J. Fluid Mech. (2000), *vol.* 409, *pp.* 243–249. Printed in the United Kingdom

A note on the two-phase Hele-Shaw problem

By SAM D. HOWISON
Mathematical Institute, Oxford University, 24–29 St Giles, Oxford, OX1 3LB, UK

(Received 15 March 1999 and in revised form 1 May 1999)

We discuss some techniques for finding explicit solutions to immiscible two-phase flow in a Hele-Shaw cell, exploiting properties of the Schwartz function of the interface between the fluids. We also discuss the question of the well-posedness of this problem.

1. Introduction

The Hele-Shaw free boundary problem has been extensively studied over the last half-century, as witnessed by a 600-paper bibliography.† Many of these papers were stimulated by the original 'fingering' paper of Saffman & Taylor (1958), and this note takes up a point made in passing in that paper.

Most theoretical and much experimental work on the Hele-Shaw problem has been for 'one-phase' flow in which a Hele-Shaw cell contains two fluids, of which one is viscous but the other is effectively inviscid. With the simplifying assumption of constant pressure at the interface between the fluids, a great deal can be done, both in constructing explicit solutions using complex-variable methods and in developing more theoretical approaches to the questions of existence and uniqueness. When the viscosity of both fluids is significant, however, much less progress has been made, largely because the interface pressure can no longer be taken to be constant. This makes progress using complex variables much more difficult, and likewise the theoretical methods cannot easily be extended to cope with the second fluid.

The main contribution of this note is to show how a class of explicit solutions can be constructed to the two-phase (or 'Muskat') problem. This is described in §4; §§2 and 3 describe the model and some of its general properties.

2. The problem

We discuss the following model (Saffman & Taylor 1958) for immiscible two-phase flow in a horizontal Hele-Shaw cell. The fluid velocity in each phase is given by

$$\boldsymbol{v}_i = -k_i \nabla p_i, \quad i = 1, 2,$$

where $p_i(x, y, t)$ is the pressure in the region Ω_i occupied by fluid i of viscosity μ_i, $k_i = h^2/12\mu_i$ are the fluid mobilities, and h is the cell gap. For incompressible flow we have

$$\nabla^2 p_i = 0 \quad \text{in } \Omega_i.$$

At the interface Γ separating the fluids, which we assume for simplicity to have only one component, we assume the simple conditions

$$p_1 = p_2 \tag{2.1}$$

† To be found at www.maths.ox.ac.uk/~howison/Hele-Shaw/.

and

$$-k_1\frac{\partial p_1}{\partial n} = -k_2\frac{\partial p_2}{\partial n} = V_n, \tag{2.2}$$

where $\partial/\partial n$ is the derivative normal to Γ and V_n is the normal velocity of Γ. Note that the effects of surface tension are ignored in these conditions. The model is complemented by appropriate singularities representing the driving mechanism for the fluid motion, and by fixed boundary conditions as necessary.

The linear stability of a planar interface is easy to establish (Saffman & Taylor 1958). A routine analysis shows that if fluid 1 is to the right of a slightly perturbed planar interface $x = Vt + \epsilon e^{\alpha t}\sin ny$ and fluid 2 to its left, then ignoring terms of $O(\epsilon^2)$,

$$\alpha = n\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}V = n\frac{k_2 - k_1}{k_2 + k_1}V. \tag{2.3}$$

An interface with $V > 0$ is therefore unstable if the less viscous fluid displaces the more viscous one, a fact to which we return below.

3. The one-phase problem

When there is just one viscous liquid, say with $\mu_2 = 0$ ($k_2 = \infty$), the pressure in the inviscid liquid is constant, equal to zero without loss of generality. We then have the commonly-studied one-phase Hele-Shaw problem for $\phi_1 = -p_1$,

$$\nabla^2\phi_1 = 0 \quad \text{in} \quad \Omega_1, \qquad \phi_1 = 0, \quad k_1\frac{\partial \phi_1}{\partial n} = V_n \quad \text{on} \quad \Gamma.$$

