to (NC) or act distally on Sub^a_G (see Theorems 4·1, 4·3 and 4·4). We also prove Proposition 4·2 for connected solvable Lie groups which will be useful in proving these theorems. After proving Theorems 4·1, we prove Theorems 4·3 and 4·4

https://doi.org/10.1017/S0305004121000694 Published online by Cambridge University Press

which show for all connected Lie groups G that if T is unipotent or if $T \in (\mathrm{Aut}(G))^0$, then the following holds: T acts distally on Sub^a_G if and only if T is contained in a compact subgroup of $\mathrm{Aut}(G)$. We also characterise connected Lie groups G which act distally on Sub^a_G (see Corollary 4·5).

THEOREM 4·1. *Let G be a connected Lie group without any compact central subgroup of positive dimension and let $T \in \mathrm{Aut}(G)$. Then the following statements are equivalent*:

(i) $T \in (\mathrm{NC})$.

(ii) *T acts distally on* Sub^a_G.

(iii) *T acts distally on* Sub_G.

(iv) *T is contained in a compact subgroup of* $\mathrm{Aut}(G)$.

Note that Theorem 4·1 does not hold for all connected Lie groups and the condition in the hypothesis is necessary as illustrated by the example in Remark 3·8 of the group $G = \mathbb{T}^n$, $n \geq 2$, for which every $T \in \mathrm{Aut}(G) \cong \mathrm{GL}(n, \mathbb{Z})$ belongs to (NC). However, it admits many distal (unipotent) and non-distal automorphisms, each of which generates a discrete infinite group in $\mathrm{Aut}(G)$.

Before proving the theorem, we prove a somewhat stronger result about distality for unipotent automorphisms on connected solvable Lie groups. Note that Proposition 4·2 (i) below will later be generalised to all connected Lie groups, see Theorem 4·3.

PROPOSITION 4·2. *Let G be a connected solvable Lie group and let $T \in \mathrm{Aut}(G)$ be unipotent. Then the following hold*:

(i) *If T acts distally on* Sub^a_G, *then* $T = \mathrm{Id}$.

(ii) *If G does not have a compact central subgroup of positive dimension and* $T \in (\mathrm{NC})$, *then* $T = \mathrm{Id}$.

Proof. ***Step 1.*** Let G be as in the hypothesis. Suppose $T \in \mathrm{Aut}(G)$ acts distally on Sub^a_G. Let K be the largest compact connected central subgroup of G. Then K is characteristic, and in particular, it is T-invariant and normal in G. If possible, suppose $T|_K \neq \mathrm{Id}$. By Proposition 3·10, there exist T-invariant compact connected (normal) nontrivial subgroups $K_1 \subset K_2$ such that T acts trivially on K_1 and on K_2/K_1 but it does not act trivially on K_2. We may, without loss of any generality, assume that K_1 is the largest connected subgroup in K_2 such that $T|_{K_1} = \mathrm{Id}$. As K_1 and K_2 are compact connected abelian Lie groups, and $\dim K_1 < \dim K_2$, we can get a nontrivial compact connected subgroup $M \subset K_2$ such that $M \cap K_1 = \{e\}$. As $M \subset K_2$, for every $x \in M$, $T(x) = xy$, for some $y \in K_1$ which depends on x. As M is monothetic, we can choose $g \in M$, such that g generates a dense subgroup in M. Let $h \in K_1$ be such that $T(g) = gh$.

We now show that h has infinite order. If possible, suppose h has finite order m (say). Then $T(g^{mn}) = g^{mn}$, $n \in \mathbb{N}$, and as $\{g^{mn} \mid n \in \mathbb{N}\}$ is dense in M, we get that T acts trivially on M. This leads to a contradiction to our assumption on K_1 as $K_1 \neq MK_1 \subset K_2$. As K_1 is compact, there exists an unbounded sequence $\{n_k\} \subset \mathbb{N}$ such that $h^{n_k} \to h$. Passing to a subsequence if necessary, we may assume that $T^{n_k}(M) \to L$ for some compact group $L \subset K_2$. As $\{g^n\}_{n \in \mathbb{N}}$ is dense in M and $T(g) = gh$, we have that $T^{n_k}(M) \subset MK_h$, where K_h is the closed subgroup generated by h in K_1. As $T^{n_k}(g) = gh^{n_k} \to gh$, we get that $gh \in L$. As M is a connected abelian Lie group, there exists a continuous real one-parameter group $\{g(t)\}_{t \in \mathbb{R}} \subset M$ such

https://doi.org/10.1017/S0305004121000694 Published online by Cambridge University Press

that $g(0)=e$ and $g(1)=g$. Let $x_k=g(1/n_k)\in M$, $k\in\mathbb{N}$. We have that $x_k\to e$ and $x_k^{n_k}=g$. Now

$$gh=T(g)=T(x_k)^{n_k}=(x_ky_k)^{n_k}=gy_k^{n_k} \text{ for some } y_k\in K_1.$$

In particular, $h=y_k^{n_k}$ for all k, and it implies that $T^{n_k}(gx_k^{-1})=gh^{n_k}h^{-1}x_k^{-1}\to g$. Therefore, $g\in L$ and hence $h\in L$. As L is a closed subgroup, we have that $M\subset L$ and $K_h\subset L$. Since $T^{n_k}(M)\subset MK_h$ for all k, we have that $L=MK_h$. As $T(MK_h)=MK_h$, this contradicts the hypothesis that T acts distally on Sub^a_G. Therefore, $T|_K=\text{Id}$. If G is compact, then $G=K$ and $T=\text{Id}$.

