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Optimal global asymptotic behaviour of the solution to a class of singular Dirichlet problems

Zhijun Zhang
School of Mathematics and Information Sciences, Yantai University, Yantai 264005, Shandong, People's Republic of China
(zhangzj@ytu.edu.cn)

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This paper is mainly concerned with the global asymptotic behaviour of the unique solution to a class of singular Dirichlet problems $-\Delta u = b(x)g(u)$, $u > 0$, $x \in \Omega$, $u|_{\partial\Omega} = 0$, where Ω is a bounded smooth domain in $\mathbb{R}^n$, $g \in C^1(0,\infty)$ is positive and decreasing in $(0,\infty)$ with $\lim_{s\to 0^+} g(s) = \infty$, $b \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$, which is positive in Ω, but may vanish or blow up on the boundary properly. Moreover, we reveal the asymptotic behaviour of such a solution when the parameters on b tend to the corresponding critical values.

Keywords: Semi-linear elliptic equations; a singular nonlinearity; boundary value problems; the unique solution; global asymptotic behaviour

2010 *Mathematics subject classification:* 35J75, 35B40

1. Introduction

This paper is concerned with the global asymptotic behaviour of the unique classical solution to the following singular Dirichlet problem:

$$-\Delta u = b(x)g(u), \quad u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0, \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, Δ is the usual Laplacian operator, b satisfies

($\mathbf{b_1}$) $b \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$ is positive in Ω and g satisfies

($\mathbf{g_1}$) $g \in C^1(0,\infty)$ is positive and decreasing in $(0,\infty)$ with $\lim_{s\to 0^+} g(s) = \infty$.

For convenience, we denote by ψ the solution to the following problem:

$$\int_0^{\psi(t)} \frac{\mathrm{d}\tau}{g(\tau)} = t, \quad \forall\, t > 0. \tag{1.2}$$

We note from ($\mathbf{g_1}$) that

$$\begin{cases} \psi(t) \to 0 & \text{if and only if} \quad t \to 0, \\ \psi(t) \to \infty & \text{if and only if} \quad t \to \infty. \end{cases} \tag{1.3}$$

The basic model of problem (1.1) is

$$-\Delta u = b(x)u^{-\gamma}, \quad u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0, \tag{1.4}$$

where $\gamma > 0$.

Problem (1.4) arises naturally in the study of the following nonlinear heat equation (see [**19**]):

$$\begin{cases} b(x)u_t = u^p \Delta u, & (x,t) \in \Omega \times (0,\infty); \\ u(x,t) = u_0(x), & (x,t) \in \Omega \times \{0\} \cup \partial\Omega \times (0,\infty), \end{cases} \tag{1.5}$$

where $p > 1$, $u_0(x) \geqslant 0,\ x \in \Omega$ and $u_0|_{\partial\Omega} = 0$.

If one considers a solution u of problem (1.5) of the form $u(x,t) = T(t)v(x)$, then

$$-T'(t) = c_0 T^{1+p}(t), \tag{1.6}$$

$$-\triangle v = c_0 b(x) v^{1-p}, \tag{1.7}$$

where $c_0 > 0$, (1.6) has a unique global positive solution in $(0,\infty)$ (for given initial data) and tends to zero as $t \to \infty$. We note that (1.7) is exactly the equation in problem (1.4). Using the maximum principle and known results for problem (1.4), it is rather easy to establish the global existence and regularity of solutions to problem (1.5).

Problem (1.1) has been discussed by many authors and in many contexts; see, for instance [**1**–**3**, **5**–**7**, **9**–**25**, **27**–**38**] and the references therein.

The following are some basic results.

For $b \in C^\alpha(\bar\Omega)$ with $b(x) > 0,\ x \in \bar\Omega$, when g satisfies $(\mathbf{g_1})$, Fulks and Maybee [**13**], Stuart [**28**], Crandall *et al.* [**10**] derived that problem (1.1) has a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar\Omega)$. Moreover, they established the following result (theorems 2.2 and 2.5 in [**10**]): if $\varphi_1 \in C[0,\delta_0] \cap C^2(0,\delta_0]$ ($\delta_0 > 0$) is the local solution to the problem

$$-\varphi_1''(t) = g(\varphi_1(t)), \quad \varphi_1(t) > 0,\ 0 < t < \delta_0,\ \varphi_1(0) = 0, \tag{1.8}$$

then there exist positive constants c_1 and c_2 such that

$$c_1\varphi_1(d(x)) \leqslant u(x) \leqslant c_2\varphi_1(d(x)) \quad \text{near } \partial\Omega, \tag{1.9}$$

where $d(x) = \operatorname{dist}(x,\partial\Omega)$, $x \in \Omega$.

In particular, when $g(u) = u^{-\gamma}$, $\gamma > 1$, u satisfies

$$c_1(d(x))^{2/(1+\gamma)} \leqslant u(x) \leqslant c_2(d(x))^{2/(1+\gamma)} \quad \text{near } \partial\Omega. \tag{1.10}$$

In [**21**], by constructing a pair of global subsolution and supersolution, Lazer and McKenna proved that (1.10) still holds on $\bar\Omega$ and u has the following properties:

($\mathbf{i_1}$) if $\gamma > 1$, then u is not in $C^1(\bar\Omega)$;

($\mathbf{i_2}$) u is not in $H_0^1(\Omega)$ if and only if $\gamma \geqslant 3$.

It is worth noting that the classical solution of problem (1.4) is not a weak solution in the case of $\gamma \geqslant 3$.

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Berhanu *et al.* [3] further proved that there exists $c_0 > 0$ such that

$$\left| \frac{u(x)}{(d(x))^{2/(1+\gamma)}} - \left(\frac{(1+\gamma)^2}{2(\gamma-1)} \right)^{1/(1+\gamma)} \right| < c_0 (d(x))^{(\gamma-1)/(1+\gamma)}, \quad \forall x \in \Omega.$$

When the function $g : (0, \infty) \to (0, \infty)$ is locally Lipschitz continuous and decreasing in $(0, \infty)$, Giarrusso and Porru [15] showed that if g satisfies the following conditions:

($\mathbf{g_{01}}$) $\int_0^1 g(s)\mathrm{d}s = \infty, \quad \int_1^\infty g(s)\mathrm{d}s < \infty$;
($\mathbf{g_{02}}$) there exist positive constants δ and M with $M > 1$ such that

$$G_1(s) < MG_1(2s), \ \forall s \in (0, \delta), \quad G_1(s) := \int_s^\infty g(\tau)\mathrm{d}\tau, \ s > 0,$$

then the unique solution u to problem (1.1) has the property

$$|u(x) - \varphi_2(d(x))| < C_0\, d(x), \quad \forall x \in \Omega,$$

where C_0 is a suitable positive constant and $\varphi_2 \in C[0, \infty) \cap C^2(0, \infty)$ is the unique solution of

$$\int_0^{\varphi_2(t)} \frac{\mathrm{d}s}{\sqrt{2G_1(s)}} = t, \quad \forall t > 0. \tag{1.11}$$

When $b \in L^\infty(\Omega)$, $b \geqslant 0$ almost everywhere and $b > 0$ on some sets of Ω of positive measure, del Pino [24] shows that

($\mathbf{i_1}$) for all $\gamma > 0$ there exists a unique solution $u \in C^{1,\alpha}(\Omega) \cap C(\bar{\Omega})$ to problem (1.4);
($\mathbf{i_2}$) assume that b satisfies

$$b(x) \leqslant \Theta(d(x)) \quad \text{for almost every } x \in \Omega,$$

for some bounded function $\Theta : [0, \infty) \to \mathbb{R}$ such that

$$\int_0^1 \left(\frac{|\Theta(s)|}{s^\gamma} \right)^p \mathrm{d}s < \infty \quad \text{for some } p > 1,$$

then

$$c_1 d(x) \leqslant u(x) \leqslant c_2\, d(x) \quad \text{in } \Omega, \tag{1.12}$$

and $|\nabla u(x)| \leqslant C_0$ in Ω, where c_1, c_2 and C_0 are positive constants.

