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CONDITIONS FOR THE SUPERSOLVABILITY OF $\mathcal{F}_S(G)$

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Abstract In this article, $\mathcal{F}_S(G)$ denotes the fusion category of G on a Sylow p-subgroup S of G where p denotes a prime. A subgroup K of G has normal complement in G if there is a normal subgroup T of G satisfying that $G = KT$ and $T \cap K = 1$. We investigate the supersolvability of $\mathcal{F}_S(G)$ under the assumption that some subgroups of S are normal in G or have normal complement in G.

Keywords: fusion system; normal complement subgroup; supersolvable

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Secondary 20E45

1. Introduction

In recent years, the study of fusion category and fusion system theory has become a meaningful direction in finite group theory. Plenty of findings exist regarding nilpotent fusion systems, as well as several established conclusions concerning saturated fusion systems, as documented in [5, 7, 10] and [11]. Lluis Puig was the first to introduce the concept of a saturated fusion system, which has been continuously studied by a cohort of scholars. They obtained the following important results, which will provide a train of thought for the proof in this article. Given a finite group G and a p-subgroup $S \leqslant G$, $\mathcal{F}_S(G)$ is used to represent the fusion category of G on S : the objects of $\mathcal{F}_S(G)$ are all subgroups of S and morphisms in $\mathcal{F}_S(G)$ are the group homomorphisms between subgroups of S induced by conjugation in G. A fusion system over S is a category $\mathcal{F}$ whose objects are all subgroups of S and whose morphisms behave as though they are induced by conjugation inside a group including S as a p-subgroup. In [2], it was shown that for a p-group P and $\mathcal{F}$, $\mathcal{F}$ is solvable when there is a series of strongly $\mathcal{F}$-closed subgroups $1 = P_0 \leqslant P_1 \leqslant P_2 \leqslant \cdots \leqslant P_n = P$ and with P_{i+1}/P_i abelian for $0 \leqslant i \leqslant n$. A saturated fusion system $\mathcal{F}$ over a p-group P is nilpotent ($\mathcal{F} = \mathcal{F}_P(P)$) if and only if $Aut_{\mathcal{F}}(S)$ is a p-group for each subgroup S of P. In [10], Linckelmann and Kessar generalized the p-nilpotent theorem to fusion systems, demonstrating that $\mathcal{F} = \mathcal{F}_P(P)$ if and only if $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$. Shen and Zhang in [14] investigated the p-supersolvable fusion systems. They gave the p-supersolvability of normal subsystems and they proved that the

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models of p-supersolvable fusion systems are p-supersolvable groups. Shen also provided a criterion for a saturated fusion system $\mathcal{F}$ to be nilpotent in [13]. The article draws on information from reference [2] for a more complete description of fusion systems, and any unfamiliar terminology or symbols can be found in that book. Additionally, Gorenstein [9] provides explanations of certain definitions that may be unfamiliar to readers and only finite groups will be considered.

Enlightened by current advances in the research of normality of subgroups, in [12], Ru and Shen consider the influence of the normality of some subgroups of S on the supersolvability of $\mathcal{F}_S(G)$. Theorems 1.1 and 1.2 are cases in point. Following this line of thought, we contrive to combine the normality of subgroups and complement subgroups and we find a few conditions that make the supersolvability of $\mathcal{F}_S(G)$ hold.

Theorem 1.1. [**12**] *Let G be a finite group and $S \leqslant G$ be a Sylow p-subgroup where p is a prime. If all subgroups of S of order p or 4 $(p = 2)$ are normal in G. Then $\mathcal{F}_S(G)$ is supersolvable.*

Theorem 1.2. [**12**] *Let G be a finite group and $S \leqslant G$ is a Sylow p-subgroup where p is an odd prime. Suppose there is a subgroup $U < S$ with $1 < |U| < |S|$ and any subgroup of S with order $|U|$ is normal in G. Then $\mathcal{F}_S(G)$ is supersolvable.*

In this article, our target is to explore how do normal subgroups or normal complement subgroups affect the supersolvability of $\mathcal{F}_S(G)$. Our main theorems are as follows. Firstly, we start with the subgroups of order p case.

