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TOPOLOGICAL RIGIDITY FOR CLOSED HYPERSURFACES OF ELLIPTIC SPACE FORMS

EDUARDO ROSINATO LONGA AND JAIME BRUCK RIPOLL

Instituto de Matemática e Estatística, Universidade Federal do Rio Grande do Sul, Porto Alegre, Brasil (eduardo.longa@ufrgs.br; jaime.ripoll@ufrgs.br)

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Abstract We prove a topological rigidity theorem for closed hypersurfaces of the Euclidean sphere and of an elliptic space form. It asserts that, under a lower bound hypothesis on the absolute value of the principal curvatures, the hypersurface is diffeomorphic to a sphere or to a quotient of a sphere by a group action. We also prove another topological rigidity result for hypersurfaces of the sphere that involves the spherical image of its usual Gauss map.

Keywords: rigidity; hypersurfaces; topology; sphere; space forms

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1. Introduction

In [7], J. H. Eschenburg defines an ε-convex hypersurface M^n immersed in a complete Riemannian manifold N^{n+1}, $n \geq 2$, $\varepsilon > 0$, as a hypersurface having all the principal curvatures with the same sign and absolute value at least ε. He then proves that if M is compact, ε-convex and N has non-negative sectional curvature, then M is the boundary of a convex body in N; in particular, M is diffeomorphic to an n-dimensional sphere. Products of spheres $\mathbb{S}^j \times \mathbb{S}^k$ in $\mathbb{S}^{n+1}$, $j + k = n$, show that the hypothesis on the sign of the principal curvatures is seemingly essential. However, there are examples in which M is an immersed sphere with nowhere zero principal curvatures and M is not ε-convex (see Remark 3.3).

Our first result gives a sufficient condition for a closed, connected and oriented hypersurface M of the round sphere $\mathbb{S}^{n+1}$ to be diffeomorphic to a sphere $\mathbb{S}^n$: the principal curvatures are required to be, in absolute value, greater than a function of the radius of a ball that contains M.

Theorem 1. *Let M^n be a closed, connected and oriented immersed hypersurface of $\mathbb{S}^{n+1}$, $n \geq 2$, and let $R \in (0, \pi)$ be the radius of the smallest geodesic ball containing M.*

If the principal curvatures λ_i *of* M *satisfy*

$$\inf_{p\in M} |\lambda_i(p)| > \tan\left(\frac{R}{2}\right), \quad \forall\, i \in \{1,\dots,n\}, \tag{1.1}$$

then M *is diffeomorphic to* $\mathbb{S}^n$.

In line with Theorem 1, Wang and Xia proved that M is diffeomorphic to a sphere assuming that the Gauss–Kronecker curvature of M does not vanish at any point and that M is contained in an open hemisphere of $\mathbb{S}^{n+1}$ [**12**, Theorem 1.1]. It is possible to prove Wang and Xia's result from Theorem 1 using Beltrami's map, in a similar way to that used in [**6**], and applying a homothetic deformation of the hypersurface (see Remark 3.4 for more details). It should be noted that in Theorem 1, not only do we allow the principal curvatures of the hypersurface to have different signs (see Remark 3.1, however), but also we do not impose any restriction on the size of the geodesic ball in which the hypersurface is contained (the radius R in the theorem can be any number in the interval $(0,\pi)$).

Our next result concerns hypersurfaces of an elliptic space form, that is, of a complete Riemannian manifold of constant sectional curvature equal to 1. The latter are known to be isometric to the quotient of $\mathbb{S}^{n+1}$ by a finite group of isometries that acts properly discontinuously on the sphere (see [**5**], for example). We give a sufficient condition for the hypersurface M to be covered by the sphere $\mathbb{S}^n$, in terms of its principal curvatures and of the distance from M to the cut locus of a certain point.

