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STABILITY THEOREMS FOR CONVEX DOMAINS OF CONSTANT WIDTH

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ABSTRACT. It is known that among all plane convex domains of given constant width Reuleaux triangles have minimal and circular discs have maximal area. Some estimates are given concerning the following associated stability problem: If K is a convex domain of constant width w and if the area of K differs at most ϵ from the area of a Reuleaux triangle or a circular disc of width w, how close (in terms of the Hausdorff distance) is K to a Reuleaux triangle or a circular disc? Another result concerns the deviation of a convex domain M of diameter d from a convex domain of constant width if the perimeter of M is close to πd.

In this note a convex domain is defined as a compact convex subset of the euclidean plane with nonempty interior. As usual, a convex domain K is said to be of constant width w if the distance between any pair of parallel support lines of K is w. In this case $w = d(K)$, where $d(K)$ denotes the diameter of K. The two best known domains of constant width are the circular disc and the Reuleaux triangle. The latter, which can be defined as a convex domain T of constant width that is bounded by three congruent circular arcs of radius $d(T)$, has area $c_o d(T)^2$, where

$$c_o = \frac{\pi - \sqrt{3}}{2}. \tag{1}$$

These two domains appear naturally as extremal cases in the following inequalities. If K is a convex domain of constant width $d(K)$ and area $a(K)$, then

$$a(K) \geqq c_o d(K)^2 \tag{2}$$

with equality if and only if K is a Reuleaux triangle, and

$$a(K) \leqq \frac{\pi}{4} d(K)^2 \tag{3}$$

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with equality if and only if K is a circular disc. (2) is known as the inequality of Blaschke-Lebesgue, and (3), which actually holds for any convex domain, is called the inequality of Bieberbach. For references regarding these inequalities and all other results on convex bodies of constant width that will be cited see [1], [2], and [4].

In the present article we concern ourselves with stability questions associated with (2) and (3). They are of the following type: If a convex domain K of constant width $d(K)$ has the property that $a(K)$ differs not more than ϵ from $c_o d(K)^2$ or $(\pi/4)d(K)^2$ what can be said about the deviation of K from a Reuleaux triangle or, respectively, from a circular disc? To measure the deviation of two convex domains M, N from each other we let $D(x, r)$ denote the circular disc of radius r centered at x and introduce the Hausdorff distance

$$h(M, N) = \inf\left\{r : M \subset \bigcup_{x \in N} D(x, r),\ N \subset \bigcup_{x \in M} D(x, r)\right\}.$$

Using the convenient normalization $d(K) = 1$ our stability statements can now be given the following precise formulation.

THEOREM 1. *Let K be a convex domain and assume that* $\epsilon \geqq 0$.

(a) *If K is of constant width* 1 *and*

$$a(K) \leqq c_o + \epsilon \tag{4}$$

(*where c_o is defined by* (1)), *then there is a Reuleaux triangle T such that*

$$h(K, T) \leqq \sqrt{10\epsilon}. \tag{5}$$

(b) *If K has diameter* 1 *and*

$$a(K) \geqq \frac{\pi}{4} - \epsilon, \tag{6}$$

then there is a circular disc D such that

$$h(K, D) \leqq \frac{1}{2}\sqrt{\epsilon}. \tag{7}$$

Furthermore, if K is of constant width 1 *there is such a D with the additional property that* $d(D) = 1$.

It is worth noting that a stability statement of the type (a) implies immediately the original inequality (2). Indeed, if one had $a(K) < c_o$ it would be possible to take $\epsilon = 0$, which would then imply $h(K, T) = 0$ and therefore $K = T$ leading to the contradiction $a(K) = c_o$. The same remark applies with regard to (b). The following corollary, which is obtained from Theorem 1 by setting, respectively $\epsilon = a(K) - c_o$ or $\epsilon = (\pi/4) - a(K)$, shows even more strikingly that the above stability statements (a) and (b) can be viewed as improvements of the classical inequalities (2) and (3).

COROLLARY 1. *If K is a convex domain of constant width* 1 *there exist a Reuleaux triangle T and a circular disc D, both of diameter* 1, *such that*

$$c_o + \frac{1}{10}h(K, T)^2 \leqq a(K) \leqq \frac{\pi}{4} - 4h(K, D)^2.$$

Another well-known theorem on convex domains of constant width is the theorem of Barbier. It states that every convex domain of constant width w has perimeter πw. Since every convex domain K is contained in a convex domain of constant width $d(K)$ it follows that

$$p(K) \leqq \pi d(K), \tag{8}$$

where $p(K)$ denotes the perimeter of K and equality holds if and only if K is of constant width. The following theorem deals with the stability problem associated with (8).