Theorem A. *Suppose G is finite and $S \leqslant G$ is a Sylow p-subgroup where p is an odd prime. If all subgroups of S with order p are normal in G or have normal complement in G. Then $\mathcal{F}_S(G)$ is supersolvable.*

Next we consider the maximal subgroup case.

Theorem B. *Let G be a finite group and $S \leqslant G$ be a Sylow p-subgroup where p is an odd prime. If all maximal subgroups of S are normal in G or have normal complement in G. Then $\mathcal{F}_S(G)$ is supersolvable.*

Theorems A and B indicate that $\mathcal{F}_S(G)$ is supersolvable. Nevertheless, G may not be p-supersolvable. See the following example.

Example. Set $G = A_5$ and $S \in Syl_5(G)$. We have S is isomorphic to C_5. Then all maximal subgroups of S are normal in G or have normal complement in G, but G is not 5-supersolvable.

In the end, we discuss a general case.

Theorem C. *Suppose G is a group and $S \in Syl_p(G)$ where p is an odd prime dividing $|G|$. Assume S has a subgroup U satisfying $1 < |U| < |S|$ and any subgroup of S with order $|U|$ or $p|U|$ is normal in G or has normal complement in G. Then $\mathcal{F}_S(G)$ is supersolvable.*

Remark. $p = 2$ is a trivial case. If $p = 2$, then G is p-nilpotent and so $\mathcal{F}_S(G)$ is supersolvable. So we suppose p is an odd prime in Theorems A, B and C.

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2. Preliminaries

In this section, we provide a plethora of lemmas that we will need later. $Z_{\mathcal{U}}(G)$ represents the greatest normal subgroup of G whose G-chief factors are cyclic. As a special case of complement subgroups, the next lemma is clear.

Lemma 2.1. [4] *Suppose G is finite and $N \trianglelefteq G$. Then $N/\Phi(N) \leqslant Z_{\mathcal{U}}(G/\Phi(N))$ if and only if $N \leqslant Z_{\mathcal{U}}(G)$.*

Lemma 2.2. *Suppose G is finite and $H \trianglelefteq G$.*
(1) *If M has normal complement in G and $M \leqslant N \leqslant G$, then M has normal complement in N.*
(2) *If M has normal complement in G and M contains H, then M/H has normal complement in G/H.*

Based on [1, Lemma 3.7], we obtain the following lemma.

Lemma 2.3. *Let S be a normal p-subgroup of G. Suppose there exists a subgroup U of S with $1 < |U| < |S|$. If all subgroups of S of order $|U|$ and $p|U|$ have normal complement in G, then $S \leqslant Z_{\mathcal{U}}(G)$.*

Lemma 2.4. *Let S be a normal p-subgroup of G where p is an odd prime. If all subgroups of S with order p are normal in G or have normal complement in G, then $S \leqslant Z_{\mathcal{U}}(G)$.*

