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THE ALPHA-MIXTURE OF SURVIVAL FUNCTIONS

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Abstract

This paper presents a flexible family which we call the α-mixture of survival functions. This family includes the survival mixture, failure rate mixture, models that are stochastically closer to each of these conventional mixtures, and many other models. The α-mixture is endowed by the stochastic order and uniquely possesses a mathematical property known in economics as the constant elasticity of substitution, which provides an interpretation for α. We study failure rate properties of this family and establish closures under monotone failure rates of the mixture's components. Examples include potential applications for comparing systems.

Keywords: Arithmetic mixture; geometric mixture; harmonic mixture; failure rate; stochastic order; weighted power mean

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1. Introduction

Mixture models are useful for analysis of data that are suspected to be generated from heterogeneous items. Let $\bar{F}_i$ and f_i denote the survival functions (SFs) and probability density functions (PDFs) or probability mass functions, respectively, and r_i defined by $r_i(x) = f_i(x)/\bar{F}_i(x)$, $\bar{F}_i(x) > 0$, $i = 1, \ldots, n$ be the failure rates (FRs) of n items. Two different mixture models are commonly used: the mixture distribution model, which can be represented in terms of SFs as

$$\bar{F}_{\text{am}}(x) = \sum_{i=1}^{n} p_i \bar{F}_i(x), \tag{1.1}$$

and the mixture FR model defined by

$$r_{\text{gm}}(x) = \sum_{i=1}^{n} p_i r_i(x), \tag{1.2}$$

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where $\sum_{i=1}^{n} p_i = 1$, $p_i > 0$. (The countable mixture is defined similarly, and the continuous mixture is defined analogously with $p = (p_1, \ldots, p_n)$ replaced by a PDF).

The SF corresponding to (1.2) is the following geometric mean of $\bar{F}_1, \ldots, \bar{F}_n$:

$$\bar{F}_{\rm gm}(x) = \prod_{i=1}^{n} \bar{F}_i^{p_i}(x). \tag{1.3}$$

(The normalization of the p_i is not necessary for the construction of (1.2) and (1.3), but we only consider the normalized case.) The FR of (1.1) is the following variable weights mixture of $r_1, \ldots, r_n$:

$$r_{\rm am}(x) = \sum_{i=1}^{n} p_i(x) r_i(x),$$

where

$$p_i(x) = \frac{p_i \bar{F}_i(x)}{\bar{F}_{\rm am}(x)},$$

provided that $\bar{F}_{\rm am}(x) > 0$. The mixtures of SFs and FRs are less restrictive than the single distribution assumption and provide interesting alternatives to nonparametric modeling.

We will consider the following classes of distributions and stochastic orders of random variables X_i, $i = 1, 2$, with SF $\bar{F}_i$.

Definition 1.1.

(a) A random variable X or its distribution is said to be increasing (decreasing) FR (IFR (DFR)) if its FR is non-decreasing (non-increasing).

(b) A nonnegative random variable X or its distribution is said to be increasing (decreasing) FR average (IFRA (DFRA)) if $-\log \bar{F}(x)/x$ is non-decreasing (non-increasing).

(c) X_1 is said to be stochastically less than or equal to X_2, denoted by $\bar{F}_1 \le_{\rm st} \bar{F}_2$, if $\bar{F}_1(x) \le \bar{F}_2(x)$ for all $x \in \mathbb{R}$.

(d) X_1 is said to be less than or equal to X_2 in hazard rate order, denoted by $\bar{F}_1 \le_{\rm hr} \bar{F}_2$, if $r_1(x) \ge r_2(x)$ for all $x \in \mathbb{R}$.

(e) X_1 is said to be less than or equal to X_2 in likelihood ratio order, denoted by $\bar{F}_1 \le_{\rm lr} \bar{F}_2$, if $f_1(x)/f_2(x)$ is non-increasing in $x \in \mathbb{R}$.

It is apparent that (1.1) and (1.3) provide two completely different models for studying lifetimes of heterogeneous items. For example, it is clear from (1.2) that if $r_i(x)$, $i = 1, \ldots, n$, are all IFR, DFR, or constant, then $\bar{F}_{\rm gm}$ is respectively an IFR, DFR, or an exponential distribution. However, it is well known that if $\bar{F}_{\rm am}$ is the mixture of two exponential distributions then $r_{\rm am}$ is decreasing. Furthermore, Wondmagegnehu *et al.* [24] showed that if $\bar{F}_{\rm am}$ is a mixture of two IFR models then $r_{\rm am}$ can have a "practical" bathtub-shaped FR (bathtub up to the tail of the mixture distribution).

Which of these two mixture models is more suitable or preferred for a problem is an open question and sometimes a subject of sharp disagreement among experts. For example, for modeling burn-in, Block & Savits [8] used the mixture of probability distributions (1.1) while Lynn & Singpurwalla [17] disputed the choice and argued in favor of the predictive FR function which is the mixture of FRs. When p is a probability vector, then (1.1) is a Bayesian predictive FR. Other examples include Aktekin [1], who used (1.1) in a Bayesian context, and Finkelstein [13], who interpreted p in (1.1) as a probability in a non-Bayesian context. In the same vein, an

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important non-Bayesian interpretation of (1.3) is the generalization of the proportional hazards model to the case when an item is operating in an unknown/random environment of specific type, a proportional effect of which is modeled by (1.3).

In many applications it is difficult to favor one of the two models (1.1) or (1.3) over the other. In this paper we propose the weighted power mean of the SFs, which through a parameter $\alpha \in \mathbb{R}$ provides a flexible family of mixture distributions. The density function version of this model has appeared in the information theory literature [5, 23] and following the former paper, we call it an α-mixture. This model gives (1.1) and (1.3) for the specific values of the parameter, as well as models that provide various degrees of compromise between the two conventional mixtures, and much more. The α-mixture inherits stochastic order from properties of the weighted power mean, and uniquely possesses a property which in economics is called the constant elasticity of substitution (CES). We study its FR properties. We obtain monotonicity results that extend the DFR result of Barlow *et al.* [7]. The α-mixture lies in the class of generalized distorted distributions studied by Navarro *et al.* [20] and Navarro & del Águila [19], and thus satisfies the results obtained by the cited authors for the generalized distorted distributions. We obtain FR order results for the α-mixture under weaker assumptions than those needed for the generalized distorted distributions.

This paper is organized as follows. Section 2 introduces the α-mixture model and its special cases. Section 3 presents the FR properties of the α-mixture model. Section 4 briefly discusses the extension of the α-mixture model to the countable and continuous cases. Section 5 concludes the paper. Proofs are given in the Appendix.

2. α-mixture model

The finite α-mixture of SFs $\bar{F}_i$, $i = 1, \ldots, n$, is defined by their weighted αth power mean as follows:

$$\bar{F}_\alpha(x) = \begin{cases} \left[\sum_{i=1}^n p_i \bar{F}_i^\alpha(x)\right]^{1/\alpha} & 0 \neq \alpha \in \mathbb{R}, \\ \bar{F}_{\text{gm}}(x) & \alpha = 0, \end{cases} \tag{2.1}$$

where $p = (p_1, \ldots, p_n)$, $p_i > 0$, $\sum_{i=1}^n p_i = 1$, and $\bar{F}_{\text{gm}}(x) = \lim_{\alpha\to 0} \bar{F}_\alpha(x)$. The α-mixture combines two very popular models: mixture and proportional hazard (PH) models. It is a PH model where the baseline model is an arithmetic mixture of PH models with different baselines and a common PH parameter α.

