Note on Whittaker's Solution of Laplace's Equation.

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§ 1. Whittaker * has shewn that a general solution of Laplace's Equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

may be put in the form

$$\int_0^{2\pi} f(z + ix \cos u + iy \sin u, u)\, du, \quad \text{...........(1·1)}$$

where $f(v, u)$ denotes an arbitrary function of the two variables u and v; such a representation is valid only in the neighbourhood of a regular point.

On account of the symmetry of the equation of Laplace, there are two other types of general solution, which are of the forms

$$\int_0^{2\pi} g(x + iy \cos u + iz \sin u, u)\, du \quad \text{...........(1·2)}$$

$$\int_0^{2\pi} h(y + iz \cos u + ix \sin u, u)\, du \quad \text{...........(1·3)}$$

A given solution of Laplace's equation may be representable in all three of these forms; but each form is valid in a different region of space, and is not transformable into either of the other forms. The simplest example of this is the solution

$$2\pi/\{x^2 + y^2 + z^2\}^{\frac{1}{2}};$$

it has the definite integral representation

$$\int_0^{2\pi} (z + ix \cos u + iy \sin u)^{-1}\, du \quad \text{.... ....(1·4)}$$

which is only valid when $z > 0$; when $z < 0$, the integral represents $-2\pi/\{x^2 + y^2 + z^2\}^{\frac{1}{2}}$; when $z = 0$, the integral vanishes. There are two other representations valid when $x > 0$, $y > 0$ respectively.

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* *Math. Ann.* **57**, (1902), 333.

The representation (1·4) is seen to be discontinuous at the plane $z = 0$, which passes through the singular point (0, 0, 0) of the solution.

In view of these facts, it seemed desirable to investigate whether the second solution of degree (− 1), viz. $Q_0(z/r)/r$, possesses definite integral representations, and to examine the connexion between the singular line $x = 0$, $y = 0$ of this solution with the region of validity of such integral representations.

§ 2. When we take polar coordinates, defined by

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\phi,$$

the second solution of degree (− 1) has the form

$$r^{-1} Q_0(\cos\theta);$$

$\theta = 0$ is the singular line.

Instead of this solution, we take the solution which has $\theta = \alpha$, $\phi = 0$, as singular line, and then intend to examine what happens as α tends to zero. The solution is

$$r^{-1} Q_0(\cos\varpi)$$

where

$$\cos\varpi = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos\phi.$$

Using the addition theorem for Legendre functions,* we have

$r^{-1} Q_0(\cos\varpi)$

$$= r^{-1} P_0(\cos\theta)\, Q_0(\cos\alpha) + 2\sum_{m=1}^{\infty} r^{-1} P_0^{-m}(\cos\theta) \cos m\phi\, Q_0^m(\cos\alpha) \ldots (2{\cdot}1)$$

In this way, we have expressed our solution as an infinite series of terms which are expressible in the form (1·1). But this representation is only valid in a region for which

$$\theta < \alpha \leq \pi/2,$$

i.e. inside a circular cone with Oz as axis and the singular line $\theta = \alpha$, $\phi = 0$ as generator.

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* See WHITTAKER and WATSON: *Modern Analysis* (3rd Edn.) 329. We are using HOBSON'S definition (*Phil. Trans.* A **187** (1896)) of the associated functions.

Now Hobson * has shewn that

$$2\pi P_0^{-m}(\cos\theta)(-1)^m\cos m\phi = \frac{1}{m!}\int_0^{2\pi}\frac{\cos mu\,du}{\cos\theta + i\sin\theta\cos(u-\phi)} \quad \ldots(2\cdot2)$$

Hence, if $z > 0$, and if $\theta < \alpha \leqq \pi/2$, we have

$$2\pi r^{-1}Q_0(\cos\varpi) = Q_0(\cos\alpha)\int_0^{2\pi}\frac{du}{z + ix\cos u + iy\sin u}$$
$$+ 2\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}Q_0^m(\cos\alpha)\int_0^{2\pi}\frac{\cos mu\,du}{z + ix\cos u + iy\sin u} \quad \ldots(2\cdot3).$$

Before we invert the order of integration and summation, it is necessary to examine the uniformity of convergence of the series

$$Q_0(\cos\alpha) + 2\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}Q_0^m(\cos\alpha)\cos mu \quad \ldots\ldots\ldots\ldots(2\cdot4)$$

It follows from some recent work † on the asymptotic expansions of the Hypergeometric Function that

$$2\frac{(-1)^m}{m!}Q_0^m(\cos\alpha) \sim \frac{i^{-m}}{m}\left[\cot^m\frac{\alpha}{2} - (-1)^m\tan^m\frac{\alpha}{2} + O\left(\frac{1}{m}\right)\right]$$

for large values of m. The coefficients of the trigonometric series (2·4) are then not bounded as $m \rightarrow \infty$, unless $\alpha = \pi/2$. If $\alpha = \pi/2$, the series (2·4) converges uniformly in the interval $(0, 2\pi)$ except at $u = \pi/2$ and $u = 3\pi/2$, where it has finite discontinuities.

It is then legitimate to invert the order of integration and summation in (2·3), only if $\alpha = \pi/2$. We have then

$$2\pi r^{-1}Q_0(\sin\theta\cos\phi) = \int_0^{2\pi}\frac{f(u)\,du}{z + ix\cos u + iy\sin u}$$

where $f(u)$ is the sum of the series

$$Q_0(0) + 2\sum_{m=1}^{\infty}\frac{(-1)}{m!}Q_0^m(0)\cos mu,$$

provided only that $z > 0$.

Hence it is impossible to represent $Q_0(z/r)/r$ in the form (1·1). The solution $Q_0(x/r)/r$ has a definite integral representation of the

* *Loc. cit.*, 499.

† G. N. Watson: *Camb. Phil. Trans.*, **22**, (1918), 277-308.

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form (1·1) valid when $z > 0$, and, by a similar argument, a representation of the form (1·3) valid when $y > 0$; in the former case, the integral represents $-Q_0(x/r)/r$ when $z < 0$, and also in the latter when $y < 0$.

§ 3. We may write these conclusions thus:—

The solution $(x^2 + y^2 + z^2)^{-\frac{1}{2}} Q_0 (z/(x^2 + y^2 + z^2)^{\frac{1}{2}})$ *cannot be represented as*

$$\int_0^{2\pi} f(z + ix \cos u + iy \sin u, u)\, du\,; \qquad \text{(1·1)}$$

it can be represented in the forms

$$\int_0^{2\pi} g(x + iy \cos u + iz \sin u, u)\, du \qquad \text{(1·2)}$$

and

$$\int_0^{2\pi} h(y + iz \cos u + ix \sin u, u)\, du \qquad \text{(1·3)}$$

The forms (1·2) *and* (1·3) *have different regions of validity, in each case bounded by a plane through the singular line* $x = 0,\ y = 0$.