Step 2. Suppose G is not compact. Let $\pi:G\to G/K$ be the natural projection and let $\overline{T}$ be the automorphism of G/K corresponding to T. Then $\overline{T}$ is also unipotent, and by Lemma 3·3, $\overline{T}\in$ (NC) in $\text{Aut}(G/K)$. Suppose (ii) holds. Since G/K does not have any compact central subgroup of positive dimension, the statement in (ii) implies that $\overline{T}=\text{Id}$. Now we have that $T|_K=\text{Id}$ and $T(xK)=xK$ for all $x\in G$. Let N be the nilradical of G, then $K\subset N$, N/K is simply connected and G/N is abelian and isomorphic to $\mathbb{R}^n\times\mathbb{T}^m$, where $m,n\geq 0$. Let R be the closed connected (solvable) normal subgroup containing N, such that $R/N=\mathbb{R}^n$, $K\subset N\subset R$ and R/K is a closed connected solvable normal subgroup of G/K without any nontrivial compact subgroup. As $G/R=\mathbb{T}^m$ is compact, we get by [17, chapter XV, theorem 3·7], that $G=CR$, where C is a maximal compact connected subgroup of G which is abelian. As $K\subset C$, we have $T(C)=C$. Here, $T|_C$ is also unipotent and from step 1, we get that $T|_C=\text{Id}$.

Now it is enough to show that $T|_R=\text{Id}$. For every $x\in R$, $T(x)\in xK$. There exists a neighbourhood U of the identity e in G such that every $x\in U$ is exponential. If $T(x)=x$ for every $x\in U\cap R$, then $T|_R=\text{Id}$, since $U\cap R$ generates R. If possible, suppose $x\in U\cap R$ is such that $x\notin K$ and $T(x)\neq x$. Since x is exponential, it can be embedded in a continuous real one-parameter subgroup $\{x(t)\}_{t\in\mathbb{R}}$ as $x=x(1)$. We know that $\{x(t)\}_{t\in\mathbb{R}}$ is unbounded (and closed) as R/K has no nontrivial compact subgroups. Note that $T(x(t))=x(t)k(t)$, $k(t)\in K$ for all $t\in\mathbb{R}$ and $k(1)=k\neq e$. Let $M'=\overline{\{k(t)\mid t\in\mathbb{R}\}}$. It is a connected compact subgroup of K. Let $C_x=\{x(t)\}_{t\in\mathbb{R}}$. It is a closed connected abelian subgroup of R. For some $t_0\in\mathbb{R}$, $k(t_0)$ has infinite order, and hence, there exists an unbounded sequence $\{n_m\}\subset\mathbb{N}$ such that $k(t_0)^{n_m}\to k(t_0)$. Passing to a subsequence, we get that $\{k(t_0/l!)^{n_m}\}$ converges for every $l\in\mathbb{N}$. As Sub^a_G is compact, passing to a subsequence if necessary, we get that $\{T^{n_m}(C_x)\}$ converges to a closed abelian subgroup H (say). Now $T^{n_m}(x(t_0))=x(t_0)k(t_0)^{n_m}\to x(t_0)k(t_0)$. Therefore, $x(t_0)k(t_0)\in H$. Also, $T^n(x(t/n))=x(t/n)k(t)$, $n\in\mathbb{N}$. Therefore, $T^{n_m}(x(t/n_m))=x(t/n_m)k(t)\to k(t)\in H$ for all t, i.e. $M'\subset H$ and hence $x(t_0)\in H$. Moreover, $T^{n_m}(x(t_0/l!))=x(t_0/l!)k(t_0/l!)^{n_m}\to x(t_0/l!)h_l$ for some $h_l\in M'$ such that $k(t_0/l!)^{n_m}\to h_l$ for all $l\in\mathbb{N}$. Hence $x(t_0/l!)\in H$ for all $l\in\mathbb{N}$. As H is a closed subgroup, we get that $C_x\subset H$, and hence that $C_xM'\subset H$. Since C_xM' is T-invariant, we get by Lemma 2·2 (iii) that $H=C_xM'$. As $C_x\neq C_xM'$, it contradicts the hypothesis that T acts distally on Sub^a_G. Therefore, $T(x)=x$ for every $x\in U\cap R$ and hence, $T|_R=\text{Id}$. This implies that $T=\text{Id}$ assuming that (ii) holds. Therefore, (i) holds if (ii) holds.

We now prove (ii). Suppose G has no compact central subgroup of positive dimension. Then its nilradical N is simply connected. Moreover, $\text{Aut}(G)$ is almost algebraic [10, theorem 1], i.e. a closed subgroup of a finite index in an algebraic subgroup of $\text{GL}(\mathcal{G})$, where $\mathcal{G}$ is the Lie algebra of G. There exists a continuous unipotent one-parameter subgroup $\{T_t\}_{t\in\mathbb{R}}\subset\text{Aut}(G)$ such that $T=T_1$. Let $G'=\{T_t\}_{t\in\mathbb{R}}\ltimes G$. Then G' is a connected solvable Lie group.

https://doi.org/10.1017/S0305004121000694 Published online by Cambridge University Press

By [24, proposition 3·11], the commutator subgroup of G' is nilpotent and, as it is contained in G, it is contained in N. This implies that each T_t acts trivially on G/N.

