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ENDOSCOPIC REPRESENTATIONS OF $\widetilde{\mathrm{Sp}}_{2n}$

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Abstract In this paper, we construct explicitly endoscopic representations of $\widetilde{\mathrm{Sp}}_{2n}$, the metaplectic cover of a symplectic group of rank n. We do this in the automorphic case, and also in the local case, over a p-adic field.

Keywords: metaplectic groups; Fourier–Jacobi coefficients; L-functions; functoriality; endoscopy

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Introduction

In this paper, we construct explicitly endoscopic representations of $\widetilde{\mathrm{Sp}}_{2n}$, the metaplectic cover of a symplectic group of rank n. We do this in the automorphic case, and also in the local case, over a p-adic field. In more detail, let K be a number field, and let $\mathbb{A}$ denote the adele ring of K. Respectively, let F denote a local non-archimedean field (of characteristic zero). Fix a non-trivial character ψ of $K\backslash\mathbb{A}$ (respectively, of F).

Global Case. *Let $\tau_1\dots\tau_r$ be pairwise inequivalent, irreducible, automorphic, cuspidal and self-dual representations of $\mathrm{GL}_{2m_1}(\mathbb{A}),\dots,\mathrm{GL}_{2m_r}(\mathbb{A})$, respectively; $n=m_1+\cdots+m_r$. Assume, for each $1\leqslant i\leqslant r$, that $L^S(\tau_i,\Lambda^2,s)$ has a pole at $s=1$, and that $L(\tau_i,\frac{1}{2})\neq 0$.*

Main (global) theorem. *There exist irreducible, automorphic, cuspidal (genuine) representations σ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which have a non-trivial ψ-Whittaker coefficient, such that the (weak) ψ-functorial lift of σ to $\mathrm{GL}_{2n}(\mathbb{A})$ is the Eisenstein series, induced from $\tau_1\otimes\cdots\otimes\tau_r$ (evaluated at $(0\cdots0)$).*

Recall that there is no canonical way to associate to σ_ν, at a place ν, where σ_ν is unramified, a conjugacy class of unramified parameters. We have to first fix a choice of a non-trivial character of K_ν. We choose ψ_ν. The ψ_ν-unramified parameters of σ_ν are the unramified parameters of $\theta_{\psi_\nu}(\sigma_\nu)$, the unramified representation of $\mathrm{SO}_{2n+1}(K_\nu)$, obtained from σ_ν, by the local theta correspondence θ_{ψ_ν}.

Main (local) theorem. *Let $\tau_1, \ldots, \tau_r$ be pairwise inequivalent, irreducible, supercuspidal and self-dual representations of $\mathrm{GL}_{2m_1}(F), \ldots, \mathrm{GL}_{2m_r}(F)$, respectively; $n = m_1 + \cdots + m_r$. Assume that, for each $1 \leqslant i \leqslant r$, $L(\tau_i, \Lambda^2, s)$ has a pole at $s = 0$. Then there exists a unique, irreducible, supercuspidal (genuine) representation σ of $\widetilde{\mathrm{Sp}_{2n}}(F)$, which has a ψ^{-1}-Whittaker model, such that (the local gamma factor) $\gamma(\sigma \otimes \tau_i, s, \psi)$ has a pole at $s = 1$, for each $1 \leqslant i \leqslant r$.*

Our two main theorems justify, in each case, the title 'endoscopic' for σ, and we note that σ is a ψ^{-1}-generic member of 'the endoscopic L-packet on $\widetilde{\mathrm{Sp}_{2n}}(\mathbb{A})$ (respectively, $\widetilde{\mathrm{Sp}_{2n}}(F)$) determined by $\tau_1 \otimes \cdots \otimes \tau_r$'.

The construction of the representation σ is by the method developed in [**5–7**]. We review this in the beginning of § 1. In brief, starting with an irreducible, automorphic, cuspidal representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$, such that $L^S(\tau, \Lambda^2, s)$ has a pole at $s = 1$ and $L(\tau, 12) \neq 0$, we considered a certain Fourier–Jacobi coefficient, stabilized by $\widetilde{\mathrm{Sp}_{2n}}(\mathbb{A})$, on $\mathrm{Res}_{s=1} E_{\tau,s}$, where $E_{\tau,s}$ is the Eisenstein series on $\mathrm{Sp}_{4n}(\mathbb{A})$, induced from $\tau \otimes |\det \cdot|^{s-(1/2)}$, on the Siegel parabolic subgroup. This Fourier–Jacobi coefficient affords an $\widetilde{\mathrm{Sp}_{2n}}(\mathbb{A})$-automorphic module $\sigma_\psi(\tau)$, which we proved to be non-trivial, cuspidal and ψ^{-1}-generic. Moreover, $\sigma_\psi(\tau)$ is a multiplicity free representation, and each of its summands is ψ^{-1}-generic and ψ-weakly lifts to τ. Any genuine, irreducible, automorphic, cuspidal, ψ^{-1}-generic representation which ψ-weakly lifts to τ has a non-trivial L^2-pairing with a (unique) summand of $\sigma_\psi(\tau)$. In particular, if V_π is an irreducible space of genuine cusp forms, orthogonal to the space of $\sigma_\psi(\tau)$, where the corresponding automorphic representation π, ψ-weakly lift to τ, then V_π has zero ψ-Whittaker coefficient. We conjecture that $\sigma_\psi(\tau)$ is actually irreducible. In this paper, we use the same construction, only now, we apply it not to a cuspidal τ, but rather to an Eisenstein series on $\mathrm{GL}_{2n}(\mathbb{A})$, induced from $\tau_1 \otimes \cdots \otimes \tau_r$ (evaluated at $(0, \ldots, 0)$.) The generalization is not automatic. In § 1, we point out the new problems that we have to face and how to solve them. The analogous local theorem is similar in nature, and is proved in § 5. In future publications, we hope to generalize the results of this paper and those of [**5–7**] to a general classical group.

Finally, let us review some of the notation we use in the paper. The elements of the symplectic group Sp_{2k} are written with respect to

$$\begin{pmatrix} & w_k \\ -w_k & \end{pmatrix}, \quad \text{where } w_k = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

($k \times k$ matrix). V_k (respectively, Z_k) denotes the standard maximal unipotent subgroup of Sp_{2k} (respectively, GL_k). We let P_k denote the Siegel parabolic subgroup of Sp_{2k}. The elements of the Levi part of P_k have the form

$$m(a) = \begin{pmatrix} a & \\ & a^* \end{pmatrix}$$

for $a \in \mathrm{GL}_k$, where $a^* = w_k {}^t a^{-1} w_k$. The elements of the unipotent radical have the form

$$\ell(x) = \begin{pmatrix} I_k & x \\ & I_k \end{pmatrix}$$

where $w_k x$ is a symmetric $(k \times k)$ matrix. We also put

$$\bar{\ell}(x) = \begin{pmatrix} I_k & \\ x & I_k \end{pmatrix}.$$

If $r_1 + r_2 + \cdots + r_e = k$, we denote by $Q_{r_1, r_2, \ldots, r_e}$ (respectively, $P_{r_1 r_2, \ldots, r_e}$) the standard parabolic subgroup of Sp_{2k} (respectively, GL_k), with Levi part isomorphic to $\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_e}$, consisting of the elements $m(a)$ (respectively, a), where

$$a = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_e \end{pmatrix}, \quad g_i \in \mathrm{GL}_{r_i}.$$

We also use the notation $Q_{\bar{r}}$ (respectively, $P_{\bar{r}}$) where $\bar{r} = (r_1, r_2, \ldots, r_e)$.

Let F be a local field, and ψ a non-trivial character of F. Denote by ψ_k the character of $V_k(F)$ defined by

$$\psi_k(v) = \psi\biggl(\sum_{i=1}^{k} v_{i,i+1}\biggr), \quad v \in V_k(F)$$

ψ_k is the standard non-degenerate (Whittaker) character of $V_k(F)$ corresponding to ψ. When we speak of a ψ-Whittaker functional, or a ψ-generic representation of $\mathrm{Sp}_{2k}(F)$ (or $\widetilde{\mathrm{Sp}}_{2k}(F)$), we refer to ψ_k. Similarly, for a number field K and its ring of adeles $\mathbb{A}$, starting with a non-trivial character ψ of $K\backslash\mathbb{A}$, we define ψ_k on $V_k(\mathbb{A})$ (trivial on $V_k(K)$), as before, and for an automorphic form f on $\mathrm{Sp}_{2k}(\mathbb{A})$ (or $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$), the ψ-Whittaker coefficient of f is

$$\int_{V_k(K)\backslash V_k(\mathbb{A})} f(n)\psi_k(n)\,\mathrm{d}n.$$

As f varies in the space of an automorphic representation η, we view the last integral as a linear functional on (the space of) η.

If U is a unipotent group, with points in a p-adic field, and χ is a character of U, we denote by $J_{U,x}$ the corresponding Jacquet functor. We also denote $J_U = J_{U,1}$. For a representation τ of a group G, we denote by V_τ a space of its realization. If τ has a central character, we denote it by ω_τ.

1. Some preliminaries and statement of the main global theorem

1.1. A review

In [**5**–**7**] we constructed explicitly the inverse to the functorial lift from $\widetilde{\mathrm{Sp}}_{2n}$ to GL_{2n}. More precisely, let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{A})$,

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such that $L^S(\tau, \Lambda^2, s)$ has a pole at $s = 1$ and $L(\tau, 12) \neq 0$. Here $\mathbb{A}$ is the adele ring of a number field K. Fix a non-trivial character ψ of $K\backslash\mathbb{A}$. Then we gave an explicit construction of an automorphic (non-trivial) cuspidal representation $\sigma_\psi(\tau)$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which is the direct sum of all (up to isomorphism) ψ^{-1}-generic automorphic, irreducible, cuspidal representations σ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, such that at almost all places ν, the ψ_ν-unramified parameters of σ_ν are those of τ_ν. Note that there is no canonical way to associate to such σ_ν a conjugacy class of unramified parameters. We have to fix first a choice of a non-trivial character ψ_ν of K_ν (see [**4**, § 3.1]). Denote by $\theta_{\psi_\nu}(\sigma_\nu)$ the unramified representation of $\mathrm{SO}_{2n+1}(K_\nu)$, associated to σ_ν by the local theta correspondence θ_{ψ_ν}, with respect to ψ_ν, from $\widetilde{\mathrm{Sp}}_{2n}(K_\nu)$ to $\mathrm{SO}_{2n+1}(K_\nu)$. Then the ψ_ν-unramified parameters of σ_ν are the unramified parameters of $\theta_{\psi_\nu}(\sigma_\nu)$. Thus, one should think of the L-group of $\widetilde{\mathrm{Sp}}_{2n}$ as that of SO_{2n+1}, i.e. $\mathrm{Sp}_{2n}(\mathbb{C})$. The ψ-weak lifting of σ above to τ is with respect to the standard embedding of L groups $\mathrm{Sp}_{2n}(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{C})$.

The construction of $V_{\sigma_\psi(\tau)}$, the space of $\sigma_\psi(\tau)$ is as follows. Let P_{2n} be the Siegel parabolic subgroup of Sp_{4n}. Consider the representation

$$\rho_{\tau,s} = \mathrm{Ind}_{P_{2n}(\mathbb{A})}^{\mathrm{Sp}_{4n}(\mathbb{A})} \tau \otimes |\det \cdot|^{s-(1/2)}, \tag{1.1}$$

where we use normalized induction. Let $\varphi_{\tau,s}$ be a holomorphic section in $\rho_{\tau,s}$. We think of $\varphi_{\tau,s}$ as a complex function on $\mathrm{Sp}_{4n}(\mathbb{A}) \times \mathrm{GL}_{2n}(\mathbb{A})$, such that $r \mapsto \varphi_{\tau,s}(g; r)$ is a cusp form in the space of τ and

$$\varphi_{\tau,s}\left(\begin{pmatrix} a & * \\ 0 & a^* \end{pmatrix} g, r\right) = |\det a|^{s+n} \varphi_{\tau,s}(g; ra). \tag{1.2}$$

Put

$$f_{\tau,s}(g) = \varphi_{\tau,s}(g; I_{2n}). \tag{1.3}$$

Consider the Eisenstein series

$$E_{\tau,s}(g) = E(g, \varphi_{\tau,s}) = \sum_{g \in P_{2n}(K)\backslash \mathrm{Sp}_{4n}(K)} f_{\tau,s}(\gamma, g).$$

The series converges absolutely for $\mathrm{Re}(s) > n+1$, and admits a meromorphic continuation to the whole plane. The assumptions on τ imply that $E_{\tau,s}$ has a simple pole at $s = 1$ [**5**, Proposition 1]. Denote

$$E_{\tau,1} = \mathrm{Res}_{s=1} E_{\tau,s}.$$

The elements of $V_{\sigma_\psi(\tau)}$ are certain Fourier–Jacobi coefficients of $E_{\tau,1}$. For this, we have to introduce more notation. Consider the following unipotent subgroups (these are unipotent radicals of standard parabolic subgroups)

$$N_i = \left\{ \begin{pmatrix} z & * & * \\ & I_{2i} & * \\ & & z^* \end{pmatrix} \in \mathrm{Sp}_{4n} \mid z \in Z_{2n-i} \right\}, \tag{1.4}$$

where Z_m is the standard maximal unipotent subgroup of GL_m. Here $0 \leqslant i < 2n$. Let χ_k be the following character of $N_{k+1}(\mathbb{A})$

$$\chi_k(v) = \psi\biggl(\sum_{j=1}^{2n-k-1} v_{j,j+1}\biggr), \quad v \in N_{k+1}(\mathbb{A}). \tag{1.5}$$

This is the restriction to $N_{k+1}(\mathbb{A})$ of the standard generic character defined by ψ. Consider the following subgroup of N_k,

$$H_k = \left\{ h = \begin{pmatrix} I_{2n-k-1} & & & & \\ & 1 & x & z & \\ & & I_{2k} & x' & \\ & & & 1 & \\ & & & & I_{2n-k-1} \end{pmatrix} \in \mathrm{Sp}_{4n} \right\}. \tag{1.6}$$

H_k is isomorphic to the Heisenberg group $\mathcal{H}_k$ on the $2k$-dimensional row vector space over K, equipped with the symplectic form defined by

$$2\begin{pmatrix} & w_k \\ -w_k & \end{pmatrix}, \quad \text{where } w_k = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}.$$

The isomorphism is given by

$$j_k(x; z) = h, \tag{1.7}$$

for h, x, z as in (1.6).

Let $\omega_\psi^{(k)}$ be the Weil representation of $\mathcal{H}_k(\mathbb{A}) \rtimes \widetilde{\mathrm{Sp}_{2k}}(\mathbb{A})$, which corresponds to the character $(0, z) \mapsto \psi(z)$ of the centre of $\mathcal{H}_k(\mathbb{A})$. $\omega_\psi^{(k)}$ acts on $S(\mathbb{A}^k)$, the space of Schwartz–Bruhat functions on $\mathbb{A}^k$. Denote, for $\phi \in S(\mathbb{A}^k)$, the corresponding theta series

$$\theta_{\psi,k}^{\phi}(h \cdot (g, \epsilon)) = \sum_{x_j \in K} \omega_\psi^{(k)}(h \cdot (g, \epsilon))\phi(x_1, \dots, x_k), \tag{1.8}$$

for $\in \mathcal{H}_k(\mathbb{A})$, $(g, \epsilon) \in \widetilde{\mathrm{Sp}_{2k}}(\mathbb{A})$. Extend j_k to an embedding of $\mathcal{H}_k \rtimes \mathrm{Sp}_{2k}$ inside Sp_{4n} by

$$j_k(g) = \begin{pmatrix} I_{2n-k} & & \\ & g & \\ & & I_{2n-k} \end{pmatrix}, \quad g \in \mathrm{Sp}_{2k}. \tag{1.9}$$

Let $V_{\sigma_{\psi,k}(\tau)}$ be the space of automorphic forms on $\widetilde{\mathrm{Sp}_{2k}}(\mathbb{A})$ generated by

$$\begin{aligned} p_k(g, \epsilon) &= p_k((g, \epsilon), \varphi_{\tau,1}, \phi) \\ &= \int_{\mathcal{H}_k(K)\backslash\mathcal{H}_k(\mathbb{A})} \int_{N_{k+1}(K)\backslash N_{k+1}(\mathbb{A})} \theta_{\psi^{-1},k}^{\phi}(h \cdot (g, \epsilon)) \\ &\qquad \times \mathrm{Res}_{s=1} E(v j_k(h \cdot g), \varphi_{\tau,s}) \chi_k^{-1}(v)\, \mathrm{d}v\, \mathrm{d}h. \end{aligned} \tag{1.10}$$

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Here $0 \leqslant k < 2n$. For $k = 0$, we define

$$\widetilde{\mathrm{Sp}}_0 = \{I\}, \qquad \mathcal{H}_0 = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}, \qquad \omega_\psi^{(0)} = \psi.$$

Note that $N_k = j_k(\mathcal{H}_k) N_{k+1}$. Denote by $\sigma_{\psi,k}(\tau)$ the representation by right translations of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ on $V_{\sigma_{\psi,k}}(\tau)$. We computed in [**5**, Theorem 8, (2.44)] the constant term of p_k along the unipotent radical

$$R_p = \left\{ \begin{pmatrix} I_p & x & y \\ & I_{2(k-p)} & x' \\ & & I_p \end{pmatrix} \in \mathrm{Sp}_{2k} \right\}, \quad 1 \leqslant p \leqslant k.$$

Take ϕ to be of the form $\phi_1 \otimes \phi_2$, where $\phi_1 \in S(\mathbb{A}^p)$, $\phi_2 \in S(\mathbb{A}^{k-p})$. Then we proved (see [**5**]) the following theorem.

Theorem 1.1. *In the above notation,*

$$\int_{R_p(k)\backslash R_p(\mathbb{A})} p_k((r,1), \varphi_{\tau,1}, \phi_1 \otimes \phi_2)\,\mathrm{d}r = \sum_{\gamma \in Z_p(k)\backslash \mathrm{GL}_p(k)} \int_{\mathcal{L}_{\mathbb{A}}} p_{k-p}(1, \hat{\gamma} x \beta \cdot \varphi_{\tau,1}, \phi_2) \phi_1(j(x))\,\mathrm{d}x. \quad (1.11)$$

Here, for $\gamma \in \mathrm{GL}_p(K)$,

$$\hat{\gamma} = \begin{pmatrix} \gamma & & \\ & I_{4n-2p} & \\ & & \gamma^* \end{pmatrix}.$$

β is the following Weyl element

$$\beta = \begin{pmatrix} & I_p & & & \\ I_{2n-k} & & & & \\ & & I_{2(k-p)} & & \\ & & & & I_{2n-k} \\ & & & I_p & \end{pmatrix}.$$

$\mathcal{L}$ is the following subgroup

$$\mathcal{L} = \left\{ x = \begin{pmatrix} I_p & & & & \\ L & I_{2n-k} & & & \\ & & I_{2(k-p)} & & \\ & & & I_{2n-k} & \\ & & & L' & I_p \end{pmatrix} \in \mathrm{Sp}_{4n} \right\}$$

and, for $x \in \mathcal{L}$,

$$j(x) = (L_{2n-k,1}, \ldots, L_{2n-k,p}).$$

From Theorem 1.1, we conclude the following theorem.

Theorem 1.2 (the tower property). *Assume that* $\sigma_{\psi,k}(\tau)=0$ *for all* $k<\ell$. *Then* $\sigma_{\psi,\ell}(\tau)$ *is either zero or cuspidal. Moreover, if* ℓ *is the first index such that* $\sigma_{\psi,\ell}(\tau)\neq 0$, *then* $\sigma_{\psi,k}(\tau)$ *is non-cuspidal, for* $k>\ell$.

In [**5**, Chapter 3], we proved the following theorem.

Theorem 1.3. *We have, for all* $k<n$,

$$\sigma_{\psi,k}(\tau)=0.$$

The proof of this theorem was based on the fact that $E_{\tau,1}$ (the residue representation) has a non-trivial period along $\mathrm{Sp}_{2n}\times\mathrm{Sp}_{2n}$, that is, if we let H denote the image in Sp_{4n} of the direct sum embedding of $\mathrm{Sp}_{2n}\times\mathrm{Sp}_{2n}$, then

$$\int_{H_k\backslash H_{\mathbb{A}}}\mathrm{Res}_{s=1}\,E(g,\varphi_{\tau,s})\,\mathrm{d}g\not\equiv 0. \tag{1.12}$$

See [**5**, Corollary 3]. We showed in [**5**, Theorem 17] that the existence of the period (1.12) 'negates' the Fourier–Jacobi model defining $\sigma_{\psi,k}(\tau)$. In this paper, we will present another proof for the vanishing of $\sigma_{\psi,k}(\tau)$, $k<n$, this time using just the self-duality of τ. This will allow us to conclude Theorem 1.3, for a larger class of automorphic representations τ. The exact details will appear right after this section.

In [**6**, § 5], we proved that our theory is not vacuous, and showed that $\sigma_{\psi,n}(\tau)\neq 0$. Our proofs there stand in a larger generality, and we summarize them as follows (see Theorems 1 and 2 and Lemmas 1 and 2 in [**6**, § 5]). For this, let us first extend the definition of $\sigma_{\psi,k}$ and apply it not only to the residue representation $E_{\tau,1}$, but rather to any automorphic module $\mathcal{E}$ of $\mathrm{Sp}_{4n}(\mathbb{A})$. Thus $\sigma_{\psi,k}(\mathcal{E})$ is the automorphic representation of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$, acting by right translations in the space $V_{\sigma_{\psi,k}(\mathcal{E})}$ generated by the Fourier Jacobi coefficients (1.10), i.e. by

$$\begin{aligned}p_k(g,\epsilon)&=p_k((g,\epsilon),\xi,\phi)\\&=\int_{\mathcal{H}_k(K)\backslash\mathcal{H}_k(\mathbb{A})}\int_{N_{k+1}(K)\backslash N_{k+1}(\mathbb{A})}\theta^{\phi}_{\psi^{-1},k}(h\cdot(g,\epsilon))\xi(vj_k(h\cdot g))\chi_k^{-1}(v)\,\mathrm{d}v\,\mathrm{d}h.\end{aligned} \tag{1.13}$$

for ξ in the space of $\mathcal{E}$. For $\alpha\in K^*$, let $\sigma_{\psi,k,\alpha}(\mathcal{E})$ be the module generated by $p_{k,\alpha}(g,\epsilon)$, where $p_{k,\alpha}(g,\epsilon)$ is given by (1.13) only that in $\theta^{\phi}_{\psi^{-1},k}$ we replace ψ^{-1} by $\psi^{-\alpha}$. Thus, we denote $\sigma_{\psi,k,1}(\mathcal{E})=\sigma_{\psi,k}(\mathcal{E})$. The only property of $E_{\tau,1}$ used in the proof of Theorem 1 in [**6**, § 5] is the fact that $\sigma_{\psi,k,\alpha}(\tau)=0$, for all $k<n$ and all $\alpha\in K^*$ (Theorem 1.3).