Theorem 2. *Let* Γ *be a non-trivial group of isometries of* $\mathbb{S}^{n+1}$, $n \geq 2$, *acting properly discontinuously, and let* $\pi : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}/\Gamma$ *be the canonical projection. For* $x_0 \in \mathbb{S}^{n+1}/\Gamma$, *let* $p_0 \in \pi^{-1}(x_0)$ *and define*

$$r = \min_{g\in\Gamma\setminus\{e\}} d(p_0, g(p_0)).$$

Let M^n *be a closed and connected hypersurface of* $\mathbb{S}^{n+1}/\Gamma$ *and suppose that*

$$d(x, C(x_0)) \leq R, \quad \forall\, x \in M,$$

where $C(x_0)$ *is the cut locus of* x_0 *and* $R \in (0, r/2)$. *If the principal curvatures* λ_i *of* M *satisfy*

$$\inf_{x\in M} |\lambda_i(x)| > \tan\left(\frac{\pi - r/2 + R}{2}\right) = \cot\left(\frac{r-2R}{4}\right), \quad \forall\, i \in \{1,\dots,n\},$$

and if $\tilde{M} := \pi^{-1}(M)$ *has* k *connected components, then there is a* $(|\Gamma|/k)$*-to-one covering map from* $\mathbb{S}^n$ *to* M *via the action of* Γ.

As an immediate consequence of Theorem 2, we have the following topological rigidity result for hypersurfaces of the projective space $\mathbb{RP}^{n+1}$.

Corollary 1. *Let M^n be a closed and connected hypersurface of $\mathbb{RP}^{n+1}$ and suppose that there exists a totally geodesic codimension-1 projective space $\mathbb{RP}^n$ of $\mathbb{RP}^{n+1}$ such that*

$$d(x, \mathbb{RP}^n) \leq R, \quad \forall\, x \in M,$$

for some $R \in (0, \pi/2)$. If the principal curvatures λ_i of M satisfy

$$\inf_{x \in M} |\lambda_i(x)| > \tan\left(\frac{\pi/2 + R}{2}\right), \quad \forall\, i \in \{1, \dots, n\},$$

then M is diffeomorphic to either $\mathbb{S}^n$ or $\mathbb{RP}^n$.

Isometric rigidity results for hypersurfaces with non-negative r-mean curvature of the sphere $\mathbb{S}^{n+1}$ have been obtained in a series of papers beginning with De Giorgi [**4**] and, independently, Simons [**10**, Theorem 5.2.1] in the minimal case, then by Nomizu and Smyth [**8**, Theorem 2] for constant mean curvature hypersurfaces, and, finally, by Alencar, Rosenberg and Santos [**2**] for constant non-negative r-mean curvature hypersurfaces. Later, a topological rigidity result was obtained by Wang and Xia [**12**, Theorem 1.2]. In all these results it is required that the image of the Gauss map is contained in a hemisphere of the sphere. Unlike these authors, we obtain a topological rigidity theorem allowing the Gauss image of the hypersurface to lie in a neighbourhood of a great hypersphere.

Theorem 3. *Let M^n be a closed, connected and oriented immersed hypersurface of $\mathbb{S}^{n+1}$, $n \geq 2$, with unit normal $\eta : M \to \mathbb{S}^{n+1}$. Suppose that there exists a point $p_0 \in \mathbb{S}^{n+1}$ such that the spherical image of η lies in a strip of width L around the totally geodesic hypersphere $T = \{x \in \mathbb{S}^{n+1} : \langle x, p_0 \rangle = 0\}$ determined by p_0, and that M is contained in the ball of radius $R \in (0, \pi)$ centred at p_0. If the principal curvatures λ_i of M satisfy*

$$\inf_{p \in M} |\lambda_i(p)| > \frac{\sin L}{1 + \cos R}, \quad \forall\, i \in \{1, \dots, n\},$$

then M is diffeomorphic to a sphere.

The technique of our paper is elementary. The results are proved by direct calculations using a Gauss map constructed from the parallel transport in $\mathbb{S}^{n+1}$.