For convenience, we denote

($\mathbf{b_2}$) there exist $\sigma \in \mathbb{R}$ and positive constants b_i $(i = 1, 2)$ such that

$$b_1(d(x))^{-\sigma} \leqslant b(x) \leqslant b_2(d(x))^{-\sigma}, \quad x \in \Omega.$$

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When b is a nonnegative bounded measurable function, which satisfies ($\mathbf{b_2}$) with $\sigma \leqslant 0$, Gui and Lin [**19**] (theorem 2.1) further established the following results to problem (1.4).

($\mathbf{i_1}$) If $-\sigma - \gamma > -1$, then (1.12) holds.

($\mathbf{i_2}$) If $-\sigma - \gamma = -1$, then there exist positive constants c_1, c_2 and c_3 such that

$$c_1 d(x)(c_3 - \ln(d(x)))^{1/(1+\gamma)} \leqslant u(x) \leqslant c_2 d(x)(c_3 - \ln(d(x)))^{1/(1+\gamma)}, \quad x \in \Omega.$$

($\mathbf{i_3}$) If $-\sigma - \gamma < -1$, then there exist positive constants c_1, c_2 such that

$$c_1(d(x))^{(2-\sigma)/(1+\gamma)} \leqslant u(x) \leqslant c_2(d(x))^{(2-\sigma)/(1+\gamma)}, \quad x \in \Omega.$$

For convenience, let λ_1 be the first eigenvalue and $\phi_1 \in C^1(\bar{\Omega}) \cap C^{2+\alpha}(\Omega)$ be the corresponding eigenfunction of the problem

$$-\Delta\phi = \lambda\phi, \ x \in \Omega, \ \phi|_{\partial\Omega} = 0. \tag{1.13}$$

It follows from the Höpfs maximum principle that $\nabla\phi_1(x) \neq 0, \ \forall x \in \partial\Omega$, and there exist $\delta_0 > 0$ and positive constants c_i $(i = 1, 2)$ such that

$$|\nabla\phi_1(x)| > 0, \ \forall x \in \bar{\Omega}_{\delta_0}; \quad c_1 d(x) \leqslant \phi_1(x) \leqslant c_2 d(x), \ \forall x \in \Omega, \tag{1.14}$$

where $\nabla\phi_1(x)$ is the gradient of $\phi_1(x)$ and $\Omega_{\delta_0} = \{x \in \Omega : d(x) < \delta_0\}$.

Without losing generality, let

$$\max_{x \in \bar{\Omega}} \phi_1(x) < \exp(-\mu), \tag{1.15}$$

where $\mu > 1$ is given as in the following ($\mathbf{b_5}$).

In this paper, we show the optimal global asymptotic behaviour of the unique solution to problem (1.1) for more general g under the following local structure conditions:

($\mathbf{g_2}$) there exists $C_g \geqslant 0$ such that

$$\lim_{s \to 0^+} g'(s) \int_0^s \frac{\mathrm{d}\tau}{g(\tau)} = -C_g;$$

($\mathbf{g_3}$) there exists $E_g \geqslant 0$ such that

$$\lim_{s \to \infty} g'(s) \int_0^s \frac{\mathrm{d}\tau}{g(\tau)} = -E_g.$$

We also give some results with regard to the nonexistence of classical solutions to problem (1.1). Moreover, we reveal the asymptotic behaviour of such a solution when the parameters on b tend to the corresponding critical values.

Some basic examples of g in ($\mathbf{g_2}$) and ($\mathbf{g_3}$) are

($\mathbf{i_1}$) When $g(s) = s^{-\gamma}$ with $\gamma > 0$ and $s > 0$, $C_g = E_g = \gamma/(1+\gamma)$. Moreover,

$$\psi(t) = ((1+\gamma)t)^{1/(1+\gamma)} \quad \text{and} \quad -tg'(\psi(t)) \equiv \frac{\gamma}{1+\gamma}, \quad \forall t > 0. \tag{1.16}$$

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($\mathbf{i_2}$) When $g(s) = s^{-\gamma_1}$ with $\gamma_1 > 0$ and $s \in (0,1)$, $C_g = \gamma_1/(1+\gamma_1)$. Meanwhile, when s is large, $g(s) = s^{-\gamma_2}$ with $\gamma_2 > 0$, $E_g = \gamma_2/(1+\gamma_2)$. Moreover, $\psi(t) = ((1+\gamma_1)t)^{1/(1+\gamma_1)}$ for sufficiently small $t > 0$, and $\psi(t) \cong ((1+\gamma_2)t)^{1/(1+\gamma_2)}$ as $t \to \infty$.

($\mathbf{i_3}$) When $g(s) = (-\ln s)^{\gamma_1}$ with $\gamma_1 > 0$ and $s \in (0,1/3)$, $C_g = 0$. Meanwhile, when s is large, $g(s) = (\ln s)^{-\gamma_2}$ with $\gamma_2 > 0$, $E_g = 0$.

($\mathbf{i_4}$) When $g(s) = \exp(s^{-\gamma_1})$ with $\gamma_1 > 0$ and $s \in (0,1)$, $C_g = 1$. Meanwhile, when s is large, $g(s) = \exp(-s^{\gamma_2})$ with $\gamma_2 > 0$, $E_g = 1$.

($\mathbf{i_5}$) When $g(s) = s^{-\gamma_1}$ with $\gamma_1 > 0$ and $s \in (0,1)$, $C_g = \gamma_1/(1+\gamma_1)$. Meanwhile, when s is large, $g(s) = (\ln s)^{-\gamma_2}$ with $\gamma_2 > 0$, $E_g = 0$.

($\mathbf{i_6}$) When $g(s) = s^{-\gamma_1}$ with $\gamma_1 > 0$ and $s \in (0,1)$, $C_g = \gamma_1/(1+\gamma_1)$. Meanwhile, when s is large, $g(s) = \exp(-s^{\gamma_2})$ with $\gamma_2 > 0$, $E_g = 1$.

($\mathbf{i_7}$) When $g(s) = (-\ln s)^{\gamma_1}$ with $\gamma_1 > 0$ and $s \in (0,1/3)$, $C_g = 0$. Meanwhile, when s is large, $g(s) = s^{-\gamma_2}$ with $\gamma_2 > 0$, $E_g = \gamma_2/(1+\gamma_2)$.

($\mathbf{i_8}$) When $g(s) = (-\ln s)^{\gamma_1}$ with $\gamma_1 > 0$ and $s \in (0,1/3)$, $C_g = 0$. Meanwhile, when s is large, $g(s) = \exp(-s^{\gamma_2})$ with $\gamma_2 > 0$, $E_g = 1$.

($\mathbf{i_9}$) When $g(s) = \exp(s^{-\gamma_1})$ with $\gamma_1 > 0$ and $s \in (0,1)$, $C_g = 1$. Meanwhile, when s is large, $g(s) = (\ln s)^{-\gamma_2}$ with $\gamma_2 > 0$, $E_g = 0$.

($\mathbf{i_{10}}$) When $g(s) = \exp(s^{-\gamma_1})$ with $\gamma_1 > 0$ and $s \in (0,1)$, $C_g = 1$. Meanwhile, when s is large, $g(s) = s^{-\gamma_2}$ with $\gamma_2 > 0$, $E_g = \gamma_2/(1+\gamma_2)$.

We notice that ($\mathbf{i_2}$)–($\mathbf{i_{10}}$) are new.

A complete characterization of g in ($\mathbf{g_2}$) and ($\mathbf{g_3}$) is provided in lemmas 2.7 and 2.8.

Our main results are summarized as follows.

THEOREM 1.1. *Let g satisfy ($\mathbf{g_1}$)–($\mathbf{g_3}$) and b satisfy ($\mathbf{b_1}$).*

($\mathbf{i_1}$) *If b satisfies the additional condition that*

($\mathbf{b_3}$) *there exist $\sigma \geqslant 2$ and positive constant b_1 such that*

$$b(x) \geqslant b_1(\phi_1(x))^{-\sigma}, \quad x \in \Omega,$$

then problem (1.1) has no classical solutions.

