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# K-STABILITY OF FANO MANIFOLDS WITH NOT SMALL ALPHA INVARIANTS

KENTO FUJITA

*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan* (fujita@kurims.kyoto-u.ac.jp)

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*Abstract* We show that any $n$-dimensional Fano manifold $X$ with $\alpha(X)=n/(n+1)$ and $n\geqslant 2$ is K-stable, where $\alpha(X)$ is the alpha invariant of $X$ introduced by Tian. In particular, any such $X$ admits Kähler–Einstein metrics and the holomorphic automorphism group $\mathrm{Aut}(X)$ of $X$ is finite.

*Keywords*: Fano varieties; K-stability; Kähler–Einstein metrics

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## 1. Introduction

Let $X$ be an $n$-dimensional *Fano manifold*, that is, a smooth projective variety $X$ over the complex number field $\mathbb{C}$ such that the anti-canonical divisor $-K_X$ is ample. It is an interesting question whether $X$ admits Kähler–Einstein metrics or not. In 1987, Tian [45] gave a sufficient condition for the problem; if the *alpha invariant* $\alpha(X)$ of $X$ is *strictly bigger than* $n/(n+1)$, then $X$ admits Kähler–Einstein metrics. For the definition of $\alpha(X)$ in this article, we use Demailly's algebraic interpretation [13] (see also [34, 49, 51]).

**Definition 1.1** [13, 45]. Let $X$ be a $\mathbb{Q}$-*Fano variety*, that is, a normal complex projective variety with at most log terminal singularities and the anti-canonical divisor $-K_X$ ample $\mathbb{Q}$-Cartier. The *alpha invariant* $\alpha(X)$ of $X$ is defined by the supremum of positive rational numbers $\alpha$ such that the pair $(X,\alpha D)$ is log canonical for any effective $\mathbb{Q}$-divisor $D$ with $D\sim_{\mathbb{Q}}-K_X$.

**Example 1.2** (See [12, Lemma 5.1] for example.). We know the equality $\alpha(\mathbb{P}^n)=1/(n+1)$.

On the other hand, it has been known that a Fano manifold $X$ admits Kähler–Einstein metrics if and only if $X$ is *K-polystable* by the works [2, 17–19, 35, 36, 44, 47] and [5–7, 48]. In this article, we focus on the conditions *K-stability* and *K-semistability*; K-stability is stronger than K-polystability and K-polystability is stronger than K-semistability.

Odaka and Sano [40, Theorem 1.4] (see also its generalizations [4, 14, 21]) proved a variant of Tian's theorem; if an $n$-dimensional $\mathbb{Q}$-Fano variety $X$ satisfies that $\alpha(X) > n/(n+1)$ (respectively $\alpha(X) \geqslant n/(n+1)$), then $X$ is K-stable (respectively K-semistable). Thus, from Odaka–Sano's theorem, it has been known the *K-semistability* of $n$-dimensional Fano manifolds $X$ with $\alpha(X) = n/(n+1)$. However, it has not been known until now the *K-stability* of those $X$. The main result in this article is to prove the K-stability of those $X$ with $n \geqslant 2$ (see [40, Conjecture 5.1]). Note that, if $n = 1$, then $X \simeq \mathbb{P}^1$, $\alpha(\mathbb{P}^1) = 1/2$, and $\mathbb{P}^1$ is not K-stable but K-semistable.

**Theorem 1.3** (Main Theorem). *If an $n$-dimensional Fano manifold $X$ satisfies that $\alpha(X) \geqslant n/(n+1)$ and $n \geqslant 2$, then $X$ is K-stable. In particular, $X$ admits Kähler–Einstein metrics and the holomorphic automorphism group* $\mathrm{Aut}(X)$ *of $X$ is a finite group.*

We note that there are many examples of $n$-dimensional Fano manifolds $X$ with $\alpha(X) = n/(n+1)$.

**Example 1.4.** Let $X$ be an $n$-dimensional Fano manifold.

(1) [41, §3], [9, Theorem 1.7] If $n = 2$, then $\alpha(X) = 2/3$ if and only if $((-K_X)^{\cdot 2}) = 4$ or $X$ is a smooth cubic surface admitting an Eckardt point.

