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THE PRIME IDEALS AND SIMPLE MODULES OF THE UNIVERSAL ENVELOPING ALGEBRA $U(\mathfrak{b} \ltimes V_2)$

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Abstract. Let $\mathfrak{b}$ be the Borel subalgebra of the Lie algebra $\mathfrak{sl}_2$ and V_2 be the simple two-dimensional $\mathfrak{sl}_2$-module. For the universal enveloping algebra $\mathcal{A} := U(\mathfrak{b} \ltimes V_2)$ of the semi-direct product $\mathfrak{b} \ltimes V_2$ of Lie algebras, the prime, primitive and maximal spectra are classified. The sets of completely prime ideals of $\mathcal{A}$ are described. The simple unfaithful $\mathcal{A}$-modules are classified and an explicit description of all prime factor algebras of $\mathcal{A}$ is given. The following classes of simple $U(\mathfrak{b} \ltimes V_2)$-modules are classified: the Whittaker modules, the $\mathbb{K}[X]$-torsion modules and the $\mathbb{K}[E]$-torsion modules.

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1. Introduction. In this paper, module means a left module, $\mathbb{K}$ is a field of characteristic zero and $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$.

Recall that the Lie algebra $\mathfrak{sl}_2 = \mathbb{K}F \oplus \mathbb{K}H \oplus \mathbb{K}E$ is a simple Lie algebra over $\mathbb{K}$, where the Lie bracket is given by the rule: $[H, E] = 2E$, $[H, F] = -2F$ and $[E, F] = H$. Let $V_2 = \mathbb{K}X \oplus \mathbb{K}Y$ be the two-dimensional simple $\mathfrak{sl}_2$-module with basis X and Y: $H \cdot X = X$, $H \cdot Y = -Y$, $E \cdot X = 0$, $E \cdot Y = X$, $F \cdot X = Y$ and $F \cdot Y = 0$. Let $\mathfrak{a} := \mathfrak{sl}_2 \ltimes V_2$ be the semi-direct product of Lie algebras where V_2 is viewed as an abelian Lie algebra. In more detail, the Lie algebra $\mathfrak{a}$ admits a basis $\{H, E, F, X, Y\}$ and the Lie bracket is as follows:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H, \quad [E, X] = 0, \quad [E, Y] = X,$$
$$[F, X] = Y, \quad [F, Y] = 0, \quad [H, X] = X, \quad [H, Y] = -Y, \quad [X, Y] = 0.$$

Let $A = U(\mathfrak{a})$ be the enveloping algebra of the Lie algebra $\mathfrak{a}$.

Let $\mathfrak{b} = \mathbb{K}H \oplus \mathbb{K}E$ be the Borel subalgebra of the Lie algebra $\mathfrak{sl}_2$. Then $\mathfrak{b} \ltimes V_2$ is a Lie subalgebra of $\mathfrak{a}$. It admits a basis $\{H, E, X, Y\}$, and the Lie bracket on $\mathfrak{b} \ltimes V_2$ is given as follows:

$$[H, E] = 2E, \quad [H, X] = X, \quad [H, Y] = -Y,$$
$$[E, X] = 0, \quad [E, Y] = X, \quad [X, Y] = 0.$$

The universal enveloping algebra $\mathcal{A} = U(\mathfrak{b} \ltimes V_2)$ of the Lie algebra $\mathfrak{b} \ltimes V_2$ is a subalgebra of A. The Lie algebra $\mathfrak{b} \ltimes V_2$ is known as the *ageing algebra*; see, e.g., [**12**, **17**, **16**]. Recently, the simple weight modules of the algebra $\mathcal{A}$ were classified in [**13**]. The quantum analogue of the algebra $\mathcal{A}$, the so-called *quantum spatial ageing algebra,* was introduced and studied in [**9**]. In [**9**], for the quantum spatial ageing algebra, its prime, primitive and maximal spectra are classified, the automorphism group was determined and the classes of simple unfaithful modules and various torsion simple modules were classified.

The paper has the following structure. In Section 2, an explicit description of the prime spectrum of the algebra $\mathcal{A}$ is given (Theorem 2.5). An explicit description of all the prime factor algebras of $\mathcal{A}$ is given in Theorem 2.5. The inclusions of primes are given in the diagram (2.9). The sets of maximal, completely prime and primitive ideals of $\mathcal{A}$ are explicitly described (Corollaries 2.6, 2.7 and Proposition 2.8, respectively). In Section 3, using the classification of all primitive ideals of the algebra $\mathcal{A}$ (Proposition 2.8) and the explicit description of primitive factor algebras of $\mathcal{A}$ (Theorem 2.5), a classification of simple unfaithful $\mathcal{A}$-modules is given (Proposition 3.3). In Section 4, a classification of simple $\mathbb{K}[X]$-torsion $\mathcal{A}$-modules is given (Corollary 4.5). In Section 5, a classification of simple Whittaker $\mathcal{A}$-module is obtained (see (5.3), Theorems 5.4, 5.7 and Proposition 5.8). In Section 6, we classify simple $\mathbb{K}[E]$-torsion $\mathcal{A}$-modules (see (6.5), Theorems 6.4, 6.6 and Proposition 6.7).

2. The prime ideals of $\mathcal{A}$. The aim of this section is to describe the prime ideals of the enveloping algebra $\mathcal{A}$ (Theorem 2.5). As a result, the sets of maximal, completely prime and primitive ideals are described (Corollaries 2.6, 2.7 and Proposition 2.8). Theorem 2.5 gives an explicit description of all prime factor algebras of $\mathcal{A}$.

The nth *Weyl algebra* $A_n = A_n(\mathbb{K})$ is an associative algebra which is generated by elements $x_1, \dots, x_n, y_1, \dots, y_n$ subject to the defining relations: $[x_i, x_j] = 0$, $[y_i, y_j] = 0$ and $[y_i, x_j] = \delta_{ij}$, where $[a, b] := ab - ba$ and δ_{ij} is the Kronecker delta function. The Weyl algebra A_n is a simple Noetherian domain of Gefand–Kirillov dimension $2n$. For an algebra R, we denote by $Z(R)$ its centre. An element r of a ring R is called a *normal element* if $Rr = rR$.

2.1. The subalgebra $\mathbb{E}$ of $\mathcal{A}$. Let $\mathbb{E}$ be the subalgebra of $\mathcal{A}$ generated by the elements E, X and Y. The generators of the algebra $\mathbb{E}$ satisfy the defining relations

$$EY - YE = X, \quad EX = XE \quad \text{and} \quad YX = XY.$$

Clearly, X is a central element of the algebra $\mathbb{E}$. The algebra $\mathbb{E}$ is isomorphic to the universal enveloping algebra of the three-dimensional Heisenberg Lie algebra. In particular, the algebra $\mathbb{E}$ is a Noetherian domain of Gelfand–Kirillov dimension 3. Let $\mathbb{E}_X$ be the localisation of the algebra $\mathbb{E}$ at the powers of the element X. Then the algebra $\mathbb{E}_X$ is the tensor product of two algebras

$$\mathbb{E}_X = \mathbb{K}[X^{\pm 1}] \otimes A_1^+,$$

where the algebra $A_1^+ := \mathbb{K}\langle EX^{-1}, Y\rangle$ is the (first) Weyl algebra since $[EX^{-1}, Y] = 1$. Since the algebra A_1^+ is a central algebra, we have $Z(\mathbb{E}_X) = \mathbb{K}[X^{\pm 1}]$. Then $Z(\mathbb{E}) = Z(\mathbb{E}_X) \cap \mathbb{E} = \mathbb{K}[X]$.

Lemma 2.1. ([**14**, Lemma 14.6.5]). *Let B be a $\mathbb{K}$-algebra, $S = B \otimes A_n$ be the tensor product of the algebra B and the Weyl algebra A_n, δ be a $\mathbb{K}$-derivation of S and $T =$*

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$S[t;\delta]$. *Then there exists an element* $s \in S$ *such that the algebra* $T = B[t';\delta'] \otimes A_n$ *is a tensor product of algebras, where* $t' = t + s$ *and* $\delta' = \delta + \mathrm{ad}_s$.

2.2. The algebra $\mathcal{A}$. Recall that the algebra $\mathcal{A}$ is the subalgebra of A generated by the elements H, E, X and Y. Then

$$\mathcal{A} = \mathbb{E}[H;\delta] \tag{2.1}$$

is an Ore extension where the $\mathbb{K}$-derivation δ of the algebra $\mathbb{E}$ is given by the rule: $\delta(E) = 2E$, $\delta(X) = X$ and $\delta(Y) = -Y$. Notice that X is a normal element of the algebra $\mathcal{A}$ since X is central in $\mathbb{E}$ and $XH = (H-1)X$. The localisation $\mathcal{A}_X$ of the algebra $\mathcal{A}$ at the powers of the element X is an Ore extension

$$\mathcal{A}_X = \mathbb{E}_X[H;\delta] = (\mathbb{K}[X^{\pm 1}] \otimes A_1^+)[H;\delta], \tag{2.2}$$

where $\delta(E) = 2E$, $\delta(X) = X$ and $\delta(Y) = -Y$. The element $s = EX^{-1}Y \in \mathbb{E}_X$ satisfies the conditions of Lemma 2.1. In more detail, the element $H^+ := H + s = H + EX^{-1}Y$ commutes with the elements of A_1^+ and

$$\mathcal{A}_X = \mathbb{K}[X^{\pm 1}][H^+;\delta'] \otimes A_1^+, \quad \text{where } \delta'(X) = X. \tag{2.3}$$

Notice that the algebra $\mathbb{K}[X^{\pm 1}][H^+;\delta']$ can be presented as a skew Laurent polynomial algebra $\mathbb{K}[H^+][X^{\pm 1};\sigma]$, where $\sigma(H^+) = H^+ - 1$. This is a central simple algebra of Gelfand–Kirillov dimension 2. Let $\partial := H^+X^{-1}$. Then $[\partial, X] = 1$ and so the subalgebra $A_1 = \mathbb{K}\langle \partial, X\rangle$ of $\mathcal{A}_X$ is the (first) Weyl algebra. Moreover, the algebra A_1 is a subalgebra of $\mathbb{K}[X^{\pm 1}][H^+;\delta']$ and the algebra $\mathbb{K}[X^{\pm 1}][H^+;\delta'] = A_{1,X}$ is the localisation of the Weyl algebra A_1 at the powers of the element X. Now,

$$\mathcal{A}_X = A_{1,X} \otimes A_1^+. \tag{2.4}$$

LEMMA 2.2.

1. *The algebra $\mathcal{A}_X$ is a central simple algebra of Gelfand–Kirillov dimension 4.*
2. $Z(\mathcal{A}) = \mathbb{K}$.

Proof.

