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The cyclicity of the period annulus of a reversible quadratic system

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We prove that perturbing the periodic annulus of the reversible quadratic polynomial differential system $\dot{x} = y + ax^2$, $\dot{y} = -x$ with $a \neq 0$ inside the class of all quadratic polynomial differential systems we can obtain at most two limit cycles, including their multiplicities. Since the first integral of the unperturbed system contains an exponential function, the traditional methods cannot be applied, except in Figuerasa, Tucker and Villadelprat (2013, *J. Diff. Equ.*, **254**, 3647–3663) a computer-assisted method was used. In this paper, we provide a method for studying the problem. This is also the first purely mathematical proof of the conjecture formulated by Dumortier and Roussarie (2009, *Discrete Contin. Dyn. Syst.*, **2**, 723–781) for $q \leqslant 2$. The method may be used in other problems.

Keywords: Perturbation of quadratic reversible centre; Abelian integral; limit cycle

2020 *Mathematics Subject Classification:* Primary 34C07, 34C08, 37G15

1. Introduction and statement of the main results

We recall that a *centre* of a planar differential system is a singular point p of the system having a neighbourhood filled up of periodic orbits with the unique exception of the point p. The *period annulus* of a centre is the maximal region filled up with the periodic orbits surrounding the centre.

There is a big programme whose objective is to find the exact upper bound for the number of limit cycles that can bifurcate from the periodic orbits of the period annuli of the quadratic polynomial differential systems under quadratic perturbations, see for instance the second part of the book of Christhopher and Li [3]. This upper bound is called *the cyclicity of the period annulus*. This programme started with Arnold [1, 2] and has produced more than 100 articles, see for instance the references of [3].

Here, we contribute to this programme determining this upper bound for the period annulus of the centre of the quadratic polynomial differential systems

$$\dot{X} = Y + aX^2, \qquad \dot{Y} = -X, \tag{1.1}$$

with $a \neq 0$. We note that to study the cyclicity of the period annulus of system (1.1) is equivalent to study the cyclicity of the period annulus of the system

$$\dot{x} = y + 4x^2, \qquad \dot{y} = -x. \tag{1.2}$$

Indeed, doing the change of variables $(X, Y) \to (x, y)$ where $X = 4x/a$ and $Y = 4y/a$ system (1.1) becomes system (1.2).

System (1.2) has the first integral

$$H(x, y) = \mathrm{e}^{8y}\left(4x^2 + y - \frac{1}{8}\right),$$

and the corresponding integrating factor $R(y) = 8\mathrm{e}^{8y}$.

The phase portrait of system (1.2) in the Poincaré disc is shown in figure 1. This phase portrait has a unique finite singular point, the centre localized at the origin of coordinates. It has two pairs of infinite singular points localized at the endpoints of the x and y axes. At the endpoint of the positive x-half-axis there is a hyperbolic stable node, at the endpoint of the negative x-half-axis there is a hyperbolic unstable node, at the endpoints of the y-axis there is a nilpotent saddle, having a hyperbolic sector at the endpoint of the positive y-half-axis and three hyperbolic sectors at the endpoint of the negative y-axis. For the definitions of first integral and integrating factor see chapter 8, for the definition of the Poincaré disc see chapter 5 and for the definitions of hyperbolic and nilpotent singular points see chapters 2 and 3 respectively of [4]. The boundary of the period annulus of the centre of system (1.2) localized at the origin of coordinates is the parabola $y = -4x^2 + 1/8$. Then the period annulus can be expressed by $\{\gamma_h : h \in (-1/8, 0)\}$, where γ_h is the periodic orbit

$$H(x, y) = \mathrm{e}^{8y}\left(4x^2 + y - \frac{1}{8}\right) = h. \tag{1.3}$$

In what follows, we will say simply *quadratic system* instead of quadratic polynomial differential system. It is known, see for instance [7], that any reversible quadratic system can be written in the complex form $\dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2$ where $z = x + iy$, or in the real form

$$\dot{x} = y + (a + b + 2)x^2 - (a + b - 2)y^2, \qquad \dot{y} = -x + 2(a - b)xy,$$

where a and b are real parameters. When $a = b = 1$ the reversible quadratic system (2.1) becomes system (1.2).