2. Gauss map

Let M^n be a closed, connected and oriented hypersurface of $\mathbb{S}^{n+1}$ with unit normal vector field $\eta : M \to \mathbb{S}^{n+1}$, and fix a point $p_0 \in \mathbb{S}^{n+1}$ such that $-p_0 \notin M$. For non-antipodal points p, q in the sphere, let $\tau_p^q : T_p\mathbb{S}^{n+1} \to T_q\mathbb{S}^{n+1}$ be the parallel transport along the unique geodesic joining p to q (we agree that τ_p^p is the identity of $T_p\mathbb{S}^{n+1}$). We define a Gauss map $\gamma : M \to \mathbb{S}^n$ by

$$\gamma(p) = \tau_p^{p_0}(\eta(p)), \quad p \in M,$$

where $\mathbb{S}^n$ is the unit sphere of $T_{p_0}\mathbb{S}^{n+1}$.

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Definition 2.1. Given $p \in \mathbb{S}^{n+1}$ and $v \in T_p\mathbb{S}^{n+1}$, define a vector field $\tilde{v}$ on $\mathbb{S}^{n+1} \setminus \{-p_0\}$ by the rule

$$\tilde{v}(q) = (\tau_{p_0}^{q} \circ \tau_{p}^{p_0})(v), \quad q \neq -p_0.$$

Let $\overline{\nabla}$ be the Riemannian connection of $\mathbb{S}^{n+1}$. Recall that the shape operator of M in the direction of η is the section A of the vector bundle $\mathrm{End}(TM)$ of endomorphisms of TM given by

$$A_p(v) = -\overline{\nabla}_v \eta, \quad p \in M, \ v \in T_pM.$$

Similarly, we define another section of $\mathrm{End}(TM)$.

Definition 2.2. The invariant shape operator of M is the section α of the bundle $\mathrm{End}(TM)$ given by

$$\alpha_p(v) = \overline{\nabla}_v \widetilde{\eta(p)}, \quad p \in M, \ v \in T_pM.$$

The proposition below establishes a relationship between γ and the extrinsic geometry of M.

Proposition 2.3. *For any $p \in M$, the following identity holds:*

$$\tau_{p_0}^{p} \circ d\gamma(p) = -(A_p + \alpha_p).$$

Proof. Fix $p \in M$ and an orthonormal basis $\{v_1, \ldots, v_{n+1}\}$ of $T_p\mathbb{S}^{n+1}$ such that $v_{n+1} = \eta(p)$. The vector fields $\tilde{v}_1, \ldots, \tilde{v}_{n+1}$ form a global orthonormal referential of $\mathbb{S}^{n+1} \setminus \{-p_0\}$, so that we can write

$$\eta = \sum_{i=1}^{n+1} a_i \tilde{v}_i \tag{2.1}$$

for certain functions $a_i \in C^\infty(M)$. Notice that $a_i(p) = 0$ for $i \in \{1, \ldots, n\}$ and that $a_{n+1}(p) = 1$.

For $y \in M$, we have

$$\gamma(y) = \tau_y^{p_0}(\eta(y)) = \tau_y^{p_0}\left(\sum_{i=1}^{n+1} a_i(y)\tilde{v}_i(y)\right) = \sum_{i=1}^{n+1} a_i(y)\tau_p^{p_0}(v_i).$$

Therefore, if $v \in T_pM$,

$$\tau_{p_0}^{p}(d\gamma(p)\cdot v) = \tau_{p_0}^{p}\left(\sum_{i=1}^{n+1} v(a_i)\tau_p^{p_0}(v_i)\right) = \sum_{i=1}^{n+1} v(a_i)v_i. \tag{2.2}$$

From (2.1) and (2.2) we obtain

$$-A_p(v) = \overline{\nabla}_v\eta = \sum_{i=1}^{n+1}\overline{\nabla}_v(a_i\tilde{v}_i) = \sum_{i=1}^{n+1}[a_i(p)\overline{\nabla}_v\tilde{v}_i + v(a_i)\tilde{v}_i(p)]$$

$$= \overline{\nabla}_v\tilde{v}_{n+1} + \sum_{i=1}^{n+1} v(a_i)v_i = \alpha_p(v) + \tau_{p_0}^{p}(d\gamma(p)\cdot v),$$

which gives the desired result. □

The next proposition gives explicit formulas for the maps τ_p^q, γ and α, obtained by straightforward computations not presented here.