($\mathbf{i_2}$) *Let b satisfy ($\mathbf{b_2}$). If*

$$\sigma = 1 \quad \text{and} \quad C_g > 0 \quad \text{or} \quad \sigma \in (1,2), \tag{1.17}$$

then problem (1.1) has a unique classical solution u_σ satisfying

$$\psi(\xi_1(2-\sigma)^{-1}\phi_1^{2-\sigma}(x)) \leqslant u_\sigma(x) \leqslant \psi(\xi_2(2-\sigma)^{-1}\phi_1^{2-\sigma}(x)), \tag{1.18}$$

where ψ is given as in (1.2), ξ_1 and ξ_2 are positive constants with $\xi_1 \leqslant \xi_2$. Moreover, we have $\lim_{\sigma\to 2^-}\min_{x\in\Omega_1} u_\sigma(x) = \infty$ and

$$\left(\frac{b_1}{C_0}\right)^{1-E_g} \leqslant \liminf_{\sigma\to 2^-} \frac{u_\sigma(x)}{\psi((2-\sigma)^{-1})} \leqslant \limsup_{\sigma\to 2^-} \frac{u_\sigma(x)}{\psi((2-\sigma)^{-1})} \leqslant \left(\frac{b_2}{c_0}\right)^{1-E_g}, \tag{1.19}$$

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uniformly for $x \in \Omega_1$*, which is an arbitrary compact subset of* Ω*, where*

$$C_0 = \max_{x\in\bar{\Omega}} \left(\lambda_1 \phi_1^2(x) + |\nabla \phi_1(x)|^2\right) \tag{1.20}$$

and

$$c_0 = \min_{x\in\bar{\Omega}} \left(\lambda_1 \phi_1^2(x) + |\nabla \phi_1(x)|^2\right). \tag{1.21}$$

In particular, when $E_g = 1$

$$\lim_{\sigma\to 2^-} \frac{u_\sigma(x)}{\psi((2-\sigma)^{-1})} = 1, \quad \forall x \in \Omega.$$

($\mathbf{i_3}$) *If* $2\sigma + 2C_g(2-\sigma) < 3$*, then* $u_\sigma \in H_0^1(\Omega)$*; and* u_σ *does not in* $H_0^1(\Omega)$ *provided* $2\sigma + 2C_g(2-\sigma) > 3$.

Remark 1.2. The existence and uniqueness of solutions to problem (1.1) follows from theorem 4.1 in [**33**].

From (1.16), we show that (3.6) (in the following proof of theorem 1.1) holds for an arbitrary $\xi > 0$, $\sigma \in (1-\gamma, 2)$ and $\beta = 2-\sigma$, i.e.

$$\left(1-\beta+\beta\Psi(\xi\beta^{-1}\phi_1^\beta(x))\right)|\nabla\phi_1(x)|^2 = \frac{\gamma-1+\sigma}{1+\gamma}|\nabla\phi_1(x)|^2 > 0, \quad x \in \Omega_{\delta_0}.$$

Thus, we obtain the following results directly.

Corollary 1.3. *When* $g(s) = s^{-\gamma}$, $s > 0$ *with* $\gamma > 0$ *in theorem* 1.1*, we have*

($\mathbf{i_1}$) *If* b *satisfies* ($\mathbf{b_3}$)*, then problem* (1.4) *has no classical solutions.*
($\mathbf{i_2}$) *If* b *satisfies* ($\mathbf{b_2}$) *with* $\sigma \in (1-\gamma, 2)$*, then problem* (1.4) *has a unique classical solution* u_σ *satisfying*

$$m_\sigma(\phi_1(x))^\theta \leqslant u_\sigma(x) \leqslant M_\sigma(\phi_1(x))^\theta, \quad x \in \Omega \tag{1.22}$$

and

$$u_\sigma \in C^\theta(\bar{\Omega}), \tag{1.23}$$

where $\theta = (2-\sigma)/(1+\gamma)$, $\theta C_\sigma m_\sigma^{1+\gamma} = b_1$, $\theta c_\sigma M_\sigma^{1+\gamma} = b_2$*, with*

$$C_\sigma = \max_{x\in\bar{\Omega}} \left(\lambda_1\phi_1^2(x) + (1-\theta)|\nabla\phi_1(x)|^2\right)$$

and

$$c_\sigma = \min_{x\in\bar{\Omega}} \left(\lambda_1\phi_1^2(x) + (1-\theta)|\nabla\phi_1(x)|^2\right).$$

Moreover, there hold

$$\lim_{\sigma\to 2^-} \min_{x\in\Omega_1} u_\sigma(x) = \infty$$

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and

$$\begin{aligned}\left(\frac{b_1(1+\gamma)}{C_0}\right)^{1/(1+\gamma)} &\leqslant \liminf_{\sigma\to 2^-}\left((2-\sigma)^{1/(1+\gamma)}u_\sigma(x)\right)\\ &\leqslant \limsup_{\sigma\to 2^-}\left((2-\sigma)^{1/(1+\gamma)}u_\sigma(x)\right)\\ &\leqslant \left(\frac{b_2(1+\gamma)}{c_0}\right)^{1/(1+\gamma)},\end{aligned}$$

uniformly for $x \in \Omega_1$*, which is an arbitrary compact subset of* Ω*.*

In particular, when $n=1$*,* $\Omega=(-R,R)$ *for some* $R>0$*, if the condition* $(\mathbf{b_2})$ *is replaced by*

$(\mathbf{b_2'})$ $b(x)=b(r)=b_0(R^2-x^2)^{-\sigma}$ $(b_0>0)$ *and* $2\sigma+\gamma=3$*,*

then

$$u_\sigma(x)=\left(\frac{b_0}{R^2}\right)^{1/(1+\gamma)}(R^2-x^2)^{1/2}$$

is the unique solution to problem (1.4)*.*
$(\mathbf{i_3})$ $u_\sigma \in H_0^1(\Omega)$ *if and only if* $\gamma+2\sigma<3$*.*

When b is in a borderline case near the boundary $\partial\Omega$, we have the following result.

THEOREM 1.4. *Let* g *satisfy* $(\mathbf{g_1})$–$(\mathbf{g_3})$ *and* b *satisfy* $(\mathbf{b_1})$*.*

$(\mathbf{i_1})$ *If* b *satisfies the additional condition that*
$(\mathbf{b_4})$ *there exist* $\mu\leqslant 1$ *and positive constants* b_1 *such that for* $x\in\Omega$

$$b(x)\geqslant b_1(\phi_1(x))^{-2}(-\ln(\phi_1(x)))^{-\mu},\quad x\in\Omega,$$

then problem (1.1) *has no classical solutions.*
$(\mathbf{i_2})$ *If* b *satisfies the additional condition that*
$(\mathbf{b_5})$ *there exist* $\mu>1$ *and positive constants* b_i $(i=1,2)$ *such that for* $x\in\Omega$

$$b_1(\phi_1(x))^{-2}(-\ln(\phi_1(x)))^{-\mu}\leqslant b(x)\leqslant b_2(\phi_1(x))^{-2}(-\ln(\phi_1(x)))^{-\mu},$$

then problem (1.1) *has a unique classical solution* u_μ *satisfying*

$$\psi(\xi_3(\mu-1)^{-1}(-\ln(\phi_1(x)))^{1-\mu})\leqslant u_\mu(x)\leqslant \psi(\xi_4(\mu-1)^{-1}(-\ln(\phi_1(x)))^{1-\mu}),\tag{1.24}$$

where ξ_3 *and* ξ_4 *are positive constants with* $\xi_3\leqslant\xi_4$*.*

Moreover, we have $\lim_{\mu\to 1^+}\min_{x\in\Omega_1}u_\mu(x)=\infty$ *and*

$$\left(\frac{b_1}{C_1}\right)^{1-E_g}\leqslant \liminf_{\mu\to 1^+}\frac{u_\mu(x)}{\psi((\mu-1)^{-1})}\leqslant \limsup_{\mu\to 1^+}\frac{u_\mu(x)}{\psi((\mu-1)^{-1})}\leqslant\left(\frac{b_2}{c_1}\right)^{1-E_g},\tag{1.25}$$

uniformly for $x\in\Omega_1$*, which is an arbitrary compact subset of* Ω*. Where* C_1 *and* c_1 *have given as in the following Corollary 1.5.*