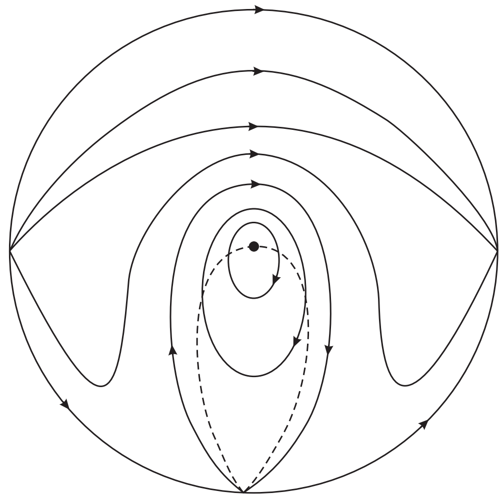

Figure 1. Phase portrait of system (1.2) in the Poincaré disc, with the parabola $y = -4x^2 + 1/8$ at the boundary of the period annulus.

Our main result is the following one.

THEOREM 1.1. *The cyclicity of the period annulus of system* (1.2) *under quadratic perturbations is two.*

Theorem 1.1 is proved in next section.

REMARK 1.2. Note that system (8) of [**5**] is just our system (1.1) with $a = -1$, hence the Abelian integrals $I_1(h), I_3(h), I_5(h)$ for $h \in (-1/8, 0)$ in lemma 2.2 are equivalent to $J_1(h), J_3(h), J_5(h)$ for $h \in (0, 1/2)$ in [**5**]. Dumortier and Roussarie formulated a conjecture on p. 726 of [**5**], that $\{J_1'(h), J_3'(h), \ldots, J_{2q+1}'(h)\}$ forms a strict Chebyshev system for $h \in (0, 1/2)$ and for any integer $q \geqslant 0$. This conjecture is obviously true for $q = 0$. We give a positive answer to this conjecture for $q = 1$ in lemma 2.4 and for $q = 2$ in lemmas 2.5–2.7, by using purely mathematical method. Note also that Figuerasa, Tucker and Villadelprat in [**6**] gave a proof of this conjecture for $q \leqslant 2$ by using theoretical analysis and computations by computer, that are based on computer-assisted techniques. For example, the computations for the proof of a lemma take six and a half hours on a desktop computer with a 2.8 GHz CPU, see remark 4.11 of [**6**].

2. Proof of theorem 1.1

We first state a result by Iliev, see statement (ii). (3) with $a = b = 1$ in theorem 2 of [**7**].

THEOREM 2.1. *The exact upper bound for the number of limit cycles produced by the period annulus of the reversible quadratic system* (1.2) *under quadratic perturbations is equal to the maximal number of zeros in the interval* $(-1/8, 0)$ *counting*

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multiplicities of the function

$$M(h) = \iint_{H(x,y)<h} 8\mathrm{e}^{8y}(\mu_1 + \mu_2 y + \mu_3 y^2)\mathrm{d}x\mathrm{d}y, \tag{2.1}$$

where μ_1, μ_2 and μ_3 are arbitrary constants, and $\mu_1^2 + \mu_2^2 + \mu_3^2 \neq 0$.

LEMMA 2.2. *For $h \in (-1/8, 0)$ function (2.1) can be expressed as*

$$M(h) = \alpha_1 I_1(h) + \alpha_2 I_3(h) + \alpha_3 I_5(h), \tag{2.2}$$

where α_1, α_2 and α_3 are arbitrary constants, $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0$, and

$$I_k(h) = \int_{\gamma_h} 8\mathrm{e}^{8y} x^k \mathrm{d}y, \quad k = 1, 3, 5. \tag{2.3}$$

Proof. It is obviously that (2.1) can be expressed as

$$\int_{\gamma_h} \mathrm{e}^{8y}(\bar{\mu}_1 + \bar{\mu}_2 y + \bar{\mu}_3 y^2)\mathrm{d}x, \tag{2.4}$$

where $\bar{\mu}_1$, $\bar{\mu}_2$ and $\bar{\mu}_3$ are arbitrary constants and $\bar{\mu}_1^2 + \bar{\mu}_2^2 + \bar{\mu}_3^2 \neq 0$. Hence, we only need to prove that each $\int_{\gamma_h} \mathrm{e}^{8y} y^k \mathrm{d}x$ (for $k = 0, 1, 2$) can be expressed as a linear combination of $I_1(h), I_3(h)$ and $I_5(h)$.