1. Since both the algebras $\mathbb{K}[X^{\pm 1}][H^+;\delta']$ and A_1^+ are central simple algebras of Gelfand–Kirillov dimension 2, statement 1 then follows from (2.3).
2. Since $\mathbb{K} \subseteq Z(\mathcal{A}) \subseteq Z(\mathcal{A}_X) = \mathbb{K}$, we have $Z(\mathcal{A}) = \mathbb{K}$. □

2.3. The factor algebra $\mathcal{B} := \mathcal{A}/(X)$. We also denote by H, E and Y the images of these elements in the factor algebra $\mathcal{B} := \mathcal{A}/(X)$. Then the algebra $\mathcal{B}$ is generated by the elements H, E and Y that satisfy the defining relations

$$[H, E] = 2E, \quad [H, Y] = -Y, \quad [E, Y] = 0.$$

Hence, the algebra $\mathcal{B}$ is an Ore extension,

$$\mathcal{B} = \mathbb{K}[E, Y][H;\delta], \quad \text{where } \delta(E) = 2E \text{ and } \delta(Y) = -Y. \tag{2.5}$$

It is clear that the element $Z := EY^2$ belongs to the centre of the algebra $\mathcal{B}$. The elements Y and E are normal elements in $\mathcal{B}$. Let $\mathcal{B}_Y$ be the localisation of the algebra $\mathcal{B}$ at the powers of element Y. Then

$$\mathcal{B}_Y = \mathbb{K}[Z] \otimes \mathbb{K}[H][Y^{\pm 1};\sigma] := \mathbb{K}[Z] \otimes \mathbb{Y}, \tag{2.6}$$

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where the skew polynomial algebra $\mathbb{Y} = \mathbb{K}[H][Y^{\pm 1}; \sigma]$ is a central simple algebra, where the $\mathbb{K}$-automorphism σ of $\mathbb{K}[H]$ is defined as follows: $\sigma(H) = H + 1$. Hence, the centre of the algebra $\mathcal{B}_Y$ is $\mathbb{K}[Z]$. The algebras $\mathcal{B}$ and $\mathcal{B}_Y$ are Noetherian domains of Gelfand–Kirillov dimension 3.

LEMMA 2.3. *$Z(\mathcal{B}) = Z(\mathcal{B}_Y) = \mathbb{K}[Z]$, where $Z = EY^2$.*

Proof. Since $\mathbb{K}[Z] \subseteq Z(\mathcal{B}) \subseteq Z(\mathcal{B}_Y) = \mathbb{K}[Z]$, we have $Z(\mathcal{B}) = \mathbb{K}[Z]$. □

2.4. The prime spectrum of the algebra $\mathcal{A}$. For an algebra R, let Spec (R) be the set of its prime ideals. The set (Spec (R), $\subseteq$) is a partially ordered set (poset) with respect to inclusion of prime ideals. Each element $r \in R$ determines two maps from R to R, $r\cdot : x \mapsto rx$ and $\cdot r : x \mapsto xr$, where $x \in R$. For an element $r \in R$, we denote by (r) the ideal of R generated by the element r. If $\{s^i \mid i \in \mathbb{N}\}$ is a left denominator set of an algebra R, we denote by R_s the localisation of R at the powers of s.

PROPOSITION 2.4([9]). *Let R be a Noetherian ring and s be an element of R such that $\mathcal{S}_s := \{s^i \mid i \in \mathbb{N}\}$ is a left denominator set of the ring R and $(s^i) = (s)^i$ for all $i \geqslant 1$ (e.g., if s is a normal element such that $\ker(\cdot s_R) \subseteq \ker(s_R\cdot)$). Then $\mathrm{Spec}\,(R) = \mathrm{Spec}(R, s) \sqcup \mathrm{Spec}_s(R)$, where $\mathrm{Spec}(R, s) := \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid s \in \mathfrak{p}\}$, $\mathrm{Spec}_s(R) = \{\mathfrak{q} \in \mathrm{Spec}\,(R) \mid s \notin \mathfrak{q}\}$ and*

(a) the map $\mathrm{Spec}\,(R, s) \to \mathrm{Spec}\,(R/(s))$, $\mathfrak{p} \mapsto \mathfrak{p}/(s)$, is a bijection with the inverse $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$, where $\pi : R \to R/(s)$, $r \mapsto r + (s)$,
(b) the map $\mathrm{Spec}_s(R) \to \mathrm{Spec}\,(R_s)$, $\mathfrak{p} \mapsto \mathcal{S}_s^{-1}\mathfrak{p}$, is a bijection with the inverse $\mathfrak{q} \mapsto \sigma^{-1}(\mathfrak{q})$, where $\sigma : R \to R_s := \mathcal{S}_s^{-1}R$, $r \mapsto \frac{r}{1}$.
(c) For all $\mathfrak{p} \in \mathrm{Spec}\,(R, s)$ and $\mathfrak{q} \in \mathrm{Spec}_s(R)$, $\mathfrak{p} \not\subseteq \mathfrak{q}$.

Let $U := U(\mathfrak{sl}_2)$ and U^+ be the 'positive part' of U, i.e., U^+ is the subalgebra of U generated by the elements H and E. Then $U^+ = \mathbb{K}[H][E; \sigma]$ is a skew polynomial algebra, where $\sigma(H) = H - 2$. The localised algebra $U_E^+ = \mathbb{K}[H][E^{\pm 1}; \sigma]$ is a central simple domain. The following diagram illustrates the idea of finding the prime spectrum of the algebra $\mathcal{A}$ by repeated application of Proposition 2.4,

$$\begin{array}{ccc}
\mathcal{A} & \longrightarrow & \mathcal{A}_X \\
\downarrow & & \\
\mathcal{B} = \mathcal{A}/(X) & \longrightarrow & (\mathcal{A}/(X))_Y = \mathcal{B}_Y \\
\downarrow & & \\
U^+ = \mathcal{A}/(X, Y) & \longrightarrow & U_E^+ \\
\downarrow & & \\
\mathbb{K}[H] = U^+/(E). & &
\end{array} \tag{2.7}$$

Using (2.7) and Proposition 2.4, we can represent the prime spectrum Spec $(\mathcal{A})$ as the disjoint union of the following subsets:

$$\mathrm{Spec}\,(\mathcal{A}) = \mathrm{Spec}\,(\mathbb{K}[H]) \sqcup \mathrm{Spec}\,(U_E^+) \sqcup \mathrm{Spec}\,(\mathcal{B}_Y) \sqcup \mathrm{Spec}\,(\mathcal{A}_X), \tag{2.8}$$

where we identify the sets of prime ideals in (2.8) via the bijections given in the statements (a) and (b) of Proposition 2.4.

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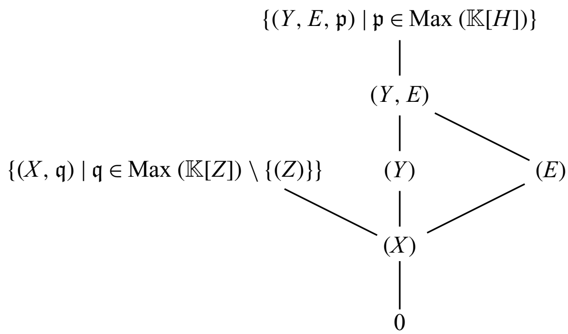
A prime ideal $\mathfrak{p}$ of an algebra R is called a *completely prime ideal* if $R/\mathfrak{p}$ is a domain. We denote by $\mathrm{Spec}_c(R)$ the set of completely prime ideals of R; it is called the *completely prime spectrum* of R. The next theorem gives an explicit description of the poset $(\mathrm{Spec}\,(\mathcal{A}),\ \subseteq)$ and of all the prime factor algebras of $\mathcal{A}$. It also shows that every prime ideal is a completely prime ideal.

THEOREM 2.5. *The prime spectrum* $\mathrm{Spec}\,(\mathcal{A})$ *of the algebra* $\mathcal{A}$ *is the disjoint union of the sets in (2.8). More precisely,*



(2.9)

where

1. $\mathrm{Spec}\,(\mathbb{K}[H]) = \{(Y, E, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\,(\mathbb{K}[H])\} = \{(Y, E)\} \ \sqcup\ \{(Y, E, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Max}\,(\mathbb{K}[H])\}$ *and* $\mathcal{A}/(Y, E, \mathfrak{p}) \simeq \mathbb{K}[H]/\mathfrak{p}$.
2. $\mathrm{Spec}\,(U_E^+) = \{(Y)\}$, $(Y) = (X, Y)$ *and* $\mathcal{A}/(Y) \simeq U^+ = \mathbb{K}[H][E; \sigma]$ *is a skew polynomial algebra which is a domain where* $\sigma(H) = H - 2$.
3. $\mathrm{Spec}\,(\mathcal{B}_Y) = \{(X),\ \ (E),\ \ (X, \mathfrak{q}) \mid \mathfrak{q} \in \mathrm{Max}(\mathbb{K}[Z]) \setminus \{(Z)\}\}$ *and*
   (a) $\mathcal{A}/(X) = \mathcal{B} = \mathbb{K}[E, Y][H; \delta]$ *is an Ore domain (see (2.5)) where* $\delta(E) = 2E$ *and* $\delta(Y) = -Y$,
   (b) $\mathcal{A}/(E) \simeq \mathbb{K}[H][Y; \sigma]$ *is a skew polynomial algebra which is a domain where* $\sigma(H) = H + 1$, *and*
   (c) $\mathcal{A}/(X, \mathfrak{q}) \simeq \mathcal{B}/(\mathfrak{q}) \simeq \mathcal{B}_Y/(\mathfrak{q})_Y \simeq L_{\mathfrak{q}} \otimes \mathbb{Y}$ *is a simple domain which is a tensor product of algebras where* $L_{\mathfrak{q}} := \mathbb{K}[Z]/\mathfrak{q}$ *is a finite field extension of* $\mathbb{K}$.
4. $\mathrm{Spec}\,(\mathcal{A}_X) = \{0\}$.

Proof. Recall that X is a normal element in the algebra $\mathcal{A}$. By Proposition 2.4,

$$\mathrm{Spec}\,(\mathcal{A}) = \mathrm{Spec}\,(\mathcal{A}/(X)) \ \sqcup\ \mathrm{Spec}\,(\mathcal{A}_X). \tag{2.10}$$

(i) *Statement 4 holds*: By Lemma 2.2.(1), the algebra $\mathcal{A}_X$ is a simple algebra. Hence, $\mathrm{Spec}\,(\mathcal{A}_X) = \{0\}$ and we are done.

Recall that Y is a normal element of the algebra $\mathcal{B} = \mathcal{A}/(X)$. By Proposition 2.4,

$$\mathrm{Spec}\,(\mathcal{A}/(X)) = \mathrm{Spec}\,(\mathcal{A}/(X, Y)) \sqcup \mathrm{Spec}\,((\mathcal{A}/(X))_Y) = \mathrm{Spec}\,(U^+) \sqcup \mathrm{Spec}\,(\mathcal{B}_Y). \tag{2.11}$$

(ii) $(Y) = (X, Y)$ *and* $(E) = (X, E)$: Both equalities follow from the relation $X = [E, Y]$.

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(iii) *Statements 1 and 2 hold*: The element E is a normal element of the algebra U^+. By Proposition 2.4,

$$\mathrm{Spec}\,(U^+) = \mathrm{Spec}\,(U^+/(E)) \ \sqcup \ \mathrm{Spec}\,(U^+_E). \tag{2.12}$$

Since $\mathbb{K}[H] = U^+/(E)$, statement 1 follows. Now, (2.8) holds by (2.10), (2.11) and (2.12). The algebra $U^+_E \simeq \mathbb{K}[H][E^{\pm 1}; \sigma]$ is a central simple domain, where $\sigma(H) = H - 2$. By the statement (ii), $\mathcal{A}/(Y) = \mathcal{A}/(X, Y) = U^+$ is a domain. The set $\mathrm{Spec}\,(U^+_E)$, as a subset of $\mathrm{Spec}\,(\mathcal{A})$, consists of the single ideal (Y), and statement 2 follows.

(iv) *Statement 3 holds*: By (2.6), $\mathcal{B}_Y = \mathbb{K}[Z] \otimes \mathbb{Y}$, where $\mathbb{Y}$ is a central simple algebra. Hence, $\mathrm{Spec}\,(\mathcal{B}_Y) = \mathrm{Spec}\,(\mathbb{K}[Z])$, and the set $\mathrm{Spec}\,(\mathcal{B}_Y)$, as a subset of $\mathrm{Spec}\,(\mathcal{A})$, is equal to $\{\mathcal{A} \cap (X)_Y,\ \mathcal{A} \cap (X, Z)_Y,\ \mathcal{A} \cap (X, \mathfrak{q})_Y \mid \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[Z]) \setminus \{(Z)\}\}$. We have to show that $\mathcal{A} \cap (X)_Y = (X)$, $\mathcal{A} \cap (X, Z)_Y = (E)$ and $\mathcal{A} \cap (X, \mathfrak{q})_Y = (X, \mathfrak{q})$.