Proof. We use the induction method on $|G|+|S|$ to solve this statement. If all minimal subgroups of S are normal in G, then $\Omega_1(S) \trianglelefteq G$ and all chief factors of G that lie below $\Omega_1(S)$ are cyclic with order p. This means $\Omega_1(S) \leqslant Z_{\mathcal{U}}(G)$. The lemma holds if $S = \Omega_1(S)$. So let $\Omega_1(S) < S$. Pick $a \in S$ such that $o(a) = p^2$ and g be any element in G. Then $\langle a^p \rangle \trianglelefteq G$ and $(a^g)^p = (a^p)^g = a^{pi} = (a^i)^p$ for some integer i. Thus we have $(a^g(a^i)^{-1})^p = 1$, which indicate $a^g(a^i)^{-1} \in \Omega_1(S)$. Then $a^g = a^i y$ where $y \in \Omega_1(S)$. Now all minimal subgroups of $S/\Omega_1(S)$ are normal in $G/\Omega_1(S)$ and so $S/\Omega_1(S) \leqslant Z_{\mathcal{U}}(G/\Omega_1(S))$ by induction on $|G| + |S|$. If follows from $\Omega_1(S) \leqslant Z_{\mathcal{U}}(G)$ that $S/\Omega_1(S) \leqslant Z_{\mathcal{U}}(G)/\Omega_1(S)$. Hence $S \leqslant Z_{\mathcal{U}}(G)$. Therefore, there exists a subgroup H of S with order p such that $H \ntrianglelefteq G$ and H has normal complement in G. Then there is a normal subgroup L of G such that $G = HL$ and $H \cap L = 1$. Note that $S = G \cap S = HL \cap S = H(L \cap S)$ and $L \cap S \trianglelefteq G$. So all minimal subgroups of $L \cap S$ are normal in G or have normal complement in G, which means $L \cap S \leqslant Z_{\mathcal{U}}(G)$ by our induction. Furthermore, since $S/L \cap S \trianglelefteq G/L \cap S$ and $|S/L \cap S| = p$, we have $S/L \cap S \leqslant Z_{\mathcal{U}}(G/L \cap S)$. If follows from $Z_{\mathcal{U}}(G/L \cap S) = Z_{\mathcal{U}}(G)/(L \cap S)$ that $S \leqslant Z_{\mathcal{U}}(G)$. □

Lemma 2.5. [3] *Suppose G is finite and $S \leqslant G$ is a Sylow p-subgroup. Assume for any proper subgroup $K < G$ satisfying $S \cap K \in Syl_p(K)$ and $O_p(G) < S \cap K$, $\mathcal{F}_{S \cap K}(K)$ is supersolvable. If $O_p(G) \leqslant Z_{\mathcal{U}}(G)$, $\mathcal{F}_S(G)$ is supersolvable.*

Lemma 2.6. [8] *Let R, S and T be subgroups of G, then the following statements are equivalent:*
(1) *$R \cap ST = (R \cap S)(R \cap T)$.*
(2) *$RS \cap RT = R(S \cap T)$.*

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Lemma 2.7. See [14] *Let $\mathcal{F}$ be a p-supersolvable fusion system on a p-group. Suppose $\mathcal{F}_1$ is a normal subsystem of $\mathcal{F}$. Then $\mathcal{F}_1$ is p-supersolvable.*

Now with these lemmas in the hand, we are able to offer the proof of our main theorems, showing the supersolvability of $\mathcal{F}_S(G)$.

3. Main results

3.1. Proof of Theorem A

Proof. If the theorem is false, we consider a counterexample $\mathcal{F} = \mathcal{F}_S(G)$ for which $|G|$ is smallest. If any subgroup of $O_p(G)$ with order p is normal in G or has normal complement in G, by Lemma 2.4, $O_p(G) \leqslant Z_{\mathcal{U}}(G)$. Next choose $L < G$ with $S \cap L \in Syl_p(L)$. Then the subgroups in $S \cap L$ with order p are normal G or have normal complement in G. Applying Lemma 2.2, these subgroups are normal in L or have normal complement in L. Then $\mathcal{F}_{S\cap L}(L)$ is supersolvable due to the minimality of $\mathcal{F}$. Now if $L < G$ with $S \cap L \in Syl_p(L)$ and $O_p(G) < S \cap L$, we have $\mathcal{F}_{S\cap L}(L)$ is supersolvable. Using Lemma 2.5, $\mathcal{F}_S(G)$ is supersolvable. □

3.2. Proof of Theorem B

Proof. Assume the theorem is wrong and let $\mathcal{F} = \mathcal{F}_S(G)$ be a counterexample for which $|G|$ is smallest.

Step 1. $O_{p'}(G) = 1$.