Proposition 2.4. *Let p and q be non-antipodal points in $\mathbb{S}^{n+1}$, with $p \in M$. In the above notation, the following formulae hold:*

(i)
$$\tau_p^q(v) = -\left[\frac{\langle v, q\rangle}{1 + \langle q, p\rangle}\right](q + p) + v, \quad v \in T_p\mathbb{S}^{n+1};$$

(ii)
$$\gamma(p) = -\left[\frac{\langle \eta(p), p_0\rangle}{1 + \langle p, p_0\rangle}\right](p + p_0) + \eta(p);$$

(iii)
$$\alpha_p(v) = \left[\frac{\langle \eta(p), p_0\rangle}{1 + \langle p, p_0\rangle}\right]v, \quad v \in T_pM.$$

3. Proofs of the theorems

We begin with Theorem 1.

Proof of Theorem 1. Let $\eta : M \to \mathbb{S}^{n+1}$ be the unit normal vector field which gives rise to the orientation of M, and let p_0 be the centre of a geodesic ball of radius R containing M. Define a function $c : M \to \mathbb{R}$ by

$$c(p) = \frac{\langle \eta(p), p_0\rangle}{1 + \langle p, p_0\rangle}, \quad p \in M,$$

and a vector field E on $\mathbb{S}^{n+1}$ by

$$E(p) = p_0 - \langle p, p_0\rangle p, \quad p \in \mathbb{S}^{n+1}.$$

Notice that $\langle \eta(p), E(p)\rangle = \langle \eta(p), p_0\rangle$ for p in M. Then, using the Cauchy–Schwarz inequality, we have the following estimate for c:

$$|c(p)| \le \frac{\|\eta(p)\|\|E(p)\|}{1 + \langle p, p_0\rangle} = \frac{\sqrt{1 - \langle p, p_0\rangle^2}}{1 + \langle p, p_0\rangle} = \sqrt{\frac{1 - \langle p, p_0\rangle}{1 + \langle p, p_0\rangle}}, \quad \forall\, p \in M.$$

Thus,

$$|c(p)| \le \sqrt{\frac{1 - \cos d(p, p_0)}{1 + \cos d(p, p_0)}} = \tan\left(\frac{d(p, p_0)}{2}\right) \le \tan\left(\frac{R}{2}\right), \quad \forall\, p \in M.$$

Fix $p \in M$. Choosing an orthonormal basis of T_pM that diagonalizes the shape operator A_p, the matrix of $-\tau_{p_0}^p \circ d\gamma(p)$ with respect to this basis is diagonal with entries $\lambda_i(p) + c(p) \neq 0$ (see Proposition 2.3). Therefore, this map is an isomorphism for each

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$p \in M$, and so is $d\gamma(p)$. Since M is compact, γ is a covering map, and since M is connected with $n \geq 2$, γ is a diffeomorphism. $\square$

Remark 3.1. It is worth pointing out that, although we allow the principal curvatures of the hypersurface to have different signs, there are topological obstructions to the number of, say, positive principal curvatures. Suppose that $M^n \subset \mathbb{S}^{n+1}$ is an immersed sphere and has k positive principal curvatures, $0 \leq k \leq n$. Reversing orientation, we may suppose that $2k \leq n$. Then [**11**, Theorem 27.16, p. 144] implies that M admits k continuous and linearly independent vector fields. But by [**1**, Theorem 1.1], we must have $k \leq \rho(n+1) - 1$, where $\rho(n+1)$ is a number depending only on n. In particular, if n is even, then all the principal curvatures of M have the same sign.

Remark 3.2. Condition (1.1) does not seem to be sharp. But it is easy to see that if we require that

$$\inf_{p\in M} |\lambda_i(p)| > \varepsilon \tan\left(\frac{R}{2}\right), \quad \forall i \in \{1, \dots, n\} \tag{3.1}$$

for $\varepsilon \in (0, \sqrt{2} - 1)$, then the result of the theorem may be false. Indeed, taking

$$M_r = \mathbb{S}^1(r) \times \mathbb{S}^{n-1}(s) = \big\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^n : \|x\| = r, \|y\| = s\big\} \subset \mathbb{S}^{n+1},$$

with $s = \sqrt{1 - r^2}$, one may prove that the radius R of the largest open geodesic ball of $\mathbb{S}^{n+1}$ that does not intersect M_r is given by

$$\cos R = \min\{r, s\}.$$

Moreover, the principal curvatures of M_r are $\lambda_1 = -\sqrt{1 - r^2}/r$ and $\lambda_2 = \cdots = \lambda_n = r/\sqrt{1 - r^2}$. A calculation shows that one can chose r so that the principal curvatures of M_r satisfy (3.1).