$\mathcal{A} \cap (X)_Y = (X)$: Let $u \in \mathcal{A} \cap (X)_Y$, then $Y^i u \in (X)$ for some $i \in \mathbb{N}$. Since $\mathcal{A}/(X) = \mathcal{B}$ is domain and $Y \notin (X)$, we must have $u \in (X)$. Hence, $\mathcal{A} \cap (X)_Y = (X)$.

$\mathcal{A} \cap (X, Z)_Y = (E)$: By the statement (ii), $(E) = (X, E)$. So, $(E)_Y = (X, E)_Y = (X, Z)_Y$. Let $u \in \mathcal{A} \cap (X, Z)_Y = \mathcal{A} \cap (E)_Y$, then $Y^i u \in (E)$ for some $i \in \mathbb{N}$. Since $\mathcal{A}/(E) = \mathcal{A}/(X, E) \simeq \mathbb{K}[H][Y; \sigma]$ is a domain where $\sigma(H) = H + 1$ and $Y \notin (E)$, we have $u \in (E)$. Therefore, $\mathcal{A} \cap (X, Z)_Y = (E)$. So, statement (b) holds and (E) is a completely prime ideal of the algebra $\mathcal{A}$.

$\mathcal{A} \cap (X, \mathfrak{q})_Y = (X, \mathfrak{q})$ *for* $\mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[Z]) \setminus \{(Z)\}$: Let us first show that the statement (c) holds. It is clear that $\mathcal{A}/(X, \mathfrak{q}) \simeq \mathcal{B}/(\mathfrak{q})$. Since $\mathfrak{q} \neq (Z)$, the non-zero element $Z = EY^2$ of $L_{\mathfrak{q}}$ is invertible in the field $L_{\mathfrak{q}}$. Hence, the element Y is invertible in the algebra $\mathcal{B}/(\mathfrak{q})$. Now, $\mathcal{B}/(\mathfrak{q}) \simeq \mathcal{B}_Y/(\mathfrak{q})_Y \simeq L_{\mathfrak{q}} \otimes \mathbb{Y}$, see (2.6). This proves the statement (c). Since $\mathcal{A}/(X, \mathfrak{q})$ is a simple algebra (by the statement (c)), the ideal $(X, \mathfrak{q})$ of $\mathcal{A}$ is a maximal ideal and $(X, \mathfrak{q}) \subseteq \mathcal{A} \cap (X, \mathfrak{q})_Y \subsetneqq \mathcal{A}$, we must have $\mathcal{A} \cap (X, \mathfrak{q})_Y = (X, \mathfrak{q})$.

(v) Clearly, we have the inclusions as in the diagram (2.9) (see the statement (ii)). It remains to show that there are no other inclusions. Recall that $Z = EY^2$. Hence, $(Z) \subseteq (E)$ and $(Z) \subseteq (Y)$. The ideals $\{(X, \mathfrak{q}) \mid \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[Z]) \setminus \{(Z)\}\}$ are maximal in $\mathcal{A}$ and $(\mathfrak{q}) + (Z) = (1)$. Therefore, none of the maximal ideals $(X, \mathfrak{q})$ contains (Y) or (E). Therefore, picture (2.9) represents the poset $(\mathrm{Spec}\,(\mathcal{A}), \subseteq)$. □

For an algebra R, let $\mathrm{Max}(R)$ be the set of its maximal ideals. The next corollary is an explicit description of the set $\mathrm{Max}(\mathcal{A})$.

Corollary 2.6. *$\mathrm{Max}\,(\mathcal{A}) = \mathcal{P} \ \sqcup \ \mathcal{Q}$, where $\mathcal{P} := \Big\{(Y, E, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Max}\,(\mathbb{K}[H])\Big\}$ and $\mathcal{Q} := \Big\{(X, \mathfrak{q}) \mid \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[Z]) \setminus \{(Z)\}\Big\}$.*

Proof. The corollary follows from (2.9). □

Corollary 2.7. *Every prime ideal of the algebra $\mathcal{A}$ is completely prime, i.e., $\mathrm{Spec}_c(\mathcal{A}) = \mathrm{Spec}\,(\mathcal{A})$.*

Proof. The corollary follows from Theorem 2.5. □

Corollary 2.7 can also be obtained directly from [**15**, Corollary 2.6].

Let R be an algebra and M be an R-module. For $a \in R$, let $a_M \cdot : M \to M,\ \ m \mapsto am$. The ideal of R, $\mathrm{ann}_R(M) := \{a \in R \mid aM = 0\}$, is called the *annihilator* of the R-module M. An R-module is called *faithful* if it has zero annihilator. The annihilator of each simple R-module is a prime ideal. Such prime ideals are called *primitive* and the set $\mathrm{Prim}\,(R)$ of

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all primitive ideals is called the *primitive spectrum* of R. The next proposition gives an explicit description of the set Prim $(\mathcal{A})$.

PROPOSITION 2.8. $\mathrm{Prim}(\mathcal{A}) = \mathrm{Max}(\mathcal{A}) \sqcup \{(Y),\ (E),\ 0\}$.

Proof. Clearly, Prim $(\mathcal{A}) \supseteq$ Max $(\mathcal{A})$. The ideals (X) and (Y, E) are not primitive ideals as the corresponding factor algebras contain the central elements Z and H, respectively.

(i) $(Y) \in$ Prim $(\mathcal{A})$: For $\lambda \in \mathbb{K}^*$, let $I(\lambda) = (Y) + \mathcal{A}(E - \lambda)$. Since $\mathcal{A}/(Y) \simeq U^+$ (see Theorem 2.5.(2)), the left $\mathcal{A}$-module $M(\lambda) := \mathcal{A}/I(\lambda) \simeq U^+/U^+(E - \lambda) \simeq \mathbb{K}[H]\bar{1}$ is a simple $\mathcal{A}$-module/U^+-module, where $\bar{1} = 1 + I(\lambda)$. By the definition of the module $M(\lambda)$, its annihilator $\mathfrak{p} := \mathrm{ann}_{\mathcal{A}}(M(\lambda))$ contains the ideal (Y) but does not contain the ideal (Y, E), since otherwise we would have $0 = E\bar{1} = \lambda\bar{1} \neq 0$, a contradiction. By (2.9), we have $\mathfrak{p} = (Y)$.

(ii) $(E) \in$ Prim $(\mathcal{A})$: For $\lambda \in \mathbb{K}^*$, let $J_\lambda = (E) + \mathcal{A}(Y - \lambda)$. Since $\mathcal{A}/(E) \simeq \mathbb{K}[H][Y; \sigma]$, where $\sigma(H) = H + 1$ (see Theorem 2.5.(3b)), the left $\mathcal{A}$-module $T(\lambda) := \mathcal{A}/J_\lambda \simeq \mathbb{K}[H]\bar{1}$ is a simple module, where $\bar{1} = 1 + J_\lambda$. Clearly, the prime ideal $\mathfrak{q} := \mathrm{ann}_{\mathcal{A}}(T(\lambda))$ contains the ideal (E) but does not contain the ideal (Y, E) since otherwise we would have $0 = Y\bar{1} = \lambda\bar{1} \neq 0$, a contradiction. By (2.9), we have $\mathfrak{q} = (E)$.

(iii) It follows from Theorem 4.4 that 0 is a primitive ideal of $\mathcal{A}$. □

The fact that (Y) and (E) are primitive ideals of $\mathcal{A}$ also follows from [**10**, Proposition 5.2].

The next lemma is a faithfulness criterion for simple $\mathcal{A}$-modules.

LEMMA 2.9. *Let M be a simple $\mathcal{A}$-module. Then M is a faithful $\mathcal{A}$-module iff* $\ker(X_M\cdot) = 0$.

Proof. The $\mathcal{A}$-module M is simple, so $\mathrm{ann}_{\mathcal{A}}(M) \in$ Prim $(\mathcal{A})$. Recall that the element X is a normal element of the algebra $\mathcal{A}$. So, $\ker(X_M\cdot)$ is a submodule of M. Then either $\ker(X_M\cdot) = 0$ or $\ker(X_M\cdot) = M$, and in the second case $\mathrm{ann}_{\mathcal{A}}(M) \supseteq (X)$. If $\ker(X_M\cdot) = 0$, then $\mathrm{ann}_{\mathcal{A}}(M) = 0$ since otherwise, by (2.9), $(X) \subseteq \mathrm{ann}_{\mathcal{A}}(M)$, a contradiction. □

3. A classification of simple unfaithful $\mathcal{A}$-modules.

In this section, all simple unfaithful $\mathcal{A}$-modules are classified (Proposition 3.3). The problem of classification is reduced to the one for algebras for which the simple modules are classified but we have to select faithful simple modules. The set $\widehat{\mathcal{A}}$ (unfaithful) of isomorphic classes of unfaithful simple $\mathcal{A}$-modules can be partitioned according to their annihilators

$$\widehat{\mathcal{A}}\,(\text{unfaithful}) = \bigsqcup_{P \in \mathrm{Prim}(\mathcal{A}) \setminus \{0\}} \widehat{\mathcal{A}}\,(P), \tag{3.1}$$

where $\widehat{\mathcal{A}}\,(P) = \{[M] \in \widehat{\mathcal{A}} \mid \mathrm{ann}_{\mathcal{A}}(M) = P\}$. By Theorem 2.5 and Proposition 2.8, there are four types of non-zero primitive ideals. We consider all four cases separately.

$\widehat{\mathcal{A}}\,(P)$ where $P = (Y, E, \mathfrak{p})$ and $\mathfrak{p} \in \mathrm{Max}(\mathbb{K}[H])$. In this case, the factor algebra $\mathcal{A}/(Y, E, \mathfrak{p})$ is isomorphic to $\mathbb{K}[H]/\mathfrak{p}$ (Theorem 2.5.(1)) and the next Lemma is obvious.

LEMMA 3.1. *For all* $\mathfrak{p} \in \mathrm{Max}(\mathbb{K}[H])$, $\widehat{\mathcal{A}/(Y, E, \mathfrak{p})} = \widehat{\mathbb{K}[H]/\mathfrak{p}} = \{\mathbb{K}[H]/\mathfrak{p}\}$.

$\widehat{\mathcal{A}}\,(P)$ where $P = (X, \mathfrak{q})$ and $\mathfrak{q} \in \mathbb{K}[Z] \setminus \{(Z)\}$. By Theorem 2.5.(3c), the algebra $\mathcal{A}/(X, \mathfrak{q})$ is isomorphic to the skew Laurent polynomial algebra $L_{\mathfrak{q}}[H][Y^{\pm 1}; \sigma]$, where σ is an $L_{\mathfrak{q}}$-automorphism of the polynomial algebra $L_{\mathfrak{q}}[H]$ over the field $L_{\mathfrak{q}}$ given by the

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rule $\sigma(H) = H + 1$. The algebra $L_{\mathfrak{q}}[H]$ is a commutative Dedekind domain. The algebra $L_{\mathfrak{q}}[H][Y^{\pm 1}; \sigma]$ is a particular case of the ring R (see in the following) for which all simple modules are classified.

3.1. Classification of simple R-modules. Let us recall the classification of simple R-modules where $R = D[t, t^{-1}; \sigma]$; here D is a commutative Dedekind domain and σ satisfies the condition (I) – see in the following. These results are very particular cases of classification of simple modules over a generalized Weyl algebra $D(\sigma, a)$ obtained in [1, 2, 3, 5, 6, 7, 8].