(2) [8, Theorem 1.3], [10, Corollary 4.10], [16, Theorem 0.2] If $X$ is a hypersurface of degree $n+1$ in $\mathbb{P}^{n+1}$, then $\alpha(X) \geqslant n/(n+1)$ holds. Moreover, $\alpha(X) = n/(n+1)$ holds if $X$ contains an $(n-1)$-dimensional cone.

From Theorem 1.3 and Example 1.4(2), we immediately get the following corollary:

**Corollary 1.5.** *Let $X$ be an arbitrary smooth hypersurface in $\mathbb{P}^{n+1}$ of degree $n+1$, where $n$ is a positive integer with $n \geqslant 2$. Then $X$ is a K-stable $n$-dimensional Fano manifold. In particular, $X$ admits Kähler–Einstein metrics.*

**Remark 1.6.** (1) The examples in Corollary 1.5 are new in general. When the hypersurface is general or of certain special type or admits some finite symmetry, it was already known that the hypersurface admits Kähler–Einstein metrics. See [45, Theorem 4.1], [49, Theorem 2.5], [38, Theorem 6.1], [46, Main Theorem], [42, Theorem 2], [1, Proposition 3.1], [11, Theore 1.4] and [15, Example 4.1].

(2) In Theorem 1.3, we assume that $X$ is smooth. In fact, we crucially use the smoothness of $X$ in order to prove the theorem. See Theorem 4.1 and Example 4.2.

For the proof of Theorem 1.3, we use a valuative criterion for K-stability and K-semistability of $\mathbb{Q}$-Fano varieties [23, 30] (see also [29]). If an $n$-dimensional Fano manifold $X$ satisfies that $\alpha(X) = n/(n+1)$, $n \geqslant 2$ and $X$ is not K-stable, then there exists a dreamy prime divisor $F$ over $X$ with $\beta(F) = 0$ (see § 2 in detail). By viewing the $F$ in detail, we can show that $X$ must be isomorphic to $\mathbb{P}^n$ (see §§ 3 and 4 in detail). This gives a contradiction since $\alpha(X) = n/(n+1)$ and $n \geqslant 2$.

In this article, we work over the category of algebraic schemes over the complex number field $\mathbb{C}$. For the theory of minimal model program, we refer the readers to the book [26]; for the theory of toric geometry, we refer the readers to the book [24]. We do not distinguish line bundles (or more generally $\mathbb{Q}$-line bundles) and Cartier divisors (or more generally $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors) if there is no confusion.

## 2. K-stability

We quickly recall the notion of K-stability and K-semistability. We remark that there are many equivalent definitions of K-(semi)stability.

**Definition 2.1** (See [18, 33, 39, 43, 47, 50]). Let $X$ be an $n$-dimensional $\mathbb{Q}$-Fano variety.

(1) A *test configuration* $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ *of* $X$ consists of the following data:

- a normal projective variety $\mathcal{X}$ together with a surjection $q\colon \mathcal{X} \to \mathbb{P}^1$;
- a $q$-ample $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$;
- an action $\mathbb{G}_m \curvearrowright (\mathcal{X}, \mathcal{L})$ such that the morphism $q$ is $\mathbb{G}_m$-equivariant with respect to the natural multiplicative action $\mathbb{G}_m \curvearrowright \mathbb{P}^1$ and there exists a $\mathbb{G}_m$-equivariant isomorphism

$$(\mathcal{X}, \mathcal{L})|_{\mathcal{X}\setminus\mathcal{X}_0} \simeq \left(X \times (\mathbb{P}^1 \setminus \{0\}), p_1^*(-K_X)\right),$$

where $\mathbb{G}_m$ is the multiplicative group, $\mathcal{X}_0$ is the scheme-theoretic fiber of $q$ at $0 \in \mathbb{P}^1$ and $p_1\colon X \times (\mathbb{P}^1 \setminus \{0\}) \to X$ is the first projection morphism.