Set $Q = O_{p'}(G)$. If $Q \neq 1$, then $SQ/Q \in Syl_p(G/Q)$. Set M/Q as a maximal subgroup of SQ/Q. Then we have a maximal subgroup S_1 of S with $M = S_1Q$. If S_1 has a normal complement in G, by Lemma 2.2, $M/Q = S_1Q/Q$ has a normal complement in G/Q. If $S_1 \trianglelefteq G$, $S_1Q/Q \trianglelefteq G/Q$. In both cases, the minimality of $\mathcal{F}$ shows that $\mathcal{F}_{SQ/Q}(G/Q)$ is supersolvable and hence $\mathcal{F}_S(G)$ is supersolvable, a contradiction.

Step 2. Let N be a minimal normal subgroup, then $\mathcal{F}_{SN/N}(G/N)$ is supersolvable.

Choose M/N as a maximal subgroup of SN/N. So we can find a maximal subgroup S_1 of S with $M = NS_1$ and $S \cap N = S_1 \cap N \in Syl_p(N)$. If S_1 has normal complement in G, there is a normal subgroup T of G satisfying $G = S_1T$ and $T \cap S_1 = 1$. Note that $G/N = (S_1N/N)(TN/N)$. Since $(|N : S_1 \cap N|, |N : T \cap N|) = 1$, we have $(S_1 \cap N)(T \cap N) = N = G \cap N = S_1T \cap N$. Therefore $TN \cap NS_1 = (S_1 \cap T)N$ by Lemma 2.6. So $(S_1N/N) \cap (TN/N) = (NS_1 \cap NT)/N = (S_1 \cap T)N/N = 1$. Then M/N has normal complement in G/N. If $S_1 \trianglelefteq G$, then $M = NS_1$ is also normal in G, which means $M/N \trianglelefteq G/N$. Therefore, the assumption in the theorem is valid for $(G/N, S/N)$. The minimal choice of $\mathcal{F}$ shows $\mathcal{F}_{SN/N}(G/N)$ is supersolvable.

Step 3. The minimal normal subgroup N of G is unique.

If not, assume that there is a minimal normal subgroup N_1 with $N \neq N_1$. Then **Step 2** tells us that $\mathcal{F}_{SN/N}(G/N)$ and $\mathcal{F}_{SN_1/N_1}(G/N_1)$ are both supersolvable. Thus we conclude from the fact $N_1 \cap N_2 = 1$ and [15, Theorem 3.2] that $\mathcal{F}_S(G)$ is supersolvable, a contradiction.

Step 4. $N \not\leq \Phi(S)$.

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If all of the maximal subgroups of S are normal in G, then all of them are normal in $\mathcal{F}_S(G)$. Therefore, we obtain from Theorem 1.2 that $\mathcal{F}_S(G)$ is supersolvable, which is impossible. Therefore, there exists some maximal subgroup S_1 of S which is not normal in G. This implies that S_1 has a normal complement K of G. Obviously K must contain N by the uniqueness of N, so that $S_1 \cap N = 1$. Hence $N \nleqslant \Phi(S)$, as required.

Step 5. $1 < N \cap S < S$.

If $N \cap S = S$, then $S \leqslant N$, contradicting to the minimality of N. Since p divides $|N|$ by **Step 1**, $N \cap S \neq 1$.

Step 6. $O_p(\mathcal{F}) = 1$.

If not, then $N \leqslant O_p(\mathcal{F})$. In view of **Step 4**, we choose a maximal subgroup S_0 of S with $S = NS_0$. If $S_0 \trianglelefteq G$, then $N \cap S_0 \trianglelefteq G$. By the minimal choice of N, $N \cap S_0 = N$ or $N \cap S_0 = 1$. If $N \cap S_0 = N$, then $N \leqslant S_0$, a contradiction. Thus $N \cap S_0 = 1$, which indicates that $|N| = p$. If follows from **Step 2** that $\mathcal{F}_S(G)$ is supersolvable, a contradiction. If S_0 has normal complement in G. Then there is a normal subgroup K of G such that $G = S_0K$ and $S_0 \cap K = 1$. By the uniqueness of N, we have $N \leqslant K$ and $|K_p| = p$, which indicates $|N| = p$. Then by **Step 2**, we have $\mathcal{F}_S(G)$ is supersolvable, a contradiction.

Step 7. Final contradiction.