Definition 1.4. Let $\mathcal{E}$ be an automorphic representation of $\mathrm{Sp}_{4n}(\mathbb{A})$. We say that $\mathcal{E}$ satisfies *the vanishing property*, if

$$\sigma_{\psi,k,\alpha}(\mathcal{E})=0,\quad\text{for all } 0\leqslant k<n \text{ and all } \alpha\in K^*.$$

Consider the unipotent subgroup E_{2n} of Sp_{4n}, defined in (4.1) of [**6**], and consider the character $\psi^{(2n)}$ of $E_{2n}(K)\backslash E_{2n}(\mathbb{A})$ (also defined in [**6**]). Their actual definition will not be needed in this paper except for the formulae that will soon be recalled. Then, if $\mathcal{E}$ has the vanishing property, (5.16) of [**6**] is valid, i.e.

$$\int_{E_{2n}(K)\backslash E_{2n}(\mathbb{A})} \xi(v)\psi^{(2n)}(v)\,\mathrm{d}v = \int_{\chi_0(\mathbb{A})}\int_{Z_{2n}(K)\backslash Z_{2n}(\mathbb{A})} \xi^{U_{2n}}(m(z)\bar{\ell}(x)\nu_0)\chi_\psi(z)\,\mathrm{d}z\,\mathrm{d}x \tag{1.14}$$

for $\xi \in V_{\mathcal{E}}$.

Here $\xi^{U_{2n}}$ is the constant term of ξ along U_{2n}, the unipotent radical of P_{2n}, the Siegel parabolic subgroup. Z_{2n} is the standard maximal unipotent subgroup of GL_{2n} and for $z \in Z_{2n}(\mathbb{A})$,

$$\left.\begin{aligned} \chi_\psi(z) &= \psi(z_{12} + z_{23} + \cdots + z_{n,n+1} - z_{n+1,n+2} - \cdots - z_{2n-1,2n}), \\ \chi_0 &= \left\{ x : \begin{pmatrix} I_{2n} & x \\ & I_{2n} \end{pmatrix} \in \mathrm{Sp}_{4n} \text{ and } x \text{ is nilpotent and upper triagular} \right\}. \end{aligned}\right\} \tag{1.15}$$

ν_0 is a certain fixed element of $\mathrm{Sp}_{4n}(K)$ ($\nu_0 = \nu a$, where a and ν and defined in (4.8), (4.9) of [**6**]). (See the introduction for $m(z)$, $\bar{\ell}(x)$.) From Theorem 2 and Lemmas 1 and 2 of [**6**, § 5], we conclude the following theorem.

Theorem 1.5. *Let $\mathcal{E}$ be an automorphic representation of $\mathrm{Sp}_{4n}(\mathbb{A})$. Assume that $\mathcal{E}$ satisfies the vanishing property. If*

$$\int_{\chi_0(\mathbb{A})}\int_{Z_{2n}(K)\backslash Z_{2n}(\mathbb{A})} \xi^{U_{2n}}(m(z)\bar{\ell}(x))\chi_\psi(z)\,\mathrm{d}z\,\mathrm{d}x \not\equiv 0,$$

as ξ varies in $V_{\mathcal{E}}$, (this is the right-hand side of (1.14)), then $\sigma_{\psi,n}(\mathcal{E}) \neq 0$. Moreover $\sigma_{\psi,n}(\mathcal{E})$ has a non-trivial ψ-Whittaker coefficient, i.e.

$$\int_{V_n(K)\backslash V_n(\mathbb{A})} p_n((v,1),\xi,\phi)\psi_n(v)\,\mathrm{d}v \not\equiv 0.$$

1.2. Statement of the main (global) theorem

The main goal of this paper is to extend the results above to the case where τ is replaced by an Eisenstein series as follows. Let $\tau_1, \tau_2, \dots, \tau_r$ be pairwise different irreducible, automorphic, cuspidal and self-dual representations of $\mathrm{GL}_{2m_1}(\mathbb{A}), \dots, \mathrm{GL}_{2m_r}(\mathbb{A})$, respectively. Assume, for each $1 \leqslant i \leqslant r$, that $L^S(\tau_i, \Lambda^2, s)$ has a pole at $s = 1$, and that $L(\tau_i, \frac{1}{2}) \neq 0$. Let $s_1, \dots, s_r$ be complex numbers. Put $\bar{s} = (s_1, \dots, s_r)$, $n = m_1 + m_2 + \cdots + m_r$. Put also $\bar{m} = (m_1, \dots, m_r)$. Let $Q_{2\bar{m}} = L_{2\bar{m}} \ltimes U_{2\bar{m}}$ be the standard parabolic subgroup of Sp_{4n}, whose Levi part $L_{2\bar{m}}$ is isomorphic to $\mathrm{GL}_{2m_1} \times \cdots \times \mathrm{GL}_{2m_r}$. Denote

$$\rho_{\bar{\tau},\bar{s}} = \mathrm{Ind}_{Q_{2\bar{m}}(\mathbb{A})}^{\mathrm{Sp}_{4n}(\mathbb{A})} \tau_1|\det\cdot|^{s_1-(1/2)} \otimes \cdots \otimes \tau_r|\det\cdot|^{s_r-(1/2)},$$

and consider the corresponding Eisenstein series

$$E(g, \varphi_{\bar{\tau},\bar{s}}) = \sum_{\gamma \in Q_{2\bar{m}}(K)\backslash \mathrm{Sp}_{4n}(K)} f_{\bar{\tau},\bar{s}}(\gamma g). \tag{1.16}$$

Here, as in (1.2), (1.3), $\varphi_{\bar\tau,\bar s}$ is a $K_{\mathrm{Sp}_{4n}}$-finite holomorphic section of $\rho_{\bar\tau,\bar s}$, regarded as a complex valued function on $\mathrm{Sp}_{4n}(\mathbb{A}) \times L_{2\bar m}(\mathbb{A})$ such that for each $g \in \mathrm{Sp}_{4n}(\mathbb{A})$, the function

$$(e_1,\dots,e_r) \mapsto \varphi_{\bar\tau,\bar s}(g;e_1,\dots,e_r),$$

on $\mathrm{GL}_{2m_1}(\mathbb{A}) \times \cdots \times \mathrm{GL}_{2m_r}(\mathbb{A})$ is a cusp form which lies in the space of $\tau_1 \otimes \cdots \otimes \tau_r$ (i.e. the space spanned by the products $\varphi_1(e_1)\cdot\cdots\cdot\varphi_r(e_r)$, where φ_i is a cusp form in τ_i). Finally, $f_{\bar\tau,\bar s}(g) = \varphi_{\bar\tau,\bar s}(g; I_{2m_1},\dots,I_{2m_r})$. The series (1.16) converges absolutely in a domain of the form $\mathrm{Re}(s_1) \gg \mathrm{Re}(s_2) \gg \cdots \gg \mathrm{Re}(s_r) \gg 0$ and it has a meromorphic continuation in $\bar s$. We will prove in the next section that this Eisenstein series has a 'simple pole' at $(1,\dots,1)$ in the sense that for each $g \in \mathrm{Sp}_{4n}(\mathbb{A})$ the function $(s_1-1)(s_2-1)\cdot\cdots\cdot(s_r-1)E(g,\varphi_{\bar\tau,\bar s})$ is holomorphic and not identically zero at $\bar 1 = (1,\dots,1)$. Denote the resulting residual representation (at $\bar 1$) by $E_{\bar\tau,\bar 1}$.

Remark 1.6. Let $P_{2m_1,\dots,2m_r}$ be the standard parabolic subgroup of GL_{2n}, where the Levi part is isomorphic to $\mathrm{GL}_{2m_1} \times \cdots \times \mathrm{GL}_{2m_r}$. Using induction by stages, we could replace $\rho_{\bar\tau,\bar s}$ by the representation of $\mathrm{Sp}_{4n}(\mathbb{A})$ induced from the Siegel parabolic subgroup and the representation of the Levi part $\mathrm{GL}_{2n}(\mathbb{A})$, given by the Eisenstein series which corresponds to

$$\mathrm{Ind}_{P_{2m_1,\dots,2m_r}(\mathbb{A})}^{\mathrm{GL}_{2n}(\mathbb{A})} \tau_1|\det|^{s_i-(1/2)} \otimes \cdots \otimes \tau_r|\det\cdot|^{s_r-(1/2)}.$$

Our main global result says that the construction described in § 1.1 can be applied to $E_{\bar\tau,\bar 1}$.

Theorem 1.7 (main (global) theorem). *$\sigma_{\psi,n}(E_{\bar\tau,\bar 1})$ is a non-trivial automorphic, cuspidal (genuine) representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. It is a multiplicity free direct sum of irreducible, automorphic, cuspidal (genuine) representations σ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which have a non-trivial ψ-Whittaker coefficient and such that at almost all places ν, the ψ_ν-unramified parameters of σ_ν are those of the unramified constituent of*

$$\mathrm{Ind}_{P_{2m_1,\dots,2m_r}(K_\nu)}^{\mathrm{GL}_{2n}(K_\nu)} \tau_{1,\nu} \otimes \cdots \otimes \tau_{r,\nu}.$$

Moreover, an irreducible, automorphic, cuspidal (genuine) representation σ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, with a non-trivial ψ-Whittaker coefficient has a non-trivial L^2-pairing with a summand of $\sigma_{\psi,n}(E_{\bar\tau,\bar 1})$, if and only if

$$L_\psi^S(\sigma \otimes \tau_i, s) \quad \text{has a pole at } s = 1, \quad \text{for all } 1 \leqslant i \leqslant r.$$

Remark 1.8. The irreducible summands of $\sigma_{\psi,n}(E_{\bar\tau,\bar 1})$ are up to isomorphism (given through a non-trivial L^2-pairing) the ψ^{-1}-generic representatives of the 'ψ-endoscopic' L-packet which lifts to

$$\mathrm{Ind}_{P_{2m_1,\dots,2m_r}(\mathbb{A})}^{\mathrm{GL}_{2n}(\mathbb{A})} \tau_1 \otimes \cdots \otimes \tau_r.$$

One of the main tools for the proof of this theorem will be the lemma of the next section, which gives a simple proof of a quite general nature for the vanishing property of $E_{\bar\tau,\bar 1}$ (see Theorem 1.3 and the definition right after (1.13)). This lemma will also be used in calculating the unramified parameters, at almost all places, of each summand of $\sigma_{\psi,n}(E_{\bar\tau,\bar 1})$.

1.3. A lemma on Fourier–Jacobi models

Let F be a local non-archimedean field. Let ψ be a non-trivial character of F. (There will be no confusion with overlapping notation in the global case.) Let π be a smooth representation of $\mathrm{Sp}_{4n}(F)$, acting in a space V_π. For $\alpha \in F^*$, we can define smooth representations $\sigma_{\psi,k}(\pi)$ and $\sigma_{\psi,k,\alpha}(\pi)$ of $\widetilde{\mathrm{Sp}}_{2k}(F)$, for $0 \leqslant k < 2n$, in complete analogy with (1.13).

$$\sigma_{\psi,k,\alpha}(\pi) = J_{\mathcal{H}_k(F)}(J_{N_{k+1}(F),\chi_k}(\pi) \otimes \omega^{(k)}_{\psi^{-\alpha}}). \tag{1.17}$$

Here $\omega^{(k)}_{\psi^{-\alpha}}$ denotes the Weil representation of $\mathcal{H}_k(F) \rtimes \widetilde{\mathrm{Sp}}_{2k}(F)$. Note that $\mathcal{H}_k(F)$ acts on the Jacquet module

$$J_{N_{k+1}(F),\chi_k}(\pi) \quad (= J_{N_{k+1}(F),\chi_k}(V_\pi))$$

through the embedding (1.7), and, similarly, $\mathrm{Sp}_{2k}(F)$ acts on $J_{N_{k+1}(F),\chi_k}(\pi)$ through the embedding (1.9).

Let $Q_{2\ell}$ be the standard parabolic subgroup of Sp_{4n}, with Levi part isomorphic to $\mathrm{GL}_{2\ell} \times \mathrm{Sp}_{4(n-\ell)}$.

Lemma 1.9. *Let η and ρ be smooth representations of $\mathrm{GL}_{2\ell}(F)$ and $\mathrm{Sp}_{4(n-\ell)}(F)$, respectively. Fix $\alpha \in F^*$. Assume that*

$$\sigma_{\psi,k,\alpha}(\rho) = 0, \quad \text{for all } 0 \leqslant k < n-\ell. \tag{1.18}$$

Then

$$\sigma_{\psi,k,\alpha}(\mathrm{Ind}_{Q_{2\ell}}^{\mathrm{Sp}_{4n}(F)} \eta \otimes \rho) = 0, \quad \text{for all } 0 \leqslant k < n-\ell. \tag{1.19}$$

Moreover, if, in addition to assumption (1.18), $\sigma_{\psi,n-\ell,\alpha}(\rho) \neq 0$, then

$$\sigma_{\psi,n-\ell,\alpha}(\mathrm{Ind}_{Q_\ell(F)}^{\mathrm{Sp}_{4n}(F)} \eta \otimes \rho) \neq 0 \quad \Leftrightarrow \quad J_{Z_{2\ell}(F),\psi}(\eta) \neq 0. \tag{1.20}$$

(In (1.20), we keep denoting by ψ the standard generic character of $Z_{2\ell}(F)$ defined by ψ.)

This lemma is very crucial for this paper. It is a special case of a more general lemma, where Sp_{4n} is replaced by any symplectic, orthogonal or unitary group and $Q_{2\ell}$ by a standard maximal parabolic subgroup. The Fourier–Jacobi model is replaced in some of the other cases by a Bessel model (see [**5**]). The proof of the general case will appear in another work of ours, which is now under preparation. For completeness sake, we bring a sketch of the proof in our present case.

Proof (sketch). We have seen in [**6**, Lemma 3.2] that

$$\sigma_{\psi,k,\alpha}(\mathrm{Ind}_{Q_{2\ell}(F)}^{\mathrm{Sp}_{4n}(F)} \eta \otimes \rho) = 0 \quad \Leftrightarrow \quad J_{N^{(k)}(F),\chi_{(k),\alpha}}(\mathrm{Ind}_{Q_{2\ell}(F)}^{\mathrm{Sp}_{4n}(F)} \eta \otimes \rho) = 0, \tag{1.21}$$

where $N^{(k)}$ is the product of N_{k+1} and the centre C of $\mathcal{H}_k$ embedded in Sp_{4n} through (1.7). $\chi_{(k),\alpha}|_{N_{k+1}} = \chi_k$ and $\chi_{(k),\alpha}(j_k(0;t)) = \psi(\alpha t)$ (see (1.7)). Note that N_{k+1} is the unipotent radical of Q_{2n-k-1}. We then first restrict

$$\mathrm{Ind}_{Q_{2\ell}(F)}^{\mathrm{Sp}_{4n}(F)} \eta \otimes \rho$$

to $Q_{2n-k-1}(F)$. The Jordan–Holder decomposition of this restriction has quotients, which are parametrized by $Q_{2\ell}\backslash \mathrm{Sp}_{4n}/Q_{2n-k-1}$. We can pick the following representatives,

$$w_{r_1,r_2}=\begin{pmatrix} I_{r_1} & & & & & & & \\ & 0 & & & & & I_{2\ell-r_1-r_2} & \\ & & 0 & I_{2n-m} & 0 & 0 & & \\ & & I_{m-2\ell+r_2} & 0 & 0 & 0 & & \\ & & 0 & 0 & 0 & I_{m-2\ell+r_2} & & \\ & & 0 & 0 & I_{2n-m} & 0 & & \\ & -I_{2\ell-r_1-r_2} & & & & & 0 & \\ & & & & & & & I_{r_1} \end{pmatrix},$$

where $m=2n-k-1$, $0\leqslant r_2\leqslant k+1$, $r_1+r_2\leqslant 2\ell$.

The corresponding quotient is

$$\Gamma_{r_1,r_2}=\mathrm{Ind}^{cQ_m(F)}_{w^{-1}_{r_1,r_2}Q_{2\ell}(F)w_{r_1,r_2}\cap Q_m(F)}(\delta^{1/2}_{Q_{2\ell}}\cdot\eta\otimes\rho)^{w_{r_1,r_2}}$$

(unnormalized and compact induction).

The group $w^{-1}_{r_1,r_2}Q_{2\ell}w_{r_1,r_2}\cap Q_m$ is isomorphic to

$$L_{r_1,r_2}=\left\{\begin{matrix} r_1\\ m-2\ell+r_2\\ 2\ell-r_1-r_2\\ r_2\\ 2(2n-m-r_2)\\ r_2\\ 2\ell-r_1-r_2\\ m-2\ell+r_2\\ r_1\end{matrix}\begin{pmatrix} a_1 & a_{12} & a_{13} & 0 & x_1 & x_2 & 0 & y_1 & y_2\\ & a_2 & a_{23} & 0 & x_3 & x_4 & 0 & y_3 & y_1'\\ & & a_3 & 0 & 0 & 0 & 0 & 0 & 0\\ & & & b & e & u & 0 & x_4' & x_2'\\ & & & & c & e' & 0 & x_3' & x_1'\\ & & & & & b^* & 0 & 0 & 0\\ & & & & & & a_3^* & a_{23}' & a_{13}'\\ & & & & & & & a_2^* & a_{12}'\\ & & & & & & & & a_1^*\end{pmatrix}\in\mathrm{Sp}_{4n}\right\},$$

and the action of $(\delta^{1/2}_{Q_{2\ell}}\cdot(\eta\otimes\rho))^{w_{r_1,r_2}}$ is

$$\left(\frac{|\det a_1|}{|\det a_3|}|\det b|\right)^{2n-\ell+(1/2)}\eta\begin{pmatrix} a_1 & & \\ & a_3^* & \\ & & b\end{pmatrix}\otimes\rho^{w'}\begin{pmatrix} a_2 & x_3 & y_3\\ & c & x_3'\\ & & a_2^*\end{pmatrix}, \tag{1.22}$$

where

$$w'=\begin{pmatrix} & I_{2n-m-r_2} & & \\ I_{m-2\ell+r_2} & & & \\ & & & I_{m-2\ell+r_2}\\ & & I_{2n-m-r_2} & \end{pmatrix}.$$

Denote the representation (1.22) of $L_{r_1,r_2}(F)$ by $\pi^{(r_1,r_2)}_{\eta,\rho}$. Thus, we have to compute

$$J_{N^{(k)}(F),\chi_{(k)},\alpha}(\mathrm{Ind}^{cQ_m(F)}_{L_{r_1,r_2}(F)}\pi^{(r_1,r_2)}_{\eta,\rho}).$$

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Restrict

$$\mathrm{Ind}^{c^{Q_m(F)}}_{L_{r_1,r_2}(F)}\pi^{(r_1,r_2)}_{\eta,\rho}$$

to $V_{2n}(F)$, where V_{2n} is the standard maximal unipotent subgroup of Sp_{4n}. The corresponding Jordan–Hölder series is parametrized by $L_{r_1,r_2}\backslash Q_m/V_{2n}$. For a Weyl element w in the last space, denote by γ_w the corresponding subquotient. Using (1.22), it turns out that $J_{N^{(k)}}(F),\chi_{(k),\alpha}(\gamma_w)=0$, unless $r_1=0$, and then, w must be of the form

$$w=w_{r_2,\epsilon}=\left(\begin{array}{cc|c|cc} & I_{m-2\ell+r_2} & & & \\ I_{2\ell-r_2} & & & & \\ \hline & & \epsilon & & \\ \hline & & & & I_{2\ell-r_2} \\ & & & I_{m-2\ell+r_2} & \end{array}\right), \tag{1.23}$$

where ϵ is a Weyl element of Sp_{2k+2} of the form

$$\epsilon={}_{r_2+1\rightarrow}\begin{pmatrix} 0 & \\ \vdots & \\ 0 & \\ 1 & * \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix}.$$

Recall that

$$\gamma_w=\mathrm{Ind}^{c^{V_{2n}(F)}}_{w^{-1}Q_{2n}(F)w\cap V_{2n}(F)}(\pi^{(r_1,r_2)}_{\eta,\rho})^w.$$

Computing the stabilizer $w^{-1}Q_{2m}w\cap V_{2n}$, for w of the form (1.23), we see that (using (1.21)) $J_{N^{(k)},\chi_{(k),\alpha}}(\gamma_w)=0$, if $\sigma_{\psi,k-r_2,\alpha}(\rho)=0$. Thus, for $0\leqslant k<n-\ell$, assumption (1.18) implies that $J_{N^{(k)},\chi_{(k),\alpha}}(\gamma_w)=0$. This proves (1.19). If $k=n-\ell$, then $k-r_2<n-\ell$, unless $r_2=0$. Thus, the only contribution to the Jacquet module, may come from w as in (1.23) with $r_2=0$, and now (1.2) can be derived by the same methods as before. □

Corollary 1.10. *Let $Q^{(2)}$ be the standard parabolic subgroup of Sp_{4n} with Levi part isomorphic to $(\mathrm{GL}_2)^n$. Let $\bar z=(z_1,\dots,z_n)\in\mathbb{C}^n$ and consider*

$$\pi_n(\bar z)=\mathrm{Ind}^{\mathrm{Sp}_{4n}(F)}_{Q^{(2)}(F)}|\det\cdot|^{z_1}\otimes\cdots\otimes|\det\cdot|^{z_n}. \tag{1.24}$$

Then $\sigma_{\psi,k,\alpha}(\pi_n(\bar z))=0$, for all $0\leqslant k<n$, and $\alpha\in F^*$.