A test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ is said to be *trivial* if $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ is $\mathbb{G}_m$-equivariantly isomorphic to $(X \times \mathbb{P}^1, p_1^*(-K_X))$ with the trivial $\mathbb{G}_m$-action, where $p_1\colon X \times \mathbb{P}^1 \to X$ is the first projection morphism.

(2) Let $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ be a test configuration of $X$. The *Donaldson–Futaki invariant* $\mathrm{DF}(\mathcal{X}, \mathcal{L})$ *of* $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ is defined by:

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) := \frac{1}{(n+1)((-K_X)^{\cdot n})}\left(n(\mathcal{L}^{\cdot n+1}) + (n+1)(\mathcal{L}^{\cdot n} \cdot K_{\mathcal{X}/\mathbb{P}^1})\right),$$

where $K_{\mathcal{X}/\mathbb{P}^1} := K_{\mathcal{X}} - q^*K_{\mathbb{P}^1}$.

(3) $X$ is said to be *K-stable* (respectively *K-semistable*) if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) > 0$ (respectively $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geqslant 0$) holds for any nontrivial test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ of $X$.

We recall a following valuative criterion for K-(semi)stability of $\mathbb{Q}$-Fano varieties [23, 30].

**Definition 2.2** [23, Definitions 1.1 and 1.3]. Let $X$ be an $n$-dimensional $\mathbb{Q}$-Fano variety and let $F$ be a prime divisor over $X$, that is, there exists a projective birational morphism $\sigma\colon Y \to X$ with $Y$ normal such that $F$ is a prime divisor on $Y$. (The divisor $F$ is not necessarily $\mathbb{Q}$-Cartier on $Y$.)

(1) For any $x \in \mathbb{R}_{\geqslant 0}$, we define

$$\begin{aligned}\mathrm{vol}_X(-K_X - xF) &:= \mathrm{vol}_Y\left(\sigma^*(-K_X) - xF\right)\\ &:= \lim_{\substack{k\to\infty\\ -kK_X:\ \mathrm{Cartier}}} \frac{\dim_{\mathbb{C}} H^0(Y, \sigma^*(-kK_X) - kxF)}{k^n/n!},\end{aligned}$$

where $H^0(Y, \sigma^*(-kK_X) - kxF)$ is the sub $\mathbb{C}$-vector space of $H^0(Y, \sigma^*(-kK_X)) = H^0(X, -kK_X)$ consisting of the sections vanishing at the generic point of $F$ at least $kx$ times. (The space does not depend on the choice of $\sigma$ as a subspace of $H^0(X, -kK_X)$.) By [27, 28], the limit exists. Moreover, the function $\mathrm{vol}_X(-K_X - xF)$ is a continuous and non-increasing function on $x \in [0, +\infty)$.

(2) We set the *pseudo-effective threshold* $\tau(F)$ of $-K_X$ with respect to $F$ as

$$\tau(F) := \sup\{\tau \in \mathbb{R}_{>0} \mid \mathrm{vol}_X(-K_X - \tau F) > 0\}.$$

We note that $\tau(F) \in \mathbb{R}_{>0}$ holds. Moreover, by [3, Theorem A], the function $\mathrm{vol}_X(-K_X - xF)$ is $\mathcal{C}^1$ on $x \in [0, \tau(F))$.

(3) We set the *log discrepancy* $A_X(F)$ of $X$ with respect to $F$ as $A_X(F) := 1 + \mathrm{ord}_F(K_{Y/X})$.

(4) We set

$$\beta(F) := A_X(F)((-K_X)^{\cdot n}) - \int_0^\infty \mathrm{vol}_X(-K_X - xF)\,dx.$$

(5) $F$ is said to be *dreamy* if the graded $\mathbb{C}$-algebra

$$\bigoplus_{k,j\in\mathbb{Z}_{\geqslant 0}} H^0(Y, \sigma^*(-kk_0K_X) - jF)$$

is finitely generated for some (hence, for any) $k_0 \in \mathbb{Z}_{>0}$ with $-k_0K_X$ Cartier.

We remark that all the above definitions do not depend on the choice of the morphism $\sigma : Y \to X$. More precisely, those are defined from only the divisorial valuation on the function field of $X$ given by $F$.

The following theorem is essential in order to prove Theorem 1.3.