Let $u_\alpha = \bar{F}_\alpha(x)$ and $u_i = \bar{F}_i(x)$, $i = 1, \ldots, n$. Then

$$u_\alpha = Q(u_1, \ldots, u_n) = \begin{cases} \left[\sum_{i=1}^n p_i u_i^\alpha\right]^{1/\alpha} & 0 \neq \alpha \in \mathbb{R}, \\ \prod_{i=1}^n u_i^{p_i} & \alpha = 0, \end{cases} \tag{2.2}$$

where $Q\colon [0, 1]^n \to [0, 1]$ is a generalized distorted distribution that is continuously increasing on $[0, 1]$ with $Q(0, \ldots, 0) = 0$ and $Q(1, \ldots, 1) = 1$ [19, 20]. Equation (2.2) represents the α-mixture (2.1) as a dual distorted distribution.

The α-mixture is a broad family of mixture distributions which includes the following models:

(a) For $\alpha = 1$ we have the usual arithmetic mixture distribution (1.1).

(b) For $\alpha = 0$ we have the SF of the mixture FR model (1.3).

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(c) For $\alpha = -1$, we have the harmonic mixture (mean) of the baseline SFs:

$$\bar{F}_{\rm hm}(x) = \left[\sum_{i=1}^{n} \frac{p_i}{\bar{F}_i(x)} \right]^{-1}, \qquad x > 0.$$

(d) For $n = 2$ and $\alpha = \frac{1}{m}$ the α-mixture is the following binomial expansion mixture:

$$\bar{F}_{\frac{1}{m}}(x) = \sum_{k=0}^{m} B_{k,m} p^{m-k} (1-p)^k \bar{F}_1^{1-k/m}(x) \bar{F}_2^{k/m}(x),$$

where $B_{k,m}$ is the binomial coefficient. In particular, for $\alpha = \frac{1}{2}$ the α-mixture gives

$$\bar{F}_{\frac{1}{2}}(x) = p^2 \bar{F}_1(x) + (1-p)^2 \bar{F}_2(x) + 2p(1-p)\sqrt{\bar{F}_1(x)\bar{F}_2(x)}. \tag{2.3}$$

This model is a weighted mean of $\bar{F}_1$, $\bar{F}_2$, and $\sqrt{\bar{F}_1 \bar{F}_2}$, and hence is similar to the Heronian mean of the two SFs. (The Heronian mean is defined by equal weights given to the three terms in (2.3) [10].) The model in (2.3) gives lower weights to $\bar{F}_1$ and $\bar{F}_2$ as compared with $\bar{F}_{\rm am}(x)$, and instead a weight of $2p(1-p)$ is given to the root of the SF of the minimum of two independent random variables with SFs $\bar{F}_1$ and $\bar{F}_2$.

This model has an interesting interpretation in terms of series systems. A product is assembled as a series system with two components by a manufacturer who uses two suppliers of the device for the components with shares p and $1-p$ and with different reliability functions $\bar{F}_i^{1/2}$, $i = 1, 2$. The products are assembled with the devices supplied by the same supplier or different suppliers. Equation (2.3) gives the reliability model of the system for a user of the product.

The α-mixture of the cumulative distribution function (CDF) can be defined similarly. For example, for $n = 2$,

$$F_\alpha(x) = \begin{cases} [pF_1^\alpha(x) + (1-p)F_2^\alpha(x)]^{\frac{1}{\alpha}} & 0 \neq \alpha \in \mathbb{R}, \\ F_{\rm gm}(x) & \alpha = 0, \end{cases}$$

where

$$F_{\rm gm}(x) = \lim_{\alpha \to 0} F_\alpha(x) = F_1^p(x) F_2^{1-p}(x).$$

Note that F_α and $\bar{F}_\alpha$ represent the same distribution for $\alpha = 1$, but different distributions for $\alpha \neq 1$.

2.1. Stochastic order

For each x, $\bar{F}_\alpha(x)$ is the weighted mean of αth order, usually of a set of nonnegative numbers [14]. The weighted power means inequality, which has been shown to be equivalent to the Hölder inequality [16], implies that, pointwise for each x,

$$\bar{F}_{\alpha_1}(x) \leq \bar{F}_{\alpha_2}(x), \qquad -\infty < \alpha_1 \leq \alpha_2 < \infty. \tag{2.4}$$

This implies the stochastic order of the α-mixture family by $\alpha \in \mathbb{R}$.

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When among the mixed components there is an SF $\bar{F}_{\max}$ which stochastically dominates the others, and an SF $\bar{F}_{\min}$ which is stochastically dominated by all others, we can define the α-mixture for all α in the extended real line as follows:

$$\begin{cases} \bar{F}_{-\infty}(x) = \lim_{\alpha\to-\infty} \bar{F}_{\alpha}(x) = \bar{F}_{\min}(x), \\ \bar{F}_{\infty}(x) = \lim_{\alpha\to\infty} \bar{F}_{\alpha}(x) = \bar{F}_{\max}(x). \end{cases}$$

These limits for each $u_i = \bar{F}_i(x)$ are known, and can be shown by the L'Hospital rule. By increasing (decreasing) α, the α-mixture moves stochastically closer to $\bar{F}_{\max}$ ($\bar{F}_{\min}$).

The stochastic order (2.4) gives

$$\bar{F}_{gm} \leq_{st} \bar{F}_{\alpha} \leq_{st} \bar{F}_{am}, \qquad 0 \leq \alpha \leq 1.$$

Thus, the α-mixture provides flexible compromises between (1.1) and (1.3), where $\alpha \in [0, 1]$ determines the extent of the compromise. When α is close to zero, $\bar{F}_{\alpha}$ is stochastically close to the mixture FRs model; when α is close to one, $\bar{F}_{\alpha}$ is stochastically close to the mixture SFs model. More formally, the stochastic distance (SD) between two distributions is defined by the variation distance (L_1-norm) between their CDFs [2, 11], which can be represented in terms of the SFs as

$$SD(\bar{F}, \bar{G}) = \int |\bar{F}(x) - \bar{G}(x)|\, dx.$$

The following proposition gives a benchmark for the extent of the compromise between (1.1) and (1.3).

Proposition 2.1. *For $n = 2$ and $p = 1/2$, $\bar{F}_{1/2}$ is stochastic equidistant from the FR mixture and SF mixture models.*

$$\begin{aligned} SD(\bar{F}_{1/2}, \bar{F}_{am}) &= SD(\bar{F}_{gm}, \bar{F}_{1/2}) \\ &= \frac{1}{4}\int \left(\sqrt{\bar{F}_1(x)} - \sqrt{\bar{F}_2(x)}\right)^2 dx \\ &= \frac{1}{2}(\mu_{am} - \mu_{gm}), \end{aligned} \tag{2.5}$$

where μ_{am} and μ_{gm} are means of the arithmetic and geometric mixtures, respectively, and the integral in (2.5) is twice the squared Hellinger distance between the two SFs.

