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SMOOTH FANO 4-FOLDS IN GORENSTEIN FORMATS

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Abstract

We construct some new deformation families of four-dimensional Fano manifolds of index one in some known classes of Gorenstein formats. These families have explicit descriptions in terms of equations, defining their image under the anticanonical embedding in some weighted projective space. They also have relatively smaller anticanonical degree than most other known families of smooth Fano 4-folds.

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1. Introduction

A projective algebraic variety X with an ample anticanonical divisor class $-K_X$ is called a Fano variety. The Fano index i is the largest integer such that $-K_X = iD$, for some ample divisor D on X. Fano varieties are one of the central topics of research in algebraic geometry, in general, and in classification problems, in particular. It is well known that there are only finitely many deformation families of smooth Fano varieties in each dimension [20]. In dimension less than or equal to three, the classification has been completed [15–17, 23, 24]. There are 1, 10 and 105 deformation families of smooth Fano varieties in dimension one, two and three, respectively.

In dimension greater than or equal to four, the full classification is still to be completed. The complete classification of smooth Fano 4-folds of index greater than or equal to two is known and there are 35 deformation families of such Fano 4-folds [9–12, 15, 17, 19, 32, 33], listed in [4]. The index one case is still not complete, although there are a number of partial classification results. The toric Fano 4-folds have been classified by Batyrev [1] who found 123 deformation families. One of the larger sets of examples was constructed by Coates, Kasprzyk and Prince: in [5], they constructed 527 new deformation families of Fano 4-folds as complete intersections in toric varieties and they constructed one more in [6] by using the Laurent inversion. Another collection of 141 deformation families has been given by Coates, Kasprzyk and Kalashnikov [18] as quiver flag zero loci in quiver flag varieties. The smooth

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Fano 4-folds of Picard rank two with hypersurface Cox ring have been classified by Hausen *et al.* [13], who found 17 new deformation families of smooth Fano 4-folds of index one. Some of the earlier index one examples were constructed by Küchle [21, 22] as sections of homogeneous vector bundles over Grassmannians and complete intersections in weighted projective spaces.

In this article, we aim to contribute to the classification of smooth Fano 4-folds of index one; this can be thought of as an extension of Küchle's lists as some of his examples also appear in our list. We construct some new deformation families of smooth Fano 4-folds of index one as weighted complete intersections of some known classes of Gorenstein formats (Definition 2.1). As a starting point, we use a computer algebra system to search for candidate families of smooth Fano 4-folds by using the algorithmic approach [3, 27]. Then we prove the existence of these 4-folds by analysing the explicit equations of the candidate varieties.

1.1. Summary of results. In total, we obtained 25 candidate families of smooth Fano 4-folds by using the computer search routine [3, 27]: four as hypersurfaces, eight as codimension two complete intersections, one each as complete intersections in codimension three and four, four in Gr(2,5) format, four in $\mathbb{P}^2 \times \mathbb{P}^2$ format and three in codimension four $\mathrm{Gr}(2,5) \cap \mathcal{H}$ format. Among these, one candidate codimension two complete intersection and one candidate example in $\mathrm{Gr}(2,5) \cap \mathcal{H}$ format failed to be smooth. All the hypersurfaces and complete intersection examples have already appeared in [22], so the new deformation families appear as noncomplete intersection Fano 4-folds.

THEOREM 1.1. *There exist at least ten families of smooth Fano 4-folds of index one whose images under the anticanonical embedding in a weighted projective space can be described as noncomplete intersection varieties, and they are given in Table 1. In four cases, they can be described by using* $\mathrm{Gr}(2,5)$ *in* $\mathbb{P}^7(w_i)$ *format, in two cases by* $\mathrm{Gr}(2,5) \cap \mathcal{H}$ *format in* $\mathbb{P}^8(w_i)$ *and in four cases by using* $\mathbb{P}^2 \times \mathbb{P}^2$ *format in* $\mathbb{P}^8(w_i)$*. The families* 1–6 *have Picard rank one and* 7–10 *have Picard rank two.*

This list of examples in not a formal complete classification of Fano 4-folds in these Gorenstein formats but it is very unlikely that there are any further such examples. In fact, we searched about half a million candidate embeddings of Fano 4-folds by using computer algebra. Examples 1 and 5 also appeared in [22] and the other examples are new deformation families of smooth Fano 4-folds.