**Theorem 2.3** (See [23, Corollary 1.5 and Theorem 1.6] and [30, Theorem 3.6]). *Let $X$ be a $\mathbb{Q}$-Fano variety. Then $X$ is K-stable* (*respectively K-semistable*) *if and only if $\beta(F) > 0$* (*respectively $\beta(F) \geqslant 0$*) *holds for any* dreamy *prime divisor $F$ over $X$.*

## 3. Dreamy prime divisors

In this section, we see some properties of dreamy prime divisors in order to prove Theorem 1.3.

The following lemma is a consequence of the results [25, Theorem 4.2], [32, Theorem 4.26], [22, § 2], [23, Claim 6] and [21, Claim 3.4].

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**Lemma 3.1.** *Let $X$ be an $n$-dimensional $\mathbb{Q}$-Fano variety and let $F$ be a dreamy prime divisor over $X$.*

(1) *We have $\tau(F) \in \mathbb{Q}_{>0}$. We can define the restricted volume (in the sense of [20])*

$$Q(x) := -\frac{1}{n}\frac{d}{dx}\,\mathrm{vol}_X(-K_X - xF)$$

*and the function $Q(x)$ is continuous, $\mathbb{R}_{>0}$-valued for any $x \in (0, \tau(F))$. Moreover, we can uniquely extend the values $Q(0)$, $Q(\tau(F)) \in \mathbb{Q}_{\geqslant 0}$ continuously. Furthermore, $Q(x)^{1/(n-1)}$ is a concave function on $x \in [0, \tau(F)]$.*

(2) *There exists a projective birational morphism $\sigma\colon Y \to X$ with $Y$ normal such that $F \subset Y$ is a prime divisor on $Y$ and $-F$ is a $\sigma$-ample $\mathbb{Q}$-Cartier divisor on $Y$. The morphism $\sigma$ is unique. (Of course, $\mathrm{vol}_X(-K_X - xF) = \mathrm{vol}_Y(\sigma^*(-K_X) - xF)$ holds for any $x \in [0, \tau(F)]$.) In particular, if $F$ is an exceptional divisor over $X$, then the exceptional set of $\sigma$ is equal to $F$ in $Y$.*

(3) *Set*

$$\varepsilon(F) := \max\{\varepsilon \in \mathbb{R}_{>0} \,|\, \sigma^*(-K_X) - \varepsilon F \text{ is nef}\},$$

*where $\sigma$ is as in (2). Then we have $\varepsilon(F) \in (0, \tau(F)] \cap \mathbb{Q}$, and*

$$Q(x) = \left((\sigma^*(-K_X) - xF)^{\cdot n-1} \cdot F\right)$$

*holds for any $x \in [0, \varepsilon(F)]$. Moreover, there exists a projective morphism $\pi\colon Y \to Z$ with $\pi_*\mathcal{O}_Y = \mathcal{O}_Z$ and an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $H_Z$ on $Z$ such that $\pi^*H_Z \sim_{\mathbb{Q}} \sigma^*(-K_X) - \varepsilon(F)F$ holds. The $\pi$ and $H_Z$ are unique. (We remark that the $\mathbb{Q}$-Cartier divisor $F$ on $Y$ is $\pi$-ample.)*

**Proof.** Let $\psi\colon \tilde{Y} \to X$ be a log resolution with $F \subset \tilde{Y}$. By [25, Theorem 4.2], there exists a sequence of rational numbers

$$0 = \tau_0 < \tau_1 < \cdots < \tau_m = \tau(F)$$

(in particular, $\tau(F) \in \mathbb{Q}_{>0}$ holds) and a mutually distinct birational contraction maps

$$\phi_i\colon \tilde{Y} \dashrightarrow Y_i$$

for $1 \leqslant i \leqslant m$ such that

- for any $x \in (\tau_{i-1}, \tau_i)$, the birational map $\phi_i$ is the ample model of $\psi^*(-K_X) - xF$; and
- for any $x \in [\tau_{i-1}, \tau_i]$, the birational map $\phi_i$ is the semiample model of $\psi^*(-K_X) - xF$.