By Proposition 3·10, either $T|_N = \mathrm{Id}$ or there exists a sequence of closed connected normal T-invariant subgroups of N; $\{e\} = G_0 \subset \cdots \subset G_m = N$, $m \geq 2$, such that T acts trivially on G_k/G_{k-1}, $1 \leq k \leq m$, and T does not act trivially on G_{k+1}/G_{k-1}, $1 \leq k \leq m-1$.

Suppose $T \in$ (NC). If possible, suppose $T|_N \neq \mathrm{Id}$. Note that $G_1 \subset G_2 \cap S_T \subsetneq G_2$, where S_T is the stabiliser of T. There exists $x \in G_2 \setminus (G_2 \cap S_T)$ such that the image of x in $G_2/(G_2 \cap S_T)$ generates an infinite group; this is true since G_2/G_1 is (simply) connected and nilpotent. Now $G_x \cap S_T = \{e\}$, where G_x is the cyclic subgroup generated by x. Moreover, $T(x) = xy$, $y \neq e$ and $y \in G_1$. As N is simply connected and nilpotent, G_1 does not contain any nontrivial compact subgroup. By Lemma 3·12, it follows that $T|_{G_2} \notin$ (NC), and hence $T|_N \notin$ (NC), which contradicts the hypothesis. Hence $T|_N = \mathrm{Id}$. If G is nilpotent, then $T = \mathrm{Id}$. Now suppose G is not nilpotent.

As observed above, T acts trivially on G/N. If possible, suppose $T \neq \mathrm{Id}$. Then $N \subset S_T \neq G$ and, as $[G, G] \subset N$ [24, proposition 3·11], we get that S_T is a proper closed normal subgroup of G and G/S_T is a nontrivial connected abelian group. There exists $x \in G \setminus S_T$ such that the image of x in G/S_T generates an infinite group. In particular, $G_x \cap S_T = \{e\}$, where G_x is the cyclic group generated by x. As T acts trivially on G/N, $T(x) = xg$ for some $g \in N$ and $g \neq e$. Since N is simply connected and nilpotent, it has no nontrivial compact subgroup. Since $T|_N = \mathrm{Id}$, we get by Lemma 3·12 that $T \notin$ (NC). This contradicts the hypothesis, and hence $T = \mathrm{Id}$.

Proof of Theorem 4·1. Let G be a connected Lie group such that it does not admit any nontrivial compact connected central subgroup and let $T \in \mathrm{Aut}(G)$. It is easy to see that (iv)$\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Now suppose (i) holds, i.e. $T \in$ (NC). To show the equivalence of (i) – (iv), it is enough to show that (iv) holds; i.e. we need to show that T is contained in a compact subgroup of $\mathrm{Aut}(G)$.

Step 1. Let $\mathcal{G}$ be the Lie algebra of G. For any $\tau \in \mathrm{Aut}(G)$, let $\mathrm{d}\,\tau$ be the corresponding Lie algebra automorphism of $\mathcal{G}$. Then we can view $\mathrm{Aut}(G)$ as a closed subgroup of $\mathrm{GL}(\mathcal{G})$. By Corollary 3·7 (a), we have that T acts distally on G. By [2, theorem 1·1], the Lie algebra automorphism $\mathrm{d}\,T$ on the Lie algebra $\mathcal{G}$ of G corresponding to T is distal and hence, all the eigenvalues of $\mathrm{d}\,T$ have absolute value 1 [1, theorem 1′]. Since G has no compact central subgroup of positive dimension, by [10, theorem 1] (see also [22]) $\mathrm{Aut}(G)$ is an almost algebraic subgroup of $\mathrm{GL}(\mathcal{G})$, i.e. a closed subgroup of finite index in an algebraic group. Let H be the smallest almost algebraic subgroup containing T in $\mathrm{Aut}(G)$. Then H is abelian and it acts distally on G [1, corollary 2·3], and $H = \mathcal{K} \times \mathcal{U}$, where $\mathcal{K}$ is a compact abelian group and $\mathcal{U}$ is unipotent and abelian, and in fact, a unipotent one-parameter group $\{T_t\}_{t \in \mathbb{R}}$ [1, corollary 2·5]. Note that all the eigenvalues of each T_t are equal to 1. Then each T_t acts distally on G [1, theorem 1′]. Let $T = T_s T_u = T_u T_s$, where $T_s \in \mathcal{K}$ and $T_u = T_{t_0} \in \mathcal{U}$, for some $t_0 \in \mathbb{R}$, $t_0 \neq 0$. By Lemma 3·4, T_u, and hence each T_t, $t \in \mathbb{R}$, belongs to (NC). Now to show that T is contained in a compact subgroup of $\mathrm{Aut}(G)$, it enough to show that $T_u = \mathrm{Id}$.