Proof. We use induction on n. If $n=1$, then $\sigma_{\psi,0,\alpha}(\pi_1(\bar z))\neq 0$, if and only if $\pi_1(\bar z)$ has a non-trivial Whittaker model (it depends on α), which is false. Assume, by induction, that $\sigma_{\psi,k,\alpha}(\pi_{n-1}(z_1,\dots,z_{n-1}))=0$ for $0\leqslant k<n-1$ and $z_j\in\mathbb{C}$. Write

$$\pi_n(\bar z)=\mathrm{Ind}^{\mathrm{Sp}_{4n}(F)}_{Q_2(F)}|\det\cdot|^{s_1}\otimes\pi_{n-1}(z_2,\dots,z_n),$$

and apply the last lemma with $\ell = 1$, $\tau = |\det\cdot|^{z_1}$ (on $\mathrm{GL}_2(F)$) $\rho = \pi_{n-1}(z_2, \dots, z_n)$ (on $\mathrm{Sp}_{4(n-1)}(F)$). Then, from the inductive assumption and from (1.16), it follows that $\sigma_{\psi,k,\alpha}(\pi_n(\bar{z})) = 0$, for $k < n-1$. For $k = n-1$, we use the end of the proof of the last lemma, where only $r_2 = 0$ contributes to the Jacquet module, and then we must have that $J_{Z_{2\ell},\psi}(\eta) \neq 0$. In our case η is a character of $\mathrm{GL}_2(F)$ ($|\det\cdot|^{z_1}$) and hence it does not have a Whittaker model. This shows that $\sigma_{\psi,n-1,\alpha}(\pi_n(\bar{z})) = 0$, as well. □

We can use Corollary 1.10 to give another proof to Theorem 1.3, where, in the notation of Theorem 1.3, *only the self-duality of* τ *and the fact that* $\omega_\tau = 1$ *are used* (even the cuspidality of τ is not necessary) and thus the vanishing statement of Theorem 1.3 applies to a much larger class of automorphic forms. Still, the idea of our first proof of Theorem 1.3, i.e. the use of the period (1.12), is useful, and, at this point, we do not know how to do without it when we deal with the analogous local theory (see [**6**, § 3.3]).

A second proof of Theorem 1.3. We go back to the notation of Theorem 1.3 (global set-up). We want to show that $\sigma_{\psi,k,\alpha}(E_{\tau,1}) = 0$, for $k < n$. Let $\pi \simeq \otimes\pi_\nu$ be an irreducible summand of $E_{\tau,1}$. Note that all such summands are nearly equivalent. At almost all places, π_ν is the unramified constituent of $\rho_{\tau_\nu,1}$ (notation analogous to (1.1)). Since τ_ν is self-dual and $\omega_{\tau_\nu} = 1$, there are unramified (quasi) characters $\chi_{1,\nu}, \dots, \chi_{n,\nu}$, such that π_ν is the unramified constituent of the representation of $\mathrm{Sp}_{4n}(K_\nu)$ induced from the standard Borel subgroup and the following character of the torus

$$\begin{aligned}\operatorname{diag}(t_1, t_2, \dots, t_{2n}; t_{2n}^{-1}, \dots, t_1^{-1})&\\ \mapsto \chi_{1,\nu}\left(\frac{t_1}{t_{2n}}\right)\chi_{2,\nu}\left(\frac{t_2}{t_{2n-1}}\right)\cdot\dots\cdot\chi_{n,\nu}\left(\frac{t_n}{t_{n+1}}\right)&|t_1\cdot t_2\cdot\dots\cdot t_{2n}|^{1/2}.\end{aligned} \tag{1.25}$$

This character can be conjugated, using a Weyl element of Sp_{4n}, to

$$\begin{aligned}\operatorname{diag}(t_1, t_2, \dots, t_{2n}; t_{2n}^{-1}, \dots, t_1^{-1})&\\ \mapsto \chi_{1,\nu}(t_1t_2)\left|\frac{t_1}{t_2}\right|^{1/2}\chi_{2,\nu}(t_3t_4)\left|\frac{t_3}{t_4}\right|^{1/2}\cdot\dots\cdot\chi_{n,\nu}(t_{2n-1}t_{2n})&\left|\frac{t_{2n-1}}{t_{2n}}\right|^{1/2}.\end{aligned} \tag{1.26}$$

See [**7**, § 2]. Thus, π_ν is the unramified constituent of $\pi_n(\bar{z}_{\chi,\nu})$, where $\chi_{i,\nu}(t) = |t|^{z_i}$ and $\bar{z}_{\chi_\nu} = (z_1, \dots, z_n)$. If $\sigma_{\psi,k,\alpha}(E_{\tau,1})$ is non-zero, for $k < n$, we may assume that $\sigma_{\psi,k,\alpha}(\pi) \neq 0$ and then clearly $\sigma_{\psi_\nu,k,\alpha}(\pi_\nu) \neq 0$ (see [**5**, § 3.3]). By exactness of Jacquet functors, $\sigma_{\psi_\nu,k,\alpha}(\pi_n(\bar{z}_{\chi,\nu})) \neq 0$. This contradicts the last lemma. □

Remark 1.11. Note again that the last proof is valid for any automorphic representation of the form $\tau = \otimes\tau_\nu$, where, for almost all ν, τ_ν is unramified, self-dual and $\omega_{\tau_\nu} = 1$. Therefore, we have the following corollary.

Corollary 1.12. *Let τ_ν be an irreducible, unramified, self-dual representation of $\mathrm{GL}_{2n}(K_\nu)$. Assume that $\omega_{\tau_\nu}=1$. Let π_{τ_ν} be the unramified constituent of $\rho_{\tau_\nu,1}$. Then*

$$\sigma_{\psi_\nu,k,\alpha}(\pi_{\tau_\nu})=0,\quad \text{for } k\leqslant n-1 \text{ and } \alpha\in K_\nu^*,$$

i.e. π_{τ_ν} satisfies the vanishing property.

We conclude, in the notation of § 1.2, the following theorem.

Theorem 1.13. *Let $\tau_1,\dots,\tau_r$ be pairwise different, irreducible, automorphic, cuspidal, self-dual automorphic representations of $\mathrm{GL}_{2m_1}(\mathbb{A}),\dots,\mathrm{GL}_{2m_r}(\mathbb{A})$, respectively. Assume that, for each $i\leqslant r$, $L^S(\tau_i,\Lambda^2,s)$ has a pole at $s=1$ and $L(\tau_i,\frac{1}{2})\neq 0$. Then $E_{\bar\tau,\bar 1}$ satisfies the vanishing property, i.e. $\sigma_{\psi,k,\alpha}(E_{\bar\tau,\bar 1})=0$, for all $k\leqslant n-1$ and all $\alpha\in K^*$.*

1.4. A preliminary lemma on Eisenstein series

The following lemma can be derived from [**8**] and we bring it for completeness sake and as a preparation for the next section. Here K is a number field as in §§ 1.1 and 1.2.

Lemma 1.14. *Let $\tau_1,\dots,\tau_r$ be irreducible, automorphic, cuspidal, unitary representations of $\mathrm{GL}_{\ell_1}(\mathbb{A}),\dots,\mathrm{GL}_{\ell_r}(\mathbb{A})$. Assume that for $i\neq j$, there is no $x\in\mathbb{C}$, such that $\tau_j=\tau_i\otimes|\det\cdot|^x$. Then the Eisenstein series on $\mathrm{GL}_n(\mathbb{A})$, $n=\ell_1+\dots+\ell_r$, corresponding to the representation induced from $\tau_1|\det\cdot|^{z_1}\otimes\dots\otimes\tau_r|\det\cdot|^{z_r}$ is holomorphic at $(z_1,\dots,z_r)$, if $\mathrm{Re}(z_1)\geqslant\dots\geqslant\mathrm{Re}(z_r)$.*

Proof. Let $P_{\ell_1,\dots,\ell_r}$ be the standard parabolic subgroup of GL_n whose Levi part L is isomorphic to $\mathrm{GL}_{\ell_1}\times\dots\times\mathrm{GL}_{\ell_r}$. By the general theory of Eisenstein series, it is enough to show that, at $(s_1,\dots,s_r)=(z_1,\dots,z_r)$ as above, all intertwining operators $M(w)$, on

$$\mathrm{Ind}_{P_{\ell_1,\dots,\ell_r}(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})}\tau_1|\det\cdot|^{s_1}\otimes\dots\otimes\tau_r|\det\cdot|^{s_r},$$

are holomorphic, for all $w\in W(L)$. $W(L)$ is the set of Weyl elements w of GL_m, of minimal length, modulo the Weyl group of L, such that wLw^{-1} is a standard Levi subgroup of GL_n. (We use the notation of [**9**, II.1.7].) We have

$$W(L)=\left\{w\in W_{\mathrm{GL}_n}\;\middle|\;\begin{matrix}w(\alpha)>0, & \text{for all positive roots } \alpha \text{ inside } L \text{ and}\\ wLw^{-1} & \text{is a standard Levi subgroup of } \mathrm{GL}_n\end{matrix}\right\}. \tag{1.27}$$

(W_{GL_n} denotes the Weyl group of GL_n.)

$W(L)$ is in bijection with the permutation group S_r. If $w\in W(L)$ corresponds to the permutation $\epsilon\in S_r$, then

$$w\begin{pmatrix}g_1&&\\&\ddots&\\&&g_r\end{pmatrix}w^{-1}=\begin{pmatrix}g_{\epsilon^{-1}(1)}&&\\&\ddots&\\&&g_{\epsilon^{-1}(r)}\end{pmatrix},\quad g_i\in\mathrm{GL}_{\ell_i}.$$

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It is easy to see, from (1.27), that if we write $w \in W(L)$ in the form $w = (w_1, w_2, \dots, w_r)$, where w_i has ℓ_i columns, then w_i has the form

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_{\ell_i} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

i.e. w_i has the block I_{ℓ_i} appearing somewhere. (We realize first W_{GL_n} as permutation matrices.) The permutation $\epsilon \in S_r$ is defined such that

$$w_{\epsilon^{-1}(1)} = \begin{pmatrix} I_{\ell_{\epsilon^{-1}(1)}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; w_{\epsilon^{-1}(2)} = \begin{pmatrix} 0 \\ I_{\ell_{\epsilon^{-1}(2)}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \}\ell_{\epsilon^{-1}(1)}, \; w_{\epsilon^{-1}(3)} = \begin{pmatrix} 0 \\ I_{\ell_{\epsilon^{-1}(3)}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \}\ell_{\epsilon^{-1}(1)}+\ell_{\epsilon^{-1}(2)},$$

$$\dots, \; w_{\epsilon^{-1}(r)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_{\ell_{\epsilon^{-1}(r)}} \end{pmatrix} \Bigg\}\ell_{\epsilon^{-1}(1)}+\cdots+\ell_{\epsilon^{-1}(r-1)}. \tag{1.28}$$

Denote (as in [**8**, p. 607]), for w corresponding to ϵ,

$$\mathrm{inv}(w) = \{(i,j) \mid 1 \leqslant i < j \leqslant r \text{ and } \epsilon(i) > \epsilon(j)\}.$$

Let $\varphi(\bar{s}) = \otimes \varphi_\nu(\bar{s})$ be a holomorphic decomposable K_{GL_n}-finite section for

$$\mathrm{Ind}_{P_{\ell_1,\dots,\ell_r}(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \tau_1 |\det\cdot|^{s_1} \otimes \cdots \otimes \tau_r |\det\cdot|^{s_r} \quad (\bar{s} = (s_1, \dots, s_r)).$$

Let S be a finite set of places, outside which all τ_i are unramified and φ_ν is the standard unramified section. Then

$$\begin{aligned} M(w)\varphi(\bar{s})(1) &= \prod_{\nu \in S} M_\nu(w)\varphi_\nu(\bar{s})(1) \prod_{(i,j)\in \mathrm{inv}(w)} \frac{L^S(\tau_i \otimes \hat{\tau}_j, s_i - s_j)}{L^S(\tau_i \otimes \hat{\tau}_j, s_i - s_j + 1)} \\ &= \prod_{\nu \in S} M^*_\nu(w)\varphi_\nu(\bar{s})(1) \prod_{(i,j)\in \mathrm{inv}(w)} \frac{L(\tau_i \otimes \hat{\tau}_j, s_i - s_j)}{L(\tau_i \otimes \hat{\tau}_j, s_i - s_j + 1)}. \end{aligned} \tag{1.29}$$

Here

$$M^*_\nu(w) = \prod_{(i,j)\in \mathrm{inv}(w)} \frac{L(\tau_{i,\nu} \otimes \hat{\tau}_{j,\nu}, s_i - s_j + 1)}{L(\tau_{i,\nu} \otimes \hat{\tau}_{j,\nu}, s_i - s_j)} M_\nu(w).$$

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Our assumptions imply that at the point in question ($s_i = z_i$),

$$\frac{L(\tau_i \otimes \hat{\tau}_j, s_i - s_j)}{L(\tau_i \otimes \hat{\tau}_j, s_i - s_j + 1)}$$

is holomorphic at $\bar{z}$, for $i < j$ (since then $\mathrm{Re}(z_i - z_j) \geqslant 0$ (see [**8**, Appendix]). By [**8**, Proposition I.10], $M^*_\nu(w)\varphi_\nu(\bar{s})(1)$ is holomorphic at $\bar{z}$, for all $\nu \in S$. This shows that $M(w)\varphi_\nu(\bar{s})(1)$ is holomorphic at $\bar{z}$, for all $w \in W(L)$. □

2. The residue representation and its constant terms

2.1. The Eisenstein series

Let $\tau_1, \dots, \tau_r$ be pairwise different, irreducible, automorphic, cuspidal and self-dual representations of $\mathrm{GL}_{2m_1}(\mathbb{A}), \dots, \mathrm{GL}_{2m_r}(\mathbb{A})$, respectively. Assume that for each $1 \leqslant i \leqslant r$, $L^S(\tau_i, \Lambda^2, s)$ has a pole at $s = 1$, and $L(\tau_i, \frac{1}{2}) \neq 0$. Recall, from [**5**, Proposition 1] that these conditions imply that the Eisenstein series on $\mathrm{Sp}_{4m_i}(\mathbb{A})$, induced from the corresponding Siegel parabolic subgroup and $\tau_i \otimes |\det\cdot|^{s-(1/2)}$ has a (simple) pole at $s = 1$. We consider the induced representation $\rho_{\bar{\tau},\bar{s}}$ of $\mathrm{Sp}_{4n}(\mathbb{A})$ ($n = m_1 + \dots + m_r$) and the corresponding Eisenstein series $E(g, \varphi_{\bar{\tau},\bar{s}})$ (1.16). The main result of this section is the following theorem.

Theorem 2.1. *Under the above assumptions (and in the notation of § 1.2)* $(s_1 - 1)(s_2 - 1) \cdot \dots \cdot (s_r - 1)E(g, \varphi_{\bar{\tau},\bar{s}})$ *is holomorphic and not identically zero at* $\bar{s} = \bar{1} = (1, 1, \dots, 1)$.

Proof. We will show this property for the constant terms of the Eisenstein series along all unipotent radicals U_k, $1 \leqslant k \leqslant 2n$, of the standard maximal parabolic subgroups $Q_k = M_k \ltimes U_k$ of Sp_{4n}:

$$Q_k = M_k \ltimes U_k = \left\{ \begin{pmatrix} g & * & * \\ & h & * \\ & & g^* \end{pmatrix} \in \mathrm{Sp}_{4n} \mid g \in \mathrm{GL}_k,\ h \in \mathrm{Sp}_{4n-2k} \right\}.$$

Denote the constant term of $E(g, \varphi_{\bar{\tau},\bar{s}})$ along U_k by $E^{U_k}(g, \varphi_{\bar{\tau},\bar{s}})$. We have (see [**9**, II.1.7]), for $g \in \mathrm{Sp}_{4n}(\mathbb{A})$ (fixed)

$$E^{U_k}(hg, \varphi_{\bar{\tau},\bar{s}}) = \sum_{w \in W(L_{2\bar{m}}, M_k)} E_{M_k}(h, M(w)(g \cdot \varphi_{\bar{\tau},\bar{s}})), \tag{2.1}$$

for $h \in M_k(\mathbb{A})$. Here $E_{M_k}(h, M(w)(g \cdot \varphi_{\bar{\tau},\bar{s}}))$ is the Eisenstein series on $M_k(\mathbb{A})$, built from $M(w)(g \cdot \varphi_{\bar{\tau},\bar{s}})|_{M_k(\mathbb{A})}$, which lies in

$$\mathrm{Ind}^{M_k(\mathbb{A})}_{wL_{2\bar{m}}(\mathbb{A})w^{-1}}\, w(\tau_1|\det\cdot|^{s_1-(1/2)} \otimes \dots \otimes \tau_r|\det\cdot|^{s_r-(1/2)}).$$

$M(w)$ is the intertwining operator corresponding to the Weyl element w and w lies in the following set

$$W(L_{2\bar{m}}, M_k) = \left\{ w \in W_{\mathrm{Sp}_{4n}} \;\middle|\; \begin{array}{ll} \text{(i) } w(\alpha) > 0, & \text{for all positive roots } \alpha \text{ inside } L_{2\bar{m}}, \\ \text{(ii) } w^{-1}(\alpha) > 0, & \text{for all positive roots } \alpha \text{ inside } M_k, \\ \text{(iii) } wL_{2\bar{m}}w^{-1} & \text{is a standard Levi subgroup inside } M_k \end{array} \right\}. \quad (2.2)$$

2.2. Description of $W(L_{2\bar{m}}, M_k)$

We realize the Weyl elements of Sp_{4n} as symplectic permutation matrices, where the non-zero elements in each row are ± 1. The non-zero elements in either the upper $2n$ rows or in columns $2n+1$ up to $4n$ are $+1$, and otherwise, they are -1. Denote the set of simple roots of $L_{2\bar{m}}$ (which correspond to upper unipotent root subgroups) by $\Delta_{L_{2\bar{m}}}$, and the corresponding positive roots by $\phi^+_{L_{2\bar{m}}}$. Similarly, consider Δ_{M_k} and $\phi^+_{M_k}$. Let w be in $W(L_{2\bar{m}}, M_k)$. For $\alpha \in \Delta_{L_{2\bar{m}}}$, (i) and (iii) in (2.2) imply

$$w(\alpha) = \sum_{\beta \in \Delta_{M_k}} a_\beta \beta, \quad \text{for some integers } a_\beta \geqslant 0.$$

Thus,

$$\alpha = \sum_{\beta \in \Delta_{M_k}} a_\beta w^{-1}(\beta). \quad (2.3)$$

By (ii) of (2.2), $w^{-1}(\beta) > 0$, for $\beta \in \Delta_{M_k}$. Since α is simple, (2.3) implies that $w(\alpha) \in \Delta_{M_k}$. We showed that $w(\Delta_{L_{2\bar{m}}}) \subset \Delta_{M_k}$. Since the elements of $\Delta_{L_{2\bar{m}}}$ have the same length, it is clear that $w(\Delta_{L_{2\bar{m}}})$ also lies inside the set of (simple) roots inside the Levi part (GL_{2n}) of P_{2n}, the Siegel parabolic subgroup. Consider the simple roots in $\Delta_{L_{2\bar{m}}}$ which lie in

$$L_{m_i} = \left\{ \begin{pmatrix} I_{2(m_1+\cdots+m_{i-i})} & & & & \\ & g & & & \\ & & I_{4(m_{i+1}+\cdots+m_r)} & & \\ & & & g^* & \\ & & & & I_{2(m_1+\cdots+m_{i-1})} \end{pmatrix} \mid g \in \mathrm{GL}_{2m_i} \right\}. \quad (2.4)$$

Denote these roots by $\Delta_i = \{\alpha_{i,1}, \ldots, \alpha_{i,m_i-1}\}$. Since Δ_i is a connected subset in the Dynkin diagram, so is $w(\Delta_i)$ inside Δ_{M_k}. This shows that $wL_{m_i}w^{-1}$ is of the form

$$L_i' = \left\{ \begin{pmatrix} I_e & & & & & & \\ & g & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & g' & \\ & & & & & & I_e \end{pmatrix} \mid g \in \mathrm{GL}_{2m_i} \right\} \subset M_k,$$

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for some appropriate e, and hence the following composition

$$\mathrm{GL}_{2m_i} \xrightarrow{\sim} L_{m_i} \to wL_{m_i}w^{-1} \xrightarrow{\sim} \mathrm{GL}_{2m_i}$$

(of the natural isomorphism of GL_{2m_i} and L_{m_i}, conjugation by w and the natural isomorphism of L_i' and GL_{2m_i}) is the identity on GL_{2m_i}. We conclude that w has the following form

$$w = (w_1, \dots, w_r, w_{r+1}, \dots, w_{2r}),$$

where $w_1, \dots, w_r$ have $2m_1, \dots, 2m_r$ columns, respectively, and $w_{r+1}, \dots, w_{2r}$ have $2m_r, \dots, 2m_1$ columns, respectively. Each w_i has the form

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pm I_{2m_i^*} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where

$$m_i^* = \begin{cases} m_i, & i \leqslant r, \\ m_{2r+1-i}, & i > r. \end{cases} \tag{2.5}$$