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In particular, when $E_g = 1$

$$\lim_{\mu\to 1^+} \frac{u_\mu(x)}{\psi((\mu-1)^{-1})} = 1, \quad \forall x \in \Omega.$$

From (1.15) and (1.16), we show that (3.11) (in the following proof of theorem 1.4) holds for an arbitrary $\xi > 0$ and $\mu > 1$, i.e.

$$\begin{aligned}
&(1 - \mu(-\ln(\phi_1(x)))^{-1} + (\mu-1)(-\ln(\phi_1(x)))^{-1} \\
&\times \Psi(\xi_4(\mu-1)^{-1}(-\ln(\phi_1(x)))^{1-\mu}))|\nabla\phi_1(x)|^2 \\
&\quad = \left(1 - \frac{\mu+\gamma}{1+\gamma}(-\ln(\phi_1(x)))^{-1}\right)|\nabla\phi_1(x)|^2 > 0, \quad \text{near } \partial\Omega.
\end{aligned}$$

Thus, we obtain the following results directly.

Corollary 1.5. *When* $g(s) = s^{-\gamma}$ *with* $\gamma > 0$ *in theorem 1.4, we have*

- $(\mathbf{i_1})$ *If* b *satisfies* $(\mathbf{b_4})$*, then problem (1.4) has no classical solutions.*
- $(\mathbf{i_2})$ *If* b *satisfies* $(\mathbf{b_5})$*, then problem (1.4) has a unique classical solution* u_μ *satisfying for* $x \in \Omega$

$$m_\mu(-\ln(\phi_1(x)))^{-(\mu-1)/(1+\gamma)} \leqslant u_\mu(x) \leqslant M_\mu(-\ln(\phi_1(x)))^{-(\mu-1)/(1+\gamma)}, \tag{1.26}$$

where

$$\begin{aligned}
M_\mu &= \left(\frac{b_2}{c_\mu}\right)^{1/(1+\gamma)} \left(\frac{1+\gamma}{\mu-1}\right)^{1/(1+\gamma))}, \\
m_\mu &= \left(\frac{b_1}{C_\mu}\right)^{1/(1+\gamma)} \left(\frac{1+\gamma}{\mu-1}\right)^{1/(1+\gamma))}, \\
C_\mu &= \max_{x\in\bar\Omega}\left(\lambda_1\phi_1^2(x) + \left(1 - \frac{\mu+\gamma}{1+\gamma}(-\ln(\phi_1(x)))^{-1}\right)|\nabla\phi_1(x)|^2\right), \\
c_\mu &= \min_{x\in\bar\Omega}\left(\lambda_1\phi_1^2(x) + \left(1 - \frac{\mu+\gamma}{1+\gamma}(-\ln(\phi_1(x)))^{-1}\right)|\nabla\phi_1(x)|^2\right).
\end{aligned} \tag{1.27}$$

Moreover, $\lim_{\mu\to 1^+}\min_{x\in\Omega_1} u_\mu(x) = \infty$*, and*

$$\begin{aligned}
(1+\gamma)^{1/(1+\gamma)}\left(\frac{b_1}{C_1}\right)^{1/(1+\gamma)} &\leqslant \liminf_{\mu\to 1^+}((\mu-1)^{1/(1+\gamma)}u_\mu(x)) \\
&\leqslant \limsup_{\mu\to 1^+}((\mu-1)^{1/(1+\gamma)}u_\mu(x)) \leqslant (1+\gamma)^{1/(1+\gamma)}\left(\frac{b_2}{c_1}\right)^{1/(1+\gamma)},
\end{aligned}$$

uniformly for $x \in \Omega_1$*, which is an arbitrary compact subset of* Ω*.*

The outline of this paper is as follows. In § 2, we give some preliminaries. The proofs of theorems 1.1 and 1.4 are provided in § 3.

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2. Some preliminaries

In this section, we present some basics of Karamata regular variation theory in order to show not only the complete characterization of g in $(\mathbf{g_1})$–$(\mathbf{g_3})$ but also the exact behaviour near zero and infinity of ψ in (1.3).

Incidentally, Cîrstea and Rădulescu [8] first introduced the theory to study the boundary behaviour of large solutions of semi-linear elliptic equations.

Definition 2.1 [26, definition 1.1]. A positive continuous function g defined on $(0, s_0]$, for some $s_0 > 0$, is called *regularly varying at zero* with index $\rho \in \mathbb{R}$, denoted by $g \in RVZ_\rho$, if for each $\xi > 0$,

$$\lim_{s\to 0^+} \frac{g(\xi s)}{g(s)} = \xi^\rho. \tag{2.1}$$

In particular, when $\rho = 0$, g is called *slowly varying at zero*.

Clearly, if $g \in RVZ_\rho$, then $L(s) := g(s)/s^\rho$ is slowly varying at zero.

Some basic examples of slowly varying functions at zero are

$(\mathbf{i_1})$ every continuous function on $(0, s_0)$ which has a positive limit at zero;
$(\mathbf{i_2})$ $(-\ln s)^\gamma$ and $(\ln(-\ln s))^\gamma$, $\gamma \in \mathbb{R}$, $s \in (0, 1/3)$;
$(\mathbf{i_3})$ $\exp((-\ln s)^\gamma)$, $0 < \gamma < 1$, $s \in (0, 1)$.

Proposition 2.2 (Uniform convergence theorem [26, theorem 1.1]). *If $g \in RVZ_\rho$, then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. Moreover, if $\rho < 0$, then uniform convergence holds on intervals of the form $[c_1, \infty)$ provided g is bounded on $[c_1, \infty)$; if $\rho > 0$, then uniform convergence holds on intervals $(0, c_2]$ for all $c_2 > 0$.*

Proposition 2.3 (The Karamata representation theorem) [26, theorem 1.4]. *A function L is slowly varying at zero if and only if it may be written in the form*

$$L(s) = l(s) \exp\left(\int_s^{s_0} \frac{y(\tau)}{\tau}\,\mathrm{d}\tau\right), \quad s \in (0, s_0], \tag{2.2}$$

where the functions l and y are continuous and for $s \to 0^+$, $y(s) \to 0$ and $l(s) \to c_0$, with $c_0 > 0$.

Definition 2.4 ([26], p. 7). We call that

$$\hat{L}(s) = c_0 \exp\left(\int_s^{s_0} \frac{y(\tau)}{\tau}\mathrm{d}\tau\right), \quad s \in (0, s_0], \tag{2.3}$$

is *normalized* slowly varying at zero, and

$$g(s) = s^\rho \hat{L}(s),\ s \in (0, s_0], \tag{2.4}$$

is *normalized* regularly varying at zero with index ρ (and denoted by $g \in NRVZ_\rho$).

Equivalently, a function $g \in NRVZ_\rho$ if and only if

$$g \in C^1(0, s_0] \text{ for some } s_0 > 0 \text{ and } \lim_{s\to 0^+} \frac{sg'(s)}{g(s)} = \rho. \tag{2.5}$$

Proposition 2.5 [4, proposition 1.3.6]. *If functions L, L_1 are slowly varying at zero, then*

($\mathbf{i_1}$) L^ρ *for every* $\rho \in \mathbb{R}$, $c_1 L + c_2 L_1$ $(c_1 \geqslant 0,\ c_2 \geqslant 0$ with $c_1 + c_2 > 0)$, $L \cdot L_1$, $L \circ L_1$ *(if* $L_1(s) \to 0$ *as* $s \to 0^+$*), are also slowly varying at zero.*
($\mathbf{i_2}$) *For every* $\varepsilon > 0$ *and* $s \to 0^+$, $s^\varepsilon L(s) \to 0$ *and* $s^{-\varepsilon} L(s) \to \infty$.
($\mathbf{i_3}$) *For* $\rho \in \mathbb{R}$ *and* $s \to 0^+$, $\ln(L(s))/\ln s \to 0$ *and* $\ln(s^\rho L(s))/\ln s \to \rho$.