Remark 3.3. We outline here a construction due to Cartan [**3**] that shows the existence of immersed 3-spheres into $\mathbb{S}^4$ with non-zero principal curvatures and which are not ε-convex. Let V be the space of traceless symmetric matrices of order 3 over $\mathbb{R}$, a vector space of real dimension 5. The group SO(3) acts on V via conjugation: if $m \in V$ and $A \in \mathrm{SO}(3)$, let $A \cdot m = AmA^{-1}$. This is an irreducible representation of SO(3), and the action described leaves invariant the (positive definite) quadratic form

$$Q(m) = \tfrac{1}{6}\mathrm{tr}(m^2)$$

as well as the cubic form

$$C(m) = \tfrac{1}{2}\det(m).$$

Let $\mathbb{S}^4 \subset V$ be the unit 4-sphere, defined by $\mathrm{tr}(m^2) = 6$. Since every $m \in V$ can be diagonalized by an element of SO(3), one can easily verify that $-1 \leq C(m) \leq 1$ for all $m \in \mathbb{S}^4$. The special immersed 3-spheres found by Cartan are the level sets $C(m) = r$ for $|r| < 1$. They are clearly SO(3)-orbits, since the only invariants of a symmetric matrix under the SO(3)-action are its eigenvalues, which are completely determined by the values of $Q(m)$ and $C(m)$ (since $\mathrm{tr}(m) = 0$).

The level set $C(m) = 0$ is a minimal hypersurface, with one of its principal curvatures (necessarily constant) equal to 0 and the other two of opposite sign. Meanwhile, as Cartan shows, the level sets $C(m) = \cos(3\theta)$, for $0 < \theta < \pi/6$, have three non-zero principal curvatures (necessarily constant) given by

$$\cot\left(\theta - \frac{\pi}{3}\right), \qquad \cot(\theta), \qquad \cot\left(\theta + \frac{\pi}{3}\right)$$

(the first one is negative and the other two are positive). Since each such orbit is diffeomorphic to $\mathrm{SO}(3)/D$, where $D \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is the finite group of order 4 consisting of the diagonal matrices, and since $\mathrm{SO}(3)$ is itself double-covered by the 3-sphere, it follows that the universal cover of each such orbit is diffeomorphic to the 3-sphere. Thus, we get an immersion of the 3-sphere into $\mathbb{S}^4$ with the properties claimed.

Remark 3.4. Theorem 1 implies Theorem 1.1 of [**12**], which states that if an immersed closed and orientable hypersurface M^n $(n \geq 2)$ of the sphere $\mathbb{S}^{n+1}$ has non-vanishing Gauss–Kronecker curvature and is contained in an open hemisphere, then it must be diffeomorphic to a sphere. We give here a sketch of the proof. To begin with, let p_0 be the north pole of $\mathbb{S}^{n+1}$ and let $\mathbb{S}^{n+1}_+$ be the open hemisphere centred at p_0. The Beltrami map $B : \mathbb{S}^{n+1}_+ \to \mathbb{R}^{n+1} \cong T_{p_0}\mathbb{S}^{n+1}$ is the diffeomorphism obtained by central projection:

$$B(p) = \left(\frac{p_1}{p_{n+2}}, \ldots, \frac{p_{n+1}}{p_{n+2}}\right), \quad p = (p_1, \ldots p_{n+2}) \in \mathbb{S}^{n+1}_+.$$

For $t > 0$, let $H_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the homothety $x \mapsto tx$. The map we are interested in is $C_t = B^{-1} \circ H_t \circ B$. After a rotation, we may suppose M is contained in $\mathbb{S}^{n+1}_+$. By Theorem 1 (with $R = \pi/2$), M would be diffeomorphic to $\mathbb{S}^n$ if all its principal curvatures were bigger than 1 in absolute value. This is not necessarily true. However, defining $M_t = C_t(M)$, it is possible to show that if t is sufficiently small, then this bound on the principal curvatures holds for M_t (actually, the absolute value of the principal curvatures of M_t go to infinity as t goes to zero). So M_t, and hence M, will be diffeomorphic to $\mathbb{S}^n$.