Step 2. Let R be the solvable radical of G. Then each $T_t|_R$ is unipotent and it belongs to (NC). Since the largest compact connected central subgroup of R, being characteristic in G, is central in G by [18, theorem 4], and hence it is trivial. Therefore, R does not have any compact central subgroup of positive dimension. By Proposition 4·2, $T_t|_R = \mathrm{Id}$. If G is solvable then $T_t = \mathrm{Id}$ for each t.

https://doi.org/10.1017/S0305004121000694 Published online by Cambridge University Press

Suppose G is not solvable. We show that T_t acts trivially on G/R. Let $G_s = G/R$. Then G_s is a connected semisimple Lie group. Moreover, $\mathrm{Inn}(G_s)$ is the connected component of the identity in $\mathrm{Aut}(G_s)$ and it is a subgroup of finite index in $\mathrm{Aut}(G_s)$. Let $\eta{:}G \to G_s$ be the natural projection and let T'_t be the automorphism of G_s corresponding to T_t for each t. We now show that each T'_t is trivial. Since $\{T'_t\}$ is unipotent and connected, there exists a Ad-unipotent one-parameter subgroup $\{u_t\}$ in G_s such that $T'_t = \mathrm{inn}(u_t)$, the inner automorphism by u_t in G_s. Suppose $\{u_t\}$ is nontrivial.

Let U be a maximal connected Ad-unipotent subgroup of G_s containing $\{u_t\}$, i.e. a maximal connected nilpotent subgroup containing $\{u_t\}$ such that for all $u \in U$, $\mathrm{Ad}(u)$ is a unipotent linear transformation of the Lie algebra of G_s. Let $G_s = KAN$ be an Iwasawa decomposition, where A is a closed abelian subgroup, N is a maximal connected Ad-unipotent subgroup of G_s which is normalised by A, and AN is a closed solvable subgroup of G_s (see [20, proof of Theorem 6·46]). Since any two maximal connected Ad-unipotent subgroups are conjugate to each other, we get that N and U are conjugate by an element in a compact set contained in K (see [11, Lemma 4·4]). If necessary, we may replace AN by its conjugate and assume that $U = N$, i.e. $u_t \in N$ for all $t \in \mathbb{R}$. Then AN is T'_t-invariant.

Let $B = \eta^{-1}(AN)$, where $\eta{:}G \to G/R$ is as above for the radical R of G. Then B is a closed connected solvable subgroup of G. Note that since AN is T'_t-invariant, we have that B is T_t-invariant for each t. Since A and N are simply connected and R has no compact central subgroup of positive dimension, we get that B has no compact central subgroup of positive dimension. As each $T_t|_B$ belongs to (NC), by Proposition 4·2, $T_t|_B = \mathrm{Id}$ for each t. This implies that u_t centralises AN for all $t \in \mathbb{R}$. Let $\mathcal{G}_s$ be the Lie algebra of G_s and let $\mathcal{A}$ and $\mathcal{N}$ be the Lie subalgebra of A and N respectively. If $X \in \mathcal{N}$ is such that $\exp tX = u_t$, $t \in \mathbb{R}$, then from above, X belongs to the center of $\mathcal{A} \oplus \mathcal{N}$, the Lie algebra of AN. This leads to a contradiction as the centraliser of $\mathcal{A}$ in $\mathcal{A} \oplus \mathcal{N}$ is $\mathcal{A}$ [20, lemma 6·50]. Therefore, $u_t = e$, and hence $T'_t = \mathrm{Id}$, for each t.

Step 3. Now we have from steps 1 and 2 that $T_t|_R = \mathrm{Id}$ and T_t acts trivially on G/R for each t, where R is the radical of G. If R is central in G, then the preceding assertions would imply that each T_t acts trivially on $\overline{[G,G]}$, the closure of the commutator subgroup of G, and as $G = \overline{[G,G]}R$, we get that $T_t = \mathrm{Id}$ in this case and, in particular, when G is semisimple. Now we want to prove that $T_t = \mathrm{Id}$, for each t in the general case.

For the natural projection $\eta{:}G \to G/R$, the subgroup AN of G/R and the subgroup $B = \eta^{-1}(AN)$ of G as in step 2, let $B_x = xBx^{-1}$, for a fixed $x \in G$. Then B_x is a closed connected solvable subgroup of G, $\eta(B_x) = \eta(x)AN\eta(x)^{-1}$ is also a closed connected solvable group of G/R, and B_x is T_t-invariant, as T_t acts trivially on G/R, for each t. Since B_x is a conjugate of B, it does not have any compact central subgroup of positive dimension. Now by Proposition 4·2, we get that the restriction of each T_t to B_x is trivial. Since this is true for every $x \in G$ and the closed subgroup generated by $H = \cup_{x\in G}\eta(B_x)$ is a closed normal subgroup containing AN in G/R, we have that H contains all the non-compact simple factors of G/R. Therefore, each T_t is trivial on a closed co-compact normal subgroup $\eta^{-1}(H)$ in G which contains R. That is, if G/R has no nontrivial compact simple factors, then $T_t = \mathrm{Id}$, for each t.

Step 4. Let S' be the product of all compact simple factors of G/R and let $G' = \eta^{-1}(S')$. Then G' is a closed characteristic (normal) subgroup in G. Let $G' = S_cR$ be the Levi decomposition of G'. Then S_c is compact and it is the product of all simple compact factors of a Levi subgroup in G. As $G = G'\eta^{-1}(H) = S_c\eta^{-1}(H)$, and T_t acts trivially on $\eta^{-1}(H)$, it is enough to show that $T_t|_{S_c} = \mathrm{Id}$.