It is well known that the plurigenera of the Hilbert series $P_X(t) = \sum h^0(-nK_X)t^n$ are deformation invariant [31]. In particular, the Reimann–Roch formula for smooth 4-folds shows that the first plurigenus $h^0(-K)$ and the anticanonical degree $(-K_X)^4$ are sufficient to distinguish between nondeformation equivalent families (see Proposition 4.1). For the known families of smooth Fano 4-folds, the list of these invariants can be found in [4–6, 13, 18, 21, 22]. We obtain at least six new deformation families of smooth Fano 4-folds of index one. The lists we obtain are conjecturally complete classifications of such varieties in these formats.

TABLE 1. Smooth Fano 4-folds in Gorenstein formats.

No.	Format	$(-K_X)^4$	$h^0(-K_X)$	Eq Degs, $w\mathbb{P}$	Weight Matrix
1	$\mathrm{Gr}(2,5)$	13	8	$X_{2,3^4} \subset \mathbb{P}(1^8)$	$\begin{matrix} 1 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 1 & 2 \\ & & & 2 \end{matrix}$
2		10	7	$X_{3^4,4} \subset \mathbb{P}(1^7,2)$	$\begin{matrix} 1 & 1 & 1 & 1 \\ & 2 & 2 & 2 \\ & & 2 & 2 \\ & & & 2 \end{matrix}$
3		7	6	$X_{2,3^4} \subset \mathbb{P}(1^6,2^2)$	$\begin{matrix} 1 & 1 & 2 & 2 \\ & 1 & 2 & 2 \\ & & 2 & 2 \\ & & & 3 \end{matrix}$
4		5	5	$X_{4^5} \subset \mathbb{P}(1^5,2^3)$	$\begin{matrix} 2 & 2 & 2 & 2 \\ & 2 & 2 & 2 \\ & & 2 & 2 \\ & & & 2 \end{matrix}$
5	$\mathrm{Gr}(2,5) \cap \mathcal{H}$	15	9	$X_{2^5,(3)} \subset \mathbb{P}(1^9)$	$\begin{matrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{matrix}$
6		10	8	$X_{2^5,(4)} \subset \mathbb{P}(1^8,2)$	$\begin{matrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{matrix}$
7	$\mathbb{P}^2 \times \mathbb{P}^2$	17	9	$X_{2^3,3^6} \subset \mathbb{P}(1^9)$	$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}$
8		11	7	$X_{3^6,4^3} \subset \mathbb{P}(1^7,2^2)$	$\begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{matrix}$
9		10	7	$X_{2,3^4,4^4} \subset \mathbb{P}(1^7,2^2)$	$\begin{matrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{matrix}$
10		5	5	$X_{4^9} \subset \mathbb{P}(1^5,2^4)$	$\begin{matrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{matrix}$

A linear section Y of each of the Fano 4-folds in Table 1 is a smooth Calabi–Yau 3-fold. Therefore, by Lefschetz's hyperplane theorem, the Picard rank of a Fano 4-fold X will be equal to that of its Calabi–Yau 3-fold section Y. The Picard ranks for the

TABLE 2. Smooth Fano 4-fold hypersurfaces and complete intersections list of [21].

Format	Codimension	$h^0(-K_X)$	$(-K_X)^4$	Fano 4-fold
Hypersurface	1	6	5	$X_5 \subset \mathbb{P}(1^6)$
		5	3	$X_6 \subset \mathbb{P}(1^5, 2)$
			2	$X_8 \subset \mathbb{P}(1^5, 4)$
		4	1	$X_{10} \subset \mathbb{P}(1^4, 2, 5)$
Comp. Int.	2	7	9	$X_{3^2} \subset \mathbb{P}(1^7)$
			8	$X_{4,2} \subset \mathbb{P}(1^7)$
		6	6	$X_{4,3} \subset \mathbb{P}(1^6, 2)$
		5	4	$X_{4^2} \subset \mathbb{P}(1^5, 2^2)$
		6	4	$X_{6,2} \subset \mathbb{P}(1^6, 3)$
		4	2	$X_{6,4} \subset \mathbb{P}(1^4, 2^2, 3)$
		3	1	$X_{6^2} \subset \mathbb{P}(1^3, 2^2, 3^2)$
	3	8	12	$X_{2^2,3} \subset \mathbb{P}(1^8)$
	4	9	16	$X_{2^4} \subset \mathbb{P}(1^9)$

corresponding Y have been calculated by using [8, Theorem 2.5] and the computer algebra package [14] in Macaulay2.