We need only the following well-known result from the many available for this problem: if Γ is written in the form $\bar{z} = g(z,t)$, where $z = x + iy$ and $g(z,t)$ is known as the *Schwarz function* for Γ (Davis 1974), then the complex potential $w_1(z,t) = -p_1 + i\psi_1 = \phi_1 + i\psi_1$ satisfies

$$k_1\frac{\partial w_1}{\partial z} = \frac{1}{2}\frac{\partial g}{\partial t}. \tag{3.1}$$

This result follows from the facts that, using s as arclength along Γ,

$$\partial z/\partial s = (\partial g/\partial z)^{-1/2}, \qquad V_n = -(i/2)(\partial g/\partial t)(\partial g/\partial z)^{-1/2},$$

so that, differentiating along Γ,

$$k_1\frac{\partial w}{\partial z} = k_1\frac{\partial w}{\partial s}\Big/\frac{\partial z}{\partial s} = -k_1\left(\frac{\partial p}{\partial s} + i\frac{\partial p}{\partial n}\right)\Big/\frac{\partial z}{\partial s} = -i\left(\frac{\partial g}{\partial z}\right)^{1/2}(-V_n) = \frac{1}{2}\frac{\partial g}{\partial t}.$$

The evolution of Γ, if known, thus determines the complex potential, i.e. the solution of the Cauchy problem for the complex potential is given by (3.1). In general, of course, the potential thus generated has singularities, and this can be exploited to generate explicit solutions of the 'direct' problem (see Cummings, Howison & King 1999 for a review of the procedure for both Hele-Shaw and Stokes flows). Here, however, we use (3.1) to generate 'indirect' solutions of the two-phase problem, in a manner we now describe.

4. Explicit solutions to the two-phase problem

There appear to be very few non-trivial explicit solutions to the Muskat problem in which both fluids are viscous. The only ones of which I am aware are the unsteady solutions of Jacquard & Séguier (1962), and the comment by Saffman & Taylor

(1958) that the air inside their fingers could be replaced by a viscous fluid without changing the interface shape. Jacquard & Séguier deal with flow in a parallel-sided channel, with fluids 1 and 2 on either side of Γ which meets both walls and divides the fluid into two domains; fluid 1 is removed at a constant rate from $x = +\infty$ and fluid 2 is injected at $x = -\infty$. The key to the solution is the remarkable observation that, if the whole strip is conformally mapped onto a half-plane $\mathrm{Re}\,\zeta > 0$ by an exponential transformation, then it is possible to satisfy *all* the boundary conditions on Γ if it is mapped onto a semi-circular arc centred on the line $\mathrm{Re}\,\zeta = 0$, provided that the centre and radius of the arc satisfy certain ordinary differential equations. In the resulting solution, in the unstable case the interface evolves from a small nearly sinusoidal perturbation of a line into a Saffman–Taylor finger occupying half the width of the channel, and it has precisely the same shape as the unsteady solutions of Saffman (1959), described in more detail below.

Saffman & Taylor (1958) found a one-parameter family of steadily-translating finger solutions to the one-phase problem for a fluid of viscosity μ_1 in a parallel-sided channel with uniform extraction from $x = +\infty$ with speed V, the parameter λ being the fraction of the channel occupied by the finger at $x = -\infty$. They noted in passing that their finger also provides a solution to the two-phase problem, in which the air inside the finger is replaced by a fluid of viscosity μ_2 moving with uniform velocity $(U, 0) = (V/\lambda, 0)$ equal to that of the finger. Only the pressure field is different; it is equal to its previous value plus the linear function $-(x - Ut)/k_2$, the latter sufficing to drive the second fluid.

This observation was the motivation for the following procedure for calculating two-phase solutions in an 'inverse' manner.†

(*a*) First, construct a complex potential $w_2(z, t)$ which is analytic not just in Ω_2 but in the *whole* domain $\Omega = \Omega_2 \cup \Gamma \cup \Omega_1$. If we knew Ω_2, this process would entail analytic continuation; the restriction in the method comes in our assumption that there are no additional singularities (this can be relaxed somewhat to allow for known singularities, as we see below).