If $\bar{F}(x) \geq \bar{G}(x)$ for all x, then their means are ordered as $\mu_F \geq \mu_G$ and $SD(\bar{F}, \bar{G}) = \mu_F - \mu_G$. Then, for $p = 1/2$, $\alpha < 1/2$ ($\alpha > 1/2$) the α-mixture is closer to (farther from) the mixture FRs model as compared to the mixture of SFs. The following example illustrates the α-mixture and the notion of SD.

Example 2.1. Figure 1 shows plots of $\bar{F}_{\alpha}(x)$ with $n = 2$, $p = 1/2$, and $\alpha = 0, .25, .5, .75, 1$, where $\bar{F}_1$ and $\bar{F}_2$ are as follows:

(a) The left panel shows the α-mixtures of two Weibull distributions with SFs $\bar{F}_1(x) = e^{-x^2}$, $x > 0$, and $\bar{F}_2(x) = e^{-4x^2}$, $x > 0$.

(b) The right panel shows the α-mixtures of a Weibull $\bar{F}_1(x) = e^{-x^2}$, $x > 0$ and a linear FR distribution (LFR) $\bar{F}_2(x) = e^{-x^2-4x}$, $x > 0$.

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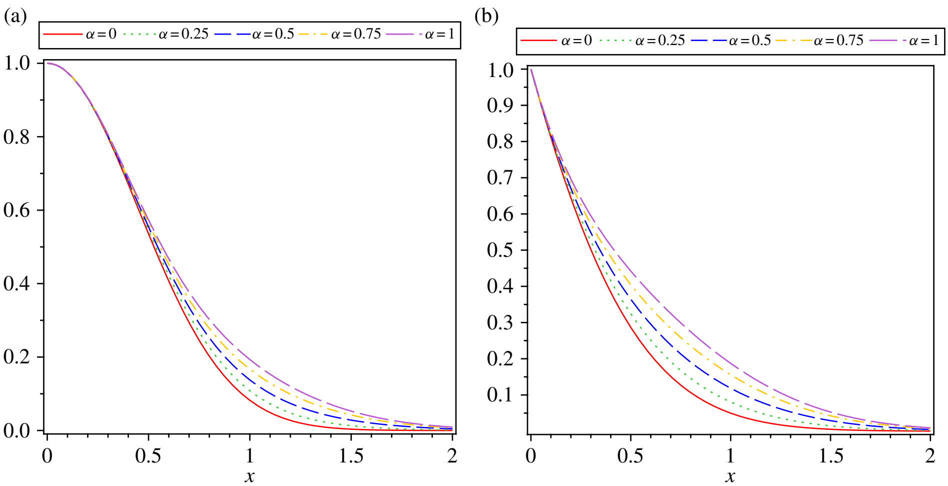

Two Weibull mixtures

α	μ_α	$SD(\bar{F}_\alpha, \bar{F}_{gm})$
0.00	0.560	0.000
0.25	0.584	0.023
0.50	0.613	**0.052**
0.75	0.640	0.080
1.00	0.665	0.104

Weibull & LFR mixtures

α	μ_α	$SD(\bar{F}_\alpha, \bar{F}_{gm})$
0.00	0.379	0.000
0.25	0.419	0.040
0.50	0.466	**0.089**
0.75	0.515	0.136
1.00	0.557	0.178

FIGURE 1: Survival functions, means, and stochastic distances of α-mixtures ($\alpha = 0$, .25, .5, .75, 1) of two Weibull distributions with a common shape parameter and different scale parameters (left) and of a Weibull distribution and a linear failure rate distribution (right).

In both panels the increasing stochastic order is evident. Due to the stochastic order the stochastic distances between the mixtures are given by the differences between the means. The means and stochastic distances between the models and $\bar{F}_{\text{gm}}$ are tabulated below each panel. The models shown in the left panel have larger means but are closer to each other than those shown in the right panel. In both panels $\bar{F}_{1/2}$ (dashed blue) is halfway between $\bar{F}_{\text{gm}}$ (solid red) and $\bar{F}_{\text{am}}$ (dashed purple), where $SD(\bar{F}_{1/2}, \bar{F}_{\text{am}}) = SD(\bar{F}_{\text{gm}}, \bar{F}_{1/2}) = .052$ and .089 for the left and right panels, respectively.

The following examples present applications of the stochastic order (2.4) to comparing series systems.

Example 2.2. Consider comparison of a certain type of product assembled as a series system with n devices according to two different processes. The devices are $m \geq 2$ types with lifetimes $X_1, \ldots, X_m$ that have SFs $\bar{F}_1, \ldots, \bar{F}_m$, respectively.

A. The systems are assembled using the same type of device for all components, where the device with lifetime X_i is used for the proportion p_i of the products, $i = 1, \ldots, m$. The reliability of a randomly selected product is

$$\bar{\mathcal{F}}_1(x) = \sum_{i=1}^{m} p_i \bar{F}_i^n(x), \qquad x > 0.$$

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B. The systems are assembled using devices drawn from a lot that contains proportion p_i of the device with lifetime X_i, $i = 1, \ldots, m$. The reliability of a randomly selected product is

$$\bar{\mathcal{F}}_2(x) = \left[\sum_{i=1}^{m} p_i \bar{F}_i(x)\right]^n, \qquad x > 0.$$

Now the question is: which product is more reliable? The monotone decreasing property of α-mixture provides the answer to this question. Noting that $\bar{\mathcal{F}}_1(x) = \bar{F}_\alpha^n(x)$ with $\alpha = n$ and $\bar{\mathcal{F}}_2(x) = \bar{F}_{\text{am}}^n(x)$, we have $\bar{\mathcal{F}}_2 \le_{\text{st}} \bar{\mathcal{F}}_1$, where the equality holds if and only if the devices have stochastically equal lifetimes: $\bar{F}_1(x) = \cdots = \bar{F}_m(x)$ for all x. This comparison illustrates that, given the weights, mixtures of series system with homogeneous components are more reliable than series systems with heterogeneous components.

Example 2.3. Consider three series systems S_k, $k = 1, 2, 3$, each with two components whose lifetimes are distributed as $\{\bar{F}_1, \bar{F}_2\}$, $\{\bar{F}_{\text{gm}}, \bar{F}_{\text{gm}}\}$, where $F_{\text{gm}}(x) = \sqrt{\bar{F}_1(x)\bar{F}_2(x)}$, and $\{\bar{F}_{-\alpha}, \bar{F}_\alpha\}$, $\alpha > 0$, respectively. Which system is more reliable? The answer depends on the weight p of $\bar{F}_1$ in S_2 and S_3.