Note that the block $\pm I_{2m_i^*}$ in w_i is either in the upper $2n$ rows or in the lower $2n$ rows. (It does not 'cross' from row $2n$ to row $2n+1$, since then w will not be a symplectic permutation matrix.) Since $wL_{2\bar{m}}w^{-1} \subset M_k$, we can find $1 \leqslant t_1, \dots, t_j \leqslant 2r$, such that

$$w_{t_1} = \begin{pmatrix} I_{2m_{t_1}^*} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad w_{t_2} = \begin{pmatrix} 0 \\ I_{2m_{t_2}^*} \\ \vdots \\ 0 \end{pmatrix}^{\}2m_{t_1}^*}, \quad w_{t_3} = \begin{pmatrix} 0 \\ I_{2m_{t_3}^*} \\ \vdots \\ 0 \end{pmatrix}^{\}2m_{t_1}^*+2m_{t_2}^*},$$

$$\dots, \quad w_{t_j} = \begin{pmatrix} 0 \\ I_{2m_{t_j}^*} \\ \vdots \\ 0 \end{pmatrix}^{\}2m_{t_1}^*+\dots+2m_{t_{j-1}}^*}, \tag{2.6}$$

and

$$2m_{t_1}^* + \dots + 2m_{t_j}^* = k. \tag{2.7}$$

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In particular, k is even. We have (in the notation (2.4)) for $i \leqslant j$,

$$wL_{m^*_{t_i}}w^{-1} = \left\{ \begin{pmatrix} I_{2m^*_{t_1}+\cdots+2m^*_{t_{i-1}}} &&&&&& \\ & g &&&&& \\ && 1 &&&& \\ &&& \ddots &&& \\ &&&& 1 && \\ &&&&& g^* & \\ &&&&&& I_{2m^*_{t_1}+\cdots+2m^*_{t_{i-1}}} \end{pmatrix} \mid g \in \mathrm{GL}_{2m^*_{t_i}} \right\}. \tag{2.8}$$

Let us show that

$$t_1 < t_2 < \cdots < t_j. \tag{2.9}$$

For this, we use property (ii) in (2.2). Consider the subgroup

$$\left\{ \begin{pmatrix} I_{2m^*_{t_1}+\cdots+2m^*_{t_{i-1}}} &&&&&&&& \\ & I_{2m^*_{t_i}} & x &&&&&& \\ && I_{2m^*_{t_{i+1}}} &&&&&& \\ &&& 1 &&&&& \\ &&&& \ddots &&&& \\ &&&&& 1 &&& \\ &&&&&& I_{2m^*_{t+1}} & x' & \\ &&&&&&& I_{2m^*_{t_i}} & \\ &&&&&&&& I_{2m^*_{t_i}+\cdot+2m^*_{t_{i-1}}} \end{pmatrix} \in \mathrm{Sp}_{4n} \right\}$$

of M_k, which is generated by positive roots in M_k. Its inverse image under w lies in V_{2n}, the standard maximal unipotent subgroup of Sp_{4n}. This forces $t_i < t_{i+1}$, for $i = 1, \ldots, j-1$. Similarly, let $1 \leqslant a_i \leqslant 2r$, $i = 1, \ldots, e$, be such that

$$w_{a_1} = \begin{pmatrix} 0 \\ I_{2m^*_{a_1}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \}k, \quad w_{a_2} = \begin{pmatrix} 0 \\ I_{2m^*_{a_2}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \}k+2m^*_{a_1} \quad , \ldots, \ w_{a_e} = \begin{pmatrix} 0 \\ I_{2m^*_{a_e}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \}k+2m^*_{a_1}+\cdots+2m^*_{a_{e-1}} \tag{2.10}$$

and

$$2m^*_{a_1} + \cdots + 2m^*_{a_e} = 2n - k. \tag{2.11}$$

As before,

$$a_1 < a_2 < \cdots < a_e. \tag{2.12}$$

Considering the inverse image under w of the subgroup

$$\left\{\begin{pmatrix} I_{2n-2m^*_{a_e}} & & & \\ & I_{2m^*_{a_e}} & x & \\ & & I_{2m^*_{a_e}} & \\ & & & I_{2n-2m^*_{a_e}} \end{pmatrix} \in \mathrm{Sp}_{4n}\right\}$$

of M_k and requiring (by (2.2) (ii)) that it lies in V_{2n}, we conclude that $a_e < 2r+1-a_e$, i.e.

$$a_e \leqslant r. \tag{2.13}$$

From (2.6), (2.7), (2.9)–(2.13), we see that w has the following form

$$w = \left(\begin{array}{cccccc|cccc} & I_{2m^*_{t_1}} & & & & & & & & \\ & & & I_{2m^*_{t_2}} & & & & & & \\ & & & & & \cdot & & & & \\ & & & & & & \cdot & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & I_{2m^*_{t_j}} & \\ I_{2m^*_{a_1}} & & & & & & & & & \\ & & I_{2m^*_{a_2}} & & & & & & & \\ & & & & I_{2m^*_{a_e}} & & & & & \\ \hline & & & & \underline{*} & & \underline{*} & & & \end{array}\right). \tag{2.14}$$

Let $0 \leqslant i \leqslant j$ be the last integer, such that $t_i \leqslant r$ (i.e. the block $I_{2m^*_{t_i}}$ lies in the left half of w, while the block $I_{2m^*_{t_{i+1}}}$ lies in the right half of w). Put

$$2m^*_{t_1} + \cdots + 2m^*_{t_i} = 2b = 2b_{w,k}. \tag{2.15}$$

Denote

$$L'_{2\bar{m}} = \left\{\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_r \end{pmatrix} \in \mathrm{GL}_{2n} \;\middle|\; \begin{matrix} g_i \in \mathrm{GL}_{2m_i}, \\ i = 1, \dots, r \end{matrix}\right\}.$$

We can find $\tilde{w} \in W_{\mathrm{GL}_{2n}}(L'_{2\bar{m}})$ (notation of § 1.4) such that for

$$w' = \begin{pmatrix} \tilde{w} & \\ & \tilde{w}^* \end{pmatrix},$$

$$ww' = \begin{pmatrix} I_{2b} & & & & \\ & 0 & 0 & I_{k-2b} & \\ & 0 & I_{4n-2k} & 0 & \\ & -I_{k-2b} & 0 & 0 & \\ & & & & I_{2b} \end{pmatrix}. \tag{2.16}$$

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Thus,

$$w = \left(\begin{array}{ccc|ccc} I_{2b} & & & & & \\ & & I_{k-2b} & & & \\ & I_{2n-k} & & & & \\ \hline & & & & I_{2n-k} & \\ & & & I_{k-2b} & & \\ & & & & & I_{2b} \end{array}\right) \times \begin{pmatrix} I_{2b+2n-k} & & & \\ & & I_{k-2b} & \\ & -I_{k-2b} & & \\ & & & I_{2b+2n-k} \end{pmatrix} w_1', \quad (2.17)$$

where

$$w_1' = \begin{pmatrix} \tilde{w}_1 & \\ & \tilde{w}_1^* \end{pmatrix}, \quad \tilde{w}_1 = \begin{pmatrix} I_{2b} & & \\ & & I_{2n-k} \\ & I_{k-2b} & \end{pmatrix}, \quad \tilde{w}^{-1} \in W_{\mathrm{GL}_{2n}}(L'_{2\bar{m}}). \quad (2.18)$$

2.3. Continuation of the proof of Theorem 2.1

By [**9**, IV.4.1], we have for $w \in W(L_{2\bar{m}}, M_k)$ as in (2.17),

$$M(w) = M \left(\begin{array}{ccc|ccc} I_{2b} & & & & & \\ & & I_{k-2b} & & & \\ & I_{2n-k} & & & & \\ \hline & & & & I_{2n-k} & \\ & & & I_{k-2b} & & \\ & & & & & I_{2b} \end{array}\right) \circ M \begin{pmatrix} I_{2b+2n-k} & & & \\ & & I_{k-2b} & \\ & -I_{k-2b} & & \\ & & & I_{2b+2n-k} \end{pmatrix} \circ M(w_1'). \quad (2.19)$$

By the lemma in § 1.4 and by (2.18), $M(w_1')$ is holomorphic at $(1, \dots, 1)$. $M(w_1')$ permutes $\tau_1|\det\cdot|^{s_1-(1/2)} \otimes \cdots \otimes \tau_r|\det\cdot|^{s_r-(1/2)}$ on $L_{2\bar{m}}(\mathbb{A})$ to

$$\tau_{t_1}|\det\cdot|^{s_{t_1}-(1/2)} \otimes \cdots \otimes \tau_{t_i}|\det\cdot|^{s_{t_i}-(1/2)} \otimes \tau_{a_1}|\det\cdot|^{s_{a_1}-(1/2)} \otimes \cdots \otimes \tau_{a_e}|\det\cdot|^{s_{a_e}-(1/2)}$$
$$\otimes \tau_{2r+1-t_{i+1}}|\det\cdot|^{s_{2r+1-t_{i+1}}-(1/2)} \otimes \cdots \otimes \tau_{2r+1-t_j}|\det\cdot|^{s_{2r+1-t_j}-(1/2)}. \quad (2.20)$$

Put

$$t'_{i+1} = 2r + 1 - t_{i+1}, \dots, t'_j = 2r + 1 - t_j.$$

We will repeatedly use the identity

$$\begin{pmatrix} I_\alpha &&& \\ && I_{\beta+\gamma} & \\ & -I_{\beta+\gamma} && \\ &&& I_\alpha \end{pmatrix} = \begin{pmatrix} I_{\alpha+\gamma} &&& \\ && I_\beta & \\ & -I_\beta && \\ &&& I_{\alpha+\gamma} \end{pmatrix} \left(\begin{array}{ccc|ccc} I_\alpha &&&&& \\ && I_\gamma &&& \\ & I_\beta &&&& \\ \hline &&&& I_\beta & \\ &&& I_\gamma && \\ &&&&& I_\alpha \end{array}\right) \times \begin{pmatrix} I_{\alpha+\beta} &&& \\ && I_\gamma & \\ & -I_\gamma && \\ &&& I_{\alpha+\beta} \end{pmatrix}. \tag{2.21}$$

We first use it for $\gamma = 2m^*_{t_j}$, $\beta = 2m^*_{t_{j-1}}$, $\alpha = 2n - (\beta + \gamma)$. We know from [**5**, Proposition 1] that

$$(s_{t_j} - 1)M \begin{pmatrix} I_{\alpha+\beta} &&& \\ && I_\gamma & \\ & -I_\gamma && \\ &&& I_{\alpha+\beta} \end{pmatrix}$$

is holomorphic and non-zero at $(1, \dots, 1)$. (Note that the inducing data come now from (2.20)). This operator (evaluated at $(1, \dots, 1)$) replaces, in (2.20), $\tau_{t'_j}$ by $\hat{\tau}_{t'_j} = \tau_{t'_j}$ and $s_{t'_j} - \frac{1}{2}$ by $-s_{t'_j} + \frac{1}{2}$. We keep using [**9**, IV.4.1] (whose conditions are easily seen to be satisfied). The next Weyl element in (2.21) gives a holomorphic intertwining operator (at $(1, \dots, 1)$) by Lemma 1.14 (where now $z_1 = \cdots = z_{r-1} = \frac{1}{2}$, $z_r = -\frac{1}{2}$, and the inducing data are given by (2.20) except that $s_{t_j} - 1/2$ is replaced by $-s_{t'_j} + \frac{1}{2}$). This operator switches the order of

$$\tau_{t'_{j-1}} \otimes |\det\cdot|^{s_{t'_{j-1}} - (1/2)} \quad \text{and} \quad \tau_{t'_j} \otimes |\det\cdot|^{-s_{t'_j} + (1/2)}$$

in the inducing data (which appears in the last parenthesis). Next consider the third Weyl element on the right-hand side of (2.21). Again, from [**5**, Proposition 1] we know that

$$(s_{t'_{j-1}} - 1)M \begin{pmatrix} I_{\alpha+\gamma} &&& \\ && I_\beta & \\ & -I_\beta && \\ &&& I_{\alpha+\gamma} \end{pmatrix}$$

is holomorphic and non-zero at $(1, \dots, 1)$. Now apply (2.21) on intertwining operators (using [**9**, IV.4.1] at each stage) for $\gamma = 2m^*_{t_{j-1}} + 2m^*_{t_j}$, $\beta = 2m^*_{t_{j-2}}$ and $\alpha = 2n - (\beta + \gamma)$,

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and then again for $\gamma = 2m^*_{t_{j-2}} + 2m^*_{t_{j-1}} + 2m^*_{t_j}$, $\beta = 2m^*_{t_{j-3}}$ and $\alpha = 2n - (\beta+\gamma)$ and so on. Note that in each step Lemma 1.14 is applicable. This shows that for w as in (2.14)

$$(s_{t'_{i+1}} - 1)(s_{t'_{i+2}} - 1)\cdot\dots\cdot(s_{t'_j - 1})M\begin{pmatrix} I_{2b+2n-k} & & & \\ & & I_{k-2b} & \\ & -I_{k-2b} & & \\ & & & I_{2b+2n-k}\end{pmatrix}M(w'_1) \tag{2.22}$$

is holomorphic and non-zero at $(1,\dots,1)$. This operator, evaluated at $(1,\dots,1)$, takes the inducing data (2.20) to

$$\begin{aligned}\tau_{t_1}|\det\cdot|^{s_{t_1}-(1/2)}\otimes\dots\otimes\tau_{t_i}|\det\cdot|^{s_{t_i}-(1/2)}\otimes\tau_{a_1}|\det\cdot|^{s_{a_1}-(1/2)}\otimes\dots\otimes\tau_{a_e}|\det\cdot|^{s_{a_e}-(1/2)}\\ \otimes\,\tau_{t'_j}|\det\cdot|^{-s_{t'_j}+(1/2)}\otimes\tau_{t'_{j-1}}|\det\cdot|^{-s_{t'_{j-1}}+(1/2)}\otimes\dots\otimes\tau_{t'_{i+1}}|\det\cdot|^{-s_{t'_{i+1}}+(1/2)}.\end{aligned} \tag{2.23}$$

Now apply the operator

$$M\left(\begin{array}{ccc|ccc} I_{2b} & & & & & \\ & & I_{k-2b} & & & \\ & I_{2n-k} & & & & \\ \hline & & & & I_{2n-k} & \\ & & & I_{k-2b} & & \\ & & & & & I_{2b}\end{array}\right),$$

which is holomorphic by Lemma 1.14, and from (2.19), we conclude that

$$M'(w) = (s_{t'_{i+1}} - 1)(s_{t'_{i+2}} - 1)\cdot\dots\cdot(s_{t'_j} - 1)M(w)\quad\text{is holomorphic and non-zero at } (1,\dots,1). \tag{2.24}$$

$M'(w)$ permutes the inducing data $\tau_1|\det\cdot|^{s_1-(1/2)}\otimes\dots\otimes\tau_r|\det\cdot|^{s_r-(1/2)}$ on $M_k(\mathbb{A})$ to

$$\begin{aligned}\tau_{t_1}|\det\cdot|^{s_{t_1}-(1/2)}\otimes\dots\otimes\tau_{t_i}|\det\cdot|^{s_{t_i}-(1/2)}\otimes\tau_{t'_j}|\det\cdot|^{-s_{t'_j}+(1/2)}\otimes\tau_{t'_{j-1}}|\det\cdot|^{-s_{t'_{j-1}}+(1/2)}\\ \otimes\dots\otimes\tau_{t'_{i+1}}|\det\cdot|^{s_{t'_{i+1}}+(1/2)}\otimes\tau_{a_1}|\det\cdot|^{s_{a_1}-(1/2)}\otimes\dots\otimes\tau_{a_e}|\det\cdot|^{s_{a_e}-(1/2)}\end{aligned} \tag{2.25}$$

on $w(M_k)(\mathbb{A})$. It remains to examine $E_{M_k}(h, M(w)(\varphi_{\bar\tau,\bar s}))$ (see equation (2.1)) on $M_k(\mathbb{A})$ at $(1,\dots,1)$. $E_{M_k}(\cdot, M'(w)(\varphi_{\bar\tau,\bar s}))$ is a sum of products of Eisenstein series on $\mathrm{GL}_k(\mathbb{A})$, induced from

$$\tau_{t_1}|\det\cdot|^{\tilde s_{t_1}}\otimes\dots\otimes\tau_{t_i}|\det\cdot|^{\tilde s_{t_i}}\otimes\tau_{t'_j}|\det\cdot|^{\widetilde{\widetilde{-s}}_{t'_j}}\otimes\dots\otimes\tau_{t'_{i+1}}|\det\cdot|^{\widetilde{\widetilde{-s}}_{t'_{i+1}}}, \tag{2.26}$$

where

$$\tilde s_t = s_t + 2n - \tfrac{1}{2}k,\qquad \widetilde{\widetilde{-s}}_t = -s_t + 2n - \tfrac{1}{2}k + 1$$

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and Eisenstein series on $\mathrm{Sp}_{4n-2k}(\mathbb{A})$, induced from the adele points of the parabolic subgroup of Sp_{4n-2k}

$$Q'_{2\bar{a}} = \left\{ \begin{pmatrix} g_1 & & & & & \\ & \ddots & & & * & \\ & & g_e & & & \\ & & & g_e^* & & \\ & & & & \ddots & \\ & & & & & g_1^* \end{pmatrix} \in \mathrm{Sp}_{4n-2k} \;\middle|\; \begin{matrix} g_i \in \mathrm{GL}_{2a_i}, \\ i = 1, \ldots, e \end{matrix} \right\},$$

and the representation $\tau_{a_1}|\det\cdot|^{s_{a_1}-(1/2)} \otimes \cdots \otimes \tau_{a_e}|\det\cdot|^{s_{a_e}-(1/2)}$. The Eisenstein series on $\mathrm{GL}_k(\mathbb{A})$ induced from (2.26) is holomorphic at $(1,\ldots,1)$ by Lemma 1.14. The Eisenstein series on $\mathrm{Sp}_{4n-2k}(\mathbb{A})$ is holomorphic at $(1,\ldots,1)$, after being multiplied by $(s_{a_1} - 1)\cdot\cdots\cdot(s_{a_e}-1)$. For this we use induction. We conclude that $(s_{t'_{i+1}}-1)\cdot\cdots\cdot(s_{t'_j}-1)(s_{a_1} - 1)\cdot\cdots\cdot(s_{a_e}-1)E_{M_k}(h, M(w)\varphi_{\bar{\tau},\bar{s}})$ is holomorphic at $(1,\ldots,1)$. In particular (see (2.1)),

$$(s_1-1)(s_2-1)\cdot\cdots\cdot(s_r-1)E^{U_k}(h, \varphi_{\bar{\tau},\bar{s}})$$

is holomorphic at $(1,\ldots,1)$ for all k. We also conclude the following corollary.

Corollary 2.2. *In the above notation, if $b > 0$ (see (2.15)), then*

$$\lim_{\bar{s}\to\bar{1}}(s_1-1)(s_2-1)\cdot\cdots\cdot(s_r-1)E_{M_k}(h, M(w)\varphi_{\bar{\tau},\bar{s}}) = 0.$$

Thus, only $w \in W(L_{2\bar{m}}, M_k)$, with $b = 0$ contribute to

$$\lim_{s\to\bar{1}}(s_1-1)(s_2-1)\cdot\cdots\cdot(s_r-1)E_{M_k}(h, M(w)\varphi_{\bar{\tau},\bar{s}}).$$

To conclude that $(s_1-1)\cdot\cdots\cdot(s_r-1)E(\cdot\varphi_{\bar{\tau},\bar{s}})$ is not identically zero at $(1,\ldots,1)$, we consider the case $k = 2n$ of the Siegel parabolic subgroup. Let $w \in W(L_{2\bar{m}}, M_{2n})$ (of the form (2.14)) with $b = 0$. From (2.14) we conclude that

$$w = \begin{pmatrix} & I_{2n} \\ -I_{2n} & \end{pmatrix}.$$

Thus, only

$$M\begin{pmatrix} & I_{2n} \\ -I_{2n} & \end{pmatrix}$$

contributes to

$$\lim_{\bar{s}\to\bar{1}}(s_1-1)\cdot\cdots\cdot(s_r-1)E^{U_{2n}}(h, \varphi_{\bar{\tau},\bar{s}}),$$

and we get from (2.1),

$$\lim_{\bar{s}\to\bar{1}}(s_1-1)\cdot\cdots\cdot(s_r-1)E^{U_{2n}}(h, \varphi_{\bar{\tau},\bar{s}}) = E_{M_{2n}}\left(h, M'\begin{pmatrix} & I_{2n} \\ -I_{2n} & \end{pmatrix}\varphi_{\bar{\tau},\bar{1}}\right), \quad h \in M_{2n}(\mathbb{A}). \tag{2.27}$$

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The right-hand side of (2.27) is non-zero. It is $|\det\cdot|^n$ times an Eisenstein series on $\mathrm{GL}_{2n}(\mathbb{A})$ induced from $\tau_r \otimes \cdots \otimes \tau_1$ evaluated at $(0,\dots,0)$. This completes the proof of Theorem 2.1. □

We denote

$$E_{\bar{\tau},\bar{1}}(g,\varphi_{\bar{\tau},\bar{1}}) = \lim_{\bar{s}\to\bar{1}}(s_1-1)\cdot\cdots\cdot(s_{r-1})E(g,\varphi_{\bar{\tau},\bar{s}}).$$

3. The tower property of $E_{\bar{\tau},\bar{1}}$

We keep the notation of §§ 1.2 and 2.

3.1. A small summary

Our goal in this section is to prove Theorem 1.1 for $E_{\bar{\tau},\bar{1}}$, i.e. prove the identity (1.11) for $E_{\bar{\tau},\bar{1}}$. We will do this only for $k \leqslant n$. (This will suffice.) Recall that we have, at this stage, Theorem 1.13 at hand, i.e. $E_{\bar{\tau},\bar{1}}$ satisfies the vanishing property, which means that $\sigma_{\psi,k,\alpha}(E_{\bar{\tau},\bar{1}}) = 0$, for all $k \leqslant n-1$ and all $\alpha \in K^*$.

3.2.