Proposition 2.6 (Asymptotic behaviour), [4, propositions 1.5.8 and 1.5.10]. *If a function* L *is slowly varying at zero, then for* $s_0 > 0$ *and* $s \to 0^+$

($\mathbf{i_1}$) $\int_0^s \tau^\rho L(\tau)\mathrm{d}\tau \cong (1+\rho)^{-1} s^{1+\rho} L(s)$, *for* $\rho > -1$;
($\mathbf{i_2}$) $\int_s^{s_0} \tau^\rho L(\tau)\mathrm{d}\tau \cong (-\rho-1)^{-1} s^{1+\rho} L(s)$, *for* $\rho < -1$.

Similarly, for a positive continuous function f defined on $[S_0, \infty)$, for some $S_0 > 0$, we can give the definitions of regularly varying and normalized regularly varying at infinity and present some basic properties. Here we omit them.

Lemma 2.7 [36, lemma 2.2]. *Let* g *satisfy* ($\mathbf{g_1}$).

($\mathbf{i_1}$) *If* g *satisfies* ($\mathbf{g_2}$), *then* $C_g \leqslant 1$.
($\mathbf{i_2}$) ($\mathbf{g_2}$) *holds with* $C_g \in (0,1)$ *if and only if* $g \in NRVZ_{-\gamma}$ *with* $\gamma > 0$. *In this case* $\gamma = C_g/(1-C_g)$.
($\mathbf{i_3}$) ($\mathbf{g_2}$) *holds with* $C_g = 0$ *if and only if* g *is normalized slowly varying at zero.*
($\mathbf{i_4}$) *If* ($\mathbf{g_2}$) *holds with* $C_g = 1$, *then* g *grows faster than any* s^{-p} $(p > 1)$ *near zero.*
($\mathbf{i_5}$) *If* $g \in C^2(0, s_0)$ *for some* $s_0 > 0$ *and*

$$g''(s) > 0, \quad \forall s \in (0, s_0); \ \lim_{s \to 0^+} \frac{g(s)g''(s)}{(g'(s))^2} = 1, \tag{2.6}$$

then g *satisfies* ($\mathbf{g_2}$) *with* $C_g = 1$.

Similarly, we have the following results.

Lemma 2.8. *Let* g *satisfy* ($\mathbf{g_1}$).

($\mathbf{i_1}$) *If* g *satisfies* ($\mathbf{g_3}$), *then* $E_g \leqslant 1$.
($\mathbf{i_2}$) ($\mathbf{g_3}$) *holds with* $E_g \in (0,1)$ *if and only if* g *is normalized regularly varying at infinity with index* $-\gamma$ *with* $\gamma > 0$. *In this case* $\gamma = E_g/(1-E_g)$.
($\mathbf{i_3}$) ($\mathbf{g_3}$) *holds with* $E_g = 0$ *if and only if* g *is normalized slowly varying at infinity.*
($\mathbf{i_4}$) *If* ($\mathbf{g_3}$) *holds with* $E_g = 1$, *then* g *grows faster than any* s^{-p} $(p > 1)$ *at infinity.*
($\mathbf{i_5}$) *If* $g \in C^2(S_0, \infty)$ *for some large* $S_0 > 0$ *and*

$$g''(s) > 0, \ \forall s \in (S_0, \infty); \quad \lim_{s \to \infty} \frac{g(s)g''(s)}{(g'(s))^2} = 1, \tag{2.7}$$

then g *satisfies* ($\mathbf{g_3}$) *with* $E_g = 1$.

For completeness, we give its proof.

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Proof. By using ($\mathbf{g_1}$), we show that

$$0 < \int_0^s \frac{\mathrm{d}\tau}{g(\tau)} \leqslant \frac{s}{g(s)}, \quad \forall s > 0,$$

i.e.

$$0 < g(s) \int_0^s \frac{\mathrm{d}\tau}{g(\tau)} \leqslant s, \quad \forall s > 0. \tag{2.8}$$

Thus

$$\lim_{s \to 0^+} g(s) \int_0^s \frac{\mathrm{d}\tau}{g(\tau)} = 0. \tag{2.9}$$

($\mathbf{i_1}$) Let

$$I(s) = -g'(s) \int_0^s \frac{\mathrm{d}\tau}{g(\tau)}, \quad \forall s > 0.$$

Integrating $I(t)$ from 0 to s and using integration by parts, we obtain by (2.9) that

$$\int_0^s I(t)\mathrm{d}t = -g(s) \int_0^s \frac{\mathrm{d}\tau}{g(\tau)} + s, \quad \forall s > 0,$$

i.e.

$$0 < \frac{g(s) \int_0^s \mathrm{d}\tau/g(\tau)}{s} = 1 - \frac{\int_0^s I(t)\mathrm{d}t}{s}, \quad \forall s > 0.$$

It follows from the l'Hospital's rule that

$$0 \leqslant \lim_{s \to \infty} \frac{g(s) \int_0^s \mathrm{d}\tau/g(\tau)}{s} = 1 - \lim_{s \to \infty} I(s) = 1 - E_g. \tag{2.10}$$

So ($\mathbf{i_1}$) holds.

($\mathbf{i_2}$) When ($\mathbf{g_3}$) holds with $E_g \in (0, 1)$, it follows from (2.10) that

$$\lim_{s \to \infty} \frac{g(s)}{sg'(s)} = \lim_{s \to \infty} \frac{g(s) \int_0^s \mathrm{d}\tau/g(\tau)}{sg'(s) \int_0^s \mathrm{d}\tau/g(\tau)} = -\frac{1}{E_g} \lim_{s \to \infty} \frac{g(s) \int_0^s \mathrm{d}\tau/g(\tau)}{s} = -\frac{1 - E_g}{E_g},$$

i.e. g is normalized regularly varying at infinity with index $-E_g/(1 - E_g)$.

Conversely, when g is normalized regularly varying at infinity with index $-\gamma$ with $\gamma > 0$, i.e. $\lim_{s \to \infty} sg'(s)/g(s) = -\gamma$ and there exist positive constant $S_0 > 0$ and $\hat{L}$, which is normalized slowly varying at infinity such that $g(s) = s^{-\gamma}\hat{L}(s), \;\; s \in$

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(S_0, ∞). By using the result similar to proposition 2.6 ($\mathbf{i_1}$), we have

$$-\lim_{s\to\infty} g'(s)\int_0^s \frac{d\tau}{g(\tau)} = -\lim_{s\to\infty}\frac{sg'(s)}{g(s)}\lim_{s\to\infty}\frac{g(s)\int_0^s d\tau/g(\tau)}{s}$$
$$= \gamma\lim_{s\to\infty} s^{-(1+\gamma)}\hat{L}_1(s)\int_0^s \tau^{\gamma}(\hat{L}_1(\tau))^{-1}d\tau = \frac{\gamma}{1+\gamma} = E_g.$$

($\mathbf{i_3}$) By $E_g = 0$ and (2.10), one can see that

$$\lim_{s\to\infty}\frac{sg'(s)}{g(s)} = \lim_{s\to\infty}\frac{sg'(s)\int_0^s d\tau/g(\tau)}{g(s)\int_0^s d\tau/g(\tau)}$$
$$= \left(\lim_{s\to\infty}\frac{g(s)}{s}\int_0^s\frac{d\tau}{g(\tau)}\right)^{-1}\lim_{s\to\infty} g'(s)\int_0^s\frac{d\tau}{g(\tau)} = 0,$$

i.e. g is normalized slowly varying at infinity.