REMARK 1.2. A pair of examples, #2 and #9, and another pair of examples, #4 and #10, have the same numerical invariants, $h^0(-K_X)$ and $(-K_X)^4$, but they lie in different codimension. We expect them to lie in different deformation families; a similar phenomenon was observed for some terminal Fano 3-folds in [2]. The corresponding Calabi–Yau 3-fold sections of pairs #2, #9 and #4, #10 have distinct Hodge numbers and lie in different deformation families, which provides evidence that the corresponding Fano 4-folds also belong to distinct deformation families.

REMARK 1.3. Our computer search routine also recovered the 13 examples of smooth Fano 4-folds of index one that are hypersurfaces or complete intersections in weighted projective spaces. However, we do not list them as a part of the theorem since they are very well-known examples that appeared in [22] and were further studied in [25]. We list them for the reader in Table 2.

REMARK 1.4. We did search for examples of smooth Fano 4-folds in some other well-known classes of Gorenstein formats, namely, in Grassmannians Gr(2,6) [29], Lagrangian Grassmannians LGr(3,6) [26], a two step flag variety in $\mathbb{C}^4$ [26], orthogonal Grassmannians OGr(5,10) [7] and a weighted homogeneous F_4 variety [28], but no new candidate examples of smooth Fano 4-folds were found.

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2. Definitions and notation

A weighted projective variety $X \hookrightarrow \mathbb{P}^N(w_i)$ of codimension c is called *wellformed* if it does not contain a singular stratum of codimension $c+1$. A format is a way of representing the equations of varieties. For example, the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ can be described as 2×2 minors of a size 3 matrix. A more formal definition of Gorenstein format is given below.

DEFINITION 2.1 [3]. A codimension c *Gorenstein format* $\mathcal{F}$ is a triple $(\widetilde{V}, \mathcal{R}, \mu)$ which consists of a codimension c affine Gorenstein variety $\widetilde{V} \subset \mathbb{A}^n$, a minimal graded free resolution $\mathcal{R}$ of $\mathcal{O}_{\widetilde{V}}$ as a graded $\mathcal{O}_{\mathbb{A}^n}$ module, and a $\mathbb{C}^*$-action μ of strictly positive weights on $\widetilde{V}$.

We only consider those Gorenstein formats where the action μ leaves the variety $\widetilde{V}$ invariant and the free resolution $\mathcal{R}$ is equivariant for the action. The varieties defined below in Definitions 2.2 and 2.3 are examples of such Gorenstein formats.

DEFINITION 2.2 [7]. Consider the Plücker embedding $\mathrm{Gr}(2,5) \hookrightarrow \mathbb{P}^9(\wedge^2\mathbb{C}^5)$ of Grassmannians of 2-planes in $\mathbb{C}^5$. For a choice of vector $w := (a_1, \ldots, a_5)$ with all $a_i \in \frac{1}{2}\mathbb{Z}$ satisfying

$$a_i + a_j > 0 \quad \text{for } 1 \le i < j \le 5,$$

one can define the weighted Grassmannian $\mathrm{wGr}(2,5) = \widetilde{\mathrm{Gr}(2,5)} \backslash \{\underline{0}\}$ as the quotient of an affine cone minus the vertex with $\mathbb{C}^\times$ action given by

$$\mu : x_{ij} \mapsto \mu^{a_i + a_j} x_{ij}.$$

This gives the embedding

$$\mathrm{wGr}(2,5) \hookrightarrow \mathbb{P}(\{w_{ij} : 1 \le i < j \le 5, w_{ij} = a_i + a_j\}). \tag{2.1}$$