Let $\mathcal{S} = D \setminus \{0\}$ and $k = \mathcal{S}^{-1}D$ be the field of fractions of the ring D. The ring R is a subring of $B := k[t, t^{-1}; \sigma]$. The ring $B = \mathcal{S}^{-1}R$ is a (left and right) localisation of R at $\mathcal{S}$. The ring B is a Euclidean ring and so is a principle left and right ideal domain. Every simple B-module is isomorphic to B/Bb for some *irreducible* element $b \in B$ (i.e., $b = ac$ implies either a or c is a unit in B). Two simple B-modules B/Ba and B/Bb are isomorphic iff the irreducible elements a and b are *similar*; that is, there exists an element $c \in B$ such that 1 is the greatest common right divisor of b and c, and ac is the least common left multiple of b and c.

Let $G := \langle \sigma \rangle$ be the subgroup of $\mathrm{Aut}(D)$ generated by σ. The group G acts on the set $\mathrm{Max}(D)$ of maximal ideals of D. For each $\mathfrak{p} \in \mathrm{Max}(D)$, $\mathcal{O}(\mathfrak{p}) := \{\sigma^i(\mathfrak{p}) \mid i \in \mathbb{Z}\}$ is the orbit of $\mathfrak{p}$. The set of all G-orbits in $\mathrm{Max}(D)$ is denoted by $\mathrm{Max}(D)/G$. The orbit $\mathcal{O}(\mathfrak{p})$ is called an *infinite* or *linear* orbit if $|\mathcal{O}(\mathfrak{p})| = \infty$; otherwise the orbit $\mathcal{O}(\mathfrak{p})$ is called a *finite* or *cyclic* orbit. If $\mathcal{O}(\mathfrak{p})$ is an infinite orbit, then the map $\mathbb{Z} \to \mathcal{O}(\mathfrak{p})$, $i \mapsto \sigma^i(\mathfrak{p})$, is a bijection, and we write $\sigma^i(\mathfrak{p}) \leqslant \sigma^j(\mathfrak{p})$ if $i \leqslant j$. So, the total ordering of $\mathbb{Z}$ is passed to $\mathcal{O}(\mathfrak{p})$. This ordering does not depend on the choice of the ideal $\mathfrak{p}$ in the orbit $\mathcal{O}(\mathfrak{p})$.

Given elements $\alpha, \beta \in D \setminus \{0\}$, we write $\alpha < \beta$ if there are no maximal ideas $\mathfrak{p}$ and $\mathfrak{q}$ that belong to the same *infinite* orbit and such that $\alpha \in \mathfrak{p}$, $\beta \in \mathfrak{q}$ and $\mathfrak{p} \geqslant \mathfrak{q}$. In particular, if $\alpha \in D^*$ is a unit of D then $\alpha < \beta$ for all $\beta \in D \setminus \{0\}$. The relation $<$ is not a partial order on $D \setminus \{0\}$ as $1 < 1$. Clearly, $\alpha < \beta$ iff $\sigma^j(\alpha) < \sigma^j(\beta)$ for some/all $j \in \mathbb{Z}$.

DEFINITION: An element $b = t^m \beta_m + t^{m+1} \beta_{m+1} + \cdots + t^n \beta_n \in R$, where $\beta_i \in D$, $m < n$ and $\beta_m, \beta_n \neq 0$, is called an *l-normal* element if $\beta_n < \beta_m$.

DEFINITION: We say that the automorphism σ of D is of (I)-*type*, if all G-orbits in $\mathrm{Max}(D)$ are infinite.

Let $\widehat{R}$ be the set of isomorphism classes $[M]$ of simple R-modules M. Then

$$\widehat{R} = \widehat{R}(D\text{-torsion}) \sqcup \widehat{R}(D\text{-torsionfree})$$

is a disjoint union, where $\widehat{R}(D\text{-torsion}) := \{[M] \in \widehat{R} \mid \mathcal{S}^{-1}M = 0\}$ and $\widehat{R}(D\text{-torsionfree}) := \{[M] \in \widehat{R} \mid \mathcal{S}^{-1}M \neq 0\}$. An R-module M is called a *weight* R-module if M is a semi-simple D-module, i.e.,

$$M = \bigoplus_{\mathfrak{p} \in \mathrm{Max}(D)} M_{\mathfrak{p}},$$

where $M_{\mathfrak{p}} = \{v \in M \mid \mathfrak{p}v = 0\}$ is called the *component* of M of *weight* $\mathfrak{p}$. The set $\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Max}(D) \mid M_{\mathfrak{p}} \neq 0\}$ is called the *support* of the weight $\mathcal{A}$-module M. It is also denoted $\mathrm{Supp}_D(M)$ and is called D-*support* if we want to stress over which ring D we consider weight modules. Every simple weight R-module is a simple D-torsion R-module, and vice versa. We denote by $\widehat{R}(D\text{-torsion, infinite})$ and $\widehat{R}(D\text{-torsion, finite})$ the sets of isomorphism classes of simple, weight R-modules with infinite and finite support, respectively.

The next theorem classifies the simple R-modules.

THEOREM 3.2. *Suppose that σ is of (I)-type. Then*

1. *[4] the map* $\mathrm{Max}(D)/G \longrightarrow \widehat{R}(D\text{-torsion})$, $\mathscr{O}(\mathfrak{p}) \mapsto R/R\mathfrak{p}$, *is a bijection with the inverse* $M \mapsto \mathrm{Supp}(M)$.
2. *[4]* $\widehat{R}(D\text{-torsionfree}) = \big\{[R/R\cap Bb] \mid b \in R \text{ is an } l\text{-normal, irreducible element of } B\big\}$. *The simple R-modules $R/R\cap Bb$ and $R/R\cap Bb'$ are isomorphic iff the B-modules B/Bb and B/Bb' are isomorphic.*

The next proposition describes the sets $\widehat{\mathcal{A}}(P)$ for all non-zero primitive ideals of $\mathcal{A}$, hence gives a classification of simple unfaithful $\mathcal{A}$-modules.

PROPOSITION 3.3.

1. *For all* $\mathfrak{p} \in \mathrm{Max}\,(\mathbb{K}[H])$, $\widehat{\mathcal{A}}\,((Y, E, \mathfrak{p})) = \widehat{\mathcal{A}/(Y, E, \mathfrak{p})} = \widehat{\mathbb{K}[H]/\mathfrak{p}} = \{\mathbb{K}[H]/\mathfrak{p}\}$.
2. *For all* $\mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[Z]) \setminus \{(Z)\}$, $\widehat{\mathcal{A}}\,((X, \mathfrak{q})) = \widehat{\mathcal{A}/(X, \mathfrak{q})} = \widehat{L_{\mathfrak{q}} \otimes \mathbb{Y}}$.
3. $\widehat{\mathcal{A}}\,((Y)) = \widehat{\mathcal{A}/(Y)}\ (\text{faithful}) = \widehat{U^+}\ (\text{faithful}) = \widehat{U^+} \setminus \Big(\bigsqcup_{\mathfrak{p}\in \mathrm{Max}\,(\mathbb{K}[H])} \widehat{\mathcal{A}}\big((Y, E, \mathfrak{p})\big)\Big)$.
4. $\widehat{\mathcal{A}}\,((E)) = \widehat{\mathcal{A}/(E)}\ (\text{faithful}) = \widehat{\mathbb{K}[H][Y;\sigma]}\ (\text{faithful}) = \widehat{\mathbb{K}[H][Y;\sigma]} \setminus \Big(\bigsqcup_{\mathfrak{p}\in \mathrm{Max}\,(\mathbb{K}[H])} \widehat{\mathcal{A}}\big((Y, E, \mathfrak{p})\big)\Big)$.

Proof. All the statements are obvious. The algebras U^+ and $\mathbb{K}[H][Y;\sigma]$ (where $\sigma(H) = H + 1$) are isomorphic to the enveloping algebra of the Borel subalgebra of $\mathfrak{sl}_2$, the simple modules of which were classified by Block [11]; see also [7]. The algebra $\mathbb{Y} = \mathbb{K}[H][Y^{\pm 1};\sigma]$ is a skew polynomial algebra with the coefficients from a Dedekind domain; the classification of simple $\mathbb{Y}$-modules can be found in [7]. □

4. A classification of simple $\mathbb{K}[X]$-torsion $\mathcal{A}$-modules. In this section, $\mathbb{K}$ is an algebraically closed field of characteristic zero. The aim of this section is to classify all the simple $\mathbb{K}[X]$-torsion $\mathcal{A}$-modules (Corollary 4.5).

The set $S = \mathbb{K}[X] \setminus \{0\}$ is a left and right Ore set of the domain $\mathcal{A}$. An $\mathcal{A}$-module M is called an *S-torsion* module or a *$\mathbb{K}[X]$-torsion* module if $S^{-1}M = 0$. Then

$$\widehat{\mathcal{A}}\,(\mathbb{K}[X]\text{-torsion}) = \widehat{\mathcal{A}}\,(\mathbb{K}[X]\text{-torsion, faithful}) \sqcup \widehat{\mathcal{A}}\,(\mathbb{K}[X]\text{-torsion, unfaithful}). \quad (4.1)$$

For each $\lambda \in \mathbb{K}$, consider the $\mathcal{A}$-module

$$V(\lambda) := \mathcal{A}/\mathcal{A}(X-\lambda) = \bigoplus_{i,j,k\geqslant 0} \mathbb{K}H^iE^jY^k\bar{1} \quad \text{where } \bar{1} := 1 + \mathcal{A}(X-\lambda). \quad (4.2)$$

The Gelfand–Kirillov dimension of the $\mathcal{A}$-module $V(\lambda)$ is 3. Since the field $\mathbb{K}$ is an algebraically closed field, each simple $\mathbb{K}[X]$-torsion $\mathcal{A}$-module is an epimorphic image of the $\mathcal{A}$-module $V(\lambda)$ for some $\lambda \in \mathbb{K}$.

LEMMA 4.1.

1. *If $\lambda \neq 0$ then the map $X\cdot : V(\lambda) \longrightarrow V(\lambda)$, $v \mapsto Xv$, is a bijection.*
2. *If $\lambda = 0$ then $V(0) = \mathcal{A}/(X)$ and* $\mathrm{ann}_{\mathcal{A}}(V(0)) = (X)$.

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Proof.

1. The $\mathcal{A}$-module $V(\lambda)$ admits two sets of bases $\{H^iE^jY^k\bar{1} \mid i, j, k \in \mathbb{N}\}$ and $\{(H-1)^iE^jY^k\bar{1} \mid i, j, k \in \mathbb{N}\}$. Now, the lemma follows from the equalities $X \cdot H^iE^jY^k\bar{1} = \lambda(H-1)^iE^jY^k\bar{1}$.
2. The element X is a normal element of $\mathcal{A}$, and so statement 2 is obvious. □

For each $\lambda \in \mathbb{K}^*$, the $A_{1,X}$-module

$$A_{1,X}(\lambda) := A_{1,X}/A_{1,X}(X-\lambda) = \bigoplus_{i\geqslant 0} \mathbb{K}\partial^i\bar{1} = \bigoplus_{i\geqslant 0} \mathbb{K}(H^+)^i\bar{1}$$

is a simple $A_{1,X}$-module. By (2.4), the algebra $\mathcal{A}_X = A_{1,X} \otimes A_1^+$ is a tensor product of algebras. So, the $\mathcal{A}_X$-module

$$V(\lambda) = A_{1,X}(\lambda) \otimes A_1^+ \tag{4.3}$$

is the tensor product of $A_{1,X}$-module $A_{1,X}(\lambda)$ and the A_1^+-module A_1^+.