We now prove Theorem 3.

Proof of Theorem 3. Notice that $\langle \eta(p), p_0 \rangle = \pm \sin d(\eta(p), T)$. So we have the following estimate for the function c defined in the proof of Theorem 1:

$$|c(p)| = \frac{|\langle \eta(p), p_0 \rangle|}{1 + \langle p, p_0 \rangle} = \frac{\sin d(\eta(p), T)}{1 + \cos d(p, p_0)} \leq \frac{\sin L}{1 + \cos R}.$$

Reasoning analogously as in the proof of that theorem, we conclude that $\gamma : M \to \mathbb{S}^n$ is a global diffeomorphism. □

Before proving Theorem 2, we need some facts about fundamental domains of a group action, following [**9**]. Let Γ be a non-trivial group of isometries of $\mathbb{S}^{n+1}$ and denote $\Gamma \setminus \{e\}$ by Γ^*. We shall make the assumption that Γ acts on the sphere properly discontinuously, meaning that each point $p \in \mathbb{S}^{n+1}$ has a neighbourhood U such that $U \cap g(U) = \emptyset$ for $g \in \Gamma^*$.

Definition 3.5. For $p \neq q \in \mathbb{S}^{n+1}$, define the sets

$$H_{p,q} = \{x \in \mathbb{S}^{n+1} : d(p,x) < d(q,x)\}$$
$$A_{p,q} = \{x \in \mathbb{S}^{n+1} : d(p,x) = d(q,x)\}.$$

The fundamental domain of Γ centred at p is

$$\Delta_p = \bigcap_{g \in \Gamma^*} H_{p,g(p)}.$$

We need the following facts.

Proposition 3.6 (Ozols [9, Proposition 3.4]). *For each $g \in \Gamma^*$ and $p \in \mathbb{S}^{n+1}$, we have that $\overline{\Delta}_p \cap \overline{\Delta}_{g(p)} \subset A_{p,g(p)}$.*

Proposition 3.7 (Ozols [9, Proposition 3.5]). *For $p \in \mathbb{S}^{n+1}$,*

$$\partial\Delta_p = \bigcup_{g \in \Gamma^*} \partial\Delta_p \cap \partial\Delta_{g(p)}.$$

From these propositions, we prove a series of lemmas.

Lemma 3.8. *For $p \in \mathbb{S}^{n+1}$, define*

$$r = \min_{g \in \Gamma^*} d(p, g(p)).$$

Then $B_{r/2}(p) \subset \Delta_p$. In particular, $B_{r/2}(p) \cap \partial\Delta_p = \emptyset$.

Proof. Suppose that this ball is not contained in the fundamental domain centred at p. Then there exists q belonging to the ball and to $\partial\Delta_p$. Then, from Proposition 3.7, there exists $g_0 \in \Gamma^*$ such that $q \in \partial\Delta_p \cap \partial\Delta_{g_0(p)}$. By Proposition 3.6, it follows that $q \in \overline{\Delta}_p \cap \overline{\Delta}_{g_0(p)} \subset A_{p,g_0(p)}$. Thus, $d(p,q) < r/2$ and $d(g_0(p),q) = d(p,q) < r/2$. Hence,

$$d(p, g_0(p)) \leq d(p,q) + d(g_0(p),q) < \frac{r}{2} + \frac{r}{2} = r,$$

contrary to the definition of r. □

Let $\mathbb{S}^{n+1}/\Gamma$ be the quotient space and denote by $\pi : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}/\Gamma$ the canonical projection. The latter is a Riemannian covering map when we endow $\mathbb{S}^{n+1}/\Gamma$ with a suitable metric.