(a) For $p = \frac{1}{2}$, the S_k, $k = 1, 2, 3$, are equally reliable:

$$\bar{F}_\alpha(x)\bar{F}_{-\alpha}(x) = \bar{F}_1(x)\bar{F}_2(x) = \bar{F}_{\text{gm}}^2(x). \tag{2.6}$$

We can write:

$$\begin{aligned}\bar{F}_\alpha(x)\bar{F}_{-\alpha}(x) &= \left(\frac{p\bar{F}_1^\alpha(x) + (1-p)\bar{F}_2^\alpha(x)}{p\bar{F}_1^{-\alpha}(x) + (1-p)\bar{F}_2^{-\alpha}(x)}\right)^{\frac{1}{\alpha}} \\ &= \left(\frac{p\bar{F}_1^\alpha(x) + (1-p)\bar{F}_2^\alpha(x)}{p\bar{F}_2^\alpha(x) + (1-p)\bar{F}_1^\alpha(x)}\right)^{\frac{1}{\alpha}} \bar{F}_1(x)\bar{F}_2(x).\end{aligned}$$

In particular, when $p = \frac{1}{2}$ we obtain (2.6). This in turn implies that with $p = \frac{1}{2}$, the geometric mean of $\bar{F}_\alpha(x)$ and $\bar{F}_{-\alpha}(x)$ equals the geometric mean of $\bar{F}_1(x)$ and $\bar{F}_2(x)$, for any α. The following inequalities can be shown similarly.

(b) If $\bar{F}_1 \le_{\text{st}} (\ge_{\text{st}})\, \bar{F}_2$ and $p > (<)\, \frac{1}{2}$, then S_3 is less reliable than S_1 and S_2:

$$\bar{F}_{-\alpha}(x)\bar{F}_\alpha(x) \le \bar{F}_1(x)\bar{F}_2(x) = (\bar{F}_{\text{gm}}(x))^2,$$

where the last equality assumes $p = \frac{1}{2}$.

(c) If $\bar{F}_1 \ge_{\text{st}} (\le_{\text{st}})\, \bar{F}_2$ and $p > (<)\, \frac{1}{2}$, then S_3 is more reliable than S_1 and S_2:

$$\bar{F}_{-\alpha}(x)\bar{F}_\alpha(x) \ge \bar{F}_1(x)\bar{F}_2(x) = (\bar{F}_{\text{gm}}(x))^2,$$

where the last equality assumes $p = \frac{1}{2}$.

2.2. The CES property

For each x, (2.2) produces an output probability based on a set of input probabilities u_i, $i = 1, \ldots, n$. The α-mixture function (2.2) uniquely possesses a mathematical property called the constant elasticity of substitution (CES) between the inputs. Since its exploration

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in economics by Nobel laureates Arrow and Solow and their collaborators [3], CES is widely used for modeling production functions and utility (consumption) functions with multiple inputs. The notion of CES can be defined for any number of inputs, but the case of $n = 2$ simplifies the exposition. For a twice differentiable (production/utility) function with two inputs $u = Q(u_1, u_2)$, the elasticity of substitution between the inputs u_1 and u_2 is defined by

$$\sigma(u_1, u_2) = \frac{\mathrm{d}\log(u_1/u_2)}{\mathrm{d}\log(c_1(u_1, u_2)/c_2(u_1, u_2))},$$

where

$$c_i(u_1, u_2) = \frac{\partial Q(u_1, u_2)}{\partial u_i}, \qquad i = 1, 2.$$

The economic interpretation of $\sigma(u_1, u_2)$ is the percentage response of the relative marginal products of the two inputs to a percentage change in the ratio of their quantities. The CES function is defined by $\sigma(u_1, u_2) = \sigma$. The CES holds if and only if

$$\log\frac{u_2}{u_1} = \sigma \log\frac{c_1(u_1, u_2)}{c_2(u_1, u_2)} + c, \tag{2.7}$$

where $c = \sigma \log(p_2/p_1)$ and p_i is the share of input u_i in the model. It is well known that if Q is a homogeneous function of degree one (that is, $Q(\lambda u_1, \lambda u_2) = \lambda Q(u_1, u_2)$), then it is CES if and only if $Q(u_1, u_2) \propto (p_1 u_1^{\alpha} + p_2 u_2^{\alpha})^{1/\alpha}$, where $\alpha = (\sigma - 1)/\sigma$. The limiting case of $\alpha = 0$ is the Cobb–Douglas production function $Q(u_1, u_2) \propto u_1^{p_1} u_2^{p_2}$.

Accordingly, the α-mixture of two SFs is characterized among the homogeneous functions of degree one by the CES between the input SFs. The left-hand side of (2.7) is the log-odds $\log[\bar{F}_2(x)/\bar{F}_1(x)]$. For the right-hand side of (2.7), we use the chain rule for the partial derivative and obtain the PDF of the α-mixture of the SFs in the form

$$f(x) = c_1(x) f_1(x) + c_2(x) f_2(x),$$

where f_i is the PDF of $\bar{F}_i$ and $c_i(x)$ are given by the partial derivatives as follows:

$$c_1(u_1, u_2) = \frac{\partial g(u_1, u_2)}{\partial u_1} = p\bar{F}_1^{\alpha-1}(x)[p\bar{F}_1^{\alpha}(x) + (1-p)\bar{F}_2^{\alpha}(x)]^{1/\alpha-1},$$
$$c_2(u_1, u_2) = \frac{\partial g(u_1, u_2)}{\partial u_2} = (1-p)\bar{F}_2^{\alpha-1}(x)[p\bar{F}_1^{\alpha}(x) + (1-p)\bar{F}_2^{\alpha}(x)]^{1/\alpha-1}.$$

Thus, the linear relationship (2.7) holds between the log-odds of the two SFs and the log-ratio of coefficients in the PDF of their α-mixture, where $\sigma[\bar{F}_1(x), \bar{F}_2(x)] = \sigma$ is free from $\bar{F}_i$, $i = 1, 2$, and $\alpha = \sigma/(\sigma - 1)$. This property provides the interpretation of α in terms of the elasticity of substitution between the two input SFs.

The CES characterization extends to $n > 2$ in terms of constant partial elasticities of substitution between every pair of inputs [22].

3. FR properties of α-mixture

The FR of the α-mixture for all $\alpha \in \mathbb{R}$ is given by

$$r_{\alpha}(x) = \sum_{i=1}^{n} p_i(x, \alpha) r_i(x), \tag{3.1}$$

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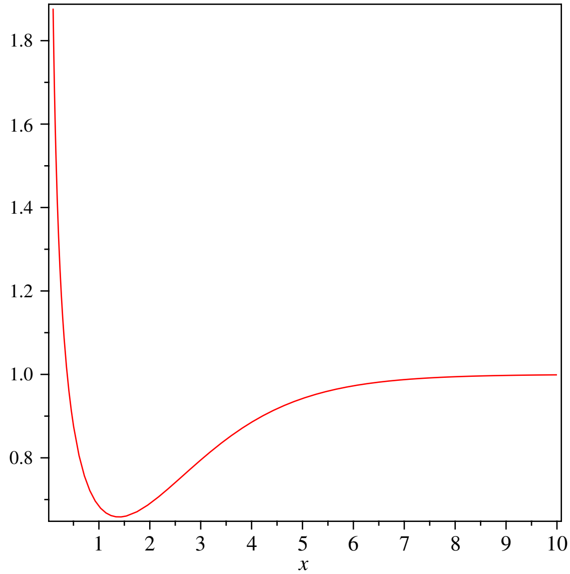

FIGURE 2: The FR of the harmonic mixture of the SFs of the exponential and Weibull distributions of Example 3.1.

where

$$p_i(x, \alpha) = p_i \left[\frac{\bar{F}_i(x)}{\bar{F}_\alpha(x)} \right]^\alpha .$$

The following theorem extends the well-known result of Barlow *et al.* [7] on the closure of the mixture of DFR distributions.