Here, $G' = S_cR$ is a closed subgroup. As G' is characteristic, it is T_t-invariant for each t. Let $[S_c, R]$ be the group generated by $\{xyx^{-1}y^{-1} \mid x \in S_c, y \in R\}$. By [17, chapter XI, theorem 3·2], $\overline{[S_c, R]} \subset N$; here N denotes the nilradical of R, which is the same as the nilradical of G. Note that $\overline{[R, R]} \subset N$ [24, proposition 3·11]. Let G'_1 be the closure of the group $[G', G']$. Then $S_c \subset G'_1 \subset S_cN$. As G'_1 is T_t-invariant, we have that $T_t(S_c) \subset S_cN$. Since each T_t acts trivially on G'/R, we have that for $x \in S_c$, $x^{-1}T_t(x) \in (S_c \cap R)N$. As $S_c \cap R$ is a finite subgroup and S_c is connected, $T_t(x) \in xN$ for all $x \in S_c$ and for all t. Note that the nilradical N of G is simply connected and has no nontrivial compact subgroups. Let K be a maximal compact connected abelian subgroup of S_c. If possible, suppose for some t, $T_t|_K \neq \mathrm{Id}$. Let S_{T_t} be the stabiliser of T_t. Then $K \cap S_{T_t}$ is a proper closed subgroup of K. Now as $K/(K \cap S_{T_t})$ is a nontrivial compact connected abelian group, it is monothetic and hence, there exists $x \in K$ such that the image of x in $K/(K \cap S_{T_t})$ generats a dense infinite group. In particular, $G_x \cap S_{T_t} = \{e\}$. As $T_t(x) \in xN$, $T_t|_N = \mathrm{Id}$ and N has no nontrivial compact subgroups, by Lemma 3·12, $T_t \notin$ (NC), which leads to a contradiction. Therefore, $T_t|_K = \mathrm{Id}$ for all t. Since the union of all maximal compact connected abelian subgroups of S_c is dense in S_c, we have that $T_t|_{S_c} = \mathrm{Id}$ for all t. As observed above, this shows that $T_t = \mathrm{Id}$ for all t. As observed at the end of step 1, this implies that T is contained in a compact subgroup of G. Therefore, (i) ⇒ (iv) and (i) – (iv) are equivalent.

We know from Remark 3·8 that all automorphisms of connected compact abelian Lie groups of dimension greater than or equal to 2 belong to class (NC) and, in particular, this hold for nontrivial unipotent automorphisms of such groups. The following theorem shows that a connected Lie group G does not admit any nontrivial unipotent automorphism which acts distally on Sub^a_G.

THEOREM 4·3. *Let G be a connected Lie group and let $T \in \mathrm{Aut}(G)$ be unipotent. Then T acts distally on Sub^a_G if and only if $T = \mathrm{Id}$.*

Proof. Let G and T be as in the hypothesis. One way implication is obvious. Now suppose T acts distally on Sub^a_G. We show that $T = \mathrm{Id}$.

Let R be the solvable radical of G. Then R is T-invariant and by Proposition 4·2, $T|_R = \mathrm{Id}$. Let C be the largest compact connected central subgroup. If C is trivial, then the assertion follows from Theorem 4·1 as the only unipotent automorphism contained in a compact subgroup of $\mathrm{Aut}(G)$ is the identity map. Suppose C is nontrivial. Let $\overline{T}$ be the automorphisms of G/C corresponding to T. Then by Lemma 3·3, $\overline{T} \in$ (NC). Note that $\overline{T}$ is also unipotent. Now by Theorem 4·1, $\overline{T}$ is contained in a compact subgroup of $\mathrm{Aut}(G/C)$, and being unipotent, $\overline{T}$ is trivial. As C is connected and central, it follows that T acts trivially on $\overline{[G, G]}$. Now as $G = \overline{[G, G]}R$, we get that $T = \mathrm{Id}$.

The following theorem shows that a part of Theorem 4·1 can be generalised to all connected Lie groups if we restrict the class of automorphisms to $(\mathrm{Aut}(G))^0$, the connected component of the identity in $\mathrm{Aut}(G)$. Example 4·7 illustrates that Theorem 4·1 can not be generalised fully even if we restrict to $(\mathrm{Aut}(G))^0$.

THEOREM 4·4. *Let G be a connected Lie group and let $T \in \mathrm{Aut}(G)$. Suppose T belongs to $(\mathrm{Aut}(G))^0$, the connected component of the identity in $\mathrm{Aut}(G)$. Then (ii) – (iv) in Theorem 4·1 are equivalent.*

Proof. Let G be a connected Lie group and let $T \in (\mathrm{Aut}(G))^0$. It is enough to show that (ii) ⇒ (iv) in Theorem 4·1. Suppose (ii) holds, i.e. T acts distally on Sub^a_G. We want to prove that T is contained in a compact subgroup of $\mathrm{Aut}(G)$.