First, using integration by parts we have

$$\int_{\gamma_h} \mathrm{e}^{8y}\mathrm{d}x = -I_1(h).$$

Next, by using (1.3) and integration by parts we have

$$\int_{\gamma_h} \mathrm{e}^{8y} y \mathrm{d}x = \int_{\gamma_h} \left[h + \mathrm{e}^{8y}\left(\frac{1}{8} - 4x^2\right)\right] \mathrm{d}x = -\frac{1}{8} I_1(h) + \frac{4}{3} I_3(h).$$

Finally, by using integration by parts we have

$$\int_{\gamma_h} \mathrm{e}^{8y} y^2 \mathrm{d}x = -2\int_{\gamma_h} xy\mathrm{e}^{8y}\mathrm{d}y - 8\int_{\gamma_h} xy^2\mathrm{e}^{8y}\mathrm{d}y = -2K_1(h) - 8K_2(h).$$

From (1.2) we have

$$x\mathrm{d}x + (y + 4x^2)\mathrm{d}y = 0, \tag{2.5}$$

Multiplying (2.5) by $x\mathrm{e}^{8y}$ we have

$$K_1(h) = -\frac{1}{2} I_3(h) - \oint_{\gamma_h} x^2 \mathrm{e}^{8y}\mathrm{d}x = -\frac{1}{6} I_3(h).$$

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Multiplying (2.5) by xye^{8y} we have

$$\begin{aligned}K_2(h) &= -4\oint_{\gamma_h} x^3 y e^{8y} \mathrm{d}y - \oint_{\gamma_h} x^2 y e^{8y} \mathrm{d}x \\ &= -4\oint_{\gamma_h} x^3 \left[d\left(\frac{1}{8} y e^{8y}\right) - \frac{1}{8} e^{8y} \mathrm{d}y\right] - \oint_{\gamma_h} x^2 y e^{8y} \mathrm{d}x \\ &= \frac{1}{2}\oint_{\gamma_h} x^2 y e^{8y} \mathrm{d}x + \frac{1}{16} I_3(h).\end{aligned}$$

Using (1.3), we change the first integral above (neglecting the fact $\frac{1}{2}$) to

$$\begin{aligned}\oint_{\gamma_h} x^2 y e^{8y} \mathrm{d}x &= \oint_{\gamma_h} x^2 \left[h + \left(\frac{1}{8} - 4x^2\right) e^{8y}\right] \mathrm{d}x \\ &= \oint_{\gamma_h} x^2 \left(\frac{1}{8} - 4x^2\right) e^{8y} \mathrm{d}x = \frac{1}{8}\oint_{\gamma_h} x^2 e^{8y} \mathrm{d}x - 4\oint_{\gamma_h} x^4 e^{8y} \mathrm{d}x \\ &= -\frac{1}{24} I_3(h) + \frac{4}{5} I_5(h).\end{aligned}$$ □

LEMMA 2.3. *For $h \in (-1/8, 0)$ the function $M(h)$ in (2.2) satisfies*

$$M'(h) = \alpha_1 I_1'(h) + \alpha_2 I_3'(h) + \alpha_3 I_5'(h), \tag{2.6}$$

where

$$I_k'(h) = \int_{\gamma_h} k x^{k-2} \mathrm{d}y, \quad k = 1, 3, 5. \tag{2.7}$$

Proof. It is easy to check by (1.3) that along γ_h one has

$$\frac{\partial x}{\partial h} = \frac{1}{8 e^{8y} x},$$

thus

$$I_k'(h) = \int_{\gamma_h} 8 e^{8y} k x^{k-1} \frac{\partial x}{\partial h} \mathrm{d}y = \int_{\gamma_h} k x^{k-2} \mathrm{d}y, \quad k = 1, 3, 5.$$ □