Theorem 3.1. *Let $\bar{F}_\alpha(x)$ be an α-mixture.*

(a) *If each $\bar{F}_i$ is IFR (DFR) then for $\alpha < 0$ ($\alpha > 0$) $\bar{F}_\alpha$ is IFR (DFR).*

(b) *If each $\bar{F}_i$ is IFRA (DFRA) then for $\alpha < 0$ ($\alpha > 0$) $\bar{F}_\alpha$ is IFRA (DFRA).*

It is known that the arithmetic mixture of an exponential distribution and an IFR Weibull distribution has a "practical" bathtub-shaped FR [24]. The following example shows that the α-mixture is not closed under the DFR property for $\alpha < 0$, and the harmonic mixture of the exponential and DFR Weibull distributions has a "practical" bathtub-shape FR.

Example 3.1. Let $\bar{F}_1(x) = \mathrm{e}^{-x}$, $x > 0$, and $\bar{F}_2(x) = \mathrm{e}^{-\sqrt{x}}$, $x > 0$. These two models are both DFR. The FR of their harmonic mixture with $p = 1/2$ is

$$r_{\mathrm{hm}}(x) = \frac{\mathrm{e}^{-\sqrt{x}}}{\mathrm{e}^{-\sqrt{x}} + \mathrm{e}^{-x}} + \frac{\mathrm{e}^{-x}}{2\sqrt{x}(\mathrm{e}^{-\sqrt{x}} + \mathrm{e}^{-x})}.$$

A plot of $r_{\mathrm{hm}}(x)$ is shown in Figure 2. It is evident from the plot that the FR of the α-mixture is decreasing for a short period of time until it attains its minimum and then starts to increase.

Let $\bar{F}_{\mathrm{rmin}}$ and $\bar{F}_{\mathrm{rmax}}$ denote the SFs corresponding to $r_{\min}(x) = \min\{r_1(x), \ldots, r_n(x)\}$ for all x and $r_{\max}(x) = \max\{r_1(x), \ldots, r_n(x)\}$ for all x, respectively. It is known that

$$\bar{F}_{\mathrm{rmax}} \leq_{\mathrm{hr}} \bar{F}_{\mathrm{gm}} \leq_{\mathrm{hr}} \bar{F}_{\mathrm{am}} \leq_{\mathrm{hr}} \bar{F}_{\mathrm{rmin}}; \tag{3.2}$$

see, for example, [21] and [4]. Next, we give some FR order results for the α-mixture.

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Theorem 3.2. *If among the components of $\bar{F}_\alpha$ there is an $\bar{F}_{\rm rmin}$ whose FR dominates the FRs of all other components, and there is an $\bar{F}_{\rm rmax}$ whose FR is dominated by the FRs of all other components, then, for all $\alpha \in \mathbb{R}$,*

$$\bar{F}_{\rm rmax} \leq_{\rm hr} \bar{F}_\alpha \leq_{\rm hr} \bar{F}_{\rm rmin}.$$

The following theorem gives further generalization of (3.2) in terms of relaxing the assumption of directional order in Theorem 3.2.

Theorem 3.3. *If the baseline FRs $r_i(x)$, $i = 1, \ldots, n$, are ordered either increasingly or decreasingly, then $r_\alpha(x)$ is decreasing in α for all $\alpha \in \mathbb{R}$.*

From Theorems 3.2 and 3.3, we have the extension of (3.2) given by the following corollary.

Corollary 3.1. *If baseline FRs are ordered either increasingly or decreasingly, then*

$$\bar{F}_{\rm hm} \leq_{\rm hr} \bar{F}_{\rm gm} \leq_{\rm hr} \bar{F}_{\rm am}.$$

The following example illustrates the FR order results for α-mixtures of three sets of three gamma distributions.

Example 3.2. Consider the gamma family $G(\beta)$ with the following SF:

$$\bar{F}(x;\beta) = \int_x^\infty \frac{1}{\Gamma(\beta)} u^{\beta-1} \mathrm{e}^{-u}\, \mathrm{d}u, \qquad x \geq 0,\ \beta > 0.$$

It is known that the shape parameter β orders the FR of the gamma family decreasingly. For $\beta < 1$ (> 1) the FR is decreasing (increasing) in x. The upper panels of Figure 3 show plots of three IFR gamma distributions (left) and a three-dimensional (3D) plot of the FR of their α-mixtures (right) as functions of (α, x), $-1 \leq \alpha \leq 1$. The middle panels show the corresponding plots for three DFR gamma distributions. The lower panels show plots of the FRs of the gamma distributions with decreasing, constant, and increasing FRs, and their α-mixtures for $\alpha = -1, 0, 1$ (left), and a 3D plot of the FR of the α-mixture (right). The following patterns are apparent:

(a) The 3D plots for the IFR(DFR) gamma distributions illustrate Theorem 3.1. The 3D plot for $\beta < 1$ ($\beta > 1$) illustrates the closure under IFR (DFR). However, these plots also confirm that the conditions on the sign of α are sufficient but not necessary.

(b) The lower two-dimensional plots illustrate the FR orders in accord with Theorem 3.2 and Corollary 3.1.

(c) All 3D plots are decreasing in α, which illustrates Theorem 3.3.

The shape of the FR of the arithmetic mixture of two Weibull distributions has been studied by many authors, for example [15, 24], and the shape of the FR of the arithmetic mixture of two linearly increasing FRs has been studied in detail in [9]. The following example illustrates the FR order results for α-mixtures of two exponential and two IFR Weibull distributions.

Example 3.3. Let $\bar{F}_1(x) = \mathrm{e}^{-x^\beta}$, $x > 0$, and $\bar{F}_2(x) = \mathrm{e}^{-\lambda x^\beta}$, $x > 0$. Then

$$r_\alpha(x) = \beta x^{\beta-1}\left[1 + \frac{(1-p)(\lambda-1)}{1-p+p\mathrm{e}^{-\alpha(1-\lambda)x^\beta}}\right].$$

Figure 4 shows plots of the FRs of the harmonic, geometric, and arithmetic mixtures where $p = 1/2$. These plots illustrate Theorem 3.2 and Corollary 3.1. The left panel shows plots