Let C be the largest compact connected central subgroup of G. Then C is characteristic. As C is compact and abelian, Aut(C) is totally disconnected and hence every element of $(\mathrm{Aut}(G))^0$ acts trivially on C. In particular, $T|_C = \mathrm{Id}$. Also, by Corollary 3·7 (*a*), T acts distally on G. Therefore, the eigenvalues of $\mathrm{d}\,T$ on the Lie algebra $\mathcal{G}$ of G are of absolute value 1. Moreover, $(\mathrm{Aut}(G))^0$ is almost algebraic as a closed subgroup of $\mathrm{GL}(\mathcal{G})$ [33, 34]; see also [10, theorem 2]. Let G_T be the smallest almost algebraic group in $(\mathrm{Aut}(G))^0$ containing T. By [1, corollary 2·5], $G_T = \mathcal{K} \times \mathcal{U}$ where $\mathcal{K}$ is compact and abelian, and $\mathcal{U}$ is unipotent and abelian. In fact, $\mathcal{U}$ is a unipotent one-parameter group $\{T_t\}$ in $(\mathrm{Aut}(G))^0$, such that $T = T_sT_u = T_uT_s$, $T_s \in \mathcal{K}$ and $T_u = T_1$ is unipotent. By [31, lemma 2·2], T_u acts distally on Sub^a_G. By Theorem 4·3, $T_u = \mathrm{Id}$. Therefore, $T = T_s$ is contained in a compact subgroup of $(\mathrm{Aut}(G))^0$, i.e. (iv) holds.

The following corollary characterises the class of connected Lie groups G which act distally on Sub^a_G. Recall that the action of G on Sub_G is the same as the action of $\mathrm{Inn}(G)$ on Sub_G, where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G.

COROLLARY 4·5. *Let G be a connected Lie group. Then the following are equivalent*:

(i) *Every inner automorphism acts distally on* Sub^a_G.

(ii) *Every inner automorphism acts distally on* Sub_G.

(iii) *G acts distally on* Sub^a_G.

(iv) *G acts distally on* Sub_G.

(v) *Either G is compact or* $G = \mathbb{R}^n \times K$, *for a compact group K and some* $n \in \mathbb{N}$.

Moreover, if G has no compact central subgroup of positive dimension, then the above statements are also equivalent to the following:

(vi) *Every inner automorphism of G belongs to* (NC).

The condition on the center of G in the above corollary is necessary for the equivalence of the statement (vi) with (i)–(v) as illustrated by Example 4·7 below.

Proof of Corollary 4·5. Let G be a connected Lie group. (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) obviously hold. We also have that (v) $\Rightarrow$ (iv), as $\mathrm{Inn}(G) = \{\mathrm{inn}(k) \mid k \in K\}$ is compact. To show that (i) – (v) are equivalent, it is enough to show that (i) $\Rightarrow$ (v). Suppose (i) holds, i.e. every inner automorphism of G acts distally on Sub^a_G. We want to show that (v) holds

Suppose G is not compact. We need to show that G is a product of $\mathbb{R}^n$ and a compact subgroup for some $n \in \mathbb{N}$. As every inner automorphism acts distally on Sub^a_G, by Corollary 3·9, G is distal. By [27, theorem 9] and [19, corollary 2·1(ii)], G/R is compact where R is the solvable radical of G. Let N be the nilradical of G. Let C be the largest compact normal subgroup of N. Then C is connected and central in G and N/C is simply connected and nilpotent. Note that every inner automorphism of G by an element of N is unipotent, i.e. $\mathrm{Ad}(x)$ is unipotent for all $x \in N$. By Theorem 4·3, $\mathrm{inn}(x) = \mathrm{Id}$ for every $x \in N$ i.e. N is abelian and central in G. In particular, since R/N is abelian, R is nilpotent, and hence $R = N$. Therefore, R is central in G. Now G is a compact extension of Z^0, the connected component of the center Z of G. Let $Z^0 = \mathbb{R}^n \times C$, where C is compact and, $n \in \mathbb{N}$ as G is non-compact. Then the subgroup $V = \mathbb{R}^n$ is a (nontrivial) vector group which is central. Using [18, lemma 3·7], we get that $G = \mathbb{R}^n \times K$, where K is the maximal compact subgroup of G, i.e. (v) holds.

If G does not have any compact central subgroup of positive dimension, by Theorem 4·1, we get that (vi) ⇔ (i), and hence the last assertion follows.

An automorphism T of a locally compact group G is said to have bounded orbits if all its orbits are relatively compact, i.e. for every $x \in G$, $\{T^n(x)\}_{n\in\mathbb{Z}}$ is relatively compact. In particular, if T is contained in a compact subgroup of $\mathrm{Aut}(G)$, then T has bounded orbits. The following result is for any automorphism on a general connected Lie group which acts distally on Sub^a_G.

COROLLARY 4·6. *Let G be a connected Lie group and let $T \in \mathrm{Aut}(G)$. If T acts distally on Sub^a_G, then T has bounded orbits.*

Proof. Suppose T acts distally on Sub^a_G. Let C be the largest compact connected central subgroup of G. Then C is characteristic in G and G/C does not have any compact central subgroup of positive dimension. Moreover, by Lemma 3·3, the automorphism $\overline{T}$ of G/C corresponding to T belongs to (NC) in $\mathrm{Aut}(G/C)$ and by Theorem 4·1, we get that $\overline{T}$ has bounded orbits. Therefore, T itself has bounded orbits, since C is compact.