THEOREM 2. *Let K be a convex domain of diameter* 1 *and $\epsilon \geqq 0$. If*

$$p(K) \geqq \pi - \epsilon,$$

then there exists a convex domain C of constant width 1 *such that $K \subset C$ and*

$$h(K, C) \leqq \frac{34}{25}\epsilon^{2/3} \tag{9}$$

The exponent 2/3 *cannot be replaced by any larger constant.*

Analogously to Corollary 1 the content of this theorem can be expressed as an improvement of (8).

COROLLARY 2. *For any convex domain of diameter* 1 *there is a convex domain C of constant width* 1 *such that $K \subset C$ and*

$$p(K) \leqq \pi - \left(\frac{25}{34}h(K, C)\right)^{3/2}.$$

For the purpose of proving these theorems we assume that the plane is equipped with the usual cartesian x, y-coordinate system having origin o. The euclidean norm will be denoted by $||\cdot||$. If K is a convex domain containing o and if $u = (u_1, u_2)$ is a unit vector forming an angle α with the positive x-axis, then K has a support line, say $L_K(\alpha)$, of the form $u_1x + u_2y = H_K(\alpha)$, where $H_K(\alpha)$ is the support function of K, i.e., the distance from o to $L_K(\alpha)$. To prove part (a) of Theorem 1 we need two lemmas. We use frequently the known fact that a convex domain of constant width has a regular circumscribed hexagon.

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LEMMA 1. *Let K be a convex domain of constant width 1 and Q a regular hexagon circumscribed about K. Assume that for some $t > 0$ there is a vertex p of Q such that*

$$D(p, t) \cap K \neq \emptyset \tag{10}$$

Let T be the Reuleaux triangle of width 1 inscribed in Q with one vertex at p. Then,

$$h(K, T) \leqq \sqrt{(2 + t)t}. \tag{11}$$

PROOF. Let v and w be the other two vertices of Q that are, together with p, the vertices of an equilateral triangle. Because of (10) there is a $q \in K$ with $||q - p|| \leqq t$. Thus, for any $z \in K$ we have $||z - p|| \leqq ||z - q|| + ||q - p|| \leqq 1 + t$ and consequently

$$K \subset D(p, 1 + t). \tag{12}$$

Let now S be the half-line starting at v and containing the vertex of Q opposite p. Because of $||p - v|| = 1$ we have $v \in D(p, 1 + t)$ and it follows that the circle $bdrD(p, 1 + t)$ intersects S in exactly one point, say s. Obviously

$$||v - s|| = \sqrt{(1 + t)^2 - 1^2} = \sqrt{(2 + t)t}.$$

Furthermore, since S contains a side of Q, there is a point of K between v and s. Hence, $K \cap D(v, \sqrt{(2 + t)t}) \neq \emptyset$, and using the same argument that led to (12) (with t replaced by $\sqrt{(2 + t)t}$) we find that

$$K \subset D(v, 1 + \sqrt{(2 + t)t}) \tag{13}$$

and analogously

$$K \subset D(w, 1 + \sqrt{(2 + t)t}). \tag{14}$$

Let us now introduce the convex domain

$$J = D(p, 1 + \sqrt{(2 + t)t}) \cap D(v, 1 + \sqrt{(2 + t)t}) \cap D(w, 1 + \sqrt{(2 + t)t}) \cap Q.$$

Because of (12), (13), (14), and since $K \subset Q$ and $t \leqq \sqrt{(2 + t)t}$ we have $K \subset J$. Since it is easily seen that every point of J is within distance $\sqrt{(2 + t)t}$ of a point of T the corresponding support functions have the property that

$$H_J(\alpha) - H_T(\alpha) \leqq \sqrt{(2 + t)t}.$$

If α is an angle such that $H_K(\alpha) \geqq H_T(\alpha)$, then

$$0 \leqq H_K(\alpha) - H_T(\alpha) \leqq H_J(\alpha) - H_T(\alpha) \leqq \sqrt{(2 + t)t}.$$

But if $H_K(\alpha) \leqq H_T(\alpha)$, then, since both K and T have constant width 1,

$$H_K(\alpha + \pi) = 1 - H_K(\alpha) \geqq 1 - H_T(\alpha) = H_T(\alpha + \pi)$$

and therefore

$$0 \leqq H_T(\alpha) - H_K(\alpha) = H_K(\alpha + \pi) - H_T(\alpha + \pi) \leqq \sqrt{(2 + t)t}.$$

Hence, for all α we have $|H_K(\alpha) - H_T(\alpha)| \leqq \sqrt{(2 + t)t}$, and this implies obviously (11).