Theorem 3.1. *For $1 \leqslant p \leqslant k \leqslant n$, $\phi_1 \in S(\mathbb{A}^p)$, $\phi_2 \in S(\mathbb{A}^{k-p})$ and an automorphic form ξ in the space of $E_{\bar{\tau},\bar{1}}$, we have for each $\alpha \in K^*$,*

$$\int_{R_p(K)\backslash R_p(\mathbb{A})} p_{k,\alpha}((r,1),\xi,\phi_1\otimes\phi_2)\,\mathrm{d}r = \sum_{\gamma\in Z_p(K)\backslash \mathrm{GL}_p(K)} \int_{\mathcal{L}_{\mathbb{A}}} p_{k-p,\alpha}(1,\hat{\gamma}x\beta\cdot\xi,\phi_2)\phi_1(j(x))\,\mathrm{d}x. \quad (3.1)$$

Here we use the notation of Theorem 1.1 and $p_{j,\alpha}$ is defined as explained right after (1.13).

Proof. We follow the proof of Theorem 8 of [**5**]. From this, it follows that for *any* automorphic form ξ on $\mathrm{Sp}_{4n}(\mathbb{A})$, we have (in the notation of (1.13))

$$\int_{R_p(K)\backslash R_p(\mathbb{A})} p_{k,\alpha}((r,1),\xi,\phi_1\otimes\phi_2)\,\mathrm{d}r = \int_{\mathcal{L}_{\mathbb{A}}}\phi_1(j(x))\int_{\mathcal{H}_{k-p}(K)\backslash\mathcal{H}_{k-p}(\mathbb{A})}\int_{N^*_{k-p,p}(K)\backslash N^*_{k-p,p}(\mathbb{A})}\theta^{\phi_2}_{\psi^{-\alpha},k-p}(h) \times \xi(vj_{k-p}(h)x\beta)\chi^{-1}_{k-p}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}x. \quad (3.2)$$

See Theorem 1.1 for the notation ($\mathcal{L}, j(x)$, etc.). Here

$$N^*_{i,j} = \left\{ v = \begin{pmatrix} I_j & a & b & c & d \\ & z & e & f & c' \\ & & I_{2(i+1)} & e' & b' \\ & & & z^* & a' \\ & & & & I_j \end{pmatrix} \in \mathrm{Sp}_{4n} \;\middle|\; \begin{matrix} z \in Z_{2n-i-j-1} \\ a \text{ has zero first column} \end{matrix} \right\}.$$

We use (3.2) for $\xi = \lim_{\bar{s}\to\bar{1}}(s_1-1)\cdot\dots\cdot(s_r-1)E(\cdot,\varphi_{\bar{\tau},\bar{s}})$. Consider the Fourier expansion of the following function on $K^p\backslash\mathbb{A}^p$:

$$t \mapsto \int_{\mathcal{H}_{k-p}(K)\backslash\mathcal{H}_{k-p}(\mathbb{A})}\int_{N^*_{k-p,p}(K)\backslash N^*_{k-p,p}(\mathbb{A})} \theta^{\phi_2}_{\psi^{-\alpha},k-p}(h)\xi(vj_{k-p}(h)\tilde{t}x\beta)\chi^{-1}_{k-p}(v)\,\mathrm{d}v\,\mathrm{d}h, \tag{3.3}$$

where

$$\tilde{t} = \begin{pmatrix} I_p & t & & & \\ & 1 & & & \\ & & I_{4n-2p-2} & & \\ & & & 1 & t' \\ & & & & I_p \end{pmatrix} \in \mathrm{Sp}_{4n}\,.$$

We claim that the trivial character contributes zero to the Fourier expansion of (3.3). Indeed the corresponding coefficient of (3.3) involves, as an inner integral, the constant term of ξ along $U_p(K)\backslash U_p(\mathbb{A})$. (Note that in [**5**, Theorem 8], this constant term was automatically zero since U_p is different from the Siegel radical, and there we induced from a cusp form on $\mathrm{GL}_{2n}(\mathbb{A})$.) From (2.1) Corollary 2.2 in § 2.3. and from (2.26) we know that, first, p must be even and, second, $\lim_{\bar{s}\to\bar{1}}(s_1-1)\cdot\dots\cdot(s_r-1)E^{U_p}(\cdot,\varphi_{\bar{\tau},\bar{s}})$ is a sum parametrized by Weyl elements $w \in W(L_{2\bar{m}}, M_p)$, such that the corresponding sum $2b_{w,\beta}$ in (2.15) is zero, and for each such w (appearing in (2.14)) the corresponding summand is a sum of products of the following form: the first factor is the value of the Eisenstein on $\mathrm{GL}_p(\mathbb{A})$ induced from $\tau_{t'_j}|\det\cdot|^{z_{t'_j}} \otimes\dots\otimes \tau_{t'_1}|\det\cdot|^{z'_{t_1}}$ evaluated at $(z_{t'_j},\dots,z_{t'_1}) = -(2n+1-\frac{1}{2}p)(1,\dots,1)$. The second factor has the form

$$E_{\bar{\tau}'} = \lim_{\bar{s}'\to(1,\dots,1)}(s_{a_1}-1)\cdot\dots\cdot(s_{a_e}-1)E_{\bar{\tau}',\bar{s}'}, \tag{3.4}$$

where $\bar{s}' = (s_{a_1},\dots,s_{a_e})$ and $E_{\bar{\tau}',\bar{s}'}$ is an Eisenstein series on $\mathrm{Sp}_{4n-2p}(\mathbb{A})$ induced from $\bar{\tau}' = \tau_{a_1}|\det\cdot|^{s_{a_1}-(1/2)} \otimes\dots\otimes \tau_{a_e}^{s_{a_e}-(1/2)}$ (using a holomorphic section). The notation $t_1,\dots,t_j,a_1,\dots,a_e$, etc. (for the element w) is explained in (2.7), (2.11), (2.14), (2.15). Thus, the contribution of the trivial character in the Fourier expansion of (3.3) is a linear combination of terms of the form

$$\int_{\mathcal{H}_{k-p}(K)\backslash\mathcal{H}_{k-p}(\mathbb{A})}\int_{N^{(2n-p)}_{k-p+1}(K)\backslash N^{(2n-p)}_{k-p+1}(\mathbb{A})} \theta^{\phi_2}_{\psi^{-\alpha},k-p}(h) \times E_{\bar{\tau}'}(vj^{(2n-p)}_{k-p}(h))(\chi^{(2n-p)}_{k-p})^{-1}(v)\,\mathrm{d}v\,\mathrm{d}h. \tag{3.5}$$

(Note that in (3.3) $x \in \mathcal{L}_{\mathbb{A}}$ is fixed.) $E_{\bar{\tau}'}$ is of the form (3.4). The superscript $(2n-p)$ marks the fact that the corresponding object is for Sp_{4n-2p}. Thus,

$$N^{(2n-p)}_{k-p+1} = \left\{ v = \begin{pmatrix} z & * & * \\ & I_{2(k-p+1)} & * \\ & & z^* \end{pmatrix} \in \mathrm{Sp}_{4n-2p} \mid z \in Z_{2n-k-1} \right\}. \tag{3.6}$$

For $h=(x,z)\in\mathcal{H}_{k-p}$,

$$j_{k-p}^{(2n-p)}(x,z)=\begin{pmatrix} I_{2n-k-1} & & & & \\ & 1 & x & z & \\ & & I_{2(k-p)} & x' & \\ & & & 1 & \\ & & & & I_{2n-k-1}\end{pmatrix}. \tag{3.7}$$

For $v\in N_{k-p+1}^{(2n-p)}(\mathbb{A})$ as in (3.6)

$$\chi_{k-p}^{(2n-p)}(v)=\psi\biggl(\sum_{j=1}^{2n-k-1}v_{j,j+1}\biggr). \tag{3.8}$$

Note that the integral (3.5) is the evaluation at the identity of $p_{k-p,\alpha}(\tilde g,E_{\bar\tau'},\phi_2)$ in the notation of (1.13). Denote by $\mathcal{E}_{\bar\tau'}$ the representation of $\mathrm{Sp}_{4n-2p}(\mathbb{A})$ generated by the elements $E_{\bar\tau'}$ of (3.4). The automorphic functions on $\widetilde{\mathrm{Sp}}_{2(k-p)}(\mathbb{A})$, $\tilde g\mapsto p_{k-p,\alpha}(\tilde g,E_{\bar\tau'},\phi_2)$, constitute the space $\sigma_{\psi,k-p,\alpha}^{(2n-p)}(\mathcal{E}_{\bar\tau'})$. Again, the superscript $(2n-p)$ is to mark that we start with Sp_{4n-2p} (p is even). Write $p=2p'$. Then, since $k\leqslant n$,

$$k-p<n-p'.$$

By Theorem 1.13, we conclude that

$$\sigma_{\psi,k-p,\alpha}^{(2n-p)}(\mathcal{E}_{\bar\tau'})=0. \tag{3.9}$$

We have shown that in the Fourier expansion of (3.3) only non-trivial characters contribute. Thus the value of (3.3) at $\tilde t=0$ is

$$\sum_{\gamma\in D(K)\backslash \mathrm{GL}_p(K)}\int_{\mathcal{H}_{k-p}(K)\mathcal{H}_{k-p}(\mathbb{A})}\int_{N^*_{k-p,p-1}(K)\backslash N^*_{k-p,p-1}(\mathbb{A})}\theta^{\phi_2}_{\psi^{-\alpha},k-p}(h)\times\xi(vj_{k-p}(h)\hat\gamma x\beta)\chi^{-1}_{k-p}(v)\,\mathrm{d}v\,\mathrm{d}h, \tag{3.10}$$

where D is the subgroup

$$\biggl\{\begin{pmatrix} g & x\\ 0 & 1\end{pmatrix}\in\mathrm{GL}_p\biggr\}$$

of GL_p. We continue in the same way for each summand of (3.10), and consider the Fourier expansion of the following function on $K^{p-1}\backslash\mathbb{A}^{p-1}$

$$t\mapsto\int_{\mathcal{H}_{k-p}(K)\backslash\mathcal{H}_{k-p}(\mathbb{A})}\int_{N^*_{k-p,p-1}(K)\backslash N^*_{k-p,p-1}}\theta^{\phi_2}_{\psi^{-\alpha},k-p}(h)\xi(v,j_{k-p}(h)\tilde t\hat\gamma x\beta)\chi^{-1}_{k-p}(v)\,\mathrm{d}v\,\mathrm{d}h,$$

where

$$\tilde t=\begin{pmatrix} I_{p-1} & t & & & \\ & 1 & & & \\ & & I_{4n-2p} & & \\ & & & 1 & t' \\ & & & & I_{p-1}\end{pmatrix}. \tag{3.11}$$

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The same argument as before shows that the trivial character does not contribute to the Fourier expansion of (3.11). Here the coefficient of (3.11) corresponding to the trivial character contains, as an inner integrand, the constant terms of ξ along U_{p-1}. As before, $p-1$ must be even (otherwise the constant terms of ξ along U_{p-1} is zero). This constant term has the form $\lim_{\bar{s}\to\bar{1}}(s_1-1)\cdot\dots\cdot(s_p-1)E^{U_{p-1}}(\cdot,\varphi_{\bar{\tau},\bar{s}})$ and hence is parametrized by $w\in W(L_{2\bar{m}},M_{p-1})$ such that $b_{w,p-1}=0$ (in (2.15)) and has the form (2.14). Thus, we get, as before, a linear combination of terms, which are evaluations at the identity of elements of $\sigma_{\psi,k-p,\alpha}^{(2n-(p-1))}(\mathcal{E}_{\bar{\tau}'})$, where $\mathcal{E}_{\bar{\tau}'}$ is the representation of $\mathrm{Sp}_{4n-2(p-1)}(\mathbb{A})$ generated by the elements $\lim_{\bar{s}'\to(1,\dots,1)}(s_{a_1}-1)\cdot\dots\cdot(s_{a_e}-1)E_{\bar{\tau}',\bar{s}'}$, similar to (3.4) only that $E_{\bar{\tau}',\bar{s}'}$ is the Eisenstein series on $\mathrm{Sp}_{4n-2(p-1)}(\mathbb{A})$ induced from $\bar{\tau}'=\tau_{a_1}|\det\cdot|^{s_{a_1}-(1/2)}\otimes\dots\otimes\tau_{a_e}|\det\cdot|^{s_{a_e}-(1/2)}$, where $(a_1,\dots,a_e)$ are determined by w (2.14) above. Since $k-p<n-\frac{1}{2}(p-1)$, we can apply the theorem at the end of § 1.3 and conclude that the trivial character does not contribute to the Fourier expansion of (3.11). We continue in this way, following the steps of the proof of [**5**, Theorem 8], until we get (3.1). □

Corollary 3.2. *For $\alpha\in K^*$, the representation $\sigma_{\psi,n,\alpha}(E_{\bar{\tau},\bar{1}})$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ is cuspidal (in the sense that the constant terms of the elements of $\sigma_{\psi,n,\alpha}(E_{\bar{\tau},\bar{1}})$ along unipotent radicals of parabolic subgroup are identically zero).*

Proof. We use (3.1) for $k=n$, and then use the theorem in § 1.3 which guarantees that $\sigma_{n-p,\psi,\alpha}(E_{\bar{\tau},\bar{1}})=0$, for $1\leqslant p\leqslant n$, so that the right-hand side of (3.1) is zero (term-wise). □

4. Endoscopic representations of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$

We are ready to prove our main (global) theorem. We keep the notation and assumptions of the previous section.

Theorem 4.1. *We have*

$$\sigma_{\psi,n}(E_{\bar{\tau},\bar{1}})\neq 0,$$

and the representation $\sigma_{\psi,n}(E_{\bar{\tau},\bar{1}})$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ has a non-trivial ψ-Whittaker coefficient.

Proof. Since $E_{\bar{\tau},\bar{1}}$ satisfies the vanishing property, it remains, by Theorem 1.5, to show that the following integral is not identically zero

$$\int_{\chi_0(\mathbb{A})}\int_{Z_{2n}(K)\backslash Z_{2n}(\mathbb{A})}\xi^{U_{2n}}(m(z)\bar{\ell}(x))\chi_\psi(z)\,\mathrm{d}z\,\mathrm{d}x, \tag{4.1}$$

as ξ varies in $E_{\bar{\tau},\bar{1}}$. We have already computed $\xi^{U_{2n}}$ in (2.27). It is an element of

$$\mathrm{Ind}_{Q_{2n}(\mathbb{A})}^{\mathrm{Sp}_{4n}(\mathbb{A})}E_{\mathrm{GL}_{2n}}(\bar{\tau},\bar{o})|\det\cdot|^{-1/2},$$

where $E_{\mathrm{GL}_{2n}}(\bar{\tau},\bar{o})$ is the Eisenstein series on $\mathrm{GL}_{2n}(\mathbb{A})$ induced from $\tau_r\otimes\dots\otimes\tau_1$. The $\mathrm{d}z$-integration in (4.1) realizes it inside

$$\mathrm{Ind}_{Q_{2n}(\mathbb{A})}^{\mathrm{Sp}_{4n}(\mathbb{A})}E_{\mathrm{GL}_{2n}}^{\chi_\psi}(\bar{\tau},\bar{o})|\det\cdot|^{-1/2},$$

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where $E^{\chi_\psi}_{\mathrm{GL}_{2n}}(\bar\tau,\bar o)$ is the χ_ψ-Whittaker model of $E_{\mathrm{GL}_{2n}}(\bar\tau,\bar o)$ (χ_ψ is the Whittaker character (1.15)). Now, we can use Lemma 2 at the end of Chapter 5 of [**6**] to conclude that (4.1) is not identically zero. □

4.1. The unramified parameters of $\sigma_{\psi,n}(E_{\bar\tau,\bar 1})$

So far, we know that $\sigma_{\psi,n}(E_{\bar\tau,\bar 1})$ is a non-trivial cuspidal (genuine) representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which admits a non-trivial ψ-Whittaker coefficients. The construction of $\sigma_{\psi,n}(E_{\bar\tau,\bar 1})$ is such that it has a non-trivial L^2-pairing with all irreducible, automorphic, cuspidal (genuine) representations σ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which admit a non-trivial ψ-Whittaker coefficient, and such that at almost all places ν, the ψ_ν-unramified parameters of σ_ν are those of the unramified constituents of

$$\mathrm{Ind}^{\mathrm{GL}_{2n}(K_\nu)}_{P_{2m_1,\ldots,2m_r}(K_\nu)}\tau_{1,\nu}\otimes\cdots\otimes\tau_{r,\nu}.$$

This follows exactly as in Remark 1 at the end of Chapter 2 of [**5**]. We note also that each irreducible summand of $\sigma_{\psi,n}(E_{\bar\tau},\bar 1)$ has a non-trivial ψ-Whittaker coefficient. This follows as in [**5**, Proposition 11]. This also implies that $\sigma_{\psi,n}(E_{\bar\tau,\bar 1})$ is a multiplicity free representation. Indeed, if σ and σ' are two isomorphic summands, acting in the subspaces V_σ and $V_{\sigma'}$, respectively, then we may choose an isomorphism $T : V_\sigma\to V_{\sigma'}$ such that the ψ-Whittaker coefficient is identically zero on $W=\{T(v)-v\mid v\in V_\sigma\}$. This follows from the uniqueness of the ψ-Whittaker model. However, W is an irreducible summand of $\sigma_{\psi,n}(E_{\bar\tau,\bar 1})$, and hence must have a non-trivial ψ-coefficient. This is a contradiction, unless $W=0$, i.e. $T=\mathrm{id}$ and so $V_\sigma=V_{\sigma'}$. Note that for the last two assertions, we have to use the theory of [**4**], where in the global integrals for $\widetilde{\mathrm{Sp}}_{2n}\times\mathrm{GL}_{2n}$ we replace the Eisenstein series (induced from Q_{2n} and a cusp form on $\mathrm{GL}_{2n}(\mathbb{A})$) by $E_{\bar\tau,\bar s}$. The theory and results of [**4**] remain the same without change. To complete the proof of the main global theorem, it remains to prove the following theorem.

Theorem 4.2. *Let σ be an irreducible summand of $\sigma_{\psi,n}(E_{\bar\tau,\bar 1})$. (We know that σ is cuspidal and admits a non-trivial ψ-Whittaker coefficient.) Then at almost all places ν, the ψ_ν-unramified parameters of σ_ν are those of the unramified constituents of*

$$\mathrm{Ind}^{\mathrm{GL}_{2n}(K_\nu)}_{P_{2m_1,\ldots,2m_r}(K_\nu)}\tau_{r,\nu}\otimes\cdots\otimes\tau_{r,\nu}.$$

Proof. The proof already lies in [**7**] almost without change. We just have to make one remark. In [**7**] we used the fact that at almost all places ν the unramified constituent π_ν of $\rho_{\tau_\nu,1}$ (where we assumed that τ is cuspidal on $\mathrm{GL}_{2n}(\mathbb{A})$) had a non-trivial $H(K_\nu)$-invariant functional, where H is the direct sum embedding of $\mathrm{Sp}_{2n}\times\mathrm{Sp}_{2n}$ inside Sp_{4n}. However, we needed this just to ensure that π_ν satisfies the vanishing property. In our case, if $\pi_{\bar\tau_\nu}$ denotes the unramified constituent of $\rho_{\bar\tau_\nu,\bar 1_\nu}$ then we know that it satisfies the vanishing property by Corollary 1.12 at the end of § 1.3. Now we can repeat the proof of [**7**]. The starting point of the proof in [**7**] was Theorem 3 (in [**7**]), which resulted from [**6**, Corollary 4.4], where again the existence of the $H(K_\nu)$ invariant functional was used just to ensure the vanishing property, which we now have. Thus, Theorem 3 of [**7**] is valid for $\pi_{\bar{\tau}_\nu}$ as well. Next, Theorem 4 in [**7**] used just the self-duality of the GL_{2n}-representation (and triviality of the central character) and hence it, as well as its corollary, Theorem 5 (of [**7**]) are valid for $\pi_{\bar{\tau}\nu}$. The material of [**7**, § 3] clearly applies to $\pi_{\bar{\tau}\nu}$ word for word. (We replace τ_ν there by $\mathrm{Ind}_{P_{2m_1,\dots,2m_r}}^{\mathrm{GL}_{2n}(F_\nu)}(K_\nu)\tau_{1,\nu}\otimes\cdots\otimes\tau_{r,\nu}$.) This completes the proof of the theorem. □

Corollary 4.3. *For each irreducible summand* σ *of* $\sigma_{\psi,n}(E_{\bar{\tau},\bar{1}})$*, the partial L-function* $L_\psi^S(\sigma\otimes\tau_i,s)$ *has a pole at* $s=1$*, for* $i=1,\dots,r$*.*

Remark 4.4. Let σ be an irreducible, automorphic, cuspidal (genuine) representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, such that it admits a non-trivial ψ-Whittaker coefficient. Assume that $L_\psi^S(\sigma\otimes\tau_i,s)$ has a pole at $s=1$, for $i=1,\dots,r$. Then σ has a non-trivial L^2-pairing with a summand of $\sigma_{\psi,n}(E_{\bar{\tau},\bar{1}})$.

Proof. As we mentioned in the beginning of this subsection, we can replace in the Shimura type integrals of [**4**], the Eisenstein series (induced from $Q_{2n}(\mathbb{A})$ and a cusp form on $\mathrm{GL}_{2n}(\mathbb{A})$) by $E_{\bar{\tau},\bar{s}}$, and these integrals will represent $L_\psi^S(\sigma\otimes\tau_1,s_1)\cdot\dots\cdot L_\psi^S(\sigma\otimes\tau_r,s_r)$, up to a denominator which is holomorphic and non-zero at $(1,\dots,1)$. The assumption on σ, means, by the structure of these Shimura integrals that the space of σ has a non-trivial L^2-pairing with the space of $\sigma_{\psi,n}(E_{\bar{\tau},\bar{1}})$. This is the same argument as in the introduction of [**5**, **6**].

5. Endoscopic representations of $\widetilde{\mathbf{Sp}}_{\mathbf{2n}}$: the local case

We present, in this section, the analogue, over a non-archimedean local field, of the global theory studied in the previous section.