(*b*) Assume that a fluid of mobility k_2 flows in all of Ω. Choose an initial curve Γ_0 and follow its evolution as a material curve under the flow generated by $w_2(z, t)$. This curve is to be Γ, and the region on one side of it is to be Ω_2.

(*c*) Calculate the correction to the potential needed to make the fluid in Ω_1, on the other side of Γ, satisfy continuity of pressure and the kinematic boundary condition on Γ. As the flow in Ω_2 automatically satisfies the kinematic condition, this means that equations (2.1) and (2.2) are both satisfied. As this step involves the solution of a Cauchy problem for the potential, singularities may be expected to occur.

It is evident from the comments made on steps (*a*) and (*c*) that the method is restricted in its scope.

Let us now consider the steps in turn. The first requires no comment; some examples are given below. To achieve the second step, we need to follow a curve $\Gamma : z = z(t)$ whose complex velocity is $k_2 \partial w_2/\partial z$, so that

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} = k_2 \frac{\partial w_2}{\partial z}. \tag{4.1}$$

Writing Γ as $\bar{z} = g(z, t)$, we have that, on Γ,

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} = \frac{\partial g}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial g}{\partial t}.$$

† It is worth noting that a similar procedure can be applied to irrotational inviscid flows.

We can substitute for $d\bar{z}/dt$ from (4.1), and dz/dt is eliminated by taking the conjugate of (4.1) to give

$$\frac{dz}{dt} = k_2 \overline{\frac{\partial w_2}{\partial z}(z,t)} = k_2 \frac{\partial}{\partial z}\bar{w}_2(\bar{z},t) = k_2 \frac{\partial}{\partial z}\bar{w}_2\left(g(z,t),t\right) = k_2 \frac{\partial \bar{w}_2}{\partial z}\left(g(z,t),t\right)\frac{\partial g}{\partial z},$$

where $\bar{w}_2(z,t) = \overline{w_2(\bar{z},t)}$ is analytic if w_2 is. We thus have that, on Γ,

$$\frac{1}{k_2}\frac{\partial g}{\partial t} + \frac{\partial \bar{w}_2}{\partial z}\left(g(z,t),t\right)\frac{\partial g}{\partial z} = \frac{\partial w_2}{\partial z}(z,t), \tag{4.2}$$

which also holds away from Γ by analytic continuation.

The characteristic equations for (4.2) are

$$dz \Big/ \frac{\partial \bar{w}_2}{\partial z}(g,t) = dg \Big/ \frac{\partial w_2}{\partial z}(z,t) = dt \Big/ \frac{1}{k_2},$$

from which we immediately find that

$$\bar{w}_2\left(g(z,t),t\right) - w_2(z,t) \quad \text{is constant along characteristics.} \tag{4.3}$$

This result, which is interesting in its own right, relates the potential on one side of Γ to its value on the other by Schwarz reflection, and it holds for any potential flow. Lastly, the remaining characteristic equation must be solved for either z or g, which is unfortunately rarely easy to do.

Having calculated the evolution of Γ, it is straightforward to find the correction to the potential in Ω_1. Writing $w_1(z,t) = w_2(z,t) + \tilde{w}_1(z,t)$, the problem for $\tilde{\phi}_1 = \mathrm{Re}\,\tilde{w}_1$ is

$$\nabla^2\tilde{\phi}_1 = 0 \quad \text{in } \Omega_1, \qquad \tilde{\phi}_1 = 0, \quad -\frac{k_2 k_1}{k_2 - k_1}\frac{\partial \tilde{\phi}_1}{\partial n} = V_n \quad \text{on } \Gamma.$$

It follows immediately from (3.1) that

$$\frac{k_2 k_1}{k_2 - k_1}\frac{\partial \tilde{w}_1}{\partial z} = \frac{1}{2}\frac{\partial g}{\partial t},$$

and since g is known, we have found $\tilde{w}_1$. Of course, $\tilde{w}_1$ is the potential for a one-phase problem with this free boundary and mobility $k_2k_1/(k_2 - k_1)$; if we already know a solution to the one-phase problem with this free boundary with mobility k_1 and potential W_1, then we need not calculate g, since we can immediately write down $\tilde{w}_1 = (k_2 - k_1)W_1/k_2$.