LEMMA 2. *Let K be a convex domain of constant width 1 which has a circumscribed regular hexagon Q with center at o and one vertex, say p, on the negative x-axis. Let T be the Reuleaux triangle of width 1 inscribed in Q with one vertex at p. If there exists an α with $-\pi/12 \leqq \alpha \leqq \pi/12$ such that*

$$0 \leqq H_K(\alpha) - H_T(\alpha) \leqq s, \tag{15}$$

then

$$D(p, 3.87\, s) \cap K \neq \emptyset. \tag{16}$$

PROOF. Let us set $I = [-\pi/12, \pi/12]$. Since (15), together with the relations

$$H_K(\alpha + \pi) = 1 - H_K(\alpha),\ H_T(\alpha + \pi) = 1 - H_T(\alpha),$$

implies

$$0 \leqq H_T(\alpha + \pi) - H_K(\alpha + \pi) \leqq s$$

it follows that for any $\alpha \in I$ the distance between the support lines $L_T(\alpha + \pi)$ and $L_K(\alpha + \pi)$ is at most s. We also note that for $\alpha \in I$ the line $L_T(\alpha + \pi)$ contains the point p and the line segment $L_K(\alpha + \pi) \cap Q$ must contain a point, say q, of K. The distance $\|p - q\|$, considered as a function of q, is clearly maximal if q is one of the endpoints of $L_K(\alpha + \pi) \cap Q$. From this fact and $\alpha \in I$ we obtain

$$\|p - q\| \leqq s/\sin \frac{\pi}{12} \leqq 3.87\, s.$$

Hence, $q \in D(p, 3.87\, s) \cap K$ and this implies obviously the desired relation (16).

PROOF OF THEOREM 1. We first prove part (a). If K is given we may assume that it is positioned so that Q, T, and p are as in Lemma 2. Let K' and T' be the convex domains obtained from K and T (respectively) by a rotation of angle π about o. Our proof proceeds by showing that either T or T' satisfies inequality (5). To establish this fact it will be shown that the validity of both

$$h(K, T) > \sqrt{10\epsilon} \tag{17}$$

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and

$$h(K', T) > \sqrt{10\epsilon} \tag{18}$$

would imply that, contrary to (4), $a(K) > c_o + \epsilon$. We may assume that

$$\sqrt{10\epsilon} \leqq 0.155 \tag{19}$$

since $H_K(\alpha) \geqq H_T(\alpha)$ implies

$$H_K(\alpha) - H_T(\alpha) \leqq H_Q(\alpha) - H_T(\alpha) \leqq (2/\sqrt{3}) - 1 < 0.155,$$

and $H_K(\alpha) \leqq H_T(\alpha)$ implies

$$\begin{aligned} H_T(\alpha) - H_K(\alpha) &= H_K(\alpha + \pi) - H_T(\alpha + \pi) \\ &\leqq H_Q(\alpha + \pi) - H_T(\alpha + \pi) \leqq (2/\sqrt{3}) - 1 < 0.155. \end{aligned}$$

It is now convenient to introduce for $n = 0, \pm 1$ the following intervals:

$$I_n = \left[-\frac{\pi}{6} + n\frac{2\pi}{3}, \frac{\pi}{6} + n\frac{2\pi}{3}\right], \tilde{I}_n = \left[-\frac{\pi}{12} + n\frac{2\pi}{3}, \frac{\pi}{12} + n\frac{2\pi}{3}\right].$$

Using this notation we can state that for all $\alpha \in I_n$ $(n = 0, \pm 1)$

$$H_K(\alpha) \geqq H_T(\alpha), \tag{20}$$

and

$$H_K(\alpha + \pi) \geqq H_T(\alpha). \tag{21}$$

For example, if $n = 0$ then (20) follows from the fact that in this case $H_K(\alpha + \pi) \leqq H_T(\alpha + \pi)$ (since the support line of T corresponding to $\alpha + \pi$ contains p and is also a support line of Q). The cases $n = \pm 1$ are obtained by suitable rotations, and (21) can be shown by applying (20) to K'.