(2) Set $Y := Y_1$. The ample model of $\psi^*(-K_X)$ is the $X$ itself. Thus there exists a projective birational morphism $\sigma\colon Y \to X$. Since $\sigma^*(-K_X) - xF$ is ample for any $x \in (0, \tau_1)$, the divisor $-F$ on $Y$ is a $\sigma$-ample $\mathbb{Q}$-Cartier divisor. Thus we have proved (2). (The uniqueness of $\sigma$ is trivial.)

(3) The $\mathbb{Q}$-divisor $\sigma^*(-K_X) - \tau_1 F$ is semiample but not ample. Thus $\tau_1 = \varepsilon(F)$ and there exists a morphism $\pi\colon Y \to Z$ and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $H_Z$ on $Z$ which satisfy the condition in (3) and they are unique. Thus we have proved (3).

(1) We already know by [3, Theorem A] the existence of the function $Q(x)$ and is continuous on $x \in (0, \tau(F))$ (see Definition 2.2(2)). Note that

$$\mathrm{vol}_X(-K_X - xF) = \left(((\phi_i)_*(\psi^*(-K_X) - xF))^{\cdot n}\right)$$

for any $x \in [\tau_{i-1}, \tau_i]$ (see [25, Remark 2.4(i)]). Thus we have

$$Q(x) = \left(((\phi_i)_*(\psi^*(-K_X) - xF))^{\cdot n-1} \cdot (\phi_i)_* F\right).$$

In particular, $Q(x) > 0$ holds for any $x \in (0, \tau(F))$ and we can naturally define the values $Q(0)$, $Q(\tau(F)) \in \mathbb{Q}_{\geqslant 0}$.

Take any $0 < \varepsilon \ll 1$ with $\varepsilon \in \mathbb{Q}$. Let us take an arbitrary complete flag

$$Y \supset Z_1 \supset \cdots \supset Z_n = \{\text{point}\}$$

in the sense of [32] with $Z_1 = F$. Consider the Okounkov body $\Delta_{Z_\bullet}(\sigma^*(-K_X) - \varepsilon F) \subset \mathbb{R}^n_{\geqslant 0}$ of $\sigma^*(-K_X) - \varepsilon F$ with respect to $Z_\bullet$ in the sense of [32]. Since $\sigma^*(-K_X) - \varepsilon F$ is ample, by [32, Theorem 4.26], $Q(x)/(n-1)!$ is equal to the restricted volume of

$$\{(\nu_1, \ldots, \nu_n) \in \Delta_{Z_\bullet}(\sigma^*(-K_X) - \varepsilon F) \mid \nu_1 = x - \varepsilon\}$$

for any $x \in [\varepsilon, \tau(F))$. Since $\Delta_{Z_\bullet}(\sigma^*(-K_X) - \varepsilon F)$ is a convex body, $Q(x)^{1/(n-1)}$ is a concave function on $x \in [\varepsilon, \tau(F))$ by the Brunn–Minkowski theorem. Thus we have proved (1). □

The following two propositions are important in this article.

**Proposition 3.2.** *Let $X$ be an $n$-dimensional $\mathbb{Q}$-Fano variety and let $F$ be a dreamy prime divisor over $X$. Let $\sigma: Y \to X$ be as in Lemma 3.1(2). Assume that*

$$\frac{n}{n+1}\tau(F)((-K_X)^{\cdot n}) \leqslant \int_0^{\tau(F)} \mathrm{vol}_Y(\sigma^*(-K_X) - xF)\, dx.$$

*Then we have the following:*

(1) *The above inequality is actually an equality.*

(2) *$\tau(F) = \varepsilon(F)$ holds, where $\varepsilon(F)$ is as in Lemma 3.1(3).*