By **Step 5**, there exists a maximal subgroup S_1 of S such that $N \cap S \leqslant S_1$. By assumption, if $S_1 \trianglelefteq G$, then $S_1 \cap N \trianglelefteq G$. By the choice of N, $S_1 \cap N = N$ or $S_1 \cap N = 1$. If $S_1 \cap N = 1$, then $|N| = p$ and hence $\mathcal{F}_S(G)$ is supersolvable. If $S_1 \cap N = N$, $N \leqslant S_1$. So N is a normal p-subgroup of G and $N \leqslant O_p(\mathcal{F}) = 1$ by **Step 6**, a contradiction. If S_1 has normal complement in G, then there is a normal subgroup L of G such that $G = S_1L$ and $S_1 \cap L = 1$. Notice that $N \cap L \trianglelefteq G$ and hence $N \cap L = 1$ or $N \cap L = N$. If $N \cap L = N$, then $N \leqslant L$. So $N \cap S \leqslant L \cap S_1 = 1$. This indicates N is a $p^{'}$-group, contrary to **Step 1**. If $N \cap L = 1$, then $N \leqslant O_p(\mathcal{F})$. This is contrary to **Step 6**, asserting $O_p(\mathcal{F}) = 1$. The contradiction ends the proof. □

3.3. Proof of Theorem C

Proof. Assume the result is false and set $\mathcal{F} = \mathcal{F}_S(G)$ be a minimal counterexample for which $|G|$ is the smallest.

Step 1. $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $SO_{p'}(G)/O_{p'}(G) \in Syl_p(G/O_{p'}(G))$. So $SO_{p'}(G)/O_{p'}(G)$ and $G/O_{p'}(G)$ satisfy the assumption in the theorem. By the minimal choice of $\mathcal{F}$, $\mathcal{F}_{SO_{p'}(G)/O_{p'}(G)}(G/O_{p'}(G))$ is supersolvable. Therefore we conclude from [6, Theorem 5.20] that $\mathcal{F}_{SO_{p'}(G)/O_{p'}(G)}(G/O_{p'}(G))$ is isomorphic to $\mathcal{F}_S(G)/(O_{p'}(G) \cap S) = \mathcal{F}_S(G)$ by the fact that $O_{p'}(G) \cap S = 1$, which implies that $\mathcal{F}_S(G)$ is supersolvable, a contradiction.

Step 2. $p < |U|$.

If $|U| \leqslant p$, then all subgroups of S with order p are normal in G or have normal complement in G. So we have that $\mathcal{F}_S(G)$ is supersolvable in view of Theorem A, which is a contradiction. So $p < |U|$.

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Step 3. $p|U| < |S|$.

By assumption in the theorem, $|S| \geq p|U|$. If $|S| = p|U|$, then all maximal subgroups of S are normal in G or have normal complement in G. Then by Theorem B, $\mathcal{F}_S(G)$ is supersolvable, a contradiction.

Step 4. Let $N \leqslant S$ be a minimal normal subgroup of G. Then $N < S$.

Since $O_{p'}(\mathcal{F}) = 1$ and $S \cap O^p(G) \in Syl_p(O^p(G))$, we have $S \cap O^p(G) \neq 1$. If any subgroup of S with order $|U|$ is normal in G, then $\mathcal{F}_S(G)$ is supersolvable by Theorem 1.2, a contradiction. So there exists a subgroup R of S with order $|U|$ satisfying $R \ntrianglelefteq G$. Then R has normal complement in G. There is a normal subgroup K of G such that $G = RK$ and G/K is a p-group. By the properties of p-groups, there exists a maximal subgroup M of G such that $K \leqslant M \trianglelefteq G$ and $|G/M| = p$. Since $|U| < |M|$ and $O^p(G) \leqslant M$, $\mathcal{F}_{S \cap O^p(G)}(O^p(G))$ is supersolvable. We obtain $S \cap O^p(G) \trianglelefteq G$ by $S \cap O^p(G)$ *char* $O^p(G) \trianglelefteq G$.