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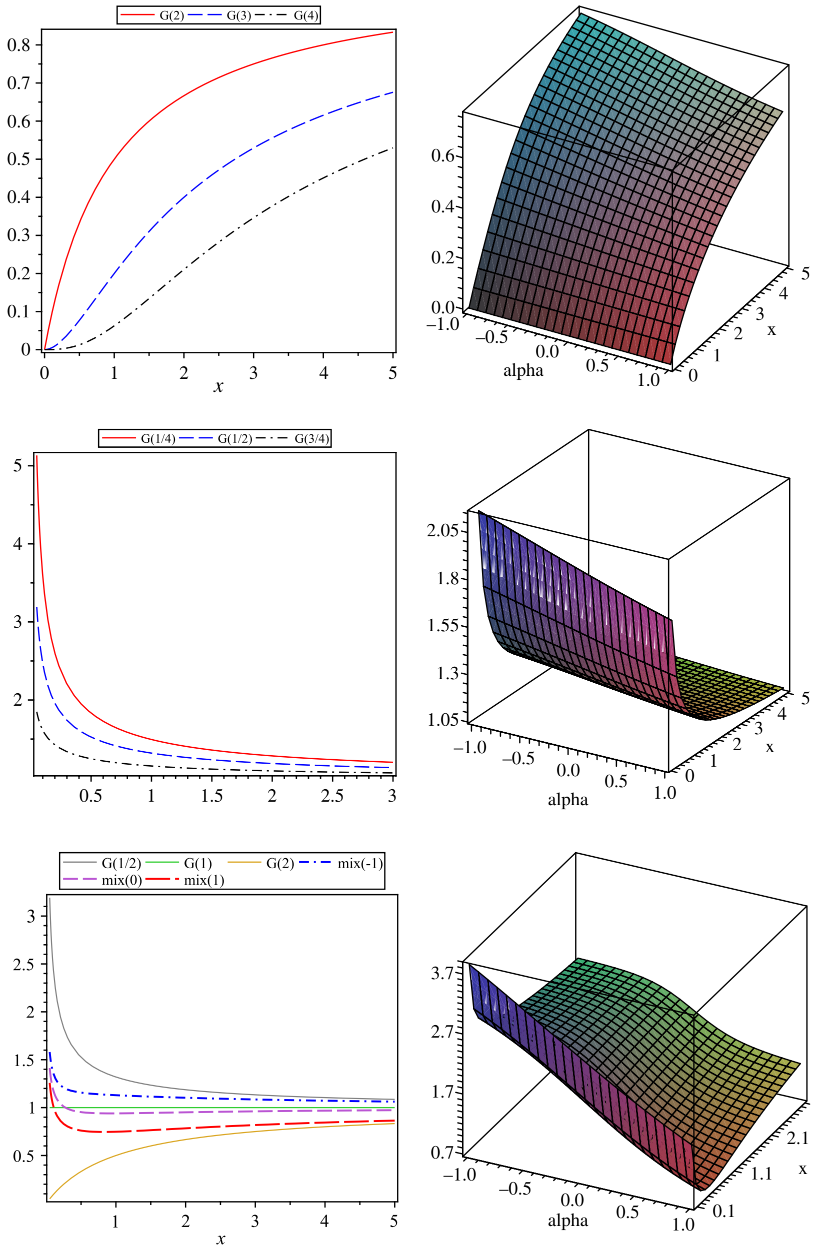

FIGURE 3: Plots of FRs of gamma distributions and their α-mixtures: three IFR gammas (upper panels); three DFR gammas (middle panels); three gamma distributions with IFR, DFR, and constant FR and their α-mixtures (lower left) and a three-dimensional plot of their mixture (lower right). All models have a common scale parameter.

for two exponential distributions where $\beta = 1$ and $\lambda = 2$, and the right panel shows plots for two Rayleigh distributions where $\beta = \lambda = 2$. In the right panel, for $\alpha = -1$ the FRs become approximately linear very early, and for $\alpha = 1$ the FR is approximately linear after $x \approx 1.5$.

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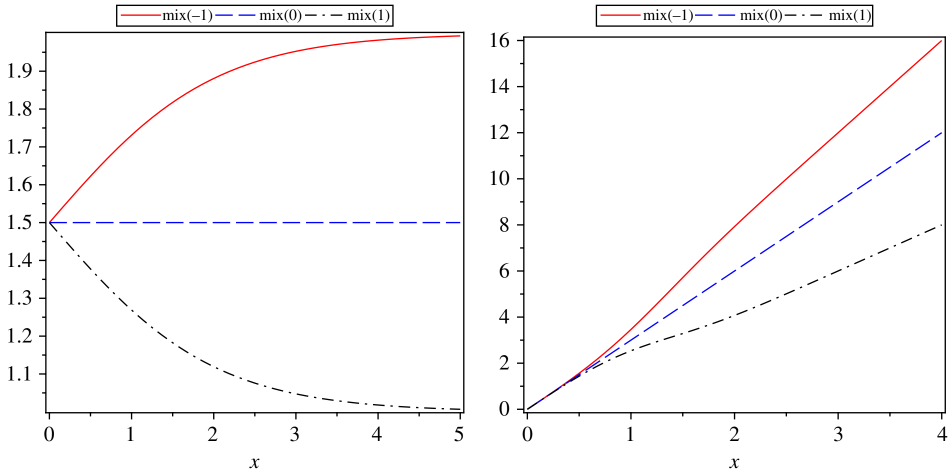

FIGURE 4: The FRs of the harmonic, geometric, and arithmetic mixtures of the SFs of two exponential (left) and two Weibull (right) distributions of Example 3.3.

Remark 3.1. It is known that if $\bar{F}_1 \le_{\text{lr}} \bar{F}_2$, then

$$\bar{F}_1 \le_{\text{lr}} \bar{F}_{\text{am}} \le_{\text{lr}} \bar{F}_2,$$

where X_{am} is a random variable with SF $\bar{F}_{\text{am}}$ (see [21]). The following example shows that the result is not necessarily true for the α-mixture, $\alpha \neq 1$; see [18] for the likelihood order of mixtures of type $\bar{F}_{\text{am}}$ and series systems which, by Example 2.3, can be interpreted as mixtures of type $\bar{F}_{\text{gm}}$.

Example 3.4. Let X_1 be distributed as exponential with PDF $f_1(x) = \mathrm{e}^{-x}$, $x > 0$, and X_2 be distributed as $f_2(x) = (1+x)^{-2}$, $x > 0$. It is easy to show that $\frac{f_2(x)}{f_1(x)}$ is increasing in x, and hence $X_1 \le_{\text{lr}} X_2$. Let X_{gm} denote the random variable with SF $\bar{F}_{\text{gm}}$. Then $X_1 \le_{\text{lr}} X_{\text{gm}}$ does not hold. This can be seen by using $p = 1/2$, where the PDF of $\bar{F}_{\text{gm}}$ is

$$f_{\text{gm}}(x) = \frac{.5\mathrm{e}^{-.5x}}{(1+x)^{.5}} + \frac{.5\mathrm{e}^{-.5x}}{(1+x)^{1.5}}.$$

The likelihood ratio is

$$\frac{f_{\text{gm}}(x)}{f_1(x)} = \frac{.5\mathrm{e}^{.5x}}{(1+x)^{.5}} + \frac{.5\mathrm{e}^{.5x}}{(1+x)^{1.5}}.$$

Its derivative is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f_{\text{gm}}(x)}{f_1(x)}\right) = \frac{.25\mathrm{e}^{.5x}[x^2+2x-2]}{(1+x)^{2.5}},$$

which has a positive root at $x = -1 + \sqrt{3} \approx 0.732$.