The converse of Corollary 4.6 does not hold as illustrated by the following example. The example also illustrates that in Theorem 4·3, the statement that a unipotent automorphism T of a connected Lie group G acts distally on Sub^a_G can not be replaced by the statement that T belongs to the class (NC). It also shows that Theorem 4·1 can not be completely generalised to any connected Lie group G even if we restrict the class of automorphisms to $(\mathrm{Aut}(G))^0$, as in Theorem 4·4. Moreover, the example also illustrates that the condition on the center in Corollary 4·5 is necessary for the statement (vi) to be equivalent to (i) – (v) in the theorem.

Example 4·7. Let $G = \mathbb{H}/D$, where $\mathbb{H}$ is the 3-dimensional Heisenberg group (the group of strictly upper triangular - unipotent - matrices in $\mathrm{SL}(3,\mathbb{R})$) and D is a discrete central subgroup of $\mathbb{H}$ isomorphic to $\mathbb{Z}$. Then G is a connected step-2 nilpotent Lie group, its center Z of G is compact and $G/Z \cong \mathbb{R}^2$. Let T be any nontrivial inner automorphism of G, i.e. $T = \mathrm{inn}(x)$ for some $x \notin Z$, Note that T has bounded orbits as $T^n(g) \in gZ$, $n \in \mathbb{Z}$, for every $g \in G$. As G is connected and nilpotent, T is nontrivial and unipotent and Theorem 4·3 implies that T does not act distally on Sub^a_G. However, T belongs to (NC). For if A is a discrete cyclic subgroup of G, it is either finite and hence central in G, in which case it is T-invariant, or its image in G/Z is a discrete infinite cyclic group and as T acts trivially on G/Z, it follows that $T \in$ (NC). Also $T \in (\mathrm{Aut}(G))^0$ and in fact, it generates a non-compact subgroup in $(\mathrm{Aut}(G))^0$. Moreover, every inner automorphism of G belongs to (NC).

The map T in the above example acts trivially on the central torus and it also acts trivially on the quotient group modulo the central torus. Such maps on connected Lie groups are known as (isotropic) shear automorphisms (see [10]).

Remark 4·8. (1) For the unit circle $\mathbb{S}^1$, $\mathrm{Aut}(\mathbb{S}^1)$ is finite. Therefore, using [22, corollary 2·5], Corollary 3·7 and Theorem 4·4, one can show that Theorem 4·1(ii)-(iv) are equivalent for a somewhat larger class of connected Lie groups whose maximal torus is central and it is of dimension at most 1. This class includes connected nilpotent Lie groups whose largest compact (connected) central subgroup is of dimension 1, e.g. G as in Example 4·7. It also includes the Lie group of the form $\mathrm{SL}(2,\mathbb{R}) \ltimes G$, where G is as in the above example. (2) It would be interesting to not only identify the spaces Sub^a_G and Sub_G, but describe the action of $\mathrm{Aut}(G)$ on them, as it is done in [8] for some subspaces of $\mathrm{Sub}_{\mathbb{H}}$, for the 3-dimensional

https://doi.org/10.1017/S0305004121000694 Published online by Cambridge University Press

Heisenberg group $\mathbb{H}$. (3) In case of compact abelian (Lie) groups G, we study the action of unipotent automorphisms. It would be interesting to study the action of a general T in Aut(G) on Sub_G for such compact groups G, as it would give more insight for the action of Aut(G) on Sub_G for a general connected Lie group G. (4) For general locally compact groups G, (especially for totally disconnected groups), a comprehensive study is needed for the action of Aut(G) on Sub_G.

We conclude with a mention of the following: using many results in this paper, further study of class (NC) and the distality of actions of automorphisms of G on Sub_G have been carried out where G is a certain type of locally compact group or a lattice in a Lie group [35, 37]. Expansivity of the action of automorphisms of G on Sub_G has also been studied for locally compact groups and lattices in Lie groups recently [36, 37].

Acknowledgments. R. Shah would like to thank: S. G. Dani for extensive discussions on the structure of semisimple Lie groups and I. Chatterji and Université Nice Sophia Antipolis, France, for hospitality during a visit in June 2019 while a part of work was carried out. R. Shah would also like to acknowledge the MATRICS research grant from DST-SERB, Govt. of India, which partially supported A. K. Yadav as a visiting scholar at JNU, New Delhi in 2018. A. K. Yadav would like to thank the Ambedkar University Delhi for an adjunct faculty position in 2018 and Krishnendu Gongopadhyay for a visiting position under a research grant from NBHM, DAE, Govt. of India, at IISER Mohali in early 2019. We thank the anonymous referee for suggestions which led to some improvements in the presentation of this paper.