Conversely, when g is normalized slowly varying at infinity, i.e. $\lim_{s\to\infty} sg'(s)/g(s) = 0$, it follows by (2.8) that

$$0 < \frac{g(s)\int_0^s d\tau/g(\tau)}{s} \leqslant 1, \quad \forall s > 0,$$

and

$$\lim_{s\to\infty} g'(s)\int_0^s\frac{d\tau}{g(\tau)} = \lim_{s\to\infty}\frac{sg'(s)}{g(s)}\frac{g(s)\int_0^s d\tau/g(\tau)}{s} = 0.$$

($\mathbf{i_4}$) By $E_g = 1$ and the proof of ($\mathbf{i_2}$), we show that $\lim_{s\to\infty} g(s)/sg'(s) = 0$, i.e. $\lim_{s\to\infty} sg'(s)/g(s) = -\infty$. Consequently, for an arbitrary $p > 1$, there exists $S_0 > 0$ such that

$$\frac{-g'(s)}{g(s)} > (p+1)s^{-1}, \quad \forall s \geqslant S_0.$$

Integrating the above inequality from S_0 to s, we obtain

$$\ln(g(S_0)) - \ln(g(s)) > (p+1)(\ln s - \ln S_0), \quad \forall s > S_0,$$

i.e.

$$0 < g(s)s^p < \frac{g(S_0)S_0^{p+1}}{s}, \quad \forall s > S_0.$$

Letting $s \to \infty$, we show that g grows faster than any s^{-p} ($p > 1$) at infinity.

($\mathbf{i_5}$) By a direct calculation and the l'Hospital's rule, we show that

$$\lim_{s\to\infty} g'(s)\int_0^s\frac{d\tau}{g(\tau)} = \lim_{s\to\infty}\frac{\int_0^s d\tau/g(\tau)}{(g'(s))^{-1}} = -\lim_{s\to\infty}\frac{(g'(s))^2}{g(s)g''(s)} = -1.$$

□

LEMMA 2.9 [**36**, lemma 2.3]. *Let g satisfy* ($\mathbf{g_1}$) *and* ($\mathbf{g_2}$). *Then we have*

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(**i₁**) $\psi'(t) = g(\psi(t))$, $\psi(t) > 0$ *for* $t > 0$, $\psi(0) = 0$, $\psi'(0) := \lim_{t\to 0^+} \psi'(t) = \lim_{t\to 0^+} g(\psi(t)) = \infty$, *and* $\psi''(t) = g(\psi(t))g'(\psi(t))$, $t > 0$;

(**i₂**) $\lim_{t\to 0^+} tg(\psi(t)) = 0$, $\lim_{t\to 0^+} tg'(\psi(t)) = -C_g$ *and* $\lim_{t\to 0} \xi tg'(\psi(\xi t)) = -C_g$ *holds uniformly for* $\xi \in [c_1, c_2]$ *with* $0 < c_1 < c_2$;

(**i₃**) $\psi \in NRVZ_{1-C_g}$ *and* $\psi' \in NRVZ_{-C_g}$.

Similarly, we have the following result.

LEMMA 2.10. *Let* g *satisfy* ($\mathbf{g_1}$) *and* ($\mathbf{g_3}$). *Then we have*

($\mathbf{i_1}$) $\lim_{t\to\infty} \psi(t) = \infty$;

($\mathbf{i_2}$) $\lim_{t\to\infty} tg'(\psi(t)) = -E_g$ *and* $\lim_{t\to\infty} \xi tg'(\psi(\xi t)) = -E_g$ *holds uniformly for* $\xi \in [c_1, c_2]$ *with* $0 < c_1 < c_2$;

($\mathbf{i_3}$) ψ *is normalized regularly varying at infinity with index* $1 - E_g$, *and* ψ' *is normalized regularly varying at infinity with index* $-E_g$.

LEMMA 2.11 [**21**, lemma]. *Let* Ω *be a bounded smooth domain in* $\mathbb{R}^n$. *We have*

$$\int_\Omega (d(x))^\lambda \, \mathrm{d}x < \infty$$

if and only if $\lambda > -1$.

LEMMA 2.12 [**33**, theorem 4.1]. *Let* b *satisfy* ($\mathbf{b_1}$). *If* g *satisfies* ($\mathbf{g_1}$), *then problem* (1.1) *has a unique solution* $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$ *if and only if the linear problem*

$$-\Delta v(x) = b(x), \quad v > 0, \ x \in \Omega, \ v|_{\partial\Omega} = 0, \tag{2.11}$$

admits a unique solution $v_0 \in C^2(\Omega) \cap C(\bar{\Omega})$.

LEMMA 2.13. *Let* b *satisfy* ($\mathbf{b_1}$).

($\mathbf{i_1}$) *If* b *satisfies* ($\mathbf{b_3}$), *then problem* (2.11) *has no solutions in* $C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$.

($\mathbf{i_2}$) *If* b *satisfies* ($\mathbf{b_4}$), *then problem* (2.11) *has no solutions in* $C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$.

Proof. ($\mathbf{i_1}$) Let $\underline{v}_\varepsilon = M_0\varepsilon^{-p}(\phi_1(x))^\varepsilon$, $x \in \Omega$, where $p, \varepsilon \in (0, 1)$ and $M_0 = b_1/C_0$, here C_0 is given as in (1.20). By using (1.15) and $\sigma \geqslant 2$, we have that

$$(\phi_1(x))^{\varepsilon-2} \leqslant (\phi_1(x))^{-\sigma}, \quad x \in \Omega$$

and

$$\begin{aligned} -\Delta\underline{v}_\varepsilon(x) &= M_0\varepsilon^{1-p}(\phi_1(x))^{\varepsilon-2}\big(\lambda_1\phi_1^2(x) + (1-\varepsilon)|\nabla\phi_1(x)|^2\big) \\ &\leqslant M_0C_0\varepsilon^{1-p}(\phi_1(x))^{\varepsilon-2} \leqslant b_1(\phi_1(x))^{-\sigma} \leqslant b(x), \quad x \in \Omega, \end{aligned}$$

i.e. $\underline{v}_\varepsilon = M_0\varepsilon^{-p}(\phi_1(x))^\varepsilon$ is a subsolution to problem (2.11) in Ω. Thus, if v were a classical solution to problem (2.11), it would then follow from ($\mathbf{g_1}$) and the maximum principle that $v(x) \geqslant M_0\varepsilon^{-p}(\phi_1(x))^\varepsilon$, $\forall x \in \Omega$. Since $\varepsilon \in (0, 1)$ is an arbitrary constant, we show that

$$\lim_{\varepsilon\to 0} v(x) = +\infty, \quad \forall x \in \Omega.$$

This is a contradiction. Thus, problem (2.11) has no classical solutions.

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($\mathbf{i_2}$) Let $\varepsilon \in (0,1)$ be an arbitrary constant with $-\ln(\phi_1(x)) > 1+\varepsilon,\ x \in \Omega$. By using $\mu \leqslant 1$, we show that

$$\min_{x\in\bar\Omega} \big(\lambda_1\phi_1^2(x) + \big(1-(1+\varepsilon)(-\ln(\phi_1(x)))^{-1}\big)|\nabla\phi_1(x)|^2\big) > 0,$$

and

$$(-\ln(\phi_1(x)))^{-\varepsilon-1} \leqslant (-\ln((\phi_1(x))))^{-\mu}, \quad x \in \Omega.$$

Let $\underline{v}_\varepsilon = M_1\varepsilon^{-p}(-\ln(\phi_1(x)))^{-\varepsilon},\ x \in \Omega$, where $p \in (0,1)$ and M_1 satisfies

$$M_1 \max_{x\in\bar\Omega}(\lambda_1\phi_1^2(x) + (1-(-\ln(\phi_1(x)))^{-1})|\nabla\phi_1(x)|^2) = b_1.$$

We have from a direct computation that

$$\begin{aligned}-\Delta\underline{v}_\varepsilon(x) &= M_1\varepsilon^{1-p}(-\ln(\phi_1(x)))^{-\varepsilon-1}(\phi_1(x)))^{-2}\\ &\quad\times \big(\lambda_1\phi_1^2(x) + \big(1-(1+\varepsilon)(-\ln(\phi_1(x)))^{-1}\big)|\nabla\phi_1(x)|^2\big)\\ &\leqslant b_1(-\ln(\phi_1(x)))^{-\mu}(\phi_1(x)))^{-2} \leqslant b(x), \quad x \in \Omega,\end{aligned}$$

i.e. $\underline{v}_\varepsilon = M_1\varepsilon^{-p}(-\ln(\phi_1(x)))^{-\varepsilon}$ is a subsolution to problem (2.11) in Ω.