Since the orientation along γ_h is clockwise, we have $I_k(h) < 0$ for $k = 1, 3, 5$ and $h \in (-1/8, 0)$. Furthermore to simplify computations, we introduce the new variable $z = y + 4x^2$, then equation (1.2) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = z, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = -x(1 - 8z) \tag{2.8}$$

and the curve γ_h has the form

$$\gamma_h = \left\{(x, z)\,|\, e^{-32x^2 + 8z}\left(z - \frac{1}{8}\right) = h, \quad h \in \left(-\frac{1}{8}, 0\right)\right\}. \tag{2.9}$$

Thus, along γ_h we have

$$\frac{\partial x}{\partial h} = -\frac{1}{64xh}, \quad \frac{\partial z}{\partial h} = -\frac{1/8 - z}{8zh}, \tag{2.10}$$

and

$$z < \frac{1}{8}, \quad \frac{\mathrm{d}z}{x} = -(1-8z)\mathrm{d}t < 0, \quad \mathrm{d}z = -x\left(\frac{1}{z} - 8\right)\mathrm{d}x. \tag{2.11}$$

By lemma 2.3 it is obvious that

$$M'(h) = \int_{\gamma_h} \frac{F(x)}{x}\,\mathrm{d}y = \int_{\gamma_h} \frac{F(x)}{x}\,\mathrm{d}z, \tag{2.12}$$

where $F(x) = \alpha_1 + 3\alpha_2 x^2 + 5\alpha_3 x^4$.

In the following we use the new variables (x, z).

Lemma 2.4. *If $\alpha_3 = 0$, then for $h \in (-1/8, 0)$ the function $M'(h)$ has at most one zero, including its multiplicity, where $M(h)$ is the linear combinations of $I_1(h), I_3(h)$ and $I_5(h)$, shown in (2.2).*

Proof. If $\alpha_3 = \alpha_2 = 0$, then $\alpha_1 \neq 0$. By (2.12) and the second equality of (2.11) we obtain

$$M'(h) = \alpha_1 \int_{\gamma_h} \frac{\mathrm{d}z}{x} \neq 0, \quad h \in (-1/8, 0).$$

If $\alpha_3 = 0, \alpha_2 \neq 0$, we can rewrite (2.2) as

$$M(h) = I_3(h) + \alpha I_1(h).$$

By using (2.7) we have

$$M'(h) = I_3'(h) + \alpha I_1'(h) = \int_{\gamma_h} \frac{3x^2 + \alpha}{x}\,\mathrm{d}z. \tag{2.13}$$

If $\alpha \geqslant 0$, then $M'(h) < 0$, because $\mathrm{d}z/x < 0$ along γ_h by (2.11).

If $\alpha < 0$, we denote x_0 the positive root of $3x^2 + \alpha = 0$. Suppose that the intersection points of the curve γ_h and the axis $\{(x, z)\,|\,z = 0\}$ are $(\pm x_M(h), 0)$, the most left and the most right points of γ_h, then by (2.9) γ_h has two branches $z = z_i(x, h)$ with

$$z_1 < 0 < z_2 < \frac{1}{8} \quad \text{for} \quad x \in (-x_M(h), x_M(h)). \tag{2.14}$$

Note that γ_h tends to the origin as $h \to -1/8^+$, monotonically expands as h increases from $-1/8$, and tends to infinity in $\pm x$ direction as $h \to 0^-$.

If $h \in (-1/8, h_0]$, where $h_0 = H(x_0, 0)$, then $x_M(h) \leqslant x_0$, the curve γ_h is located in the strip $\{(x, z)\,|\,3x^2 + \alpha \leqslant 0\}$, see figure 2(i). Hence $M'(h) > 0$, because $\mathrm{d}z/x < 0$ along γ_h by (2.11).