5.1. Some preliminaries

Let F be a non-archimedean local field of characteristic zero. Let $\tau_1,\dots,\tau_r$ be irreducible, supercuspidal, self-dual representations of $\mathrm{GL}_{2m_1}(F),\dots,\mathrm{GL}_{2m_r}(F)$, respectively. Assume that these representations are pairwise inequivalent, and that $L(\tau_i,\Lambda^2,s)$ has a pole at $s=0$, for $i=1,\dots,r$. Here, it will be convenient to denote

$$\tau=\mathrm{Ind}_{P_{2m_1,\dots,2m_r}(F)}^{\mathrm{GL}_{2n}(F)}\tau_1\otimes\cdots\otimes\tau_r. \tag{5.1}$$

This is an irreducible, self-dual tempered representation of $\mathrm{GL}_{2n}(F)$. Denote

$$\rho_{\tau,s,}=\mathrm{Ind}_{Q_{2n}(F)}^{\mathrm{Sp}_{4n}(F)}\tau|\det\cdot|^{s-(1/2)}. \tag{5.2}$$

Let π_τ be the Langlands quotient of $\rho_{\tau,1}$. Note that $\rho_{\tau,1}$ is reducible, since

$$\mathrm{Ind}_{Q_{2n_i}(F)}^{\mathrm{Sp}_{4n}(F)}\tau_i|\det\cdot|^{1/2}$$

is reducible, by [**11**]. The last representation has two irreducible constituents: one generic subrepresentation and one (non-generic) irreducible quotient. Clearly, π_τ is not generic. Our goal in this section is to study the following $\widetilde{\mathrm{Sp}}_{2n}(F)$-module

$$\sigma_{\psi,n}(\pi_\tau)=J_{\mathcal{H}_n(F)}(J_{N_{n+1}(F),\chi_n}(\pi_\tau)\otimes\omega_{\psi^{-1}}^{(n)}). \tag{5.3}$$

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Here, ψ is a fixed non-trivial character of F. $\mathcal{H}_n(F)$ and $N_{n+1}(F)$ are as in the previous sections (see (1.4), (1.6)) and χ_n is defined by (1.5). $\omega_{\psi^{-1}}^{(n)}$ is the Weil representation of $\widetilde{\mathrm{Sp}}_{2n}(F)$. Recall that J_U (respectively, $J_{U,\chi}$) denotes the Jacquet functor with respect to the unipotent group U and the trivial character (respectively, the character χ). We studied this module in [**6**] for the case $r = 1$, and there we showed the following theorem.

Theorem 5.1. *In the above notation, assume that $r = 1$. Then $\sigma_{\psi,n}(\pi_\tau)$ is a non-trivial, irreducible, supercuspidal, genuine and ψ^{-1}-generic representation of $\widetilde{\mathrm{Sp}}_{2n}(F)$. It is the unique such representation σ, such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$.*

Our goal in this section is to prove this theorem for $r > 1$ as well. Our proof will follow the steps of [**6**].

5.2. Existence of a pole of order r at $s = 1$ of $\gamma(\sigma \times \tau, s, \psi)$

Let σ be an irreducible, supercuspidal, genuine and ψ^{-1}-generic representation of $\widetilde{\mathrm{Sp}}_{2n}(F)$. The global Shimura-type integrals of [**4**] yield a corresponding local theory, which centres around the local functional equation (τ is self dual)

$$\gamma(\sigma \times \tau, s, \psi) J(W, \phi, \varphi_{\tau,s}) = L(\tau, \Lambda^2, 2(1-s)) \tilde{J}(W, \phi, M_s^*(\varphi_{\tau,s})). \tag{5.4}$$

Here we use the notation of [**6**, (1.16)]. Let us recall this. W is a Whittaker function in the ψ^{-1}-Whittaker model of σ. ϕ is a Schwartz–Bruhat function on F^n. $\varphi_{\tau,s}$ is a holomorphic section for $\rho_{\tau,s}$, realized as a smooth, complex function on $\mathrm{Sp}_{4n}(F) \times \mathrm{GL}_{2n}(F)$, such that for $a \in \mathrm{GL}_{2n}(F)$.

$$\varphi_{\tau,s}\left(\begin{pmatrix} a & * \\ 0 & a^* \end{pmatrix} g, I_{2n}\right) = |\det a|^{s+n} \varphi_{\tau,s}(g, a),$$

and $a \mapsto \varphi_{\tau,s}(g, a)$ lies in the Whittaker model of τ with respect to the character given by [**6**, (1.2)]. M_s is the local intertwining operator on $\rho_{\tau,s}$, defined with respect to the Weyl element

$$\tilde{w}_{2n} = \begin{pmatrix} & I_{2n} \\ -I_{2n} & \end{pmatrix},$$

and

$$M_s^* = \frac{\epsilon(\tau, s - \frac{1}{2}, \psi) \epsilon(\tau, \Lambda^2, 2s - 1, \psi)}{L(\tau, \Lambda^2, 2s - 1)} M_s. \tag{5.5}$$

The local $\Lambda^2 - L$ and ϵ-factors are those defined by Shahidi (see [**10**, **11**]). We have

$$J(W, \phi, \varphi_{\tau,s}) = \int_{V_n(F) \backslash \mathrm{Sp}_{2n}(F)} W(g) J_{\psi,n}(\omega_\psi^{(n)}(g)\phi, \rho_{\tau,s}(j_n(g))\varphi_{\tau,s}) \, \mathrm{d}g \tag{5.6}$$

($j_n(g)$ is given by (1.9)). The precise form of $J_{\psi,n}$ is given in [**6**, (1.11)]. Suffice it to say that $J_{\psi,n}(\phi, \varphi_{\tau,s})$ is given by an integral which stabilizes on large compact open subgroups (of a certain unipotent subgroup, and hence is holomorphic in s) [**6**, Proposition 11], and that

$$J_{\psi,n}(\omega_\psi^{(n)}(u \cdot h)\phi, \rho_{\tau,s}(v j_n(u \cdot h))\varphi_{\tau,s}) = \psi_n(u) \chi_n^{-1}(v) J_{\psi,n}(\phi, \varphi_{\tau,s}), \tag{5.7}$$

for $v \in N_{n+1}(F)$, $h \in \mathcal{H}_n(F)$, $v \in V_n(F)$. ψ_n is the standard non-degenerate character of $V_n(F)$ defined by ψ. $\tilde{J}(W,\phi,\varphi_{\tau,s})$ has the form (5.6), with $\tilde{J}_{\psi,n}$ $(\omega_\psi^{(n)}(g)\phi, \rho_{\tau,1-s}(j_n(g))M_s^*(\varphi_{\tau,s}))$ replacing $J_{\psi,n}(\cdots)$. $\tilde{J}_{\psi,n}(\phi,\tilde{\varphi}_{\tau,1-s})$ has exactly the same structure as $J_{\psi,n}$, except for a very slight modification (to adjust a certain Whittaker character), so that $\tilde{J}_{\psi,n}(\phi,\tilde{\varphi}_{\tau,1-s})$ is holomorphic and satisfies (5.7).

Proposition 5.2. *$J(W,\phi,\varphi_{\tau,s})$ and $\tilde{J}(W,\phi,M_s^*(\varphi_{\tau,s}))$ are holomorphic.*

Proof. Since σ is supercuspidal, $W(g)$ has a support which is compact module $V_n(F)$. The integral (5.6) is then absolutely convergent and holomorphic, since $J_{\psi,n}(\omega_\psi^{(n)}(g)\phi, \rho_{\tau,s}(j_n(g))\varphi_{\tau,s})$ is holomorphic and smooth. The same proof works for $\tilde{J}(W,\phi,M_s^*(\varphi_{\tau,s}))$, provided we know that $M_w^*(\varphi_{\tau,s})$ is holomorphic. This is indeed the case by Theorem 5.1 of [**3**]. (Note that the conditions of this theorem are satisfied in our case (see, for example, [**3**, Theorem 3.4]).) □

Corollary 5.3. *Assumptions are as above. The only possible poles of $\gamma(\sigma\times\tau,s,\psi)$ occur among those of $L(\tau,\Lambda^2,2(1-s))$, i.e. on the line $\mathrm{Re}(s)=1$. $\gamma(\sigma\times\tau,s,\psi)$ has a pole of order r at $s=1$, if and only if*

$$\int_{V_n(F)\backslash\mathrm{Sp}_{2n}(F)} W(g)\tilde{J}_{\psi,n}(\omega_\psi^{(n)}(g)\phi,\rho_{\tau,0}(j_n(g))M_1^*(\varphi_{\tau,1}))\,\mathrm{d}g \not\equiv 0. \tag{5.8}$$

Proof. From the last proposition, the only poles of the right-hand side of (5.4) lie among those of $L(\tau,\Lambda^2,2(1-s))$ (τ is self-dual). We showed in [**4**, Proposition 6.6] that data $(W,\phi,\varphi_{\tau,s})$ can be chosen so that $J(W,\phi,\varphi_{\tau,s})=1$, for all s. The first assertion follows now from the functional equation (5.4). Note that (see [**10**])

$$L(\tau,\Lambda^2,z)=\prod_{1\leqslant i<j\leqslant r}L(\tau_i\times\tau_j,z)\prod_{i=1}^{r}L(\tau_i,\Lambda^2,z)=\prod_{i=1}^{r}L(\tau_i,\Lambda^2,z),$$

since, by our assumption $L(\tau_i\times\tau_j,z)=1$ for $i\neq j$. Thus $L(\tau,\Lambda^2,z)$ has a pole of order r at $z=0$. This implies the second assertion. □

As in [**6**, (1.21)], we consider the following space of functions on $\widetilde{\mathrm{Sp}_{2k}}(F)$ $(k<2n)$. Let θ be an irreducible, generic representation of $\mathrm{GL}_{2n}(F)$, and let $\pi(\theta)$ be a subrepresentation of $\rho_{\theta,(1/2)}$. Then we consider

$$V_{\tilde{\sigma}_{\psi,k}(\pi(\theta))}=\mathrm{Span}\left\{(g,\epsilon)\to J_{k,\psi}(\omega_\psi^{(k)}(g,\epsilon)\phi,\rho_{\theta,(1/2)}(j_k(g))\varphi)\,\middle|\,\begin{matrix}\phi\in S(F^k),\\ \varphi\in V_{\pi(\theta)}\end{matrix}\right\} \tag{5.9}$$

(j_k is defined in (1.9)). As in (5.7) this is a space of Whittaker functions, with respect to ψ_k,on $\widetilde{\mathrm{Sp}_{2k}}(F)$. It affords a representation $\tilde{\sigma}_{\psi,k}(\pi(\theta))$ of $\widetilde{\mathrm{Sp}_{2k}}(F)$ (by right translations). Clearly, we have a surjective $\widetilde{\mathrm{Sp}_{2k}}(F)$-morphism

$$\sigma_{\psi^{-1},k}(\pi(\theta))=J_{\mathcal{H}_k(F)}(J_{N_{k+1}(F),\chi_k^{-1}}\pi(\theta)\otimes\omega_\psi^{(k)})\to\tilde{\sigma}_{\psi,k}(\pi(\theta)), \tag{5.10}$$

(N_{k+1},χ_k and $\mathcal{H}_k$ are defined in (1.4), (1.5), (1.6)). This follows from (5.7). It will be convenient to introduce $\sigma_{\psi^{-1},k,\alpha}$ and $\tilde{\sigma}_{\psi,k,\alpha}$ of $\pi(\theta)$, by replacing $\omega_\psi^{(k)}$ by $\omega_{\psi^\alpha}^{(k)}$ in (5.9), (5.10) (see (1.17)). The case which interests us is $\theta = \tau \otimes |\det\cdot|^{-1/2}$ and $\pi(\theta) = \pi_\tau$, which is the unique irreducible subrepresentation of $\rho_{\theta,(1/2)} = \rho_{\tau,0}$. π_τ is the image of M_1^* applied to $\rho_{\tau,1}$. We know from [**6**, Theorem 1.3] that $\tilde{\sigma}_{\psi,n}(\pi(\theta)) \neq 0$, and hence $\sigma_{\psi^{-1},n}(\pi(\theta)) \neq 0$. Replacing ψ by ψ^{-1}, we get $\sigma_{\psi,n}(\pi(\theta)) \neq 0$, for any θ as above. In particular,

$$\sigma_{\psi,n}(\pi_\tau) \neq 0. \tag{5.11}$$

Since σ is supercuspidal, it is projective, and then condition (5.8) says that $\gamma(\sigma \times \tau, s, \psi)$ has a pole of order r at $s = 1$, if and only if $\hat{\sigma}$ is a summand of $\tilde{\sigma}_{\psi,n}(\pi_\tau)$. This corollary is valid if we replace n by $k < 2n$, and we conclude the following corollary.

Corollary 5.4. *Let σ be an irreducible, supercuspidal, genuine and ψ^{-1}-generic representation of $\widetilde{\mathrm{Sp}}_{2k}(F)$ ($k < 2n$). Then, for τ as above, the only possible poles of $\gamma(\sigma \times \tau, s, \psi)$ occur among those of $L(\tau, \Lambda^2, 2(1-s))$, i.e. on the line $\mathrm{Re}(s) = 1$. $\gamma(\sigma \times \tau, s, \psi)$ has a pole of order r at $s = 1$, if and only if $\hat{\sigma}$ is a summand of $\tilde{\sigma}_{\psi,k}(\pi_\tau)$.*

Remark 5.5. Although we did not prove the multiplicativity of gamma factors for $\widetilde{\mathrm{Sp}}_{2k} \times \mathrm{GL}_m$, it certainly is true with proof similar to the case $\mathrm{SO}_{2k+1} \times \mathrm{GL}_m$ as in [**12**]. In our case here, by embedding σ in a global automorphic cuspidal (compatibly) generic representation of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ which is unramified at all finite places except that of F, and by embedding each τ_i in a global automorphic representation of $\mathrm{GL}_{2m_i}(\mathbb{A})$, which is unramified at all finite places, except that of F, we can compare, as we did in [**6**, § 6.3],

$$\gamma(\sigma \times \mathrm{Ind}_{P_{2m_1,\dots,2m_r}(F)}^{\mathrm{GL}_{2n}(F)} \tau_1|\det\cdot|^{s_i-(1/2)} \otimes \cdots \otimes \tau_r|\det\cdot|^{s_r-(1/2)}, 0, \psi)$$

and $\prod_{i=1}^r \gamma(\sigma \times \tau_i, s_i, \psi)$ and obtain that they are equal at least up to an exponential. Thus, $\gamma(\sigma \times \tau, s, \psi)$ has a pole of order r at $s = 1$, if and only if, for all $1 \leqslant i \leqslant r$, $\gamma(\sigma \times \tau_i, s, \psi)$ has a (simple) pole at $s = 1$.

5.3.

Theorem 5.6 (the main local theorem). *Let $\tau_1, \dots, \tau_r$ be irreducible, pairwise inequivalent, supercuspidal, self-dual representations of $\mathrm{GL}_{2m_1}(F), \dots, \mathrm{GL}_{2m_r}(F)$, respectively. Assume that $L(\tau_i, \Lambda^2, s)$ has a pole at $s = 0$, for each $1 \leqslant i \leqslant r$. Let τ be as in (5.1). Then $\sigma_{\psi,n}(\pi_\tau)$ is a non-trivial, irreducible, supercuspidal (genuine) and ψ^{-1}-generic representation of $\widetilde{\mathrm{Sp}}_{2n}(F)$, such that $\gamma(\sigma_{\psi,n}(\pi_\tau) \times \tau, s, \psi)$ has a pole of order r at $s = 1$. $\sigma_{\psi,n}(\pi_\tau)$ is unique with these properties, i.e. if σ is an irreducible, supercuspidal, genuine and ψ^{-1}-generic representation of $\widetilde{\mathrm{Sp}}_{2n}(F)$, such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole of order r at $s = 1$, then $\sigma \cong \sigma_{\psi,n}(\pi_\tau)$.*

5.4. Main steps of the proof

The proof goes along the same lines of the proof for the case $r = 1$ in [**6**]. We will prove the following theorem.

Theorem 5.7. *We have, for $\alpha \in F^*$ and $0 \leqslant k < n$,*

$$\sigma_{\psi,k,\alpha}(\pi_\tau) = 0.$$

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Here the idea is similar to [**6**, Chapter 3]. We will show that $\rho_{\tau,1}$ admits non-trivial $\mathrm{Sp}_{2n}(F)\times\mathrm{Sp}_{2n}(F)$ invariant functionals. We do not know how to give here another proof, which is 'uniform' as Theorem 1.13 (end of § 1.3).

Theorem 5.8. *The representation $\sigma_{\psi,n}(\pi_\tau)$ is supercuspidal.*

Here, we will compute Jacquet modules of $\sigma_{\psi,n}(\pi_\tau)$ along unipotent radicals of parabolic subgroups of $\mathrm{Sp}_{2n}(F)$. We will see that these Jacquet modules depend on $\sigma_{\psi,k}(\pi_\tau)$, for $k < n$, and on certain Jacquet modules of π_τ along unipotent radicals of parabolic subgroups of $\mathrm{Sp}_{4n}(F)$. We will prove that these are zero, using an analysis similar to that in § 2. This together with Theorem 1.1 will prove the theorem. (Recall, from (5.11), that $\sigma_{\psi,n}(\pi_\tau)$ is non-trivial.) The irreducibility and ψ^{-1}-genericity of $\sigma_{\psi,n}(\pi_\tau)$ will now follow almost exactly as in [**6**] using analysis of Jacquet modules of π_τ.

5.5. Proof of Theorem 5.7

We denote by H the image of $\mathrm{Sp}_{2n}\times\mathrm{Sp}_{2n}$ inside Sp_{4n} under the direct sum embedding. We first prove the following proposition.

Proposition 5.9. *The representation $\rho_{\tau,1}$ admits non-trivial $H(F)$-invariant functionals.*

Proof. We will show that τ (in (5.1)) admits a non-trivial $\mathrm{GL}_n(F)\times\mathrm{GL}_n(F)$-invariant functional, where $\mathrm{GL}_n\times\mathrm{GL}_n$ is embedded in GL_{2n} by

$$(g_1,g_2)\mapsto\begin{pmatrix} g_1 & \\ & g_2\end{pmatrix}.$$

Once we have this, the argument is exactly the one used in the end of [**6**, Theorem 3.3.2]. (Let ℓ be such a functional on τ. Then ℓ defines a non-trivial $H(F)$-morphism

$$T_\ell:\rho_{\tau,1}\to\mathrm{Ind}_{P_n(F)\times P_n(F)}^{\mathrm{Sp}_{2n}(F)\times\mathrm{Sp}_{2n}(F)}\delta_{P_n\times P_n}^{1/2}$$

(normalized induction) by $T_\ell(f)(g_1,g_2)=\ell[f(g_1,g_2)]$, thinking now of elements f of the space of $\rho_{\tau,1}$, as V_τ-valued functions on $\mathrm{Sp}_{4n}(F)$. Here P_n is the Siegel parabolic subgroup of Sp_{2n}. Since $1_{\mathrm{Sp}_{2n}(F)\times\mathrm{Sp}_{2n}(F)}$ is a quotient of

$$\mathrm{Ind}_{P_n(F)\times P_n(F)}^{\mathrm{Sp}_{2n}(F)\times\mathrm{Sp}_{2n}(F)}\delta_{P_n\times P_n}^{1/2},$$

we get, by composition with T_ℓ, an element of $\mathrm{Hom}_{H(F)}(\rho_{\tau,1}1_{H(F)})$, which is easily seen to be non-trivial.) We will show that τ admits non-trivial $\mathrm{GL}_n(F)\times\mathrm{GL}_n(F)$-invariant functionals by induction on r. We know this for $r=1$. This is the heart of the proof of [**6**, Theorem 3.3.2]. Let

$$\tau'=\mathrm{Ind}_{P_{2m_1,\dots,2m_{r-1}}}^{\mathrm{GL}_{2n'}(F)}\tau_1\otimes\cdots\otimes\tau_{r-1}.$$

Assume by induction, that $V_{\tau'}$ admits a non-trivial $\mathrm{GL}_{n'}(F)\times\mathrm{GL}_{n'}(F)$-invariant functional T'. Again, by the case $r=1$, V_{τ_r} admits a non-trivial $\mathrm{GL}_{m_r}(F)\times\mathrm{GL}_{m_r}(F)$-invariant functional T''. Think of τ as

$$\rho'_{\tau',\tau_r}=\mathrm{Ind}_{P_{2n',2m_r}(F)}^{\mathrm{GL}_{2n}(F)}\tau'\otimes\tau_r.$$

An element f in $V_{\rho'_{\tau',\tau_r}}$ is a smooth $V_{\tau'} \otimes V_{\tau_r}$-valued function on $\mathrm{GL}_{2n}(F)$, such that

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} h\right) = \frac{|\det a_1|^{m_r}}{|\det a_2|^{n'}}(\tau'(a_1) \otimes \tau_r(a_2))(f(h)), \tag{5.12}$$

where $a_1 \in \mathrm{GL}_{2n},(F)$, $a_2 \in \mathrm{GL}_{2m_r}(F)$. Consider the following embedding i of $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ in $\mathrm{GL}_{2n}(F)$. Write $h_i \in \mathrm{GL}_n(F)$, $i = 1, 2$, as

$$h_i = \begin{pmatrix} a_i & x_i \\ y_i & b_i \end{pmatrix},$$

where $a_i \in M_{n' \times n'}(F)$, $x_i \in M_{n \times m_r}(F)$, $y_i \in M_{m_r \times n'}(F)$, $b_i \in M_{m_n \times m_r}(F)$. Then

$$i(h_1, h_2) = \begin{pmatrix} a_1 & & & x_1 \\ & a_2 & x_2 & \\ & y_2 & b_2 & \\ y_1 & & & b_1 \end{pmatrix}.$$