If we now set $t = 4.97\epsilon$ and use (19) then $\sqrt{(2 + t)t} \leqq \sqrt{10\epsilon}$ and Lemma 1 shows that under the assumption (17) $D(p, 4.97\,\epsilon) \cap K = \emptyset$. Hence, if Lemma 2 is applied with

$$s = \frac{4.97}{3.87}\epsilon$$

we find that (17) implies that for all $\alpha \in \tilde{I}_0$

$$H_K(\alpha) - H_T(\alpha) > \frac{4.97}{3.87}\epsilon \geqq 1.28\,\epsilon. \tag{22}$$

Since corresponding statements hold of course for the other three vertices of T we can state that (22) is valid for all $\alpha \in \tilde{I}_n$ $(n = 0, \pm 1)$. Analogously we obtain from (18) that for all $\alpha \in \tilde{I}_n$ $(n = 0, \pm 1)$

$$H_K(\alpha + \pi) - H_T(\alpha) > 1.28\,\epsilon. \tag{23}$$

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We can now complete the proof using an idea of Eggleston [3], [4]. It depends on the fact that the radius of curvature, say $\rho(\alpha)$, of K (at the point $K \cap L_K(\alpha)$) exists for all α with at most countably many exceptions and has the property that $\rho(\alpha) + \rho(\alpha + \pi) = 1$ and

$$a(K) = \frac{1}{2} \int_{-\pi}^{\pi} \rho(\alpha) H_K(\alpha) d\alpha.$$

In particular, if $K = T$ this shows that

$$a(T) = \frac{1}{2} \sum \int_{I_n} H_T(\alpha) d\alpha$$

(where $\sum$ indicates summation over $n = 0, -1, 1$). Hence (20), (21), (22), and (23) implies that

$$\begin{aligned} a(K) &= \frac{1}{2} \sum \int_{I_n} (\rho(\alpha) H_K(\alpha) + \rho(\alpha + \pi) H_K(\alpha + \pi)) d\alpha \\ &> \frac{1}{2} \sum \int_{\tilde{I}_n} (\rho(\alpha)(H_T(\alpha) + 1.28\,\epsilon) + \rho(\alpha + \pi)(H_T(\alpha) + 1.28\,\epsilon)) d\alpha \\ &+ \frac{1}{2} \sum \int_{I_n \setminus \tilde{I}_n} (\rho(\alpha) H_T(\alpha) + \rho(\alpha + \pi) H_T(\alpha)) d\alpha \\ &= \frac{1}{2} \sum \left(\int_{I_n} H_T(\alpha) d\alpha + \int_{\tilde{I}_n} 1.28\,\epsilon d\alpha \right) = a(T) + \frac{\pi}{4} 1.28\,\epsilon \geqq c_o + \epsilon. \end{aligned}$$

Part (b) of Theorem 1 is an almost immediate consequence of a theorem of Bonnesen (cf. [2], p. 83) which states that there are two concentric circular discs, say $D(p, r)$ and $D(p, R)$ with the property that $D(p, r) \subset K \subset D(p, R)$ and

$$\frac{1}{4\pi} p(K)^2 - a(K) \geqq (R - r)^2.$$

Because of $p(K) \leqq \pi$ this can also be written as $(R - r)^2 \leqq (\pi/4) - a(K)$. Thus, if (6) holds and if one writes

$$D = D\left(p, \frac{R + r}{2}\right),$$

then $h(K, D) \leqq (R - r)/2$ and we obtain immediately the desired inequality (7).