If $h \in (h_0, 0)$, then $x_M(h) > x_0$, the curve γ_h must cut the straight lines $\{(x, z)|x = \pm x_0\}$, see figure 2(ii). The curve γ_h is symmetry with respect to the

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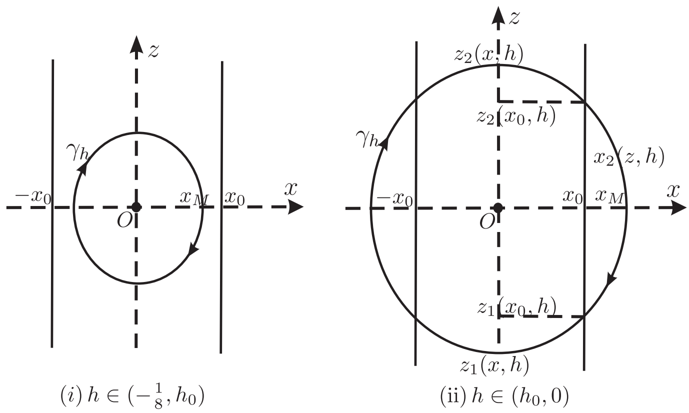

Figure 2. Relative positions of γ_h and the straight lines $\{(x,z)|x = \pm x_0\}$.

z-axis, so we will only consider the side $x \geqslant 0$. We divide the integral form of $M'(h)/2$ into two parts as follows:

$$\frac{M'(h)}{2} = \int_0^{x_0} (3x^2+\alpha)\left(\frac{1}{z_1} - \frac{1}{z_2}\right) \mathrm{d}x - \int_{z_1(x_0,h)}^{z_2(x_0,h)} \frac{3x_2^2+\alpha}{x_2} \mathrm{d}z,$$

where $z_i = z_i(x,h), i = 1,2$; we use the last quality of (2.11) in the first integral to change $\mathrm{d}z$ to $\mathrm{d}x$; in the second integral, $x_2 = x_2(z,h)$ is the positive root of $\mathrm{e}^{-32x^2+8z}(z-1/8)= h$ for any $z \in [z_1(x_0,h), z_2(x_0,h)]$, hence $x_2 \geqslant x_0 > 0$.

Making one more derivative with respect to h by using (2.10), we have

$$\begin{aligned}\frac{M''(h)}{2} &= \int_0^{x_0} \frac{3x^2+\alpha}{8h}\left(\frac{1/8-z_1}{z_1^3} - \frac{1/8-z_2}{z_2^3}\right)\mathrm{d}x \\ &\quad - \frac{3x_0^2+\alpha}{x_0}\left(\frac{\partial z_2(x_0,h)}{\partial h} - \frac{\partial z_1(x_0,h)}{\partial h}\right) + \int_{z_1(x_0,h)}^{z_2(x_0,h)} \frac{3x_2^2-\alpha}{64hx_2^3}\mathrm{d}z < 0,\end{aligned}$$

because $3x^2+\alpha<0$ for $x \in (0,x_0)$, $z_1 < 0 < z_2 < 1/8$ (see (2.14)), $h<0$, and $3x_2^2 - \alpha > 0$. Besides, $3x_0^2+\alpha = 0$ and $\left|\frac{\partial z_2(x_0,h)}{\partial h} - \frac{\partial z_1(x_0,h)}{\partial h}\right|$ is bounded by (2.10), so the above middle term is zero and we will directly omit the similar terms in further calculations.

Thus, $M'(h)$ has at most one zero on $(h_0, 0)$. As we have proved that $M'(h) > 0$ on $h \in (-1/8, h_0]$, hence $M'(h)$ has at most one zero on $(-1/8, 0)$. □

If $\alpha_3 \neq 0$ in (2.2), without loss of generality we consider

$$M(h) = \alpha I_1(h) + \beta I_3(h) + I_5(h), \quad h \in \left(-\frac{1}{8}, 0\right). \tag{2.15}$$

where α and β are arbitrary constants. From (2.7) we have

$$M'(h) = I_5'(h) + \beta I_3'(h) + \alpha I_1'(h) = \int_{\gamma_h} \frac{F(x)}{x}\,\mathrm{d}z, \tag{2.16}$$

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where

$$F(x) = 5x^4 + 3\beta x^2 + \alpha. \tag{2.17}$$

LEMMA 2.5. *If $\beta \geqslant 0$, then $M'(h)$ has at most one zero on $(-1/8, 0)$, including its multiplicity.*

Proof. When $\alpha \geqslant 0$, it is obvious that $M'(h) < 0$, because $\mathrm{d}z/x < 0$ along γ_h. Hence we suppose $\alpha < 0$ in the rest part, and denote the only positive root of $F(x) = 0$ by x_0, and $h_0 = H(x_0, 0)$.