Let R be an algebra and S be a non-empty subset of R. The algebra $C_R(S) = \{r \in R \mid rs = sr \text{ for all } s \in S\}$ is called the *centraliser* of S in R. The next lemma describes the centraliser of the element X in $\mathcal{A}$.

Lemma 4.2. $C_{\mathcal{A}}(X) = \mathbb{E}$.

Proof. Clearly, $\mathbb{E} \subseteq C_{\mathcal{A}}(X)$ and $XH^i = (H-1)^iX$ for all $i \geqslant 0$. So, the result follows from the equality $\mathcal{A} = \mathbb{E}[H; \delta]$; see (2.1). □

The element X is a central element of the algebra $C_{\mathcal{A}}(X) = \mathbb{E}$ (Lemma 4.2). For $\lambda \in \mathbb{K}^*$, the factor algebra $\mathbb{E}(\lambda) := \mathbb{E}/\mathbb{E}(X-\lambda)$ is isomorphic to the Weyl algebra since $[E, Y] \equiv \lambda \mod \mathbb{E}(X-\lambda)$. Using the equality $\mathcal{A} = \mathbb{E}[H; \delta]$ (see (2.1)), the $\mathcal{A}$-module $V(\lambda)$ (where $\lambda \neq 0$) can be written as

$$V(\lambda) = \mathbb{K}[H] \otimes \mathbb{E}(\lambda) = \bigoplus_{i\geqslant 0} H^i \otimes \mathbb{E}(\lambda), \tag{4.4}$$

where $\mathbb{K}[H] \otimes \mathbb{E}(\lambda)$ is a tensor product of vector spaces. The next proposition gives an explicit description of all submodules of the $\mathcal{A}$-module $V(\lambda)$.

Proposition 4.3. *Let* $\lambda \in \mathbb{K}^*$.

1. *The set* $\{\mathbb{K}[H] \otimes I \mid I$ *is a left ideal of the Weyl algebra* $\mathbb{E}(\lambda)\}$ *is the set of all distinct submodules of the* $\mathcal{A}$*-module* $V(\lambda)$.
2. *The set* $\{\mathbb{K}[H] \otimes I \mid I$ *is a maximal left ideal of* $\mathbb{E}(\lambda)\}$ *is the set of all distinct maximal submodules of the* $\mathcal{A}$*-module* $V(\lambda)$.

Proof.

1. Let M be a submodule of the $\mathcal{A}$-module $V(\lambda) = \bigoplus_{i\geqslant 0} H^i \otimes \mathbb{E}(\lambda)$; see (4.4). We have to show that $M = \mathbb{K}[H] \otimes I$ for some left ideal I of the algebra $\mathbb{E}(\lambda)$. The $\mathcal{A}$-module $V(\lambda) = \bigcup_{i\geqslant 0} V(\lambda)_{\leqslant i}$ is the union of the vector spaces $V(\lambda)_{\leqslant i} = \{\sum_{j=0}^{i} H^j \otimes r_j \mid r_j \in \mathbb{E}(\lambda)\}$. Then $M \cap V(\lambda)_{\leqslant 0} = M \cap \mathbb{E}(\lambda) = I$, where I is a left ideal of $\mathbb{E}(\lambda)$. We claim that $M = \mathbb{K}[H] \otimes I$. We have to show that, for all $i \geqslant 0$,

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$M\cap V(\lambda)_{\leqslant i}=M_i:=\bigoplus_{j=0}^{i}H^j\otimes I$. To prove this, we use induction on i. The initial case when $i=0$ is trivial. So, let $i>0$, and we assume that the result holds for all $i'<i$. Clearly, $(X-\lambda)V(\lambda)_{\leqslant i}\subseteq V(\lambda)_{\leqslant i-1}$ for all $i\geqslant 0$ (where $V(\lambda)_{\leqslant -1}:=0$) since, for all $r\in\mathbb{E}(\lambda)$, $(X-\lambda)H^i\otimes r\bar{1}=[X,H^i]\otimes r\bar{1}=\lambda((H-1)^i-H^i)\otimes r\bar{1}=\lambda(-iH^{i-1}+\cdots)\otimes r\bar{1}$, where the three dots mean a polynomial of degree $<i-1$. Moreover,

$$X-\lambda:\frac{V(\lambda)_{\leqslant i}}{V(\lambda)_{\leqslant i-1}}\to\frac{V(\lambda)_{\leqslant i-1}}{V(\lambda)_{\leqslant i-2}},\ H^i\otimes r\bar{1}+V(\lambda)_{\leqslant i-1}\mapsto -\lambda iH^{i-1}\otimes r\bar{1}+V(\lambda)_{\leqslant i-2}. \tag{4.5}$$

It follows from (4.5) and induction on i that if $w=\sum_{j=0}^{i}H^j\otimes r_j\bar{1}\in M\cap V(\lambda)_{\leqslant i}$, then $-\lambda ir_i\in I$, i.e., $r_i\in I$. Hence, $H^i\otimes r_i\in M_i$, and so $\sum_{j=0}^{i-1}H^j\otimes r_j=w-H^i\otimes r_i\in M\cap V(\lambda)_{\leqslant i-1}=M_{i-1}$, by induction. Therefore, $w\in M_i$, i.e., $M\cap V(\lambda)_{\leqslant i}=M_i$, as required.

2. Statement 2 follows from statement 1. □

Let V be a vector space. A linear map $f:V\to V$ is called a *locally nilpotent* linear map if $V=\bigcup_{n\geqslant 1}\ker(f^n)$. We also say that the map f acts *locally nilpotently* on V. The next theorem gives a classification of simple faithful $\mathbb{K}[X]$-torsion $\mathcal{A}$-modules.

THEOREM 4.4. *$\widehat{\mathcal{A}}(\mathbb{K}[X]\text{-torsion, faithful})=\{\mathbb{K}[H]\otimes M\mid [M]\in\widehat{\mathbb{E}(\lambda)},\ \lambda\in\mathbb{K}^*\}$ and the simple $\mathcal{A}$-modules $\mathbb{K}[H]\otimes M$ and $\mathbb{K}[H]\otimes M'$ (where $[M]\in\widehat{\mathbb{E}(\lambda)}$ and $[M']\in\widehat{\mathbb{E}(\lambda')}$) are isomorphic iff $\lambda=\lambda'$ and $M\simeq M'$ as $\mathbb{E}(\lambda)$-modules.*

Proof. Since X is a normal element of the algebra $\mathcal{A}$, all simple factor modules of the $\mathcal{A}$-module $V(0)=\mathcal{A}/\mathcal{A}X=\mathcal{A}/(X)$ are annihilated by (X) and, therefore, are unfaithful. Let $\mathbb{S}=\{\mathbb{K}[H]\otimes M\mid [M]\in\widehat{\mathbb{E}(\lambda)},\ \lambda\in\mathbb{K}^*\}$. By Proposition 4.3.(2), $\widehat{\mathcal{A}}(\mathbb{K}[X]\text{-torsion, faithful})\subseteq\mathbb{S}$. Conversely, let $\mathcal{M}=\mathbb{K}[H]\otimes M\in\mathbb{S}$, where $[M]\in\widehat{\mathbb{E}(\lambda)}$ for some $\lambda\neq 0$. Clearly, the $\mathcal{A}$-module $\mathcal{M}$ has the form $\mathbb{K}[H]\otimes\frac{\mathbb{E}(\lambda)}{I}$, where I is a maximal left ideal of the Weyl algebra $\mathbb{E}(\lambda)$. By Proposition 4.3.(2), the $\mathcal{A}$-module $\mathcal{M}$ is simple since $\mathcal{M}\simeq\frac{\mathbb{K}[H]\otimes\mathbb{E}(\lambda)}{\mathbb{K}[H]\otimes I}$. By (2.9), the $\mathcal{A}$-module $\mathcal{M}$ is faithful since $X\mathcal{M}=\lambda\mathcal{M}=\mathcal{M}\neq 0$. Therefore, $\widehat{\mathcal{A}}(\mathbb{K}[X]\text{-torsion, faithful})=\mathbb{S}$.

The element $(X-\lambda)$ acts locally nilpotently on the $\mathcal{A}$-module $\mathcal{M}=\mathbb{K}[H]\otimes M$, where $M\in\widehat{\mathbb{E}(\lambda)}$ since $(X-\lambda)H^i\otimes m=\lambda((H-1)^i-H^i)\otimes m$ for $i\geqslant 1$ on $m\in M$. But then, for all $\mu\neq\lambda$, the element $X-\mu$ acts bijectively on $\mathcal{M}$. So, the scalar λ is a unique scalar for the $\mathcal{A}$-module $\mathcal{M}$ such that the element $(X-\lambda)$ acts locally nilpotently. Let $M\in\mathbb{E}(\lambda)$ and $[M']\in\mathbb{E}(\lambda')$, where $\lambda,\lambda'\in\mathbb{K}^*$. The simple $\mathcal{A}$-modules $\mathbb{K}[H]\otimes M$ and $\mathbb{K}[H]\otimes M'$ are isomorphic iff $\lambda=\lambda'$ and the $\mathbb{E}(\lambda)$-modules $\ker(X-\lambda)_{\mathbb{K}[H]\otimes M}=M$ and $\ker(X-\lambda)_{\mathbb{K}[H]\otimes M'}=M'$ are isomorphic. □

By Proposition 2.8 and (2.9), all the non-zero primitive ideals of $\mathcal{A}$ contain the ideal (X). Hence,

$$\widehat{\mathcal{A}}(\mathbb{K}[X]\text{-torsion, unfaithful})=\widehat{\mathcal{A}}(\text{unfaithful}). \tag{4.6}$$

COROLLARY 4.5. *Theorem 4.4 and Proposition 3.3 give a classification of all simple $\mathbb{K}[X]$-torsion $\mathcal{A}$-modules.*

Proof. The proof follows from (4.1) and (4.6). □

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5. Whittaker $\mathcal{A}$-modules. In this section, $\mathbb{K}$ is an algebraically closed field. In this section, a classification of Whittaker $\mathcal{A}$-modules is given (see (5.3), Theorems 5.4, 5.7 and Proposition 5.8).

Let $h := H^+X = HX + EY$. Then the Ore extension $A_{1,X} = \mathbb{K}[X^{\pm 1}][H^+; \delta']$, where $\delta'(X) = X$ (see (2.4)), can be written as the Ore extension

$$A_{1,X} = \mathbb{K}[X^{\pm 1}][h; \delta] \quad \text{where } \delta(X) = X^2 \;\; ([h, X] = X^2). \tag{5.1}$$

LEMMA 5.1.

1. $C_{\mathcal{A}_X}(Y) = A_{1,X} \otimes \mathbb{K}[Y]$.
2. *The centraliser of the element Y in $\mathcal{A}$, $C_{\mathcal{A}}(Y) = \mathbb{K}[Y] \otimes R$, is a tensor product of algebras, where $R := \mathbb{K}[X][h; \delta]$ is an Ore extension, $h = H^+X = HX + EY$ and $\delta(X) = X^2$.*
3. *The centre of the algebra $C_{\mathcal{A}}(Y)$ is $\mathbb{K}[Y]$.*

Proof.