(3) *$\sigma(F)$ is a point $p \in X$.*

**Proof.** The proof is similar to the one in [21, Theorem 3.2]. Since

$$\mathrm{vol}_Y(\sigma^*(-K_X) - xF) = n \int_x^{\tau(F)} Q(y)\, dy,$$

we have

$$\int_0^{\tau(F)} \mathrm{vol}_Y(\sigma^*(-K_X) - xF)\, dx = n \int_0^{\tau(F)} yQ(y)\, dy$$

and

$$((-K_X)^{\cdot n}) = n \int_0^{\tau(F)} Q(y)\, dy.$$

Set

$$b := \frac{\int_0^{\tau(F)} yQ(y)\, dy}{\int_0^{\tau(F)} Q(y)\, dy}.$$

From the assumption, we have $b \in [(n/(n+1))\tau(F), \tau(F))$. By Lemma 3.1(1) (concavity of $Q(x)^{1/(n-1)}$), we have the following:

- $Q(x) \geqslant Q(b)(x/b)^{n-1}$ holds for any $x \in [0, b]$;
- $Q(x) \leqslant Q(b)(x/b)^{n-1}$ holds for any $x \in [b, \tau(F)]$.

Thus we get

$$\begin{aligned} 0 &= \int_{-b}^{\tau(F)-b} yQ(y+b)\,dy \leqslant \int_{-b}^{\tau(F)-b} yQ(b)\frac{(y+b)^{n-1}}{b^{n-1}}dy \\ &= \frac{Q(b)\tau(F)^n}{b^{n-1}}\left(\frac{\tau(F)}{n+1} - \frac{b}{n}\right). \end{aligned}$$

This implies that $b \leqslant (n/(n+1))\tau(F)$. Therefore, we have $b = (n/(n+1))\tau(F)$ and $Q(x) = Q(b)(x/b)^{n-1}$ holds for any $x \in [0, \tau(F)]$. By Lemma 3.1(3),

$$Q(x) = \sum_{i=0}^{n-1}\binom{n-1}{i}x^i\left(\sigma^*(-K_X)^{\cdot n-1-i}\cdot(-F)^{\cdot i}\cdot F\right)$$

holds for any $x \in [0, \varepsilon(F)]$. Set $c := \dim \sigma(F)$.

**Claim 3.3.** *We have $c = 0$, that is, $\sigma$ maps $F$ to a point $p \in X$.*

**Proof of Claim 3.3.** We may assume that $n \geqslant 2$. Since

$$0 = (\sigma^*(-K_X)^{\cdot n-1}\cdot F) = ((-K_X)^{\cdot n-1}\cdot \sigma_* F),$$

we have $c \leqslant n-2$, that is, $F$ is exceptional over $X$. Take any $\varepsilon' \in (0, \varepsilon(F)) \cap \mathbb{Q}$ and take a sufficiently divisible $k \in \mathbb{Z}_{>0}$ such that both $-kK_X$ and $\sigma^*(-kK_X) - k\varepsilon' F$ are very ample Cartier divisors. Take $A_1, \dots, A_c \in |-kK_X|$ and $A'_1, \dots, A'_{n-2-c} \in |\sigma^*(-kK_X) - k\varepsilon' F|$ generally. By Bertini's theorem, $S := \sigma^*A_1 \cap \cdots \cap \sigma^*A_c \cap A'_1 \cap \cdots \cap A'_{n-2-c}$ is a normal surface and $C := F|_S$ is a nonzero effective $\mathbb{Q}$-Cartier divisor on $S$ such that $C$ is contracted by the birational morphism $\sigma|_S \colon S \to \sigma(S)$. By the Hodge index theorem, we have

$$0 > (C^{\cdot 2})_S = k^{n-2}\left(\sigma^*(-K_X)^{\cdot c}\cdot(\sigma^*(-K_X) - \varepsilon' F)^{\cdot n-2-c}\cdot(-F)^{\cdot 2}\right).$$

If $c \geqslant 1$, then the right-hand side of the above inequality must be equal to zero. Thus we get $c = 0$. □

We have

$$\begin{aligned} Q(x) &= x^{n-1}((-F)^{\cdot n-1}\cdot F), \\ \mathrm{vol}_Y(\sigma^*(-K_X) - xF) &= ((-K_X)^{\cdot n}) - x^n((-F)^{\cdot n-1}\cdot F) \end{aligned}$$

for any $x \in [0, \tau(F)]$. Hence we have the equality $\varepsilon(F) = \tau(F)$ by [31, Lemma 10]. □

**Remark 3.4.** Odaka and Sano showed in [40, § 5] that, if an $n$-dimensional Fano manifold $X$ satisfies that $\alpha(X) = n/(n+1)$ and $X$ is not K-stable, then any destabilizing flag ideal has zero-dimensional support. The assertion in Proposition 3.2(3) looks similar to their observation. In fact, they conjectured in [40, Conjecture 5.1] that $X$ might be isomorphic to $\mathbb{P}^n$ under the assumption from the observation.