We will use $\mathrm{w}\mathcal{G}$ to denote $\mathrm{wGr}(2,5)$. The image of $\mathrm{w}\mathcal{G}$ under the embedding (2.1) can be defined by five maximal Pfaffians of the 5×5 skew symmetric matrix

$$\begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ & x_{23} & x_{24} & x_{25} \\ & & x_{34} & x_{35} \\ & & & x_{45} \end{pmatrix},$$

where we omit the diagonal of zeros and the lower triangular part of the skew symmetric matrix. If $\mathrm{w}\mathcal{G}$ does not contain a 5-dimensional singular locus of $\mathbb{P}(w_i)$, then the canonical divisor class is given by

$$K_{\mathrm{w}\mathcal{G}} = \Big(-\frac{1}{2} \sum_{1 \le i < j \le 5} w_{ij}\Big) H, \tag{2.2}$$

for an ample divisor H. Another variant of this format is the $\mathrm{Gr}(2,5) \cap \mathcal{H}$ format, which is a nonquasilinear hypersurface section of the $\mathrm{Gr}(2,5)$ format.

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DEFINITION 2.3 [2, 30]. Let Σ be the Segre embedding of

$$\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8(x_{ij}) \quad \text{for } 1 \leq i, j \leq 3,$$

and consider a pair of half integer vectors $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$ satisfying

$$b_i + c_j > 0, \quad b_i \leq b_j \quad \text{and} \quad c_i \leq c_j \quad \text{for } 1 \leq i \leq j \leq 3.$$

Then the quotient of the punctured affine cone $\widetilde{\Sigma} \backslash \{\underline{0}\}$ by $\mathbb{C}^\times$, with the action

$$\mu : x_{ij} \mapsto \mu^{b_i + c_j} x_{ij} \quad \text{for } 1 \leq i, j \leq 3,$$

is called a weighted $\mathbb{P}^2 \times \mathbb{P}^2$ variety, which we will denote by $\mathrm{w}\mathcal{P}$. For a choice of b, c, written together as a single input parameter $p = (b_1, b_2, b_3; c_1, c_2, c_3)$, we have the embedding

$$\mathrm{w}\mathcal{P} \hookrightarrow \mathbb{P}^8(w_{ij} : w_{ij} = b_i + c_j \text{ for } 1 \leq i, j \leq 3).$$

The equations of the image are the well-known 2×2 minors of the 3×3 matrix, which we usually refer to as the weight matrix, written as

$$\begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}, \quad \text{where } w_{ij} = b_i + c_j \text{ for } 1 \leq i, j \leq 3.$$

If $\mathrm{w}\mathcal{P}$ is wellformed, then the canonical divisor class is given by

$$K_{\mathrm{w}\mathcal{P}} = \Big(- \sum_{i=1}^{3} w_{ii} \Big) H,$$

for an ample divisor H.

3. Families of Fano smooth 4-folds

In this section, we give a proof of Theorem 1.1 by providing details of the calculations in three cases. The remaining cases can be checked by similar calculations.

3.1. Example #3. For $w = \frac{1}{2}(1, 1, 1, 3, 3)$, we get the embedding $\mathrm{w}\mathcal{G} \hookrightarrow \mathbb{P}(1^3, 2^6, 3)$. Let x_1, x_2, x_3 be weight one variables, let $y_1, \ldots, y_6$ be weight two variables and let z be the weight three variable. Now it is evident that $\mathrm{w}\mathcal{G}$ does not contain any 5-dimensional singular locus of the ambient $\mathbb{P}^9(w_i)$, the weight three locus is just an orbifold point and the weight two locus describes a cubic 3-fold in $\mathbb{P}^5$, so it is wellformed. Thus, by (2.2), $K_{\mathrm{w}\mathcal{G}} = \mathcal{O}(-9)$. Let $Y_1 \subset \mathbb{P}(1^6, 2^6, 3)$ be a projective cone over $\mathrm{w}\mathcal{G}$ with vertex $\mathbb{P}^2$, that is, we add three variables of weight one to the ambient $\mathbb{P}^9(w_i)$ which are not involved in any defining equations of $\mathrm{w}\mathcal{G}$. Then Y_1 is a 9-dimensional variety with $K_{Y_1} = \mathcal{O}(-12)$.