Lemma 3.9. *The restriction of π to a fundamental domain Δ_p is an isometry onto its image.*

Proof. Since π is a local isometry, it suffices to prove that the restriction of π to Δ_p is injective. Suppose $\pi(q_1) = \pi(q_2)$, with $q_i \in \Delta_p$. Without loss of generality, suppose $d(p,q_1) \leq d(p,q_2)$. There exists $g \in \Gamma$ such that $g(q_1) = q_2$. If $g \neq e$, then we would have

$$d(p,q_2) < d(g(p),q_2) = d(g(p),g(q_1)) = d(p,q_1),$$

contrary to our assumption. Thus, $g = e$ and $q_1 = q_2$. □

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Lemma 3.10. *For $p \in \mathbb{S}^{n+1}$, the antipodal point of p does not belong to $\overline{\Delta}_p$.*

Proof. Suppose the contrary. Then either $-p \in \Delta_p$ or $-p \in \partial\Delta_p$. The first case cannot occur, otherwise

$$\pi = d(p, -p) < d(g(p), -p)$$

for $g \in \Gamma^*$. So we must have $-p \in \partial\Delta_p$. By Propositions 3.6 and 3.7, there exists $g_0 \in \Gamma^*$ such that $-p \in \partial\Delta_p \cap \partial\Delta_{g_0(p)} \subset A_{p,g_0(p)}$. Thus,

$$\pi = d(p, -p) = d(g_0(p), -p)$$

which implies that $g_0(p) = p$. This is absurd, since no element of Γ^* has a fixed point. □

From Lemma 3.10 the next fact, from [9], follows.

Proposition 3.11 (Ozols [9, Corollary 3.11]). *If $\overline{\Delta}_p \cap C(p) = \emptyset$, then $C(\pi(p)) = \pi(\partial\Delta_p)$, where $C(\cdot)$ denotes the cut locus.*

Lemma 3.12. *For $p \in \mathbb{S}^{n+1}$,*

$$\pi^{-1}(\pi(\partial\Delta_p)) = \bigcup_{g\in\Gamma} \partial\Delta_{g(p)}.$$

Proof. This follows from the easily verifiable fact that $g(\partial\Delta_p) = \partial\Delta_{g(p)}$ (see [9, Proposition 3.2, (3)]). □

We are now ready to prove Theorem 2.

Proof of Theorem 2. Since π is a local isometry, the principal curvatures of M and $\tilde{M}$ coincide. Due to Theorem 1, it thus suffices to prove that the open ball $B_{r/2-R}(p_0)$ does not intersect $\tilde{M}$, for then the ball $B_{\pi-r/2+R}(-p_0)$ contains any connected component of $\tilde{M}$. We argue by contradiction. Suppose q lies both in $B_{r/2-R}(p_0)$ and in $\tilde{M}$. Then $d(q, p_0) < r/2 - R$ and $d(\pi(q), C(x_0)) \leq R$. From Lemmas 3.8 and 3.9, we have that $d(\pi(q), x_0) < r/2 - R$. Thus,

$$\begin{aligned} d(x_0, C(x_0)) &\leq d(x_0, \pi(q)) + d(\pi(q), C(x_0)) \\ &< \left(\frac{r}{2} - R\right) + R \\ &= \frac{r}{2}. \end{aligned}$$

So there exists $y \in C(x_0)$ such that $d(x_0, y) < r/2$. By Lemma 3.9, $d(\pi|_{\Delta_{p_0}}^{-1}(y), p_0) < r/2$, and by Proposition 3.11 and Lemma 3.12,

$$\pi|_{\Delta_{p_0}}^{-1}(y) \in \pi^{-1}(y) \subset \pi^{-1}(C(x_0)) \subset \pi^{-1}(\pi(\partial\Delta_{p_0})) = \bigcup_{g\in\Gamma} \partial\Delta_{g(p_0)}.$$

This contradicts Lemma 3.8, since $B_{r/2}(p_0) \cap \partial\Delta_{p_0} = \emptyset$. This concludes the proof. □

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