1. By (2.4), $\mathcal{A}_X = A_{1,X} \otimes A_1^+$ and $Y \in A_1^+$. Then $C_{\mathcal{A}_X}(Y) = A_{1,X} \otimes C_{A_1^+}(Y) = A_{1,X} \otimes \mathbb{K}[Y]$.
2. Now, $C_{\mathcal{A}}(Y) = \mathcal{A} \cap C_{\mathcal{A}_X}(Y) = \mathcal{A} \cap A_{1,X} \otimes \mathbb{K}[Y] \overset{(5.1)}{=} \mathcal{A} \cap \mathbb{K}[X^{\pm 1}][h; \delta] \otimes \mathbb{K}[Y] = \mathbb{K}[X][h; \delta] \otimes \mathbb{K}[Y]$ (since $h = HX + EY$ and X is a normal element of $\mathcal{A}$) and so the result.
3. By statement 2, $Z\Big(C_{\mathcal{A}}(Y)\Big) = \mathbb{K}[Y] \otimes Z(R) = \mathbb{K}[Y] \otimes \mathbb{K} = \mathbb{K}[Y]$. $\square$

For an $\mathcal{A}$-module M, an element $m \in M$ is called a *Whittaker vector* if $Ym = \lambda m$ for some $\lambda \in \mathbb{K}$. An $\mathcal{A}$-module M is called a *Whittaker module* if there is a Whittaker vector $m \in M$ which generates the $\mathcal{A}$-module M, i.e., $M = \mathcal{A}m$. For a given $\lambda \in \mathbb{K}$, the *universal Whittaker module* $W(\lambda)$ is defined as follows:

$$W(\lambda) := \mathcal{A}/\mathcal{A}(Y - \lambda) = \bigoplus_{i,j,k \geqslant 0} \mathbb{K}H^iE^jX^k\bar{1} \quad \text{where } \bar{1} := 1 + \mathcal{A}(Y - \lambda). \tag{5.2}$$

Any Whittaker module M is an epimorphic image of the $\mathcal{A}$-module $W(\lambda)$ for some $\lambda \in \mathbb{K}$, and vice versa.

LEMMA 5.2. *Let M be a Whittaker $\mathcal{A}$-module, i.e., M is an epimorphic image of the $\mathcal{A}$-module $W(\lambda)$ for some $\lambda \in \mathbb{K}$. Then*

1. *the element $Y - \lambda$ acts locally nilpotently on M but the elements $Y - \mu$, where $\mu \in \mathbb{K} \setminus \{\lambda\}$, act bijectively on M.*
2. *If $V \subseteq M$ is a non-zero submodule of M, then V contains a non-zero Whittaker vector.*

Proof.

1. The inner derivation ad_Y acts locally nilpotently on $\mathcal{A}$. Recall that $W(\lambda) = \mathcal{A}\bar{1}$, where $\bar{1} = 1 + \mathcal{A}(Y - \lambda)$. It follows from the equalities (where $a \in \mathcal{A}$) $(Y - \lambda)a\bar{1} = [Y, a]\bar{1} + a(Y - \lambda)\bar{1} = \mathrm{ad}_Y(a)\bar{1}$ that the element $Y - \lambda$ acts locally nilpotently on M. Then for all $\mu \neq \lambda$, the element $Y - \mu$ acts bijectively

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on M since $(Y-\mu)_M^{-1} = \Big(\lambda-\mu+(Y-\lambda)\Big)_M^{-1} = (\lambda-\mu)^{-1}\Big(1-(\mu-\lambda)^{-1}(Y-\lambda)\Big)_M^{-1} = (\lambda-\mu)^{-1}\sum_{i\geqslant 0}(\mu-\lambda)^{-i}(Y-\lambda)_M^i$ is the inverse map of $(Y-\mu)_M : M\to M,\ m\mapsto (Y-\mu)m$.

2. Let V be a non-zero submodule of M and $0\neq v\in V$. Then there exists $i\geqslant 0$ such that $v' := (Y-\lambda)^i v\neq 0$ and $(Y-\lambda)^{i+1}v=0$. Then v' is a Whittaker vector in V. □

Let $\widehat{\mathcal{A}}$ (Whittaker) be the set of isomorphism classes of simple Whittaker $\mathcal{A}$-modules. Then, by Lemma 5.2.(1), the set

$$\widehat{\mathcal{A}}\,(\text{Whittaker}) = \bigsqcup_{\lambda\in\mathbb{K}} \widehat{\mathcal{A}}\,(\text{Whittaker},\lambda) \tag{5.3}$$

is a disjoint union, where $\widehat{\mathcal{A}}\,(\text{Whittaker},\lambda) := \{[M]\in \widehat{\mathcal{A}}\,(\text{Whittaker}) \mid \ker(Y-\lambda)_M\neq 0\}$.

The sets $\widehat{\mathcal{A}}\,(\text{Whittaker},\lambda)$ where $\lambda\in\mathbb{K}^*$. It follows from the equality $h=HX+EY$ that $E=(h-HX)Y^{-1}$, and so the localisation of $\mathcal{A}$ at the powers of Y

$$\mathcal{A}_Y = R[Y,Y^{-1}][H;\delta] = C_{\mathcal{A}_Y}(Y)[H;\delta] \tag{5.4}$$

is an Ore extension, where $R[Y,Y^{-1}]=R\otimes\mathbb{K}[Y,Y^{-1}]$ is a Laurent polynomial algebra with coefficient in the algebra R and the derivation δ of the algebra $R[Y,Y^{-1}]$ is defined as $\delta(X)=X,\ \delta(h)=h$ and $\delta(Y)=-Y$. Let $\lambda\in\mathbb{K}^*$. By Lemma 5.2.(1) and (5.4),

$$W(\lambda)=W(\lambda)_Y=\mathcal{A}_Y/\mathcal{A}_Y(Y-\lambda)=\mathbb{K}[H]\otimes R\bar{1}=\bigoplus_{i\geqslant 0} H^i\otimes R\bar{1}, \tag{5.5}$$

where $\mathbb{K}[H]\otimes R$ is the tensor product of vector spaces. The next proposition gives an explicit description of all the submodules of the $\mathcal{A}$-module $W(\lambda)$, where $\lambda\in\mathbb{K}^*$.

PROPOSITION 5.3. *Let $\lambda\in\mathbb{K}^*$.*

1. *The set $\{\mathbb{K}[H]\otimes I\bar{1} \mid I$ is a left ideal of $R\}$ is the set of all distinct submodules of the $\mathcal{A}$-module $W(\lambda)$.*
2. *The set $\{\mathbb{K}[H]\otimes I\bar{1} \mid I$ is a maximal left ideal of $R\}$ is the set of all distinct maximal submodules of the $\mathcal{A}$-module $W(\lambda)$.*

Proof.

1. Let M be a submodule of the $\mathcal{A}$-module $W(\lambda)=\bigoplus_{i\geqslant 0}H^i\otimes R\bar{1}$; see (5.5). We have to show that $M=\mathbb{K}[H]\otimes I\bar{1}$ for some left ideal I of the algebra R. The $\mathcal{A}$-module $W(\lambda)=\bigcup_{i\geqslant 0}W(\lambda)_{\leqslant i}$ is the union of the vector spaces $W(\lambda)_{\leqslant i} = \{\sum_{j=0}^i H^j\otimes r_j\bar{1} \mid r_j\in R\}$. Then $M\cap W(\lambda)_{\leqslant 0} = M\cap R\bar{1} = I\bar{1}$, where I is a left ideal of R (since ${}_R R\bar{1}\simeq {}_R R$). We claim that $M=\mathbb{K}[H]\otimes I\bar{1}$. We have to show that, for all $i\geqslant 0$, $M\cap W(\lambda)_{\leqslant i} = M_i := \bigoplus_{j=0}^i H^j\otimes I\bar{1}$. To prove this, we use induction on i. The initial case when $i=0$ is trivial. So, let $i>0$, and we assume that the result holds for all $i'<i$. Clearly, $(Y-\lambda)W(\lambda)_{\leqslant i}\subseteq W(\lambda)_{\leqslant i-1}$ for all $i\geqslant 0$ (where $W(\lambda)_{\leqslant -1}:=0$) since, for all $r\in R$, $(Y-\lambda)H^i\otimes r\bar{1} = [Y,H^i]\otimes r\bar{1} = \lambda((H+1)^i-H^i)\otimes r\bar{1} = \lambda(iH^{i-1}+\cdots)\otimes r\bar{1}$, where the three dots mean a polynomial of degree $<i-1$. Moreover,

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$$Y-\lambda : \frac{W(\lambda)_{\leqslant i}}{W(\lambda)_{\leqslant i-1}} \to \frac{W(\lambda)_{\leqslant i-1}}{W(\lambda)_{\leqslant i-2}}, \; H^i \otimes r\bar{1} + W(\lambda)_{\leqslant i-1} \mapsto \lambda i H^{i-1} \otimes r\bar{1} + W(\lambda)_{\leqslant i-2}. \tag{5.6}$$

It follows from (5.6) and induction on i that if $w = \sum_{j=0}^{i} H^j \otimes r_j\bar{1} \in M \cap W(\lambda)_{\leqslant i}$, then $\lambda i r_i \in I$, i.e., $r_i \in I$. Hence, $H^i \otimes r_i\bar{1} \in M_i$, and so $\sum_{j=0}^{i-1} H^j \otimes r_j\bar{1} = w - H^i \otimes r_i\bar{1} \in M \cap W(\lambda)_{\leqslant i-1} = M_{i-1}$, by induction. Therefore, $w \in M_i$, i.e., $M \cap W(\lambda)_{\leqslant i} = M_i$, as required.

2. Statement 2 follows from statement 1. □

The next theorem gives an explicit description of the sets $\widehat{\mathcal{A}}$ (Whittaker, λ) for all $\lambda \in \mathbb{K}^*$.

Theorem 5.4. *Let $\lambda \in \mathbb{K}^*$. Then $\widehat{\mathcal{A}}\,(\text{Whittaker}, \lambda) = \left\{ \left[\mathbb{K}[H] \otimes M \right] \mid [M] \in \widehat{R} \right\}$ and the simple $\mathcal{A}$-modules $\mathbb{K}[H] \otimes M$ and $\mathbb{K}[H] \otimes M'$ (where $[M], [M'] \in \widehat{R}$) are isomorphic iff the R-modules M and M' are isomorphic.*

Proof. By Proposition 5.3.(2), every simple module in the set $\widehat{\mathcal{A}}$ (Whittaker, λ), where $\lambda \neq 0$, is isomorphic to the factor module

$$\frac{W(\lambda)}{\mathbb{K}[H] \otimes I\bar{1}} \simeq \frac{\mathbb{K}[H] \otimes R\bar{1}}{\mathbb{K}[H] \otimes I\bar{1}} \simeq \mathbb{K}[H] \otimes R/I \simeq \mathbb{K}[H] \otimes M,$$

where I is a maximal left ideal of R and $M := R/I$ is a simple R-module. If simple $\mathcal{A}$-modules $\mathbb{K}[H] \otimes M$ and $\mathbb{K}[H] \otimes M'$ are isomorphic (where M and M' are simple R-modules) then the R-modules $\ker(Y-\lambda)_{\mathbb{K}[H]\otimes M} = M$ and $\ker(Y-\lambda)_{\mathbb{K}[H]\otimes M'} = M'$ are isomorphic (see (5.6)). □

Lemma 5.5. *Suppose that T is a ring, x is a normal element of T and M is a simple T-module. Then either $xM = 0$ or, otherwise, the map $x_M : M \to M$, $m \mapsto xm$, is a bijection.*

Proof. The element x is normal. Hence, $\ker(x_M)$ and $\operatorname{im}(x_M)$ are submodules of M. So, either $\ker(x_M) \neq 0$, or, otherwise, $\operatorname{im}(x_M)$ is a non-zero submodule of the simple module M, i.e., $\operatorname{im}(x_M) = M$ – here x_M is a bijection. □

The set $\widehat{\mathcal{A}}$ (Whittaker, 0)**.** Clearly,

$$\widehat{\mathcal{A}}\,(\text{Whittaker}, 0) = \widehat{\mathcal{A}}\,(\text{Whittaker}, 0, \text{faithful}) \sqcup \widehat{\mathcal{A}}\,(\text{Whittaker}, 0, \text{unfaithful}). \tag{5.7}$$

The set $\widehat{\mathcal{A}}$ (Whittaker, 0, faithful)**.** The element X is a normal element of $\mathcal{A}$. By Lemma 5.5, for every $[M] \in \widehat{\mathcal{A}}$ (Whittaker, 0, faithful), the map $X_M : M \to M$, $m \mapsto Xm$, is a bijection. Therefore, the $\mathcal{A}$-module M coincides with its localisation M_X at $\{X^i \mid i \geqslant 0\}$, i.e., $M = M_X$ is an $\mathcal{A}_X$-module. By (2.4), the algebra $\mathcal{A}_X$ is the tensor product $A_{1,X} \otimes A_1^+$ of algebras. Recall that $Y \in A_1^+$ and $C_{\mathcal{A}_X}(Y) = A_{1,X} \otimes \mathbb{K}[Y]$ (Lemma 5.1.(1)). The A_1^+-module $V := A_1^+/A_1^+Y = \mathbb{K}[EX^{-1}]\bar{1} = \bigoplus_{i\geqslant 0} \mathbb{K}(EX^{-1})^i\bar{1}$ is a simple A_1^+-module, where $\bar{1} = 1 + A_1^+Y$. The $\mathcal{A}_X$-module $\mathcal{V} := \mathcal{A}_X/\mathcal{A}_XY \simeq A_{1,X} \otimes V$ is a tensor product of the $A_{1,X}$-module $A_{1,X}$ and the A_1^+-module V. The next proposition describes all the submodules of the $\mathcal{A}_X$-module $\mathcal{V}$ and its simple factor modules.