Define for $f \in V_{\rho'_{\tau',\tau_r}}$, and $(h_1, h_2) \in \mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$

$$L(f)(h_1, h_2) = (T' \otimes T'')(f(i(h_1, h_2))).$$

Then, from (5.12)

$$\begin{aligned} L(f)&\left(\begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix}(h_1, h_2)\right) \\ &= (T' \otimes T'')\left(f\left(\begin{array}{cc|cc} a_1 & & & x_1 \\ & a_2 & x_2 & \\ \hline & & b_2 & \\ & & & b_1 \end{array}\right)(h_1, h_2)\right) \\ &= \frac{|\det a_1 a_2|^{m_r}}{|\det b_1 b_2|^{n'}} L(f)(h_1, h_2) \\ &= \frac{|\det a_1|^{m_r}}{|\det b_1|^{n'}} \cdot \frac{|\det a_2|^{m_r}}{|\det b_2|^{n'}} L(f)(h_1, h_2), \end{aligned}$$

for $a_i \in \mathrm{GL}_{n'}(F)$, $b_i \in \mathrm{GL}_{m_r}(F)$, $x_i \in M_{n' \times m_r}(F)$. This shows that L defines an $i(\mathrm{GL}_n(F) \times \mathrm{GL}_n(F))$-map

$$\rho'_{\tau',\tau_r} \to \mathrm{Ind}_{P_{n',m_r}(F)}^{\mathrm{GL}_n(F)} \delta_{P_{n',m_r}}^{1/2} \otimes \mathrm{Ind}_{P_{n',m_r}}^{\mathrm{GL}_n(F)} \delta_{P_{n',m_r}}^{1/2}. \tag{5.13}$$

The right-hand side of (5.13) has $1_{\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)}$ as a quotient, and composition with the quotient map gives an $i(\mathrm{GL}_n(F) \times \mathrm{GL}_n(F))$-invariant functional on ρ'_{τ',τ_r}, and hence on τ, which is easily seen to be non-trivial. Since $i(\mathrm{GL}_n(F) \times \mathrm{GL}_n(F))$ is conjugate within $\mathrm{GL}_{2n}(F)$ to

$$\left\{\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} \mid g_i \in \mathrm{GL}_n(F)\right\},$$

our assertion is proved. □

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We now conclude the proof of Theorem 5.7. Frobenius reciprocity and Proposition 5.9 imply the existence of a non-trivial morphism $\rho_{\tau,1} \to \mathrm{Ind}_{H(F)}^{\mathrm{Sp}_{4n}(F)} 1$, and dualizing this map, we get a non-trivial morphism

$$R : \mathrm{Ind}_{H(F)}^{c\mathrm{Sp}_{4n}(F)} 1 \to \rho_{\tau,0}.$$

Since π_τ is the unique irreducible subrepresentation of $\rho_{\tau,0}$, and R is non-trivial, it is clear that π_τ is a subrepresentation of $\mathrm{Im}(R)$. By exactness of Jacquet functors, it follows that if $\sigma_{\psi,k,\alpha}(\pi_\tau) \neq 0$, for $k < n$ (and $\alpha \in F^*$), then $\sigma_{\psi,k,\alpha}(\mathrm{Im}\, R) \neq 0$, and hence

$$\sigma_{\psi,k,\alpha}(\mathrm{Ind}_{H(F)}^{c\mathrm{Sp}_{4n}(F)} 1) \neq 0.$$

This is impossible by [**5**, Theorem 16]. We conclude that $\sigma_{\psi,k,\alpha}(\pi_\tau) = 0$, for all $k < n$. This completes the proof of Theorem 5.7. □

5.6. Proof of Theorem 5.8

We have to show that $J_{R_p(F)}(\sigma_{\psi,n}(\pi_\tau)) = 0$, for all $1 \leqslant p \leqslant n$. Here

$$R_p = \left\{ \begin{pmatrix} I_p & z & y \\ & I_{2(n-p)} & x' \\ & & I_p \end{pmatrix} \in \mathrm{Sp}_{2n} \right\}.$$

We have seen in [**6**, Proposition 2.3.1] that there is a vector space isomorphism

$$J_{R_p(F)}(V_{\pi_\tau}) \cong J_{D_p(F),\chi_{n-p-1}}(V_{\pi_\tau^*}). \tag{5.14}$$

Here

$$D_p = \left\{ v = \begin{pmatrix} I_p & u & * & * & * \\ & z & * & * & * \\ & & I_{2(n-p)} & * & * \\ & & & z^* & u' \\ & & & & I_p \end{pmatrix} \in \mathrm{Sp}_{4n} \;\middle|\; \begin{array}{l} z \in Z_n \text{ and} \\ \text{the first column} \\ \text{of } u \text{ is zero} \end{array} \right\}, \tag{5.15}$$

χ_{n-p-1} is defined by (1.5): for $v \in D_p(F)$, as in (5.15)

$$\chi_{n-p-1}(v) = \psi\left(\sum_{j=1}^{n-1} z_{j,j+1}\right).$$

The representation π_τ^* of $D_p(F)$ acts on $V_{\pi_\tau} \otimes S(F^{n-p})$ by

$$\pi_\tau^*(v)(\xi \otimes \phi) = \pi_\tau(v)\xi \otimes \omega_{\psi^{-1}}^{(n-p)}(v')\phi, \tag{5.16}$$

where, for $v \in D_p(F)$, as in (5.15)

$$v' = (v_{n+p,n+p+1}, v_{n+p,n+p+2}, \ldots, v_{n+p,3n-p+1}) \in \mathcal{H}_{n-p}(F). \tag{5.17}$$

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Note that the isomorphism (5.14) is valid even if we replace π_τ by any smooth representation of $\mathrm{Sp}_{4n}(F)$. See Remark 1 at the end of § 2.3 in [**6**]. (Note also the isomorphism appearing right after (2.9) in [**6**].) We continue as in [**6**] right after (2.23). Consider the right-hand side of (5.14) as an E-module, where

$$E = \left\{ \begin{pmatrix} m & x & & & \\ & 1 & & & \\ & & I_{2(2n-p-1)} & & \\ & & & 1 & x' \\ & & & & m^* \end{pmatrix} \in \mathrm{Sp}_{4n}(F) \right\}.$$

E is isomorphic to the parabolic subgroup of $\mathrm{GL}_{p+1}(F)$ of type $(p,1)$ (the so-called mirabolic subgroup). By [**1**] the Jordan–Hölder decomposition over E of $J_{D_p(F),\chi_{n-p-1}}(V_{\pi_\tau^*})$ is expressed through Jacquet modules of π_τ along the unipotent radicals $U_{p-\ell}$ in Sp_{4n}, where $0 \leqslant \ell < p$. (See [**1**] for the notion of derivatives of smooth representations of the mirabolic subgroup of $\mathrm{GL}_{p+1}(F)$.) The one derivative of $J_{D_p(F),\chi_{n-p-1}}(V_{\pi_\tau^*})$, which does not involve a Jacquet module (with respect to the trivial character) along a unipotent radical of Sp_{4n} is

$$\mathrm{Ind}^{c^E}_{Z_{p+1}(F)}(J_{Z_{p+1}(F),\psi}(J_{Dp(F),\chi_{n-p-1}}(V_{\pi_\tau^*}))), \tag{5.18}$$

where $Z_{p+1}(F)$ is embedded naturally in E

$$\left(\text{by } z \mapsto \begin{pmatrix} z & & \\ & I_{2(2n-p-1)} & \\ & & z^* \end{pmatrix} \right)$$

and ψ still denotes the standard generic character of $Z_{p+1}(F)$, defined by ψ. By definition, (5.18) is isomorphic to

$$\mathrm{Ind}^{c^E}_{Z_{p+1}(F)} V_{\sigma_{\psi,n-p}(\pi_\tau)}$$

which is zero by Theorem 5.7. The derivative which involves $J_{U_{p-\ell}(F)}(V_{\pi_\tau})$, $0 \leqslant \ell < p$, has the following form

$$\mathrm{Ind}^{c^E}_{Z'_{\ell+1}}(J_{Z'_{\ell+1},\psi'}(J_{D_p(F),\chi_{n-p-1}}(V_{\pi_\tau^*}))), \tag{5.19}$$

where $Z'_{\ell+1}(F)$ is the image in E of the following subgroup of $Z_{p+1}(F)$

$$\left\{ v = \begin{pmatrix} I_{p-1} & * \\ & z \end{pmatrix} \in Z_{p+1}(F) \right\}, \tag{5.20}$$

and for v of the form (5.20), $\psi'(v) = \psi(z_{12} + z_{13} + \cdots + z_{\ell,\ell+1})$. It is clear from the definitions that the space (5.19) is isomorphic to

$$\mathrm{Ind}^{c^E}_{Z'_{\ell+1}}[(1 \otimes \sigma^{(2n-(p-\ell))}_{\psi,n-p})(J_{U_{p-\ell}(F)}(\pi_\tau))]. \tag{5.21}$$

Here, when we consider $J_{U_{p-\ell}(F)}(\pi_\tau)$ as a representation of $\mathrm{GL}_{p-\ell}(F) \times \mathrm{Sp}_{2(2n-(p-\ell))}(F)$, $1 \otimes \sigma^{(2n-(p-\ell))}(\cdot)$ means that we apply the Jacquet functor $\sigma^{(2n-(p-\ell))}_{\psi,n-p}$ to the second

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factor and do not touch the first factor. The superscript $(2n-(p-\ell))$ in $\sigma_{\psi,n-p}^{(2n-(p-\ell))}$ marks the fact that this Jacquet functor is applied to a representation of $\mathrm{Sp}_{2(2n-(p-\ell))}(F)$. Thus, it remains to show that

$$(1\otimes\sigma_{\psi,n-p}^{(2n-(p-\ell))})(J_{U_{p-\ell}(F)}(\pi_\tau))=0, \tag{5.22}$$

for $0\leqslant\ell<p$.

Put, for short, $k=p-\ell$. Of course, $1\leqslant k\leqslant p\leqslant n$. We have to analyse $J_{U_k(F)}(\pi_\tau)$. We will prove the following theorem.

Theorem 5.10. *$J_{U_k(F)}(\pi_\tau)$ is non-zero, if and only if k is of the form $k=2(m_{i_1}+m_{i_2}+\cdots+m_{i_t})$, for $1\leqslant i_1<i_2<\cdots<i_t\leqslant r$. In this case the semisimplification* (ss) *of $J_{U_k(F)}(\pi_\tau)$ is as follows*

$$\begin{aligned}\mathrm{ss}\,J_{U_k(F)}(\pi_\tau)\\ =\bigoplus_{\substack{1\leqslant i_1<\dots<i_t\leqslant r\\ k=2(m_{i_1}+\cdots+m_{i_t})}}[|\det\cdot|^{2n-(k/2)}(\mathrm{Ind}_{P_{2(m_{i_1},\dots,m_{i_t})}(F)}^{\mathrm{GL}_k(F)}\tau_{i_1}\otimes\cdots\otimes\tau_{i_t})\otimes\pi_\tau(i_1,\dots,i_t)].\end{aligned} \tag{5.23}$$

Here $\pi_\tau(i_1,\dots,i_t)$ is the representation of $\mathrm{Sp}_{4n-2k}(F)$, which is the Langlands quotient of

$$\mathrm{Ind}_{Q_{2n-k}(F)}^{\mathrm{Sp}_{4n-2k}(F)}\tau(i_1,\dots,i_t)\otimes|\det\cdot|^{1/2},$$

where

$$\tau(i_1,\dots,i_t)=\mathrm{Ind}_{P_{2(m_{j_1},\dots,m_{j_\ell})}(F)}^{\mathrm{GL}_{2n-k}(F)}\tau_{j_1}\otimes\cdots\otimes\tau_{j_\ell},$$

and $\{j_1,\dots,j_\ell\}$ is the complement of $\{i_1,\dots,i_t\}$ inside $\{1,\dots,r\}$.

Once we have this theorem, then the semisimplification of the left-hand side of (5.22) becomes

$$\bigoplus_{\substack{1\leqslant i_1<\cdots<i_t\leqslant r\\ k=p-\ell=2(m_{i_1}+\cdots+m_{i_t})}}[|\det\cdot|^{2n-(k/2)}(\mathrm{Ind}_{P_{2(m_{i_1},\dots,m_{i_t})}(F)}^{\mathrm{GL}_k(F)}\tau_{i_1}\otimes\cdots\otimes\tau_{i_t})\otimes\sigma_{\psi,n-p}^{(2n-k)}(\pi_\tau(i_1,\dots,i_t))]. \tag{5.24}$$

Each summand of (5.24) is zero, since $\sigma_{\psi,n-p}^{(2n-k)}(\pi_\tau(i_1,\dots,i_t))=0$. This follows from Theorem 5.7, since $n-p<n-\frac{1}{2}k$. This will achieve the proof of Theorem 5.8.

5.7. Proof of Theorem 5.10

We first prove a special case.

Proposition 5.11. *We have*

$$J_{U_{2n}(F)}(\pi_\tau)\cong|\det\cdot|^n\tau. \tag{5.25}$$

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Proof. We prove (5.25) by induction on r. For $r = 1$, we know [**11**] that $\rho_{\tau,1}$ has exactly two irreducible constituents: the Langlands quotient π_τ and an irreducible generic subrepresentation. A direct computation (which we will soon repeat in larger generality) shows that $J_{U_{2n}}(F)(\rho_{\tau,1})$ has two constituents: $|\det\cdot|^{n+1}\tau$ and $|\det\cdot|^{n}\tau$. (Here we use the supercuspidality of τ, in case $r = 1$, and the fact that τ is self-dual.) Since π_τ is the image of the intertwining operator M_1^*, (5.25) becomes clear. Assume now that $r > 1$. Recall that $\pi_\tau(1)$ denotes the irreducible representation of $\mathrm{Sp}_{4(n-m_1)}(F)$, which is the Langlands quotient of

$$\mathrm{Ind}_{Q_{2(m_2,\ldots,m_r)}}^{\mathrm{Sp}_{4(n-m_1)}} \tau_2|\det\cdot|^{1/2} \otimes \cdots \otimes \tau_r|\det\cdot|^{1/2}.$$

$\pi_\tau(1)$ is also the unique irreducible subrepresentation of

$$\mathrm{Ind}_{Q_{r(m_2,\ldots,m_r)}(F)}^{\mathrm{Sp}_{4(n-m_1)}(F)} \tau_2|\det\cdot|^{-1/2} \otimes \cdots \otimes \tau_r|\det\cdot|^{-1/2}.$$

Clearly π_τ is the unique irreducible subrepresentation of

$$\beta_{\tau_1} = \mathrm{Ind}_{Q_{2m_1}(F)}^{\mathrm{Sp}_{4n}(F)} \tau_1|\det\cdot|^{-1/2} \otimes \pi_\tau(1).$$

We first compute $J_{U_{2n}(F)}(\beta_{\tau_1})$. As usual, we consider the restriction $\beta_{\tau_1}|_{Q_{2n}(F)}$, which has a filtration parametrized by $Q_{2m_1}\backslash \mathrm{Sp}_{4n}/Q_{2n}$. This set of double cosets can be described by the Weyl elements

$$w_j = \begin{pmatrix} I_j & & & & \\ & 0 & 0 & I_{2m_1-j} & \\ & 0 & I_{4(n-m_1)} & 0 & \\ & -I_{2m_1-j} & 0 & 0 & \\ & & & & I_j \end{pmatrix}.$$

The corresponding quotients in the filtration above are

$$\Gamma_j = \mathrm{Ind}_{w_j^{-1}Q_{2m_1}(F)w_j \cap Q_{2n}(F)}^{c^{Q_{2m}(F)}} (\delta_{Q_{2m_1}}^{1/2} \cdot \tau_1 \cdot |\det|^{-1/2} \otimes \pi_\tau(1))^{w_j} \quad \text{(unnormalized induction)}.$$

We have

$$\begin{aligned} L_j &= w_j^{-1} Q_{2m_1} w_j \cap Q_{2n} \\ &= \left\{ \begin{matrix} j \\ 2-m_1-j \\ 2(n-m_1) \\ \\ \\ \\ \end{matrix} \left(\begin{array}{ccc|ccc} a_1 & a_{12} & a_{13} & y_{11} & y_{12} & y_3 \\ & a_2 & 0 & 0 & 0 & y_{12}' \\ & a_{32} & a_3 & y_{31} & 0 & y_{11}' \\ \hline & & & a_3^* & 0 & a_{13}' \\ & & & a_{32}' & a_2^* & a_{12}^* \\ & & & & & a_1^* \end{array} \right) \in \mathrm{Sp}_{4n}(F) \right\}. \end{aligned} \tag{5.26}$$

The representation $(\delta_{Q_{2m_1}}^{1/2} \tau_1 \cdot |^{1/2} \otimes \pi_\tau(1))^{w_j}$ sends an element of the form (5.26) in $L_j(F)$ to

$$\left(\frac{|\det a_1|}{|\det a_2|}\right)^{2n-m_1} \tau_1 \begin{pmatrix} a_1 & y_{12} \\ & a_2^* \end{pmatrix} \otimes \pi_\tau(1) \begin{pmatrix} a_3 & y_{31} \\ & a_3^* \end{pmatrix}. \tag{5.27}$$

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It is clear, from (5.27), that $J_{U_{2n}(F)}(\Gamma_j)=0$, unless $j=2m_1$ or $j=0$. This follows from the supercuspidality of τ_1. If $j=2m_1$, then $w_{2m_1}=I_{4n}$, and

$$L_{2m_1}=\left\{\begin{matrix}2m_1\\2(n-m_1)\\ \\ \\\end{matrix}\quad\left(\begin{array}{cc|cc}a_1 & a_{13} & y_{11} & y_{13}\\ & a_3 & y_{31} & y'_{11}\\ \hline & & a_3^* & a'_{13}\\ & & 0 & a_1^*\end{array}\right)\in \mathrm{Sp}_{4n}\right\}=Q_{2(m_1,n-m_1)}.$$

By induction on r,

$$J_{U_{2(n-m_1)}(F)}(\pi_\tau(1))\cong|\det\cdot|^{n-m_1}\tau(1).$$

(Recall that $\tau(1)=\mathrm{Ind}_{P_{2(m_2,\dots,m_r)}(F)}^{\mathrm{GL}_{2(n-m_1)}(F)}\tau_2\otimes\cdots\otimes\tau_r$.) We conclude from (5.27) that (and now writing induction in normalized form)

$$J_{U_{2n}(F)}(\Gamma_{2m_1})\cong|\det\cdot|^n\,\mathrm{Ind}_{P_{2(m_1,n-m_1)}(F)}^{\mathrm{GL}_{2n}(F)}\tau_1\otimes\tau(1)\cong|\det\cdot|^n\tau. \tag{5.28}$$

If $j=0$, then

$$L_0=\left\{\begin{matrix}2m_1\\2(n-m_1)\\ \\ \\\end{matrix}\quad\left(\begin{array}{cc|cc}a_2 & 0 & 0 & 0\\ a_{j2} & a_3 & y_{31} & 0\\ \hline & & a_3^* & 0\\ & & a'_{32} & a*_2\end{array}\right)\in \mathrm{Sp}_{4n}\right\},$$

and we get, as before, (writing induction in normalized form)

$$J_{U_{2n}(F)}(\Gamma_0)\cong|\det\cdot|^n\,\mathrm{Ind}_{P_{2(m_1,n-m_1)}(F)}^{\mathrm{GL}_{2n}(F)}\tau_1|\det\cdot|\otimes\tau(1). \tag{5.29}$$

Thus, the semisimplification of $J_{U_{2n}}(F)(\beta_{\tau_1})$ has two irreducible constituents: (5.28) and (5.29). Since each irreducible subquotient of $\rho_{\tau,0}$ has a non-trivial Jacquet module along $U_{2n}(F)$ (see [**13**, Remark 3.5]), we conclude that β_{τ_1} has at most two irreducible constituents. If β_{τ_1} is irreducible, then it equals π_τ. This is impossible, since then $J_{U_{2n}(F)}(\pi_\tau)$ will have (5.28) and (5.29) as its two irreducible constituents. But, we could repeat the same calculation with

$$\beta_{\tau_2}=\mathrm{Ind}_{Q_{2m_2}(F)}^{\mathrm{Sp}_{4n}(F)}\tau_2|\det\cdot|^{-1/2}\otimes\pi_\tau(2)$$

instead of β_{τ_1} (π_τ is the unique irreducible subrepresentation of β_{τ_2}) and get that the constituents of $J_{U_{2n}(F)}(\pi_\tau)$ are $|\det\cdot|^n\tau$ and

$$|\det\cdot|^n\,\mathrm{Ind}_{P_{2(m_2,n-m_2)}}^{\mathrm{GL}_{2n}(F)}\tau_2|\det\cdot|\otimes\tau(2),$$

which is clearly not isomorphic to (5.29). This is a contradiction. We conclude from this that β_{τ_1} has two irreducible constituents and that (5.28) captures $J_{U_{2n}(F)}(\pi_\tau)$, i.e. $J_{U_{2n}(F)}(\pi_\tau)\cong|\det\cdot|^n\tau$. This proves the proposition. □

As a corollary from the proof, we get the following proposition.

Proposition 5.12. *The representation* $\mathrm{Ind}_{Q_{2m_1}(F)}^{\mathrm{Sp}_{4n}(F)} \tau_1|\det\cdot|^{-1/2}\otimes\pi_\tau(1)$ *has exactly two irreducible constituents: one irreducible subrepresentation, which is* π_τ *and an irreducible quotient* w_{τ_1}. *We have*

$$J_{U_{2n}(F)}(\pi_\tau) = |\det\cdot|^n\tau$$

and

$$J_{U_{2n}(F)}(w_{\tau_1}) = |\det\cdot|^n \mathrm{Ind}_{P_{(m_1,n-m_1)}(F)}^{\mathrm{GL}_{2n}(F)} \tau_1|\det\cdot|\otimes\tau(1).$$

We are now ready to start proving Theorem 5.10. Since $\tau_1,\dots,\tau_r$ are supercuspidal, the Weyl elements, representing $Q_{2(m_1,\dots,m_r)}\backslash \mathrm{Sp}_{4n}/Q_k$, which parametrize the quotients of the standard filtration of

$$\mathrm{Res}_{Q_k(F)}(\mathrm{Ind}_{Q_{2(m_1,\dots,m_r)}(F)}^{\mathrm{Sp}_{4n}(F)} \tau_1|\det\cdot|^{-1/2}\otimes\cdots\otimes\tau_r|\det\cdot|^{-1/2}) \tag{5.30}$$

need to be taken only in $W(L_{2\bar m},M_k)$ described in § 2.2. For a Weyl element w representing a coset of $Q_{2\bar m}\backslash \mathrm{Sp}_{4n}/Q_k$, the corresponding quotient of (5.30) is

$$\varGamma_w = \mathrm{Ind}_{w^{-1}Q_{2\bar m}(F)w\cap Q_k(F)}^{c^{Q_k(F)}}(\delta_{Q_{2\bar m}}^{1/2}\cdot(\tau_1|\det\cdot|^{-1/2}\otimes\cdots\otimes\tau_r|\det\cdot|^{-1/2}))^w$$

(unnormalized induction).