We take a complete intersection of Y_1 with four general quadrics to get a Fano 5-fold

$$Y_2 \subset \mathbb{P}(1^6, 2^2, 3) \quad \text{with } K_{Y_2} = \mathcal{O}(-12 + 2 \times 4) = \mathcal{O}(-4),$$

by using the adjunction formula. Now the base locus of the linear system of quadrics $|\mathcal{O}(2)|$ contains a single point which is a coordinate point of the variable z. Moreover,

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$Y_2 \cap \mathbb{P}(2,2)$ is the empty locus and the only singular point on Y_2 is the quotient singularity $\frac{1}{3}(1,1,1,2,2)$. As a final step, we take an intersection of Y_2 with a general cubic to get a Fano 4-fold,

$$X \subset \mathbb{P}(1^6, 2^2) \quad \text{with } K_X = \mathcal{O}(-1).$$

As X does not contain any singular point of $\mathbb{P}(1^6, 2^2)$, it follows that X is a smooth Fano 4-fold of index one. By using the Hilbert series one can easily show that $(-K_X)^4 = 7$, and $h^0(-K_X) = 6$ is evident from the embedding.

3.2. Example #6. The Grassmannian Gr(2, 5) has an embedding in $\mathbb{P}^9(x_1, \ldots, x_{10})$. It is a 6-fold with $K_X = \mathcal{O}(-5)$. Let Y_1 be the variety obtained by taking a cone of weight two over it, that is, we have the embedding

$$Y_1 = C^2\mathrm{Gr}(2,5) \hookrightarrow \mathbb{P}^{10}(1^9, 2),$$

where the new variable y of weight two is not involved in any defining equations. Then Y_1 is a singular 7-fold with $K_{Y_1} = \mathcal{O}(-7)$. Now we take a general quartic section

$$Q_4 = y^2 + f_4(x_i, y) \quad \text{for } 1 \le i \le 10,$$

of Y_1 to get a 6-fold

$$Y_2 \subset \mathbb{P}(1^{10}, 2) \quad \text{with } K_{Y_2} = \mathcal{O}(-3).$$

The linear system $|\mathcal{O}(4)|$ has empty base locus. The 6-fold Y_2 is a codimension four smooth variety since the weight two points have been removed by the quartic section. Now we take two hyperplane sections of Y_2 to get a smooth Fano 4-fold $X \subset \mathbb{P}(1^8, 2)$ of index one with $h^0(-K_X) = 8$ and $-K_X^4 = 10$.

3.3. Example #10. For the choice of parameter $w = (1,1,1;1,1,1)$, we find that $\mathrm{w}\mathcal{P} \hookrightarrow \mathbb{P}(2^9)$, which is *a priori* a nonwellformed 4-fold. Consider a projective cone Y_1 over $\mathrm{w}\mathcal{P}$ with vertex $\mathbb{P}^4$. Then we have a 9-fold

$$Y_1 \subset \mathbb{P}(1^5, 2^9) \quad \text{with } K_{Y_1} = \mathcal{O}(-11).$$

The 9-fold Y_1 is wellformed, although it contains the orbifold locus of dimension four defined by the weight two variables and a further $\mathbb{P}^4$ given by the cone variables. The quasilinear section of Y_1 with five general quadrics is a Fano 4-fold

$$X = Y_1 \cap \{\cap_{i=1}^5 Q_i\} \subset \mathbb{P}(1^5, 2^4) \quad \text{with } K_X = \mathcal{O}(-11+10) = \mathcal{O}_X(-1),$$

that is, X is a Fano 4-fold of index one. The intersection $X \cap \mathbb{P}(1^5, 2^4)$ is empty and also the base locus of the linear system $|\mathcal{O}(2)|$ of quadrics is empty. Thus, X is a smooth Fano 4-fold.