If $N = S$, then **Step 3** shows $|S| = |N| > p|U|$. Choose $T < N$ with order $|U|$. By assumption, if $T \trianglelefteq G$, then $T = 1$ by the minimality of N, a contradiction. Thus T has normal complement in G. So there exists a normal subgroup K of G satisfying $G = TK$ and $T \cap K = 1$. Note that $G = NK$ and $N \cap K$ is also normal in G. If $N \cap K = N$, then $N \leqslant K$. This shows that $G = NK = K$, a contradiction. If $N \cap K = 1$, then $T = N$, a contradiction. So **Step 4** holds.

Step 5. $|U| > |N|$ for an arbitrary minimal normal subgroup N of G contained in $S \cap O^p(G)$.

Suppose $p < |U| < |N|$. Since $N < S$, then N and G meet the assumption in Lemma 2.3, so we have $N \leqslant Z_{\mathcal{U}}(G)$. Therefore $|N| = p \geq p|U|$, which indicates that $|U| = 1$. This contradicts **Step 2**.

Suppose $|U| = |N|$. So the assumption in Theorem A is valid for $(G/N, S/N)$ by Lemma 2.2. Then $\mathcal{F}_{S/N}(G/N)$ is supersolvable. Pick R/N as a minimal normal subgroup of S/N and then $|R/N| = p$. We denote $R = N\langle a \rangle$, where $a \notin N$ and $a^p \in N$. If $N = \Phi(R)$, $R = \langle a \rangle$ is cyclic and so N is cyclic. This means $|N| = p$ and so $\mathcal{F}_S(G)$ is supersolvable, a contradiction. Then $N > \Phi(R)$. Since $\Phi(R)$ *char* $R \trianglelefteq S$, $\Phi(R) \trianglelefteq S$. We choose N_1 as a maximal subgroup of N such that $\Phi(R) \leqslant N$ and $N_1 \trianglelefteq S$. Write $H = N_1\langle a \rangle$. It follows from $a^p \in \Phi(R) \leqslant N_1$ that $|N| = |H| = |U|$. If $H \trianglelefteq G$, by the minimality of N, $H = 1$, a contradiction. Therefore, H has normal complement in G. There is a normal subgroup T of G such that $G = HT$ and $H \cap T = 1$. Since $N \leqslant O^p(G) \leqslant T$, $N_1 = H \cap N \leqslant H \cap O^p(G) \leqslant H \cap T = 1$. This means N is of order p and so $\mathcal{F}_S(G)$ is supersolvable, a contradiction.

Step 6. Complete the proof.

Since $|U| > |N|$, by Lemma 2.2, S/N and G/N satisfy the assumption in this theorem and so $\mathcal{F}_{S/N}(G/N)$ is supersolvable, by the minimal choice of $\mathcal{F}$. Note that $N \nleqslant \Phi(G)$. We choose a maximal subgroup V of G such that $G = NV$. In addition, $S = N(S \cap V)$ and $S \cap V \neq 1$. We pick S_1 as a maximal subgroup of S containing $S \cap V$. So $S = NS_1$ and $N \cap S_1 < N$. If $N \cap S_1 = 1$, then $|N|$ is a prime and hence $\mathcal{F}_S(G)$ is supersolvable, a contradiction. Then $N \cap S_1 \neq 1$. We select a subgroup E of S_1 containing $N \cap S_1$ satisfying $|E| = |U|$ and $E \trianglelefteq S$. Then $N \cap E = N \cap S_1 \neq 1$. By assumption, if E is normal in G, then $N \cap E \trianglelefteq G$. Since $N \cap E = N \cap S_1 < N$, by the minimality of N, $N \cap E = 1$, a contradiction. If E has normal complement in G, then there is a normal

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subgroup F of G such that $G = EF$ and $E \cap F = 1$. Now $N \cap E \leqslant O^p(G) \cap E \leqslant F \cap E = 1$ and thus $N \cap E = 1$. This is a contradiction. □

Data availability statement. The authors declare that data supporting the findings of this study are available within the article.

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