If K is of constant width 1 one may take for R the circumradius and for r the inradius of K (since for convex domains of constant width the circumcircle and incircle are concentric). These radii are known to have the property that $R + r = 1$ and it follows that

$$D = D\left(p, \frac{R + r}{2}\right) = D\left(p, \frac{1}{2}\right).$$

PROOF OF THEOREM 2. Let C be a convex domain of constant width 1 such that $K \subset C$. Setting $h(K, C) = h$ we are going to prove that under the assumption $p(K) \geqq \pi - \epsilon$ we have $h \leqq (34/25)\epsilon^{2/3}$. There is an angle α such that the two corresponding support lines $L_K = L_K(\alpha)$ and $L_C = L_C(\alpha)$ have mutual distance h. Let L_K^+ and L_C^+ denote, respectively, the corresponding support half-spaces with the property that $K \subset K_K^+$, $C \subset L_C^+$ and therefore $L_K^+ \subset L_C^+$. If we set

$$M = C \cap L_K^+,$$

then $h(M, C) = h(K, C) = h$ and

$$p(M) \geqq p(K) \geqq \pi - \epsilon. \tag{24}$$

The set $C \backslash M$ is bounded by an arc $(L_C^+ \backslash L_K^+) \cap bdrC$, whose length will be denoted by s, and a line segment $C \cap L_K$, whose length will be denoted by t. Because of (24) we have

$$s - t = p(C) - p(M) = \pi - p(M) \leqq \epsilon. \tag{25}$$

If one inscribes in $C \backslash M$ a triangle with two vertices at the endpoints of $C \cap L_K$ and the third vertex at a point $q \in C \cap L_C$, then we see that

$$s \geqq \sqrt{h^2 + x^2} + \sqrt{h^2 + (t - x)^2},$$

where $0 \leqq x \leqq t$. Hence, $s \geqq \sqrt{4h^2 + t^2}$, and because of (25) we find $\sqrt{4h^2 + t^2} - t \leqq \epsilon$. This can also be written as

$$4h^2 \leqq \epsilon(\sqrt{4h^2 + t^2} + t) \tag{26}$$

If q^* is the point on $bdrC$ with $||q - q^*|| = 1$, then $C \subset D(q^*, 1)$ and consequently $C \cap L_K \subset D(q^*, 1)$. Since this fact implies

$$t \leqq 2\sqrt{1^2 - (1 - h)^2} \leqq \sqrt{8h}$$

it follows from (26) that $2h^2 \leqq \epsilon(\sqrt{(h^2 + 2h)} + \sqrt{2h})$ and therefore

$$h^{3/2} \leqq \frac{\epsilon}{2}(\sqrt{h + 2} + \sqrt{2}).$$

The desired result (9) is now an immediate consequence of this inequality and $h \leqq 1$.

Finally, to show that the exponent $3/2$ in (9) cannot be improved let P_m denote the regular m-gon (m odd) of diameter 1, and Q_m the convex domain obtained from P_m by replacing $m - 1$ sides of P_m by circular arcs of radius 1 with centers at the vertices opposite the respective sides. Then, there is only one convex domain of constant width 1 that contains Q_m, namely the Reuleaux polygon, say R_m, that is obtained from Q_m by replacing the remaining straight side by a circular arc. A simple calculation shows that

$$h(Q_m, R_m) = 1 - \cos(\pi/2m)$$

and

$$p(R_m) - p(Q_m) = (\pi/m) - 2\sin(\pi/2m).$$

Thus, setting $\epsilon = \pi - p(Q_m) = p(R_m) - p(Q_m)$ we find

$$\lim_{m\to\infty} \frac{\epsilon^2}{h(Q_m, R_m)^3} = 108.$$

It is therefore impossible that $h(Q_m, R_m) \leqq c\epsilon^{\beta}$ with some constant c and $\beta > 2/3$.

We conclude with some remarks concerning generalizations of our results to euclidean n-dimensional space E^n. In view of the fact that it is not even known which convex bodies in E^3 of constant width 1 have minimal volume it is presently impossible to prove an exact analogue of part (a) of Theorem 1. Since the proof of part (b) of Theorem 1 is essentially based on a stability version of the isoperimetric inequality and since such stability statements can be proved for E^n (cf. Osserman [5]) it is possible to derive analogues of part (b) of Theorem 1 that are valid for all n. However, it appears not to be possible to obtain a result as strong as (7) by restricting these general statements to the case $n = 2$. A possible way to generalize Theorem 2 is to estimate the Hausdorff distance between a convex body and a nearest convex body of constant width if the diameter of the body differs at most ϵ from its mean width. Professor P. Goodey has noted (private communication) that results of this kind can be obtained from an inequality of Vitale [6] concerning metrics on classes of convex bodies defined in terms of L_1 and L_∞ norms. Again, if these results are specialized to the two-dimensional case the resulting inequalities are weaker than Theorem 2.

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