Proposition 5.6. *Let $\mathcal{I}_l(A_{1,X})$ be the set of all left ideals of the algebra $A_{1,X}$. Then*

1. $\{I \otimes V \mid I \in \mathcal{I}_l(A_{1,X})\}$ is the set of distinct submodules of the $\mathcal{A}_X$-module $\mathcal{V}$.

2. *$\{M\otimes V \mid [M]\in \widehat{A_{1,X}}\}$ is the set of all simple factor modules of the $\mathcal{A}_X$-module $\mathcal{V}$. The $\mathcal{A}_X$-modules $M\otimes V$ and $M'\otimes V$ are isomorphic iff the $A_{1,X}$-modules M and M' are isomorphic.*

Proof.

1. Let M be an $\mathcal{A}_X$-submodule of $\mathcal{V}=A_{1,X}\otimes V=\bigoplus_{i\geqslant 0}A_{1,X}\otimes (EX^{-1})^i\bar{1}$. We have to show that $M=I\otimes V$ for some left ideal of the algebra $A_{1,X}$. We may assume that $M\neq 0$. Each element m of M is a unique sum $m=\sum_{i=0}^n a_i\otimes (EX^{-1})^i\bar{1}$ for unique elements $a_i\in A_{1,X}$. Let I be the left ideal of the algebra $A_{1,X}$ generated by all the elements a_i for all $m\in M$. Then M is a submodule of the $\mathcal{A}_X$-module $M':=I\otimes V$. It remains to show that $M'\subseteq M$. To prove this, it suffices to show that for all elements $m=\sum_{i=0}^n a_i\otimes (EX^{-1})^i\bar{1}$ of M, we have $a_i\otimes \bar{1}\in M$ (since then $M\supseteq A_1^+a_i\otimes\bar{1}=a_i\otimes A_1^+\bar{1}=a_i\otimes V$, hence $M\supseteq \sum A_{1,X}a_i\otimes V=I\otimes V$).

To prove this statement, we use induction on the degree $n:=\deg(m)=\max\{i\mid a_i\neq 0\}$ of the element m. The case $n=0$ is obvious. So, let $n\geqslant 1$ and we assume that the statement holds for all elements of degree $<n$. The element

$$Ym=-\sum_{i=1}^n a_i\otimes i(EX^{-1})^{i-1}\bar{1}\in M$$

has degree $n-1$. Hence, by induction, $a_1\otimes\bar{1},\dots,a_n\otimes\bar{1}\in M$. Then $m':=\sum_{i=1}^n a_i\otimes (EX^{-1})^i\bar{1}\in M$, and so $a_0\otimes\bar{1}=m-m'\in M$, as required.

2. Statement 2 follows from statement 1. □

The algebra $A_{1,X}$ contains the skew polynomial algebra $\Lambda=\mathbb{K}[H^+][X;\sigma]$, where $\sigma(H^+)=H^+-1$. The element X is a normal element of Λ and $\Lambda_X=A_{1,X}$. By Lemma 5.5, for a simple Λ-module M, the following conditions are equivalent:

$$\ker(X_M)=0\Leftrightarrow XM\neq 0\Leftrightarrow X_M:M\to M,\ \ m\mapsto Xm \text{ is a bijection}\Leftrightarrow \mathrm{ann}_\Lambda(M)=0. \quad (\star)$$

Let $\widehat{\Lambda}\,(\star)$ be the set of all simple Λ-modules that satisfy one of the equivalent conditions $(\star)$.

The next theorem gives an explicit description of the set $\widehat{\mathcal{A}}\,$(Whittaker, 0, faithful).

THEOREM 5.7. *$\widehat{\mathcal{A}}\,$(Whittaker, 0, faithful) $=\{M\otimes V\mid [M]\in\widehat{\Lambda}\,(\star)\ \}$ and simple $\mathcal{A}$-modules $M\otimes V$ and $M'\otimes V$ are isomorphic (where $[M],[M']\in\widehat{\Lambda}\,(\star)$) iff the Λ-modules M and M' are isomorphic.*

Proof. Let $[N]\in\widehat{\mathcal{A}}\,$(Whittaker, 0, faithful). The element X is a normal element of $\mathcal{A}$. By Lemma 5.5, $N=N_X$. Hence, N_X is a simple factor module of the $\mathcal{A}_X$-module $\mathcal{V}$. By Proposition 5.6.(2), $N_X\simeq M\otimes V$ for some $[M]\in\widehat{A_{1,X}}$.

Claim: If M' is a Λ-submodule of M, then $M'\otimes V$ is an $\mathcal{A}$-submodule of $M\otimes V$:

$$XM'\otimes V=(XM')\otimes V\subseteq M'\otimes V,$$
$$YM'\otimes V=M'\otimes YV\subseteq M'\otimes V,$$
$$EM'\otimes V=XM'\otimes (EX^{-1})V\subseteq M'\otimes V,$$
$$HM'\otimes V=(H^+-EX^{-1}Y)M'\otimes V\subseteq H^+M'\otimes V-M'\otimes EX^{-1}YV\subseteq M'\otimes V.$$

By the Claim and the simplicity of the $\mathcal{A}$-module $M \otimes V$, we must have $M = M'$, i.e., $M \in \widehat{\Lambda}\,(\star)$.

Conversely, let $L := M \otimes V$ for some $[M] \in \widehat{\Lambda}\,(\star)$. The element X is a normal element of Λ. By Lemma 5.5, the element X acts bijectively on M. So, $M = M_X$ is a simple $A_{1,X}$-module, since $A_{1,X} = \Lambda_X$. Hence, L is an $\mathcal{A}_X$-module and $\mathcal{A}$-module. Let us show that the $\mathcal{A}$-module L is simple. It suffices to show that $\mathcal{A}u = L$ for all non-zero elements $u \in L$. Fix a non-zero element $u \in L$. Then $u = \sum_{i=0}^{n} m_i \otimes (EX^{-1})^i \bar{1}$, where $m_i \in M$ and $m_n \neq 0$. Then $Y^n u = (-1)^n n! m_n \otimes \bar{1} \in M = M \otimes \bar{1}$. The action of the element H on $M \otimes \bar{1}$ *coincides* with the action of the element $H^+ = H + EX^{-1}Y$ since $H^+ m \otimes \bar{1} = Hm \otimes \bar{1} + m \otimes EX^{-1}Y\bar{1} = Hm \otimes \bar{1}$ (since $Y\bar{1} = 0$), where $m \in M$. Therefore, to say *that* $M \otimes \bar{1}$ *is a* Λ*-module is the same as to say that* $M \otimes \bar{1}$ *is a* $\tilde{\Lambda}$*-module, where* $\tilde{\Lambda} = \mathbb{K}[H][X;\sigma]$ is a skew polynomial algebra, where $\sigma(H) = H - 1$. Since M is a simple Λ-module, it is a simple $\tilde{\Lambda}$-module. Hence, $\mathcal{A}u \supseteq \mathcal{A}m_n \otimes \bar{1} \supseteq \tilde{\Lambda} m_n \otimes \bar{1} = \Lambda m_n \otimes \bar{1} = M \otimes \bar{1}$. Now, for all $i \geqslant 0$,

$$\mathcal{A}u \supseteq E^i M \otimes \bar{1} = (EX^{-1})^i X^i M \otimes \bar{1} = X^i M \otimes (EX^{-1})^i \bar{1} = M \otimes (EX^{-1})^i \bar{1}.$$

Therefore, $\mathcal{A}u = M \otimes V$, as required. □

The set $\widehat{\mathcal{A}}$ (Whittaker, 0, unfaithful)**.** Notice that

$$\widehat{\mathcal{A}}\,(\text{Whittaker, 0, unfaithful}) = \bigsqcup_{P \in \text{Prim}(\mathcal{A}) \setminus \{0\}} \widehat{\mathcal{A}}\,(\text{Whittaker, 0, } P), \tag{5.8}$$

where $\widehat{\mathcal{A}}\,(\text{Whittaker, 0, } P) := \{[M] \in \widehat{\mathcal{A}}\,(\text{Whittaker, 0}) \mid \text{ann}_{\mathcal{A}}(M) = P\}$. An explicit description of the set $\widehat{\mathcal{A}}$ (Whittaker, 0, unfaithful) is given in the following proposition.

Proposition 5.8. $\widehat{\mathcal{A}}$ (Whittaker, 0, unfaithful) $=$ $\widehat{\mathcal{A}/(Y)}$ (faithful) $\bigsqcup_{\mathfrak{p} \in \text{Max}(\mathbb{K}[H])} \widehat{\mathcal{A}/(Y, E, \mathfrak{p})}$. *In more detail,*

- *(a)* $\widehat{\mathcal{A}}\,(\text{Whittaker, 0, } (Y)) = \widehat{\mathcal{A}/(Y)}$ (faithful).
- *(b) For all* $\mathfrak{p} \in \text{Max}(\mathbb{K}[H])$, $\widehat{\mathcal{A}}\,(\text{Whittaker, 0, } (Y, E, \mathfrak{p})) = \widehat{\mathcal{A}/(Y, E, \mathfrak{p})} = \widehat{\mathbb{K}[H]/\mathfrak{p}}$.
- *(c)* $\widehat{\mathcal{A}}\,(\text{Whittaker, 0, } (E)) = \emptyset$.
- *(d)* $\widehat{\mathcal{A}}\,(\text{Whittaker, 0, } (X, \mathfrak{q})) = \emptyset$ *for all* $\mathfrak{q} \in \text{Max}(\mathbb{K}[Z]) \setminus \{(Z)\}$.