4. Countable and continuous α-mixture

The general case of α-mixture is defined by

$$\bar{F}_\alpha(x) = \left[\int_{\mathcal{A}} \bar{F}_\theta^\alpha(x)\,\mathrm{d}G(\theta)\right]^{1/\alpha}, \tag{4.1}$$

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where $\mathcal{A}$ is an index set for F_θ, $\theta \in \mathcal{A}$, $\mathrm{d}G(\theta) \geq 0$, and $\int_{\mathcal{A}} \mathrm{d}G(\theta) = 1$. In this representation, the distribution of X depends on a covariate or a latent parameter θ with an associated probability mass function $\mathrm{d}G(\theta) = p_i = \mathrm{P}(\theta = \theta_i)$ or a PDF $\mathrm{d}G(\theta) = g(\theta)\mathrm{d}\theta$. For example, when X is the lifetime of a product and θ represents the environment in which the product operates, $\mathrm{d}G(\theta)$ depicts an expert opinion or a prior distribution. Theorem 3.1 holds for the general case (4.1); see Appendix A.1.

The following example gives an interesting application of Theorem 3.1 for the case of a countable mixture.

Example 4.1. Assume that $X_1, \dots X_N$ are independent and identically distributed random variables with SF $\bar{F}(x)$, and N is a random variable independent of the X_i. Let N have a truncated Poisson distribution with probability function

$$g(n) = \frac{\mathrm{e}^{-\lambda}\lambda^n}{n!(1 - \mathrm{e}^{-\lambda})}, \qquad n = 1, 2, \dots, \ \lambda > 0.$$

Then, given $N = n$, $Y_n = \min(X_1, \dots, X_n)$ has SF $\bar{F}^n$. The harmonic mixture of the distributions of minima when the mixing distribution is the truncated Poisson given above is

$$\begin{aligned} [\bar{F}_{\mathrm{hm}}(x)]^{-1} &= \sum_{n=1}^{\infty} \frac{g(n)}{\bar{F}^n(x)} \\ &= \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\lambda}\lambda^n}{n!\,\bar{F}^n(x)(1 - \mathrm{e}^{-\lambda})} \\ &= \frac{\mathrm{e}^{-\lambda}}{1 - \mathrm{e}^{-\lambda}} \sum_{n=1}^{\infty} \frac{[\lambda/\bar{F}(x)]^n}{n!} \\ &= \frac{\mathrm{e}^{-\lambda}}{1 - \mathrm{e}^{-\lambda}} (\mathrm{e}^{\lambda/\bar{F}(x)} - 1). \end{aligned}$$

This implies that

$$\bar{F}_{\mathrm{hm}}(x) = \frac{\mathrm{e}^{\lambda} - 1}{\mathrm{e}^{\lambda/\bar{F}(x)} - 1}, \qquad x > 0.$$

According to Theorem 3.1, if $\bar{F}$ is IFR (IFRA) then so is $\bar{F}_{\mathrm{hm}}$.

For a continuous θ, the α-mixture of the SF $\bar{F}_\theta(x)$ is defined by

$$\bar{F}_\alpha(x) = \begin{cases} [\int \bar{F}_\theta^\alpha(x) g(\theta)\,\mathrm{d}\theta]^{1/\alpha} & \alpha \neq 0, \\ \bar{F}_{\mathrm{gm}}(x), & \alpha = 0, \end{cases} \tag{4.2}$$

where

$$\bar{F}_{\mathrm{gm}}(x) = \lim_{\alpha \to 0} \bar{F}_\alpha(x) = \mathrm{e}^{-\mathrm{E}_g[R_\theta(x)]} \tag{4.3}$$

is the geometric mixture of the conditional baseline SF, $R_\theta(x) = -\log \bar{F}_\theta(x)$ is the conditional cumulative hazard function, and E_g denotes the expectation with respect to g, assumed to exist. For $\alpha = 1$, (4.2) gives the continuous mixture of the distribution,

$$\bar{F}(x) = \int \bar{F}_\theta(x) g(\theta)\,\mathrm{d}\theta,$$

and for $\alpha = -1$ it gives the continuous version of the harmonic mixture of SF.

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Example 4.2. When the baseline distribution is proportional hazards, $\bar{F}_\theta(x) = [\bar{F}(x)]^\theta$ with a prior $g(\theta)$, then

$$\begin{aligned}\bar{F}_{\mathrm{gm}}(x) &= \mathrm{e}^{\int \log[\bar{F}(x)]^\theta g(\theta)\,\mathrm{d}\theta} \\ &= \mathrm{e}^{\log \bar{F}(x) \int \theta g(\theta)\,\mathrm{d}\theta} \\ &= [\bar{F}(x)]^{\mathrm{E}(\theta)}.\end{aligned}$$

This shows that when $\bar{F}_\theta(x)$ for each θ is proportional hazards, then the geometric mixture model $\bar{F}_{\mathrm{gm}}$ is the proportional hazards model where the parameter of the model is the expectation of θ.

The case of (4.3) is the SF of the mixture FR model

$$r_{\mathrm{gm}}(x) = \int r_\theta(x) g(\theta)\,\mathrm{d}\theta,$$

where $r_\theta(x)$ is the FR corresponding to $\bar{F}_\theta(x)$. The behavior of the FR of $\bar{F}_{\mathrm{gm}}(x)$ depends on the behavior of the FR of $\bar{F}_\theta(x)$. For example, if $\bar{F}_\theta(x)$ is IFR (DFR) then so is $\bar{F}_{\mathrm{gm}}(x)$. The FR of the continuous α-mixture (4.2) for $\alpha \neq 0$ is as follows:

$$\begin{aligned}r_\alpha(x) &= \frac{f_\alpha(x)}{\bar{F}_\alpha(x)} \\ &= \frac{1}{\alpha}\frac{\int \alpha f_\theta(x)\bar{F}_\theta^{\alpha-1}(x) g(\theta)\,\mathrm{d}\theta}{\int \bar{F}_\theta^\alpha(x) g(\theta)\,\mathrm{d}\theta} \\ &= \int r_\theta(x) g_\alpha(\theta \mid x)\,\mathrm{d}\theta,\end{aligned}$$

where

$$g_\alpha(\theta \mid x) = \frac{g(\theta)\bar{F}^\alpha(x \mid \theta)}{\int \bar{F}^\alpha(x \mid u) g(u)\,\mathrm{d}u}$$

is the conditional density function of θ given that $X_\alpha > x$, in which X_α is a random variable with the proportional hazard SF $\bar{F}^\alpha(x)$.

Example 4.3. Let $\bar{F}_\theta(x) = \mathrm{e}^{-\theta x}$, $x \geq 0$, and consider the gamma prior $G(\lambda, \beta)$. Then

$$\begin{aligned}\bar{F}_\alpha(x) &= \left[\int_0^\infty \mathrm{e}^{-\alpha\theta x}\frac{\lambda^\beta}{\Gamma(\beta)}\theta^{\beta-1}\mathrm{e}^{-\lambda\theta}\,\mathrm{d}\theta\right]^{1/\alpha} \\ &= \left(\frac{\lambda}{\lambda + \alpha x}\right)^{\beta/\alpha}, \qquad x \geq 0,\ \alpha \geq 0 \quad (0 \leq x \leq -\lambda/\alpha,\ \alpha < 0).\end{aligned}$$

This is the SF of the generalized Pareto distribution, which gives the following models:

(a) For $\alpha > 0$, $\bar{F}_\alpha(x)$, $x \geq 0$, is Pareto, which is DFR.