4.1. *Examples*

The main difficulty with the procedure outlined above is that it is necessary to solve (4.3) for either g or z, and in practice this is usually difficult. Nevertheless, we can give some examples.

Travelling-wave solutions. If the interface translates uniformly with velocity $(U, 0)$, then its Schwarz function is $\bar{z} = Ut + g_0(z - Ut)$, where $g_0(z)$ is the Schwarz function at $t = 0$. If the fluid in region 2 also has velocity $(U, 0)$, then $w_2(z,t) = \bar{w}_2(z,t) = Uz/k_2$, and then the correction potential $\tilde{w}_1$ can be that of any travelling-wave Hele-Shaw flow. If, for example, the free boundary is $x = Ut$, we have $g_0(z) = -z$, $\tilde{w}_1 = U(z - Ut)(k_2 - k_1)/k_2k_1$, and $w_1 = \tilde{w}_1 + w_2 = U(z - Ut)/k_1 + U^2t/k_2$ as required. Less trivially, suppose that the free boundary is the Saffman–Taylor finger with parameter λ, and that the fluid velocity at $+\infty$ is $(V, 0)$. Writing W_1 for the potential of this flow with mobility k_1 as above, we have that $\tilde{w}_1 = (k_2 - k_1)W_1/k_2$. Now we know that as $x \to +\infty$, $W_1 \sim \lambda Uz/k_1$ and we require that $w_1 \sim Vz/k_1$. Substituting into

$w_1 = w_2 + \tilde{w}_1$, we see that the finger speed U and the extraction speed V are related by

$$U = \frac{k_2 V}{(1-\lambda)k_1 + \lambda k_2}, \quad 0 < \lambda < 1,$$

which is simply conservation of mass from $x = -\infty$ to $x = +\infty$. Note that for $k_2 < \infty$, U is bounded as $\lambda \to 0$, which it is not in the one-phase case; we return to this below. Of course, any other travelling-wave solution of the one-phase problem, such as the Ivantsov parabola in which the free boundary is a parabola translating along its axis, can also be realised in the two-phase case.

Radially symmetric solutions. These are trivial to calculate and we omit the details.

Stagnation point flow. Suppose that fluid 2 has the stagnation-point potential $w_2(z,t) = Az^2/2k_2$, and that the initial interface is the line $x = -a$, $a > 0$. Then $g_0(z) = g(z,0) = 2a - z$, and the equation for $g(z,t)$ is

$$\frac{\partial g}{\partial t} + Ag\frac{\partial \bar{g}}{\partial z} = Az,$$

so that $g(z,t) = 2a\mathrm{e}^{At} - z$. The interface remains a straight line moving towards the origin (but never reaching it) for $A < 0$. (If the initial interface is a line not parallel to one of the coordinate axes, it remains straight and rotates as well as translating.) The correction to the potential in Ω_1 is the linear function $\tilde{w}_1 = (k_2 - k_1)a\mathrm{e}^{At}(z - A\mathrm{e}^{At})$.

The solution of Jacquard & Séguier. This solution to the two-phase problem has a free boundary whose shape at each time is the same as that of Saffman's (1959) solution with $\lambda = \frac{1}{2}$, but whose time evolution is different. If there is just one phase, the potential W_1 for Saffman's solution is related to z by the map

$$z = \frac{k_1 W_1}{V} + d(t) + \log \tfrac{1}{2}(1 + a(t)\mathrm{e}^{-k_1 W_1/V}) = F(W_1, t), \text{ say,}$$

and since this is also a mapping from the right-hand half-plane onto Ω_1 extended periodically across the channel walls, we have that $g(z,t) = \bar{F}(-W_1, t)$. After some simplification, we find that

$$\frac{k_1 W_1(z,t)}{V} = \log(2\mathrm{e}^{z-d} - a) = z + \log(2\mathrm{e}^{-d} - a\mathrm{e}^{-z}),$$

$$g(z,t) = d - z - \log(2\mathrm{e}^{-d} - a\mathrm{e}^{-z}) + \log \tfrac{1}{2}(1 - a^2 + 2a\mathrm{e}^{z-d}).$$