The following set is a set of representatives of $Q_{2\bar m}\backslash \mathrm{Sp}_{4n}/Q_k$ (see [**2**, Proposition 2.7.3])

$$\tilde W(L_{2\bar m},M_k) = \left\{ w\in W_{\mathrm{Sp}_{4n}} \;\middle|\; \begin{matrix} w(\alpha)>0, & \text{for all positive roots } \alpha \text{ inside } L_{2\bar m}, \\ w^{-1}(\alpha)>0, & \text{for all positive roots } \alpha \text{ inside } M_k \end{matrix}\right\}.$$

It is easy to see that for $w\in\tilde W(L_{2\bar m},M_k)\backslash W(L_{2\bar m},M_k)$, we have $J_{U_k(F)}(\varGamma_w)=0$, due to the supercuspidality of $\tau_1,\dots,\tau_r$. From (2.7), we see that k must have the form $2(m_{i_1}+\cdots+m_{i_t})$, for $1\leqslant i_1<\cdots<i_t\leqslant r$. From now on, we assume that k is of this form. We will prove (5.23) by induction on r. If $r=1$, then $k=2m_1=2n$, and (5.23) is a special case of (5.25). Assume that $r>1$. We may assume that $m_1=\min\{m_1,\dots,m_r\}$ (otherwise, we rearrange the indices $1,\dots,r$). We start by considering $J_{U_k(F)}(\beta_{\tau_1})$. Consider again $\mathrm{Res}_{Q_k(F)}\beta_{\tau_1}$ and its standard filtration, whose quotients are parametrized by $Q_{2m_1}\backslash \mathrm{Sp}_{4n}/Q_k$. The representatives of these double cosets can be chosen to be the Weyl elements w_{r_1,r_2} as in § 1.3, i.e.

$$w_{r_1,r_2} = \begin{pmatrix} I_{r_1} & & & & & & & \\ & 0 & & & & & I_{2m_1-r_1-r_2} & \\ & & 0 & I_{2n-k} & 0 & 0 & & \\ & & I_{r_2+k-2m_1} & 0 & 0 & 0 & & \\ & & 0 & 0 & 0 & I_{r_2+k-2m_1} & & \\ & & 0 & 0 & I_{2n-k} & 0 & & \\ & -I_{2m_1-r_1-r_2} & & & & & 0 & \\ & & & & & & & I_{r_1} \end{pmatrix}, \tag{5.31}$$

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where

$$0 \leqslant r_1 \leqslant 2m_1, \quad 0 \leqslant r_2 \leqslant 2n-k, \quad r_1+r_2 \leqslant 2m_1. \tag{5.32}$$

The corresponding quotient (in the above filtration) is

$$\Gamma_{r_1,r_2} = \mathrm{Ind}_{\mathcal{L}_{r_1,r_2}(F)}^{c^{Q_k(F)}} (\delta_{Q_{2m_1}}^{1/2} \cdot \tau_1 |\det\cdot|^{1/2} \otimes \pi_\tau(1))^{w_{r_1,r_2}} \quad \text{(unnormalized induction)},$$

where

$$\mathcal{L}_{r_1,r_2} = w_{r_1,r_2}^{-1} Q_{2m_1} w_{r_1,r_2} \cap Q_k.$$

The elements of $\mathcal{L}_{r_1,r_2}$ have the form

$$\begin{matrix} r_1 \\ 2m_1-(r_1+r_2) \\ r_2+k-2m_1 \\ r_2 \\ 4n-2k-2r_2 \\ r_2 \\ r_2+k-2m_1 \\ 2m_1-(r_1+r_2) \\ r_1 \end{matrix} \left(\begin{array}{ccc|ccc|ccc} a_1 & a_{12} & a_{13} & x_{11} & x_{12} & x_{13} & y_{11} & y_{12} & y_{13} \\ 0 & a_2 & 0 & 0 & 0 & x_{23} & 0 & 0 & y'_{12} \\ 0 & a_{32} & a_3 & 0 & x_{32} & x_{33} & y_{31} & 0 & y'_{11} \\ \hline & & & b & e & u & x'_{33} & x'_{23} & x'_{13} \\ & & & & c & e' & x'_{32} & 0 & x'_{12} \\ & & & & & b^* & 0 & 0 & x'_{11} \\ \hline & & & & & & a_3^* & 0 & a'_{13} \\ & & & & & & a_{32}^* & a_2^* & a'_{12} \\ & & & & & & 0 & 0 & a_1^* \end{array}\right). \tag{5.33}$$

The action of

$$\pi_{r_1,r_2} = (\delta_{Q_{2m_1}}^{1/2} \cdot \tau_1 |\det\cdot|^{-1/2} \otimes \pi_\tau(1))^{w_{r_1,r_2}}$$

on the element in $\mathcal{L}_{r_1,r_2}(F)$, of the form (5.33) is given by

$$\left(\frac{|\det a_1|}{|\det a_2|} |\det b|\right)^{2n-m_1} \tau_1 \begin{pmatrix} a_1 & y_{12} & x_{11} \\ 0 & a_2^* & 0 \\ 0 & x'_{23} & b \end{pmatrix} \otimes \pi_\tau(1)^{w'} \begin{pmatrix} a_3 & x_{32} & y_{31} \\ & c & x'_{32} \\ & & a_3^* \end{pmatrix}, \tag{5.34}$$

where

$$w' = \left(\begin{array}{cc|cc} & I_{2n-k-r_2} & & \\ I_{r_2+k-2m_1} & & & \\ \hline & & & I_{r_2+k-2m_1} \\ & & I_{2n-k-r_2} & \end{array}\right).$$

It is now clear, due to the supercuspidality of τ_1, that $J_{U_k(F)}(\Gamma_{r_1,r_2}) = 0$, unless $r_1 = 2m_1$, or $r_1 = 0$. Let us examine each case.

Assume that, $r_1 = 2m_1$. From (5.32), we must have $r_2 = 0$. Note that

$$\mathcal{L}_{2m_1,0} = \left\{ \begin{matrix} 2m_1 \\ k-2m_1 \\ 4n-2k \\ \\ \\ \end{matrix} \quad \left(\begin{array}{cc|c|cc} a_1 & a_{13} & x_{12} & y_{11} & y_{13} \\ 0 & a_3 & x_{32} & y_{31} & y'_{11} \\ \hline & & c & x'_{32} & x'_{12} \\ \hline & & & a_3^* & a'_{13} \\ & & & 0 & a'_1 \end{array}\right) \right\}, \tag{5.35}$$

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and $\pi_{2m_1,0}$ is given by (in the notation of (5.35))

$$|\det a_1|^{2n-m_1}\tau_1(a_1)\otimes\pi_\tau(1)^{w'}\begin{pmatrix} a_3 & x_{32} & y_1\\ & c & x'_{32}\\ & & a_3^*\end{pmatrix}. \tag{5.36}$$

By induction,

$$\begin{aligned}&\mathrm{ss}\,J_{U_{k-2m_1}(F)}(\pi_\tau(1))\\ &\quad=\bigoplus_{\substack{k-2m_1=2(m_{i'_1}+\cdots+m_{i'_{t'}})\\ 2\leqslant i'_1<\cdots<i'_{t'}\leqslant r}}[(|\det\cdot|^{2(n-m_1)-((k-2m_1)/2)}\cdot\mathrm{Ind}^{\mathrm{GL}_{k-2m_1}(F)}_{P_{2(m_{i'_1},\dots,m_{i'_{t'}})}(F)}\tau_{i'_1}\otimes\cdots\otimes\tau_{i'_{t'}})\\ &\qquad\qquad\otimes\pi_\tau(1,i'_1,\dots,i'_{t'})].\end{aligned} \tag{5.37}$$

We conclude from (5.36) and (5.37) that

$$\begin{aligned}&\mathrm{ss}\,J_{U_k(F)}(\Gamma_{2m_1,0})\\ &\quad=\bigoplus_{\substack{k=2(m_1+m_{i'_1}+m_{i'_1}+\cdots+m_{i'_{t'}})\\ 2\leqslant i'_1<\cdots<i'_{t'}\leqslant r}}[(|\det\cdot|^{2n-(k/2)}\cdot\mathrm{Ind}^{\mathrm{GL}_k(F)}_{P_{(2m_1,k-2m_1)}(F)}\\ &\qquad\qquad\times(\tau_1\otimes\mathrm{Ind}^{\mathrm{GL}_{k-2m_1}(F)}_{P_2(m_{i'_1},\dots,m_{i'_{t'}})(F)}\tau_{i'_1}\otimes\cdots\otimes\tau_{i'_{t'}}))\otimes\pi_\tau(1,i'_1,\dots,i'_{t'})]\\ &\quad=\bigoplus_{\substack{k=2(m_{i_1}+\cdots+m_{i_t})\\ 1=i_1<i_2<\cdots<i_t\leqslant r}}[(|\det\cdot|^{2n-(k/2)}\cdot\mathrm{Ind}^{\mathrm{GL}_k(F)}_{P_{2(m_{i_1},\dots,m_{i_t})}(F)}\tau_{i_1}\otimes\cdots\otimes\tau_{i_t})\otimes\pi_\tau(i_1,\dots,i_t)].\end{aligned} \tag{5.38}$$

Assume that $r_1=0$. Again, from the supercuspidality of τ_1 and (5.34), we must have $r_2=0$, or $r_2=2m_1$. In case $r_1=r_2=0$, we have

$$\mathcal{L}_{0,0}=\left\{\begin{matrix}2m_1\\ k-2m_1\\ 4n-2k\\ \\ \\ \end{matrix}\quad\left(\begin{array}{cc|c|cc} a_2 & 0 & 0 & 0 & 0\\ a_{32} & a_3 & x_{32} & y_{31} & 0\\ \hline & & c & x'_{32} & 0\\ \hline & & & a_3^* & 0\\ & & & a_{32}^* & a_2^*\end{array}\right)\in\mathrm{Sp}_{4n}\right\}, \tag{5.39}$$

and $\pi_{0,0}$ is given by (in the notation of (5.39))

$$|\det a_2|^{-(2n-m_1)}\tau_1^*(a_2)\otimes\pi_\tau(1)^{w'}\begin{pmatrix} a_3 & x_{32} & y_{31}\\ & c & x_{32}\\ & & a_3^*\end{pmatrix}. \tag{5.40}$$

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Here $\tau_1^*(a_2) = \tau_1(a_2^*)$. Since τ_1 is self-dual, $\tau_1^* \cong \tau_1$. Using induction, we calculate that

$$\begin{aligned}
&\mathrm{ss}\, J_{U_k(F)}(\Gamma_{0,0}) \\
&\quad\cong \bigoplus_{\substack{k-2m_1=2(m_{i'_1}+\cdots+m_{i'_{t'}})\\ 2\leqslant i'_1<\cdots<i'_{t'}\leqslant r}} [|\det\cdot|^{2n-(k/2)}\,\mathrm{Ind}_{\bar{P}_{2(m_1,k-2m_1)}}^{\mathrm{GL}_k(F)} \\
&\qquad\qquad\times(\tau_1|\det\cdot|\otimes \mathrm{Ind}_{P_2(m_{i'_1},\dots,m_{i'_{t'}})(F)}^{\mathrm{GL}_{k-2m_1}(F)}\tau_{i'_1}\otimes\cdots\otimes\tau_{i'_{t'}})\otimes\pi_\tau(1,i'_1,\dots,i'_{t'})] \\
&\quad\cong \bigoplus_{\substack{k=2(m_{i_1}+\cdots+m_{i_t})\\ 1=i_1<i_2<\cdots<i_t=\leqslant r}} [(|\det\cdot|^{2n-(k/2)}\cdot\mathrm{Ind}_{P_{2(m_{i_1},\dots,m_{i_t})}(F)}^{\mathrm{GL}_k(F)}\tau_1|\det\cdot|\otimes\tau_{i_2}\otimes\cdots\otimes\tau_{i_t}) \\
&\qquad\qquad\otimes\pi_\tau(i_1,\dots,i_t)]. \qquad (5.41)
\end{aligned}$$

Finally, assume that $r_1 = 0$ and $r_2 = 2m_1$. Here,

$$\mathcal{L}_{0,2m_1} = \left\{\begin{matrix} k \\ 2m_1 \\ 4n-2k-4m \\ \\ \\ \end{matrix} \quad \begin{pmatrix} a_3 & 0 & x_{32} & x_{33} & y_{31} \\ & b & e & u & x'_{33} \\ & & c & e' & x_{32'} \\ & & & b^* & 0 \\ & & & & a_3^* \end{pmatrix} \in \mathrm{Sp}_{4n} \right\}, \qquad (5.42)$$

and (in the notation of (5.42)) the action of $\pi_{0,2m_1}$ is given by

$$|\det b|^{2n-m_1}\tau_1(b)\otimes\pi_\tau(1)^{w'}\begin{pmatrix} a_3 & x_{32} & y_{31} \\ & c & x'_{32} \\ & & a_3^* \end{pmatrix}.$$

Using induction, we calculate,

$$\begin{aligned}
\mathrm{ss}\, J_{U_k(F)}(\Gamma_{0,2m_1}) \cong \bigoplus_{\substack{k=2(m_{i'_1}+\cdots+m_{i'_{t'}})\\ 2\leqslant i'_1<\cdots<i'_{t'}\leqslant r}} &[(|\det\cdot|^{2n-(k/2)}\cdot\mathrm{Ind}_{P_{2(m_{i'_1},\dots,m_{i'_{t'}})}(F)}^{\mathrm{GL}_k(F)}\tau_{i'_1}\otimes\cdots\otimes\tau_{i'_{t'}}) \\
&\otimes(\mathrm{Ind}_{Q_{2m_1}(F)}^{\mathrm{Sp}_{4n-2k}(F)}\tau_1|\det\cdot|^{-1/2}\otimes\pi_\tau(1,i'_1,\dots,i'_{t'}))]. \qquad (5.43)
\end{aligned}$$

By Proposition 5.12 (in this section)

$$\mathrm{Ind}_{Q_{2m_1}(F)}^{\mathrm{Sp}_{4n-2k}(F)}\tau_1|\det\cdot|^{-1/2}\otimes\pi_\tau(1,i'_1,\dots,i'_{t'})$$

has two irreducible constituents: the irreducible subrepresentation $\pi_\tau(i'_1,\dots,t'_{t'})$ and the irreducible quotient, denote it now by $w_{\tau_1}(i'_1,\dots,i'_{t'})$. Recall, also that

$$\begin{aligned}
&J_{U_{2n-k}(F)}(w_{\tau_1}(i'_1,\dots,i'_{t'})) \\
&\qquad= |\det\cdot|^{n-(k/2)}\,\mathrm{Ind}_{P_{2(m_1,n-(1/2)k-m_1)}(F)}^{\mathrm{GL}_{2n-k}(F)}\tau_1|\det\cdot|\otimes\tau(1,i'_1,\dots,i'_{t'}). \qquad (5.44)
\end{aligned}$$

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Thus, $\mathrm{ss}\, J_{U_k(F)}(\Gamma_{0,2m_1})$ is the sum of the following two sums:

$$\bigoplus_{\substack{k=2(m_{i_1}+\cdots+m_{i_t})\\ 2\leqslant i_1<\cdots<i_t\leqslant r}} [(|\det\cdot|^{2n-(k/2)} \operatorname{Ind}_{P_{2(m_{i_1},\dots,m_{i_t})}(F)}^{\mathrm{GL}_k(F)} \tau_{i_1}\otimes\cdots\otimes\tau_{i_t})\otimes\pi_\tau(i_1,\dots,i_t)] \tag{5.45}$$

and

$$\bigoplus_{\substack{k=2(m_{i_1}+\cdots+m_{i_t})\\ 2\leqslant i_1<\cdots<i_t\leqslant r}} [(|\det\cdot|^{2n-(k/2)} \operatorname{Ind}_{P_{2(m_{i_1},\dots,m_{i_t})}(F)}^{\mathrm{GL}_k(F)} \tau_{i_1}\otimes\cdots\otimes\tau_{i_t})\otimes w_{\tau_1}(i_1,\dots,i_t)]. \tag{5.46}$$

We have proved that $\mathrm{ss}\, J_{U_k(F)}(\beta_{\tau_1})$, which is also $\mathrm{ss}\, J_{U_k(F)}(\pi_\tau)\oplus \mathrm{ss}\, J_{U_k(F)}(w_{\tau_1})$, is the sum of (5.38), (5.41), (5.45) and (5.46). It remains to show that $\mathrm{ss}\, J_{U_k(F)}(w_{\tau_1})$ is the sum of (5.41) and (5.46), and $\mathrm{ss}\, J_{U_k(F)}(\pi_\tau)$ is the sum of (5.38) and (5.45). To see this, let us take a further Jacquet module with respect to the Siegel radical $U_{2n-k}^{(2n-k)}(F)$ on the $\mathrm{Sp}_{4n-2k}(F)$ factor. By Proposition 5.12, we should then replace $\pi_\tau(i_1,\dots,i_k)$ in (5.38), (5.41) and (5.45) by $|\det\cdot|^{n-(k/2)}\tau(i_1,\dots,i_t)$, while in (5.46), we should replace $w_{\tau_1}(i_1,\dots,i_t)$ by (5.44),

$$J_{U_{2n-k}^{(2n-k)}(F)}(J_{U_k(F)}(\beta_{\tau_1})) = J_{V_{(k,2n-k)}(F)}(J_{U_{2n}(F)}(\beta_{\tau_1})), \tag{5.47}$$

where

$$V_{(k,2n-k)} = \left\{ \begin{pmatrix} I_k & x & & \\ & I_{2n-k} & & \\ & & I_{2n-k} & x' \\ & & & I_k \end{pmatrix} \in \mathrm{Sp}_{4n} \right\}.$$

Again, by Proposition 5.12,

$$\mathrm{ss}(J_{U_{2n}(F)}(\beta_{\tau_1})) = |\det\cdot|^n\tau \oplus |\det\cdot|^n \operatorname{Ind}_{P_{2(m_1,n-m_1)}(F)}^{\mathrm{GL}_{2n}(F)} \tau_1|\det\cdot|\otimes\tau(1),$$

and (5.47) shows that

$$J_{U_{2n-k}^{(2n-k)}(F)}(\pi_\tau) = J_{V_{(k,2n-k)}(F)}(|\det\cdot|^n\tau)$$

(we identify $V_{(k,2n-k)}$ and its image in GL_{2n}). This easily implies what we want, and the proof of Theorem 5.10 is now complete. □

5.8. Conclusion of the proof of the main local theorem

It remains to prove that $\sigma_{\psi,n}(\pi_\tau)$ is irreducible and ψ^{-1}-generic. This follows exactly as in the outline given in § 4.2 of [**6**]. Indeed, Theorem 4.2.1 in [**6**] is valid for π_τ, in our case, since in the proof (given in [**6**, § 4.3]), we use the fact that $\sigma_{\psi,k,\alpha}(\pi_\tau)=0$, for $0\leqslant k<n$, which is Theorem 5.7. (Note that although we require in [**6**, Theorem 4.3] that the representation π admits non-trivial $\mathrm{Sp}_{2n}(F)\times\mathrm{Sp}_{2n}(F)$-invariant functionals, we

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use this just to ensure that $\sigma_{\psi,k,\alpha}(\pi) = 0$, for $0 \leqslant k < n$.) Thus, for the unipotent group E_{2n} and its character $\psi^{(2n)}$ introduced in (4.1), we have

$$J_{E_{2n},\psi^{(2n)}}(\pi_\tau) \cong J_{V_{2n}(F),\tilde{\psi}}(\pi_\tau), \tag{5.48}$$

where V_{2n} is the standard maximal unipotent subgroup of Sp_{4n}, and $\tilde{\psi}$ is the character of $V_{2n}(F)$, which is trivial on $U_{2n}(F)$ and is the standard non-degenerate character defined by ψ on $m(Z_{2n}(F))$. Using Proposition 5.7.1, we conclude that $\dim J_{V_{2n}(F),\tilde{\psi}}(\pi_\tau) = 1$, and hence

$$\dim J_{E_{2n},\psi^{(2n)}}(\pi_\tau) = 1. \tag{5.49}$$

The next step is to see that Proposition 4.2 and its proof in [**6**] hold here as well without any change (except that here we take ψ^{-1} instead of ψ). Thus, we see that each irreducible summand of $\sigma_{\psi,n}(\pi_\tau)$ is ψ^{-1}-generic. (Note that Theorem 6.2(c) in [**6**] applies for any τ.) Next, Theorem 4.2.2 in [**6**] is completely general (see the proof in [**6**, § 4.4]). This theorem implies, using (4.6) of [**6**], that the dimension of the space of ψ^{-1}-Whittaker functions on $V_{\sigma_{\psi,n}(\pi_\tau)}$ equals $\dim J_{E_{2n},\psi^{(2n)}}(\pi_\tau)$, which is 1, by (5.49). This proves the irreducibility of $\sigma_{\psi,n}(\pi_\tau)$ (and of course the ψ^{-1}-genericity of $\sigma_{\psi,n}(\pi_\tau)$).

We conclude, as in [**6**] that (see (5.9), (5.10))

$$\sigma_{\psi,n}(\pi_\tau) \cong \tilde{\sigma}_{\psi^{-1},n}(\pi_\tau) \tag{5.50}$$

and hence Corollary 1.12 in § 5.2 shows that there is a unique irreducible, supercuspidal, genuine representation σ of $\widetilde{\mathrm{Sp}}_{2n}(F)$, which is ψ^{-1}-generic, and such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole of order r at $s = 1$. σ is the representation $\sigma_{\psi,n}(\pi_\tau)$. The proof of the main local theorem is now complete. □

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