The rest of the proof is similar to ($\mathbf{i_1}$) and the proof is omitted here. □

3. Global asymptotic behaviour

In this section, we prove theorems 1.1 and 1.4.

Firstly, we introduce a sub-supersolution method with the boundary restriction to the following more general problem:

$$-\Delta u = f(x,u), \quad u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0, \tag{3.1}$$

where $f(x,s)$ is locally Hölder continuous in $\Omega \times (0,\infty)$ with exponent $\alpha \in (0,1)$ and continuously differentiable with respect to the variables s.

Definition 3.1. A function $\underline{u} \in C^{2+\alpha}(\Omega) \cap C(\bar\Omega)$ is called a subsolution to problem (3.1) if

$$-\Delta\underline{u} \leqslant f(x,\underline{u}), \quad \underline{u} > 0,\ x \in \Omega,\ \underline{u}|_{\partial\Omega} = 0. \tag{3.2}$$

Definition 3.2. A function $\bar u \in C^{2+\alpha}(\Omega) \cap C(\bar\Omega)$ is called a supersolution to problem (3.1) if

$$-\Delta\bar u \geqslant f(x,\bar u), \quad \bar u > 0,\ x \in \Omega,\ \ \bar u|_{\partial\Omega} = 0. \tag{3.3}$$

The basis for our subsequent discussions is the following lemma, which is formulated in terms of supersolution and subsolution.

Lemma 3.3 [**12**, lemma 3]. *Suppose problem (3.1) has a supersolution $\bar u$ and a subsolution $\underline{u}$ such that $\underline{u} \leqslant \bar u$ on Ω, then problem (3.1) has at least one solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar\Omega)$ in the order interval $[\underline{u}, \bar u]$.*

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For convenience, let

$$\Psi(t) := -tg'(\psi(t)), \quad t > 0, \tag{3.4}$$

we show from lemmas 2.9 and 2.10 that $\Psi(t)$ is positive and bounded on $(0, \infty)$.

Next, we prove theorem 1.1.

Proof of theorem 1.1. The result $(\mathbf{i_1})$ follows from lemmas 2.12 and 2.13 $(\mathbf{i_1})$ directly.

$(\mathbf{i_2})$ Let $\beta = 2 - \sigma$. It follows from lemma 2.9 $(\mathbf{i_2})$ and (3.4) that

$$\lim_{t \to 0} \Psi(\xi \beta^{-1} t) = \lim_{d(x) \to 0} \Psi(\xi \beta^{-1} \phi_1^{\beta}(x)) = C_g \tag{3.5}$$

holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.

From (1.14), we show that there is a sufficiently small $\delta_1 \in (0, \delta_0)$, which is independent of $\xi \in [c_1, c_2]$, such that for $x \in \Omega_{\delta_1}$,

$$(1 - \beta + \beta \Psi(\xi \beta^{-1} \phi_1^{\beta}(x))) |\nabla \phi_1(x)|^2 > 0. \tag{3.6}$$

Moreover, we have from lemma 2.10 $(\mathbf{i_2})$ that

$$0 \leqslant \inf_{t > 0} \Psi(t) \leqslant \Psi(t) \leqslant \sup_{t > 0} \Psi(t) < \infty, \quad t > 0.$$

Denote

$$\bar{\Psi}(x) = \begin{cases} \Psi(\xi \beta^{-1} \phi_1^{\beta}(x)), \ d(x) \leqslant \dfrac{\delta_1}{2}, \\ \sup\limits_{t > 0} \Psi(t), \ d(x) \geqslant \dfrac{\delta_1}{2}; \end{cases} \tag{3.7}$$

and

$$\underline{\Psi}(x) = \begin{cases} \Psi(\xi \beta^{-1} \phi_1^{\beta}(x)), \ d(x) \leqslant \dfrac{\delta_1}{2}; \\ \inf\limits_{t > 0} \Psi(t), \ d(x) \geqslant \dfrac{\delta_1}{2}. \end{cases} \tag{3.8}$$

Let $\bar{u}_\sigma = \psi(\xi_2 \beta^{-1} \phi_1^{\beta}(x)), \ x \in \Omega$, where ξ_2 satisfies

$$\xi_2 \inf_{x \in \bar{\Omega}} \big(\lambda_1 \phi_1^2(x) + \big(1 - \beta + \beta \underline{\Psi}(x)\big) |\nabla \phi_1(x)|^2\big) = b_2.$$

By using $(\mathbf{b_2})$ and a direct computation, we show that for $x \in \Omega$

$$\begin{aligned} -\Delta \bar{u}_\sigma(x) &= \xi_2 \phi_1^{-\sigma}(x) g(\psi(\xi_2 \beta^{-1} \phi_1^{\beta}(x))) \big(\lambda_1 \phi_1^2(x) + \big(1 - \beta \\ &\quad + \beta \Psi(\xi_2 \beta^{-1} \phi_1^{\beta}(x))\big) |\nabla \phi_1(x)|^2\big) \\ &\geqslant b_2 \phi_1^{-\sigma}(x) g(\psi(\xi_2 \beta^{-1} \phi_1^{\beta}(x))) \geqslant b(x) g(\bar{u}_\sigma(x)), \end{aligned}$$

i.e. $\bar{u}_\sigma = \psi(\xi_2 \beta^{-1} \phi_1^{\beta}(x))$ is a supersolution to problem (1.1) in Ω.

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In a similar way, we can show that $\underline{u}_\sigma = \psi(\xi_1\beta^{-1}\phi_1^{\beta}(x))$ is a subsolution to problem (1.1) in Ω, where ξ_1 satisfies

$$\xi_1 \sup_{x\in\bar{\Omega}} \left(\lambda_1\phi_1^2(x) + \left(1-\beta+\beta\bar{\Psi}(x)\right)|\nabla\phi_1(x)|^2\right) = b_1.$$

Obviously, $\bar{u} \geqslant \underline{u}$ on Ω. Hence lemma 3.3 and remark 1.2 imply that problem (1.1) has a unique solution $u_\sigma \in C^{2+\alpha}(\Omega)\cap C(\bar{\Omega})$ in the order interval $[\underline{u}, \bar{u}]$, i.e. (1.18) holds. Using lemma 2.10 ($\mathbf{i_3}$), we show that

$$\lim_{s\to\infty} \frac{\psi(\xi s)}{\psi(s)} = \xi^{1-E_g}, \tag{3.9}$$

holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$, and thus

$$\lim_{\sigma\to 2^-} \frac{\psi(\xi_2(2-\sigma)^{-1}\phi_1^{2-\sigma}(x))}{\psi((2-\sigma)^{-1}\phi_1^{2-\sigma}(x))} = \left(\frac{b_2}{c_0}\right)^{1-E_g}, \quad \forall x \in \Omega,$$

i.e. (1.19) holds.