(b) The limiting case of $\alpha \to 0$, $\bar{F}_{\mathrm{gm}}(x)$, $x \geq 0$, is exponential, which is constant FR.

(c) For $\alpha < 0$, $\bar{F}_\alpha(x)$, $0 \leq x \leq -\lambda/\alpha$, is rescaled beta, which is IFR.

5. Conclusion

Mixtures of SFs and FRs provide two alternative models for the lifetime of heterogeneous items. The α-mixture introduced in this paper is a flexible general family that contains both these alternatives as special cases. In addition, with $0 \le \alpha \le 1$, it provides various degrees of compromise between these well-known conventional mixtures, as well as models that are way beyond these two types with $\alpha \in \mathbb{R}$. Other examples of the α-mixture family include the harmonic mixture of distributions with $\alpha = -1$ and a Heronian type mixture of two distributions with $\alpha = 1/2$. The α-mixture of two SFs with $p = \alpha = 1/2$ is stochastic equidistant from the two conventional mixture models, hence $\bar{F}_{1/2}$ provides a benchmark for the compromise between them. As such, the α-mixture provides a flexible tool for modeling the lifetime of heterogeneous items. As the weighted power mean of order α, the family is stochastic ordered by α. Examples of applications of the stochastic order for comparison of systems are presented. The α-mixture uniquely possesses the CES property, which provides an interpretation for α.

Some hazard rate properties of the α-mixture family have been explored. A result provided the following extension of the well-known result of Barlow *et al.* [7] on the closure of the mixture of DFR distributions: the α-mixture family with $\alpha < 0$ ($\alpha > 0$) is IFR (DFR) when all components of the mixture are IFR (DFR). A similar closure property holds in terms of IFRA (DFRA). Another result states that if the FR of a component of the mixture of SFs dominates the FRs of all other components and the FR of another component is dominated by the FRs of all other components, then the FR of the α-mixture is bounded between the dominated and dominator FRs. A further result states that if the FRs of the components of the mixture of SFs are ordered either increasingly or decreasingly, then the FR of the α-mixture family is decreasingly ordered by α. These results were illustrated through examples of mixtures of well-known survival models.

Appendix A. Proofs

A.1. Proof of Theorem 3.1

Following [12] and [6], we provide the proof for the general case (4.1).

Definition A.1. The hazard transform of the α-mixture is defined by [6, 12]

$$\eta_\alpha(\{u_\theta\}) = -\frac{1}{\alpha} \log \int_{\mathcal{A}} \mathrm{e}^{-\alpha u_\theta} \, \mathrm{d}G(\theta), \qquad 0 \le u_\theta \le \infty, \ \alpha \in (-\infty, \infty). \tag{A.1}$$

Using the Hölder inequality, we have the following extension of [6, Theorem 4, p. 162].

Lemma A.1. *The hazard transform $\eta(\{u_\theta\})$ of the mixture is concave for $\alpha > 0$ and convex for $\alpha < 0$. That is, for $\alpha > 0$ ($\alpha < 0$),*

$$\eta_\alpha[\beta\{u_\theta\} + (1-\beta)\{v_\theta\}] \ge (\le) \ \beta\eta_\alpha(\{u_\theta\}) + (1-\beta)\eta_\alpha(\{v_\theta\})$$

for any $0 \le \beta \le 1$, $0 \le u_\theta$, $v_\theta \le \infty$, and $\theta \in \mathcal{A}$.

Following a lemma of [6], if in (A.1) we let $u_\theta = R_\theta(x)$ be the hazard function associated with $\bar{F}_\theta$, then

$$R_\alpha(x) = \eta_\alpha(\{R_\theta(x)\}) \equiv -\frac{1}{\alpha} \log \int_{\mathcal{A}} \mathrm{e}^{-\alpha R_\theta(t)} \, \mathrm{d}G(\theta), \qquad 0 \le x < \infty. \tag{A.2}$$

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Proof. The proof of Theorem 3.1 proceeds as follows.

(a) This part follows from Lemma A.1 and (A.2).

(b) As $\bar{F}_\theta$ is IFRA (DFRA), and η_α is increasing, the hazard function $R_\alpha(x)$ of the α-mixture satisfies

$$R_\alpha(\beta x) = \eta_\alpha(\{R_\theta(\beta x)\}) \leq (\geq)\ \eta_\alpha(\{\beta R_\theta(x)\}), \qquad 0 \leq \beta \leq 1.$$

Now, choosing $\{v_\theta\} = 0$ in the result of Lemma A.1, we obtain

$$\eta_\alpha(\beta\{R_\theta(x)\}) \leq (\geq)\ \beta\eta_\alpha(\{R_\theta(x)\}). \tag{A.3}$$

That is, $R_\alpha(\beta x) \leq (\geq)\ \beta R_\alpha(x)$, and hence $\bar{F}_\alpha$ is IFRA (DFRA). □

A.2. Proof of Theorem 3.2

Proof. The hazard order is equivalently defined by $\frac{\bar{F}_2(x)}{\bar{F}_1(x)}$ and is increasing in $x \in \mathbb{R}$. Denote the SFs corresponding to $r_{\max}(x)$ by $\bar{F}_j$. Then

$$\begin{aligned}\frac{\bar{F}_\alpha(x)}{\bar{F}_j(x)} &= \frac{1}{\bar{F}_j(x)}\left[p_j\bar{F}_j^\alpha(x) + \sum_{i=1, i\neq j}^{n} p_i\bar{F}_i^\alpha(x)\right]^{1/\alpha} \\ &= \left[p_j + \sum_{i=1, i\neq j}^{n} p_i\left(\frac{\bar{F}_i(x)}{\bar{F}_j(x)}\right)^\alpha\right]^{1/\alpha}.\end{aligned}$$

This is increasing for all $\alpha \in \mathbb{R}$, by the assumption. The implication $r_{\min}(x) \leq r_\alpha(x)$ can be established similarly. □

A.3. Proof of Theorem 3.3

Proof. After some algebraic manipulations we find the derivative of (3.1) as follows:

$$\frac{\partial r_\alpha(x)}{\partial\alpha} = \frac{1}{[\bar{F}_\alpha(x)]^{2\alpha}} \sum_{i=1}^{n-1}\sum_{j>i}^{n} p_i p_j \bar{F}_i^\alpha(x)\bar{F}_j^\alpha(x)\left[(r_j(x) - r_i(x))\log\frac{\bar{F}_j(x)}{\bar{F}_i(x)}\right].$$

Let $r_1(x) \leq \cdots \leq r_n(x)$ for all $x > 0$. Then $\bar{F}_1(x) \geq \cdots \geq \bar{F}_n(x)$ for all $x > 0$, and the expression in the square brackets is negative, implying that the derivative is negative. Hence, $r_\alpha(x)$ is decreasing in α. If $r_1(x) \geq \cdots \geq r_n(x)$ for all $x > 0$, then $\bar{F}_1(x) \leq \cdots \leq \bar{F}_n(x)$ for all $x > 0$, implying the same conclusion. □

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