If $h \in (-1/8, h_0]$, then $x_M(h) \leqslant x_0$, the curve γ_h is located in the strip $\{(x, y) \,|\, F(x) < 0\}$. Hence $M'(h) > 0$.

If $h \in (h_0, 0)$, then $x_M(h) > x_0$, the proof is completely similar to the proof of lemma 2.4, so we only list some different computations. We first rewrite $M'(h)$ as

$$\frac{M'(h)}{2} = \int_0^{x_0} \left(\frac{F(x)}{z_1} - \frac{F(x)}{z_2} \right) \mathrm{d}x - \int_{z_1(x_0,h)}^{z_2(x_0,h)} \frac{F(x_2)}{x_2} \,\mathrm{d}z. \tag{2.18}$$

Then by (2.10) we obtain

$$\begin{aligned} \frac{M''(h)}{2} &= \int_0^{x_0} \frac{F(x)}{8h} \left(\frac{1/8 - z_1}{z_1^3} - \frac{1/8 - z_2}{z_2^3} \right) \mathrm{d}x \\ &\quad + \int_{z_1(x_0,h)}^{z_2(x_0,h)} \frac{G(x_2)}{64hx_2^3} \,\mathrm{d}z, \end{aligned} \tag{2.19}$$

where

$$G(x) = 15x^4 + 3\beta x^2 - \alpha. \tag{2.20}$$

Now $\beta \geqslant 0, \alpha < 0$, by using (2.17) (together with the definition of x_0), (2.14), (2.20) and $h < 0$ we obtain $M''(h) < 0$, hence $M'(h)$ has at most one zero on $(-1/8, 0)$. □

Now suppose $\beta < 0$ and we start from the simple case $\alpha \leqslant 0$.

LEMMA 2.6. *If $\beta < 0, \alpha \leqslant 0$, then $M'(h)$ has at most one zero on $(-1/8, 0)$, including its multiplicity, where $M(h)$ is shown in (2.15).*

Proof. In this case, $F(x) = 0$ has exactly one positive root x_0. Similar to the proof of lemma 2.5, we denote $h_0 = H(x_0, 0)$, if $h \in (-1/8, h_0]$, then $x_M(h) \leqslant x_0$, $F(x) \leqslant 0$ along γ_h, hence $M'(h) > 0$.

If $h \in (h_0, 0)$, then $x_M(h) > x_0$, we get the same expressions (2.18) and (2.19). Now $\beta < 0, \alpha \leqslant 0$, hence $F(x) < 0$ on $(0, x_0)$; when $x \geqslant x_0$, we have $F(x) \geqslant 0$, then $G(x) = 3F(x) - 6\beta x^2 - 4\alpha > 0$, hence $M''(h) < 0$. In any case we obtain that $M'(h)$ has at most one zero for $h \in (-1/8, 0)$. □

Finally we consider the most complicated case $\beta < 0$ and $\alpha > 0$.

LEMMA 2.7. *If $\beta < 0$ and $\alpha > 0$, then $M'(h)$ has at most two zeros on $(-1/8, 0)$, including their multiplicities, where $M(h)$ is shown in (2.15).*

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Proof. Since $\beta < 0$ and $\alpha > 0$, $F(x) = 0$ may have no positive root, a double positive root, or two different positive roots. In the first two cases, it is obviously $M'(h) < 0$ for all $h \in (-1/8, 0)$, because $F(x) > 0$ for all x, except for the possible double root of F, and $\mathrm{d}z/x < 0$ along γ_h.

It remains to consider the case that $F(x) = 0$ has two different positive roots. In this case we denote the smaller root by x_0, then

$$x_0^2 = \frac{-3\beta - \sqrt{9\beta^2 - 20\alpha}}{10}, \quad 9\beta^2 - 20\alpha > 0. \tag{2.21}$$

We will prove two assertions, and denote $h_0 = H(x_0, 0)$.

Assertion 1: $M'(h) < 0$ for $h \in (-1/8, h_0]$.

Since $x_M(h) \leqslant x_0$ for $h \in (-1/8, h_0]$, the curve γ_h is located in the strip $\{(x, z) \,|\, |x| \leqslant x_0\}$. Notice $F(x) > 0$ on $(0, x_0)$, because $\alpha > 0$. Similar to the proof of lemma 2.5, we can get $M'(h) < 0$. This assertion is proved.