The first of these formulae shows that the fluid moves under the potential of a uniform stream plus a source at $z = d + \log(a/2)$, outside Ω_1. It also shows that the pressure due to this source term alone varies linearly in x on the isobars of the combined pressure. The second shows that g has a singularity within the fluid, and the constancy of this in time (it must not contribute to $\partial g/\partial t$ which is proportional to $\partial W_1/\partial z$) leads to two relations between a and d, and thence to the interface shape as

$$\cos y = \frac{\sqrt{1 - a_0^2}}{a_0}\mathrm{e}^{-Vt}\sinh(x - Vt), \tag{4.4}$$

where a_0 is the initial value of a and $d(0)$ has been chosen appropriately. The interface is symmetric about $x = Vt$.

When there are two phases, motivated by the presence of two source-type singularities in $g(z,t)$ for the one-phase problem, we seek a solution in which the interface has the same shape as, but different time evolution from, the one-phase case, and in which each phase moves under a potential equivalent to a uniform stream together with a moving source lying outside the fluid. The sources for phase 2 lie on the channel walls, and those for phase 1 on the real axis, and they are symmetrically disposed with respect to the interface. Taking into account the periodicity of the flow in y, imposed by the zero-flux conditions on the channel walls $y = \pm\pi$, the potentials have the forms

$$w_1(z) = \frac{Vz}{k_1} + Q_1 \log(\mathrm{e}^{-z} + D_1(t)) + F_1(t), \quad w_2(z) = \frac{Vz}{k_2} + Q_2 \log(\mathrm{e}^{-z} - D_2(t)) + F_2(t)$$

(in fact both the functions of time vanish with an appropriate choice of origin). Matching of the pressure on the interfaces, taking into account the linear behaviour mentioned above and the disposition of the sources, shows that $Q_1 + Q_2 = V/k_1 - V/k_2$ (which can also be deduced from the argument that follows). Furthermore, as the interface is an isobar of $z + \log(2\mathrm{e}^{-d} - a\mathrm{e}^{-z})$, its symmetry about its midline $x = d + \log 2 + \frac{1}{2}\log(1 - a^2)$ shows that it is also an isobar of $-z + \log(1 - a^2 + 2a\mathrm{e}^{z-d})$. The normal velocities at the interface can therefore be balanced as well as the pressures provided that we take $D_1 = 2\mathrm{e}^{-d}/a$, $D_2 = (1 - a^2)\mathrm{e}^{d}/2a$, and $Q_1/k_1 = Q_2/k_2$. The easiest way to find the time-dependence is to recall that

$$\frac{\partial w_1}{\partial z} - \frac{\partial w_2}{\partial z} = \frac{1}{2}\frac{k_2 - k_1}{k_2 k_1}\frac{\partial g}{\partial t},$$

from which comparison of coefficients verifies the relations already found, and gives differential equations for a and d, whose solution is

$$\frac{\sqrt{1 - a^2}}{a} = \frac{\sqrt{1 - a_0^2}}{a_0}\mathrm{e}^{-Vt(k_2 - k_1)/(k_2 + k_1)}, \quad d + \tfrac{1}{2}\log(1 - a^2) = Vt + \log 2.$$

Lastly the interface shape is

$$\cos y = \frac{\sqrt{1 - a_0^2}}{a_0}\mathrm{e}^{-Vt(k_2 - k_1)/(k_2 + k_1)}\sinh(x - Vt).$$

The one-phase case can be recovered in the limit $k_2 \to \infty$. Jacquard & Séguier remark of their solution ‘c’est là un hasard que rien ne pouvait laisser prévoir’. We have derived it using a more systematic approach, however, which may allow further solutions to be constructed.

5. Discussion

Although explicit solutions have intrinsic interest, our main reason for trying to construct them is the hope that they may shed some light on the question of well-posedness for the Muskat problem, about which very little appears to be known. In the one-phase (Hele-Shaw) case, it may loosely be stated that problems in which the fluid domain is expanding are well-posed, while those in which it contracts are not. This is true for weak solutions as treated by, say, Elliott & Janovsky (1981), and although classical solutions can have singular behaviour (for example when the topology changes), the same dichotomy is generally observed. In particular, most contraction problems lead to blow-up via some kind of singularity in the free boundary; often, this is a cusp. Likewise, the speed of a Saffman–Taylor finger becomes infinite as $\lambda \to 0$.