**Proposition 3.5.** *Let $X$ be an $n$-dimensional $\mathbb{Q}$-Fano variety and $F$ be a dreamy prime divisor over $X$. Let $\sigma: Y \to X$, $\pi: Y \to Z$ and $\varepsilon(F) \in \mathbb{Q}_{>0}$ be as in Lemma 3.1. Assume that $A_X(F) \geqslant (n/(n+1))\tau(F)$ and $\beta(F) \leqslant 0$. Then we have the following:*

(1) *$A_X(F) = n$, $\tau(F) = \varepsilon(F) = n+1$ holds. Moreover, $\sigma(F)$ is a point $p \in X$.*

(2) *$\dim Z = n-1$ and $\pi|_F : F \to Z$ is an isomorphism. Moreover, any fiber of $\pi$ is one-dimensional and a general fiber $l$ of $\pi$ is isomorphic to $\mathbb{P}^1$ and $(F \cdot l) = 1$ holds.*

**Proof.** Take an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $H_Z$ on $Z$ as in Lemma 3.1(3). Since

$$\begin{aligned} 0 &\geqslant A_X(F)((-K_X)^{\cdot n}) - \int_0^{\tau(F)} \mathrm{vol}_Y(\sigma^*(-K_X) - xF)\,dx \\ &\geqslant \frac{n}{n+1}\tau(F)((-K_X)^{\cdot n}) - \int_0^{\tau(F)} \mathrm{vol}_Y(\sigma^*(-K_X) - xF)\,dx, \end{aligned}$$

we have $A_X(F) = (n/(n+1))\tau(F) = (n/(n+1))\varepsilon(F)$ and $\sigma$ maps $F$ to a point $p \in X$ by Proposition 3.2. Since

$$\pi^* H_Z \sim_{\mathbb{Q}} \sigma^*(-K_X) - \tau(F)F$$

is not big and

$$(\pi^* H_Z^{\cdot n-1} \cdot F) = \varepsilon(F)^{n-1}((-F)^{\cdot n-1} \cdot F) > 0,$$

we have $\dim Z = n-1$. Moreover, any curve in $F$ intersects $F$ negatively since $-F$ is $\sigma$-ample. Thus the morphism $\pi|_F : F \to Z$ is a finite morphism. Hence any fiber of $\pi$ is one-dimensional since $F$ is $\mathbb{Q}$-Cartier. Let $l \subset Y$ be a general fiber of $\pi$. The set of singular points of $Y$ is at most $(n-2)$-dimensional. Thus $Y$ is smooth and $F$ is Cartier around a neighborhood of $l \subset Y$. Note that

$$\begin{aligned} -(K_Y + F) &= \sigma^*(-K_X) - \frac{n}{n+1}\varepsilon(F)F, \\ \pi^* H_Z &\sim_{\mathbb{Q}} -K_Y - \left(1 + \frac{1}{n+1}\varepsilon(F)\right)F. \end{aligned}$$

In particular, $-(K_Y + F)$ is ample. Hence $l \simeq \mathbb{P}^1$, $(-K_Y \cdot l) = 2$ and $(F \cdot l) = 1$. In particular, the morphism $\pi|_F : F \to Z$ is an isomorphism since it is finite and birational with $Z$ normal. Moreover, we have

$$0 = (\pi^* H_Z \cdot l) = 2 - \left(1 + \frac{1}{n+1}\varepsilon(F)\right).$$

This implies that $\varepsilon(F) = \tau(F) = n+1$ and $A_X(F) = n$. □

## 4. A characterization of the projective space

In this section, we give a characterization of the projective space and prove Theorem 1.3.