REFERENCES

[1] H. ABELS. Distal affine transformation groups. *J. Reine Angew. Math.* **299/300** (1978), 294–300.
[2] H. ABELS. Distal automorphism groups of Lie groups. *J. Reine Angew. Math.* **329** (1981), 82–87.
[3] R. ARENS. Topologies for homeomorphism groups. *Amer. J. Math.* **68** (1946), 593–610.
[4] M. ABERT, N. BERGERON, I. BIRINGER, T. GELANDER, N. NIKOLOV, J. RAIMBAULT and I. SAMET. On the growth of L^2-invariants for sequences of lattices in Lie groups. *Ann. of Math.* (2) 185 (2017), 711–790.
[5] H. BAIK and L. CLAVIERL. The space of geometric limits of one-generator closed subgroups of $PSL_2(\mathbb{R})$. *Algebr. Geom. Topol.* **13** (2013), 549–576.
[6] H. BAIK and L. CLAVIER. The space of geometric limits of abelian subgroups of $PSL_2(\mathbb{C})$. *Hiroshima Math. J.* **46** (2016), 1–36.
[7] R. BENEDETTI and C. PETRONIO. *Lectures on Hyperbolic Geometry* (Springer-Verlag, 1992).
[8] M. R. BRIDSON, P. DE LA HARPE and V. KLEPTSYN. The Chabauty space of closed subgroups of the three-dimensional Heisenberg group. *Pacific J. Math.* **240** (2009), 1–48.
[9] C. CHABAUTY. Limite d'ensemble et géométrie des nombres. *Bull. Soc. Math. France* **78** (1950), 143–151.
[10] S. G. DANI. On automorphism groups of connected Lie groups. *Manuscripta Math.* **74** (1992), 445–452.
[11] S. G. DANI and R. SHAH. Contractible measures and Lévy's measures on Lie groups. In: Probability on Algebraic Structures (Ed. G. Budzban P. Feinsilver and A. Mukherjea), *Contemporary Math.* 261 (2000), 3–13.
[12] R. ELLIS. Distal transformation Groups. *Pacific J. Math.* **8** (1958), 401–405.
[13] H. FURSTENBERG. The structure of distal flows. *Amer. J. Math.* **85** (1963), 477–515.
[14] H. HAMROUNI and B. KADRI. On the compact space of closed subgroups of locally compact groups. *J. Lie Theory* **23** (2014), 715–723.
[15] W. HAZOD and E. SIEBERT. Automorphisms on a Lie group contracting modulo a compact subgroup and applications to semistable convolution semigroups. *J. Theoret. Probab.* **1** (1988), 211–225.

[16] G. HOCHSCHILD. The automorphism group of a Lie group. *Trans. Amer. Math. Soc.* **72** (1952), 209–216.
[17] G. HOCHSCHILD. *The Structure of Lie Groups* (Holden-Day,Inc., 1965).
[18] K. IWASAWA. On some type of topological groups. *Annals of Math.* **II**, 50 (1949), 507–558.
[19] J. JENKINS. Growth of connected locally compact groups. *J. Functional Analysis* **12** (1973), 113–127.
[20] A. W. KNAPP. *Lie Groups Beyond an Introduction*. Second Edition. Progr. Math. **140** (BirkhÄuser Boston Inc., 2002).
[21] C. C. MOORE. Distal affine transformation groups. *Amer. J. Math.* **90** (1968), 733–751.
[22] W. H. PREVITS and S. T. WU. On the group of automorphisms of an analytic group. *Bull. Austral. Math. Soc.* **64** (2001), 423–433.
[23] I. POUREZZA and J. HUBBARD. The space of closed subgroups of $\mathbb{R}^2$. *Topology* **18** (1979), 143–146.
[24] M. S. RAGHUNATHAN. *Discrete Subgroups of Lie Groups* (Springer-Verlag, 1972).
[25] C. R. E. RAJA and R. SHAH. Distal actions and shifted convolution property. *Israel J. Math.* **177** (2010), 301–318.
[26] C. R. E. RAJA and R. SHAH. Some properties of distal actions on locally compact groups. *Ergodic Theory and Dynam. Systems* **39** (2019), 1340–1360.
[27] J. ROSENBLATT. A distal property of groups and the growth of connected locally compact groups. *Mathematika* **26** (1979), 94–98.
[28] R. SHAH. Orbits of distal actions on locally compact groups. *J. Lie Theory* **22** (2010), 586–599.
[29] R. SHAH. Expansive automorphisms on locally compact groups. *New York J. Math.* **26** (2020), 285–302.
[30] R. SHAH and A. K. YADAV. Dynamics of certain distal actions on spheres. *Real Analysis Exchange* **44** (2019), 77–88.
[31] R. SHAH and A. K. YADAV. Distality of certain actions on p-adic spheres. *J. Australian Math. Soc.* **109** (2020), 250–261.
[32] M. STROPPEL. *Locally Compact Groups*. EMS Textbooks in Mathematics (European Math. Soc. - EMS, 2006).
[33] D. WIGNER. On the automorphism group of a Lie group. *Proc. Amer. Math. Soc.* **45** (1974), 140–143.
[34] D. WIGNER. Erratum to "On the automorphism group of a Lie group". *Proc. Amer. Math. Soc.* **60** (1976), 376.
[35] R. PALIT and R. SHAH. Distal actions of automorphisms of nilpotent groups G on Sub_G and applications to lattices in Lie groups. *Glasgow Math. Journal*, **63** (2021), 343–362.
[36] M. B. PRAJAPATI and R. SHAH. Expansive actions of automorphisms of locally compact groups G on Sub_G. *Monatsh. Math.* (2020), 193 (2020), 129–142.
[37] R. PPALIT, M. B. PRAJAPATI and R. SHAH. Dynamics of actions of automorphisms of discrete groups G on Sub_G and applications to lattices in Lie groups. *Groups, Geometry, and Dynamics* (to appear).