Unfortunately it is much less easy to construct weak solutions to the two-phase problem (see Otto 1999 for an approach in which a phase function is used to ‘smear out’ the distinction between the two fluids) and evidence concerning its well-posedness or otherwise is hard to come by. Insofar as there is a general view, it is probably fair to say that the ‘folklore’ has it that if a Muskat problem has a moving boundary in which a less viscous fluid displaces a more viscous one, it is probably ill-posed, at least as far as classical solutions are concerned. The only concrete evidence to back up this view is the linear stability result (2.3), in which the large growth rate of short-wavelength disturbances might be thought to promote blow-up: even though the result is only local, since it is present at all length scales its influence might be more global, as it is in the one-phase problem. Set against this, though, is the fact that the maximum velocity of a Saffman–Taylor finger with two fluids is bounded when both mobilities are finite. This implies that the speed of a parabolic interface (the inner solution near the tip of a finger with $\lambda \ll 1$) is bounded (Otto 1997 shows that the 'mixing zone' for a weak solution in the Saffman–Taylor geometry grows linearly in time with the same bound for its growth). If one were to contemplate the development of a cusp in a two-phase free boundary, before blow-up its tip would also be approximately parabolic and have bounded speed. It is not clear whether this shows that cusps cannot form (in finite or infinite time); but it does indicate that if they do so, they have finite speed at their formation, leading one to speculate that they may propagate with finite speed rather than existing instantaneously with infinite tip speed. This argument is backed up by the physical observation that the blow-up of a one-phase problem is due to a feedback: as a nascent cusp in a retreating boundary protrudes further into the fluid than neighbouring points, the constancy of the pressure on the free boundary forces the pressure gradient directly ahead of the cusp to increase by more than that nearby, thereby accelerating the instability. In a two-phase problem, the second fluid occupies the interior of the protuberance, and can absorb some of the pressure gradient, mitigating the tendency of the protuberance to grow.

In summary, we have discussed some approaches to finding explicit solutions to Muskat problems, and we have made some speculations concerning the global behaviour of solutions in general. It remains to be seen whether explicit solutions can be as helpful in this respect as they have been for the Hele-Shaw problem.

This paper is dedicated to Philip Saffman, in recognition of his enormous contribution to fluid mechanics in general, and the Hele-Shaw problem in particular. I am grateful for helpful discussions with Peter Howell and John Ockendon.

REFERENCES

Cummings, L., Howison, S. D. & King, J. R. 1999 Two-dimensional Stokes and Hele-Shaw flow with free surfaces. *Eur. J. Appl. Maths* **10**, 681–706.

Davis, P. J. 1974 *The Schwarz Function and its Applications.* Carus Mathematical Monographs, vol. 17. Math. Assoc. of America.

Elliott, C. M. & Janovsky, V. 1981 A variational inequality approach to Hele-Shaw flow with a moving boundary. *Proc. R. Soc. Edinb.* A **88**, 93–107.

Jacquard, P. & Séguier, P. 1962 Mouvement de deux fluides en contact dans un milieu poreux. *J. de Méc.* **1**, 367–394.

Otto, F. F. 1997 Viscous fingering: an optimal bound on the growth rate of the mixing zone. *SIAM J. Appl. Maths* **57**, 982–990.

Otto, F. F. 1999 Evolution of microstructure in unstable porous media flow: a relaxational approach. *Commun. Pure Appl. Maths* **52**, 873–915.

Saffman, P. G. 1959 Exact solutions for the growth of fingers from a flat interface between two fluids in a porous medium or Hele-Shaw cell. *Q. J. Mech. Appl. Maths* **12**, 146–150.

Saffman, P. G. & Taylor, G. I. 1958 The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. *Proc. R. Soc. Lond.* A **245**, 312–329.