Assertion 2: $M'(h)$ has at most two zeros (including the multiplicities) for $h \in (h_0, 0)$.

In this case $x_M(h) > x_0$. We get the same forms (2.18) and (2.19). It is easy to see from (2.20) that $G(x)$ has exactly one positive root for $\beta < 0, \alpha > 0$, and it is a simple root. Besides, $G(0) = -\alpha < 0$, and by using (2.20) and (2.21) we have

$$G(x_0) = 20x_0^4 + 6\beta x_0^2 = -2x_0^2\sqrt{9\beta^2 - 20\alpha} < 0.$$

Hence the unique positive root of $G(x)$, denoted by x_1, satisfies $x_1 > x_0 > 0$.

If $h \in (h_0, h_1]$, where $h_1 = H(x_1, 0)$ then $x_0 < x_M(h) \leqslant x_1$. Obviously $G(x) < 0$ on (x_0, x_1). From (2.19) we have $M''(h) > 0$, because $F(x) > 0$ on $(0, x_0)$, $h < 0$, $z_1 < 0 < z_2 < 1/8$, and $x_2 > 0$.

If $h \in (h_1, 0)$, then $x_M(h) > x_1$, we divide the second integral in (2.19) into two parts, use the last equality of (2.11) in the first part, and move $(-h)$ to the left side in the whole equality, we have

$$\begin{aligned}\frac{(-h)M''(h)}{2} &= \int_0^{x_0} \frac{F(x)}{8}\left(\frac{1/8 - z_2}{z_2^3} - \frac{1/8 - z_1}{z_1^3}\right)\mathrm{d}x \\ &\quad - \int_{x_0}^{x_1} \frac{G(x)}{64x^2}\left(\frac{1}{z_2} - \frac{1}{z_1}\right)\mathrm{d}x - \int_{z_1(x_1,h)}^{z_2(x_1,h)} \frac{G(x_2)}{64x_2^3}\,\mathrm{d}z.\end{aligned}$$

Making one more derivative with respect to h by (2.10), we obtain

$$\begin{aligned}\frac{[(-h)M''(h)]'}{2} &= \int_0^{x_0} \frac{F(x)}{32h}\left(\frac{(1/8 - z_2)(3/16 - z_2)}{z_2^5} - \frac{(1/8 - z_1)(3/16 - z_1)}{z_1^5}\right)\mathrm{d}x \\ &\quad - \int_{x_0}^{x_1} \frac{G(x)}{8^3hx^2}\left(\frac{1/8 - z_2}{z_2^3} - \frac{1/8 - z_1}{z_1^3}\right)\mathrm{d}x \\ &\quad + \int_{z_1(x_1,h)}^{z_2(x_1,h)} \frac{15x_2^4 - 3\beta x_2^2 + 3\alpha}{64^2hx_2^5}\,\mathrm{d}z < 0,\end{aligned}$$

because $F(x) > 0$ on $(0, x_0)$, $h < 0$, $z_1 < 0 < z_2 < 1/8$, $G(x) < 0$ on (x_0, x_1), $x_2 > 0$, and $15x_2^4 - 3\beta x_2^2 + 3\alpha > 0$ for $\beta < 0$ and $\alpha > 0$.

Thus, $(-h)M''(h)$, hence $M''(h)$, has at most one zero on $(h_1, 0)$. Since we have proved that $M''(h) > 0$ on $(h_0, h_1]$, we get that $M''(h)$ has at most one zero on $(h_0, 0)$, hence $M'(h)$ has at most two zeros on $(h_0, 0)$. assertion 2 is proved.

Supping up the results in assertions 1 and 2, we obtain that $M'(h)$ has at most two zeros on $(-1/8, 0)$. All multiplicities of the zeros are taken into account. □

Proof of theorem 1.1. We claim that $M(h)$ has at most two zeros on $(-1/8, 0)$, including their multiplicities. Otherwise, if $M(h)$ has at least three zeros on $(-1/8, 0)$, then since $M(-1/8) = 0$, $M'(h)$ would have at least three zeros on $(-1/8, 0)$, which contradicts lemmas 2.4–2.7. □

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