4. Geography of smooth Fano 4-folds

The deformation type of a smooth Fano variety X depends on the plurigenera $h^0(-nK_X)$ of the Hilbert series $\sum_{n\ge 0} h^0(-nK_X)t^n$ of X, as they are invariant under

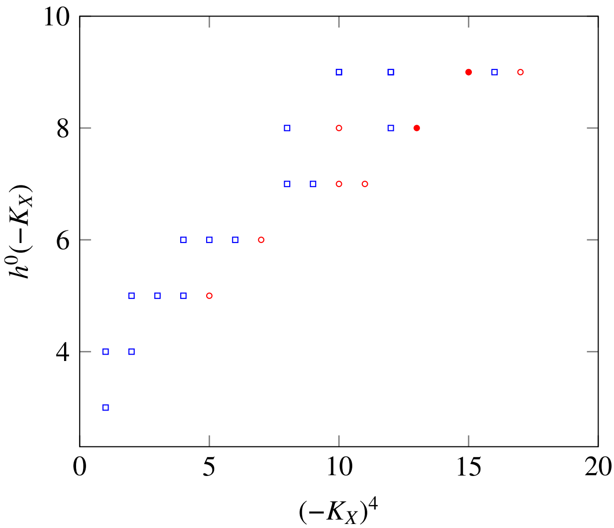

FIGURE 1. Fano 4-folds with small invariants satisfying (4.1). The red circles represent new examples constructed in this paper, blue squares represent the examples from [13, 21, 22] and the filled red dots represent examples which appear in this paper and in [22].

smooth projective deformations [31]. Therefore, if two varieties have different plurigenera, then they represent two distinct different deformation families of Fano 4-folds. By using various vanishing theorems, one can derive the form of the Riemann–Roch formula for smooth Fano 4-folds of index one [22, page 48] given by

$$\chi(-nK_X) = h^0(-nK_X) = 1 + \frac{n(n+1)}{24}(-K_X)^2 c_2(X) + \frac{n^2(n+1)^2}{24}(-K_X)^4.$$

In particular,

$$h^0(-K_X) = 1 + \frac{(-K_X)^2 c_2(X)}{24} + \frac{(-K_X)^4}{6}.$$

Thus, the intersection number $(-K_X)^2 c_2(X)$ is determined easily from the first term if we can compute $h^0(-K_X)$ and $(-K_X)^4$. In our case, these two invariants can be readily computed from the Hilbert series of X. Therefore we have the following result.

PROPOSITION 4.1. *Let X and Y be two smooth Fano 4-folds of index one such that $h^0(-K_X) \neq h^0(-K_Y)$ and $(-K_X)^4 \neq (-K_Y)^4$. Then X and Y belong to two distinct deformation families of smooth Fano 4-folds.*

4.1. Geography with respect to $h^0(-K_X)$ and $(-K_X)^4$. In total, there are at least 987 known deformation families of smooth Fano 4-folds with distinct anticanonical degree $(-K_X)^4$ and $h^0(-K_X)$. They can be found in [4–6, 13, 18, 21, 22]. Among these, there are 13 examples of Fano 4-folds of index one which are hypersurfaces or complete

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intersections in weighted projective spaces [22]. For all of these examples, the first plurigenus $h^0(-K_X)$ and the anticanonical degree $(-K_X)^4$ satisfy

$$1 \leq -K^4 \leq 17 \quad \text{and} \quad 3 \leq h^0(-K_X) \leq 9. \tag{4.1}$$

In what follows, we say that a smooth Fano 4-fold has small invariants if its invariants satisfy (4.1). There are very few families of smooth Fano 4-folds with such small invariants, other than hypersurfaces and complete intersections in weighted projective spaces. In total, excluding the hypersurfaces or complete intersections, only 7 out of the approximately 970 remaining known examples have small invariants. Among these, three appeared in [21, 22], three are listed in [4] and three are in [13], but two of these have the same invariants as those in [22]. All our new families of Fano 4-folds have small invariants and thus they lie in the lower-left-hand corner of the graph of the geography of smooth Fano 4-folds which is a graph in the positive quadrant with $(-K_X)^4$ on the x-axis and $h^0(-K_X)$ on the y-axis. In Figure 1, we show the known smooth families of Fano 4-folds with small invariants.

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