($\mathbf{i_3}$) Note that when L is slowly varying at zero, we have

$$\int_0^{s_0} s^\rho L(s)\mathrm{d}s \begin{cases} < \infty & \text{if} \quad \rho > -1, \\ = \infty & \text{if} \quad \rho < -1. \end{cases} \tag{3.10}$$

In fact, when $\rho > -1$, let $\rho_0 \in (-1, \rho)$, using proposition 2.5 ($\mathbf{i_2}$), we show that $\lim_{s\to 0^+} s^{\rho-\rho_0}L(s) = 0$, and there exists a sufficiently small $\delta \in (0, s_0)$ such that $s^{\rho-\rho_0}L(s) < 1, \quad \forall s \in (0, \delta]$. It follows that

$$\int_0^{s_0} s^\rho L(s)\mathrm{d}s = \int_0^{s_0} s^{\rho_0} s^{\rho-\rho_0} L(s)\mathrm{d}s \leqslant \int_0^{\delta} s^{\rho_0}\mathrm{d}s + \int_\delta^{s_0} s^\rho L(s)\mathrm{d}s < \infty.$$

Similarly, we can show $\int_0^{s_0} s^\rho L(s)\mathrm{d}s = \infty$ provided $\rho < -1$.

In addition, from lemma 2.9, we show that $\psi(t)g(\psi(t))$ is normalized regularly varying at zero with index $1-2C_g$ and $\psi(t)g(\psi(t)) = t^{1-2C_g}\hat{L}(t)$, here $\hat{L}$ is normalized slowly varying at zero.

So

$$(\phi_1(x))^{-\sigma}\psi(\phi_1^{2-\sigma}(x))g(\psi(\phi_1^{2-\sigma}(x))) = \phi_1^{(2-\sigma)(1-2C_g)-\sigma}\hat{L}(\phi_1^{2-\sigma}(x)).$$

Since $u_\sigma \in H_0^1(\Omega)$ if and only if

$$-\int_\Omega u_\sigma(x)\Delta u_\sigma(x)\mathrm{d}x = \int_\Omega |\nabla u_\sigma(x)|^2\mathrm{d}x = \int_\Omega b(x)u_\sigma(x)g(u_\sigma(x))\mathrm{d}x,$$

we show from ($\mathbf{b_2}$), (1.14), (1.18), (3.10) and lemma 2.11 that

$$\begin{aligned} &\int_\Omega b(x)u_\sigma(x)g(u_\sigma(x))\mathrm{d}x \\ &\quad \leqslant b_2\int_\Omega (\phi_1(x))^{-\sigma}\psi(\xi_2(2-\sigma)^{-1}\phi_1^{2-\sigma}(x))g(\psi(\xi_1(2-\sigma)^{-1}\phi_1^{2-\sigma}(x)))\mathrm{d}x \\ &\quad < \infty \end{aligned}$$

provided $(2-\sigma)(1-2C_g) > \sigma - 1$.

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Similarly, since

$$b_1\int_\Omega(\phi_1(x))^{-\sigma}\psi(\xi_1(2-\sigma)^{-1}\phi_1^{2-\sigma}(x))g(\psi(\xi_2(2-\sigma)^{-1}\phi_1^{2-\sigma}(x)))\mathrm{d}x$$
$$\leqslant b_1\int_\Omega(\phi_1(x))^{-\sigma}u_\sigma(x)g(u_\sigma(x))\mathrm{d}x\leqslant\int_\Omega b(x)u_\sigma(x)g(u_\sigma(x))\mathrm{d}x,$$

one can see that

$$\int_\Omega b(x)u_\sigma(x)g(u_\sigma(x))\mathrm{d}x=\infty$$

provided

$$(2-\sigma)(1-2C_g)<\sigma-1.$$

The proof is finished. □

Finally, we prove theorem 1.4.

Proof of theorem 1.4. The result $(\mathbf{i_1})$ follows from lemmas 2.12 and 2.13 $(\mathbf{i_2})$ directly.

$(\mathbf{i_2})$ It follows from (1.15) that

$$-\ln(\phi_1(x))>\mu,\quad\forall x\in\bar\Omega.$$

Let $\bar u_\mu=\psi(\xi_4(\mu-1)^{-1}(-\ln(\phi_1(x)))^{1-\mu}),\ x\in\Omega$, where ξ_4 satisfies

$$\xi_4\min_{x\in\bar\Omega}\Big(\lambda_1\phi_1^2(x)+\Big(1-\mu(-\ln(\phi_1(x)))^{-1}$$
$$+(\mu-1)(-\ln(\phi_1(x)))^{-1}\inf_{t>0}\Psi(t)\Big)|\nabla\phi_1(x)|^2\Big)=b_2.$$

Since

$$\lim_{d(x)\to0}\Big(1-\mu(-\ln(\phi_1(x)))^{-1}+(\mu-1)(-\ln(\phi_1(x)))^{-1}$$
$$\Psi(\xi_4(\mu-1)^{-1}(-\ln(\phi_1(x)))^{1-\mu})\Big)=1$$

holds uniformly for $\xi_4\in[c_1,c_2]$ with $0<c_1<c_2$.

We show that there is a sufficiently small $\delta_1\in(0,\delta_0)$, which is independent of $\xi\in[c_1,c_2]$, such that for $x\in\Omega_{\delta_1}$,

$$|\nabla\phi_1(x)|^2(1-\mu(-\ln(\phi_1(x)))^{-1}+(\mu-1)(-\ln(\phi_1(x)))^{-1}$$
$$\times\Psi(\xi_4(\mu-1)^{-1}(-\ln(\phi_1(x)))^{1-\mu}))>0. \tag{3.11}$$

It follows that for $x\in\Omega$

$$\lambda_1\phi_1^2(x)+(1-\mu(-\ln(\phi_1(x)))^{-1}+(\mu-1)(-\ln(\phi_1(x)))^{-1}$$
$$\times\Psi(\xi_4(\mu-1)^{-1}(-\ln(\phi_1(x)))^{1-\mu}))|\nabla\phi_1(x)|^2>0.$$

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By using ($\mathbf{b_5}$) and a direct computation, we have that for $x \in \Omega$

$$\begin{aligned}-\Delta \bar{u}_\mu(x) &= \xi_4(\phi_1(x))^{-2}(-\ln(\phi_1(x)))^{-\mu} g(\psi(\xi_4(\mu-1)^{-1}(-\ln(\phi_1(x)))^{1-\mu}))\\ &\quad \times \Big(\lambda_1\phi_1^2(x) + \Big(1-\mu(-\ln(\phi_1(x)))^{-1} + (\mu-1)(-\ln(\phi_1(x)))^{-1}\\ &\quad \times \Psi(\xi_4(\mu-1)^{-1}(-\ln(\phi_1(x)))^{1-\mu})\Big)|\nabla\phi_1(x)|^2\Big)\\ &\geqslant b_2(\phi_1(x))^{-2}(-\ln(\phi_1(x)))^{-\mu} g(\psi(\xi_4(\mu-1)^{-1}(-\ln(\phi_1(x)))^{1-\mu}))\\ &\geqslant b(x)g(\bar{u}_\mu(x)),\end{aligned}$$

i.e. $\bar{u}_\mu = \psi(\xi_4(\mu-1)^{-1}(-\ln(\phi_1(x)))^{1-\mu})$ is a supersolution to problem (1.1) in Ω.

In a similar way, we can show that $\underline{u}_\mu = \psi(\xi_3(\mu-1)^{-1}(-\ln(\phi_1(x)))^{1-\mu})$ is a subsolution to problem (1.1) in Ω, where ξ_3 satisfies

$$\begin{aligned}&\xi_3 \max_{x\in\bar{\Omega}} \Big(\lambda_1\phi_1^2(x) + \Big(1-\mu(-\ln(\phi_1(x)))^{-1}\\ &\quad + (\mu-1)(-\ln(v(x)))^{-1}\sup_{t>0}\Psi(t)\Big)|\nabla\phi_1(x)|^2\Big) = b_1.\end{aligned}$$

Obviously, $\bar{u}_\mu \geqslant \underline{u}_\mu$ on Ω. The rest of the proof is similar to that of theorem 1.1 and the proof is omitted here. □

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