**Theorem 4.1.** *Let $X$ be an $n$-dimensional Fano manifold and $F$ be a dreamy prime divisor over $X$. Assume that $A_X(F) \geqslant (n/(n+1))\tau(F)$ and $\beta(F) \leqslant 0$. Then $X$ is isomorphic to $\mathbb{P}^n$.*

**Proof.** We apply Proposition 3.5; $\sigma$ maps $F$ to a point $p \in X$ and $A_X(F) = n$. Note that the point $p \in X$ is a smooth point. Thus $F$ is given by the ordinary blowup (see [26, Corollary 2.31(2) and Lemma 2.45]). Thus the morphism $\sigma$ is given by the ordinary blowup along the point $p \in X$ by the uniqueness of $\sigma$ (see Lemma 3.1(2)). In particular, $Y$ is smooth and $F \simeq \mathbb{P}^{n-1}$ with $\mathcal{N}_{F/Y} \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Since $-K_Y$ and $F$ are $\pi$-ample Cartier divisors, any fiber of $\pi$ is one-dimensional and a general fiber $l$ of $\pi$ satisfies that $l \simeq \mathbb{P}^1$ and $(F \cdot l) = 1$, any fiber of $\pi$ is scheme-theoretically isomorphic to $\mathbb{P}^1$ and $F$ is a section of $\pi$. In particular, $Y \simeq \mathbb{P}_Z(\pi_* \mathcal{O}_Y(F))$ holds. Let us consider the $\pi_*$ of the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(F) \to \mathcal{N}_{F/Y} \to 0.$$

Then we get $Y \simeq \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O} \oplus \mathcal{O}(-1))$. In particular, we have $X \simeq \mathbb{P}^n$. □

We remark that Theorem 4.1 is not true if $X$ is not smooth. See the following example.

**Example 4.2.** Fix a lattice $N := \mathbb{Z}^{\oplus 2}$ and set $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Let us consider the complete fan $\Sigma$ in $N_{\mathbb{R}}$ such that

$$\{(1,0),\ (0,1),\ (-1,0),\ (-2,-3)\}$$

is the set of the generators of the one-dimensional cones in $\Sigma$. Let $Y$ be the projective toric surface associated to the fan $\Sigma$. Let $F \subset Y$ be the torus invariant curve associated to the cone $\mathbb{R}_{\geqslant 0}(-1,0) \in \Sigma$. Then there exists a projective toric birational morphism $\sigma\colon Y \to \mathbb{P}(1,2,3)$ such that $\sigma$ maps $F$ to a point, where $\mathbb{P}(1,2,3)$ is the weighted projective plane with the weights $1, 2, 3$. Set $X := \mathbb{P}(1,2,3)$. We can check that $A_X(F) = 2$. On the other hand, there is a projective toric contraction morphism $\pi\colon Y \to \mathbb{P}^1$ such that $\sigma^*(-K_X) - 3F \sim_{\mathbb{Q},\mathbb{P}^1} 0$. Thus we have $\varepsilon(F) = \tau(F) = 3$. Moreover, we get

$$\beta(F) = 2 \cdot 6 - \int_0^3 \left(6 - \frac{2}{3}x^2\right) dx = 0.$$

Of course, $X \not\simeq \mathbb{P}^2$, $\alpha(X) = 1/6(< 2/3)$ by [12, Lemma 5.1], and $X$ is not K-semistable by [22, Lemma 9.2].

**Proof of Theorem 1.3.** Assume that $X$ is not K-stable. By Theorem 2.3, there exists a dreamy prime divisor $F$ over $X$ such that $\beta(F) \leqslant 0$. By [21, Lemma 3.3], we have the inequality $A_X(F) \geqslant (n/(n+1))\tau(F)$. Thus we get $X \simeq \mathbb{P}^n$ by Theorem 4.1. This gives a contradiction since $\alpha(X) \geqslant n/(n+1)$ and $n \geqslant 2$ (see Example 1.2). Thus $X$ is K-stable. The remaining assertions are proved from [5–7, 48] and [37] (see also [40, § 1]). □

**Remark 4.3.** The author expects that Theorem 1.3 holds for $\mathbb{Q}$-Fano varieties even if Theorem 4.1 does not hold.

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