Proof. In view of the disjoint union (5.8) and the explicit description of primitive ideals of the algebra $\mathcal{A}$ given in Proposition 2.8, it suffices to show the statements (a)–(d) hold. The statements (a) and (b) are obvious. By Theorem 2.5.(3b), $\mathcal{A}/(E) \simeq L := \mathbb{K}[H][Y;\sigma]$ is a skew polynomial algebra, where $\sigma(H) = H + 1$. Since Y is a normal element of the algebra L, the annihilator of the L-module $L/LY = L/(Y) \simeq \mathbb{K}[H]$ is equal to $(Y) \neq 0$. So, all simple factor modules of the L-module L/LY are unfaithful L-modules, and so statement (c) follows. Finally, for all $\mathfrak{q} \in \text{Max}(\mathbb{K}[Z]) \setminus \{(Z)\}$, the algebra $\mathcal{A}/(X, \mathfrak{q})$ is isomorphic to the algebra $\mathbb{Y}$ since $\mathbb{K}$ is an algebraically closed field; see Theorem 2.5.(3c). The element E is a unit of the algebra $\mathcal{A}/(X, \mathfrak{q})$, since the element $Z = EY^2$ is a non-zero element of the field $\mathbb{K}$. Hence, the statement (d) follows. □

6. A classification of simple $\mathbb{K}[E]$-torsion $\mathcal{A}$-modules. In this section, $\mathbb{K}$ is an algebraically closed field of characteristic zero. The aim of this section is to give a classification of simple $\mathbb{K}[E]$-torsion $\mathcal{A}$-modules (see (6.5), Theorems 6.4, 6.6 and Proposition 6.7).

Using the equality $[E, YX^{-1}] = 1$, we see that the subalgebra $A_1' := \mathbb{K}\langle E, YX^{-1}\rangle$ of $\mathcal{A}_X$ is the (first) Weyl algebra. Then $\mathbb{E}_X = \mathbb{K}[X^{\pm 1}] \otimes A_1^+ = \mathbb{K}[X^{\pm 1}] \otimes A_1'$ is the tensor product

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of algebras. By (2.2), $\mathcal{A}_X = (\mathbb{K}[X^{\pm 1}] \otimes A'_1)[H; \delta]$, where δ is as in (2.2). By Lemma 2.1, the algebra

$$\mathcal{A}_X = R' \otimes A'_1 \tag{6.1}$$

is a tensor product of algebras where $R' := \mathbb{K}[X^{\pm 1}][H'; \delta']$ is an Ore extension, $H' := H + 2YX^{-1}E$ and $\delta'(X) = X$. Then $h' := H'X = HX + 2YE \in \mathcal{A}$ and

$$R' = \mathbb{K}[X^{\pm 1}][h'; \delta], \quad \text{where } \delta(X) = X^2. \tag{6.2}$$

Notice that the elements $H'X^{-1} = h'X^{-2}$ and X of R' satisfy the commutation relation $[H'X^{-1}, X] = 1$. Therefore, the subalgebra $A_1 := \mathbb{K}\langle H'X^{-1}, X\rangle$ of R' is the (first) Weyl algebra and the algebra $R' = A_{1,X}$ is the localisation of the Weyl algebra A_1 at the powers of the element X. In particular, the algebra R' is a central simple domain.

LEMMA 6.1.

1. *$C_{\mathcal{A}_X}(E) = R' \otimes \mathbb{K}[E]$.*
2. *The centraliser of the element E in $\mathcal{A}$, $C_{\mathcal{A}}(E) = \mathbb{K}[E] \otimes \mathcal{R}$, is a tensor product of algebras where $\mathcal{R} := \mathbb{K}[X][h'; \delta]$ is an Ore extension, $h' = H'X = HX + 2YE$ and $\delta(X) = X^2$.*
3. *The centre of the algebra $C_{\mathcal{A}}(E)$ is $\mathbb{K}[E]$.*

Proof.

1. By (6.1), $\mathcal{A}_X = R' \otimes A'_1$ and $E \in A'_1$. Then $C_{\mathcal{A}_X}(E) = R' \otimes C_{A'_1}(E) = R' \otimes \mathbb{K}[E]$.
2. Now, $C_{\mathcal{A}}(E) = \mathcal{A} \cap C_{\mathcal{A}_X}(E) = \mathcal{A} \cap R' \otimes \mathbb{K}[E] \overset{(6.2)}{=} \mathcal{A} \cap \mathbb{K}[X^{\pm 1}][h'; \delta] \otimes \mathbb{K}[E] = \mathbb{K}[X][h'; \delta] \otimes \mathbb{K}[E]$ (since $h' = HX + 2EY$ and X is a normal element of $\mathcal{A}$) and so the result.
3. By statement 2, $Z(C_{\mathcal{A}}(E)) = \mathbb{K}[E] \otimes Z(R) = \mathbb{K}[E] \otimes \mathbb{K} = \mathbb{K}[E]$. □

The set $S := \mathbb{K}[E] \setminus \{0\}$ is a left and right Ore set in $\mathcal{A}$. An S-torsion $\mathcal{A}$-module is called a $\mathbb{K}[E]$-*torsion* $\mathcal{A}$-module. We aim to classify all simple $\mathbb{K}[E]$-torsion $\mathcal{A}$-modules (see (6.5), Theorems 6.4, 6.6 and Proposition 6.7).

For each $\lambda \in \mathbb{K}$, consider the $\mathcal{A}$-module

$$U(\lambda) := \mathcal{A}/\mathcal{A}(E-\lambda) = \bigoplus_{i,j,k\geqslant 0} \mathbb{K}H^iX^jY^k\bar{1} \quad \text{where } \bar{1} := 1 + \mathcal{A}(E-\lambda). \tag{6.3}$$

The Gelfand–Kirillov dimension of the $\mathcal{A}$-module $U(\lambda)$ is 3. Since the field $\mathbb{K}$ is an algebraically closed field, each simple $\mathbb{K}[E]$-torsion $\mathcal{A}$-module is an epimorphic image of the $\mathcal{A}$-module $U(\lambda)$ for some $\lambda \in \mathbb{K}$.

LEMMA 6.2. *Let $\lambda \in \mathbb{K}$. Then the element $E - \lambda$ acts locally nilpotently on $U(\lambda)$ but the elements $E - \mu$, where $\mu \in \mathbb{K} \setminus \{\lambda\}$, act bijectively on $U(\lambda)$.*

Proof. Repeat the proof of Lemma 5.2.(1). □

It follows from the equality $h' = HX + 2YE$ that $Y = \frac{1}{2}(h' - HX)E^{-1}$, and so the localisation of $\mathcal{A}$ at the powers of the element E

$$\mathcal{A}_E = \mathcal{R}[E, E^{-1}][H; \delta] = C_{\mathcal{A}_E}(E)[H; \delta] \tag{6.4}$$

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is an Ore extension, where $\mathcal{R}[E, E^{-1}] = \mathcal{R} \otimes \mathbb{K}[E, E^{-1}]$ is a Laurent polynomial algebra with coefficients in the algebra $\mathcal{R}$ and the derivation δ of the algebra $\mathcal{R}[E, E^{-1}]$ is defined as $\delta(X) = X$, $\delta(h') = h'$ and $\delta(E) = 2E$.

By Lemma 6.2, the set

$$\widehat{\mathcal{A}}\,(\mathbb{K}[E]\text{-torsion}) = \bigsqcup_{\lambda\in\mathbb{K}} \widehat{\mathcal{A}}\,(\mathbb{K}[E]\text{-torsion}, \lambda) \tag{6.5}$$

is a disjoint union, where $\widehat{\mathcal{A}}\,(\mathbb{K}[E]\text{-torsion}, \lambda) := \{[M] \in \widehat{\mathcal{A}}\,(\mathbb{K}[E]\text{-torsion}) \mid \ker(E - \lambda)_M \neq 0\}$.

The set $\widehat{\mathcal{A}}\,(\mathbb{K}[E]\text{-torsion}, \lambda)$ where $\lambda \in \mathbb{K}^*$. Let $\lambda \in \mathbb{K}^*$. By Lemma 6.2 and (6.4),

$$U(\lambda) = U(\lambda)_E = \mathcal{A}_E/\mathcal{A}_E(E - \lambda) = \mathbb{K}[H] \otimes \mathcal{R}\bar{1} = \bigoplus_{i\geqslant 0} H^i \otimes \mathcal{R}\bar{1}, \tag{6.6}$$

where $\mathbb{K}[H] \otimes \mathcal{R}$ is the tensor product of vector spaces. The next proposition is an explicit description of all the submodules of the $\mathcal{A}$-module $U(\lambda)$, where $\lambda \in \mathbb{K}^*$.

PROPOSITION 6.3. *Let $\lambda \in \mathbb{K}^*$ and $\mathcal{I}_l(\mathcal{R})$ be the set of all left ideals of the algebra $\mathcal{R}$.*

1. *The set $\{\mathbb{K}[H] \otimes I\bar{1} \mid I \in \mathcal{I}_l(\mathcal{R})\}$ is the set of distinct submodules of the $\mathcal{A}$-module $U(\lambda)$.*
2. *$\{\mathbb{K}[H] \otimes I\bar{1} \mid I$ is a maximal left ideal of $\mathcal{R}\}$ is the set of all maximal submodules of the $\mathcal{A}$-module $U(\lambda)$.*

Proof.

1. Let M be a submodule of the $\mathcal{A}$-module $U(\lambda) = \bigoplus_{i\geqslant 0} H^i \otimes \mathcal{R}\bar{1}$; see (6.6). We have to show that $M = \mathbb{K}[H] \otimes I\bar{1}$ for some left ideal I of the algebra $\mathcal{R}$. The $\mathcal{A}$-module $U(\lambda) = \bigcup_{i\geqslant 0} U(\lambda)_{\leqslant i}$ is the union of the vector spaces $U(\lambda)_{\leqslant i} = \{\sum_{j=0}^{i} H^j \otimes r_j\bar{1} \mid r_j \in \mathcal{R}\}$. Then $M \cap U(\lambda)_{\leqslant 0} = M \cap \mathcal{R}\bar{1} = I\bar{1}$ for some left ideal I of the algebra $\mathcal{R}$ (since ${}_{\mathcal{R}}\mathcal{R}\bar{1} \simeq {}_{\mathcal{R}}\mathcal{R}$). We claim that $M = \mathbb{K}[H] \otimes I\bar{1}$. We have to show that, for all $i \geqslant 0$, $M \cap U(\lambda)_{\leqslant i} = M_i := \bigoplus_{j=0}^{i} H^j \otimes I\bar{1}$. To prove this we use induction on i. The initial case when $i = 0$ is trivial. So, let $i > 0$, and we assume that the result holds for all $i' < i$. Clearly, $(E - \lambda)U(\lambda)_{\leqslant i} \subseteq U(\lambda)_{\leqslant i-1}$ for all $i \geqslant 0$ (where $U(\lambda)_{\leqslant -1} := 0$) since, for all $r \in \mathcal{R}$, $(E - \lambda)H^i \otimes r\bar{1} = [E, H^i] \otimes r\bar{1} = \lambda((H - 2)^i - H^i) \otimes r\bar{1} = \lambda(-2iH^{i-1} + \cdots) \otimes r\bar{1}$ where the three dots means a polynomial of degree $< i - 1$. Moreover,

$$E - \lambda : \frac{U(\lambda)_{\leqslant i}}{U(\lambda)_{\leqslant i-1}} \to \frac{U(\lambda)_{\leqslant i-1}}{U(\lambda)_{\leqslant i-2}},$$
$$H^i \otimes r\bar{1} + U(\lambda)_{\leqslant i-1} \mapsto -2\lambda i H^{i-1} \otimes r\bar{1} + U(\lambda)_{\leqslant i-2}. \tag{6.7}$$

   It follows from (6.7) and the induction on i that if $w = \sum_{j=0}^{i} H^j \otimes r_j\bar{1} \in M \cap U(\lambda)_{\leqslant i}$, then $-2\lambda i r_i \in I$, i.e., $r_i \in I$. Hence, $H^i \otimes r_i\bar{1} \in M_i$, and so $\sum_{j=0}^{i-1} H^j \otimes r_j\bar{1} = w - H^i \otimes r_i\bar{1} \in M \cap U(\lambda)_{\leqslant i-1} = M_i$, by induction. Therefore, $w \in M_i$, i.e., $M \cap U(\lambda)_{\leqslant i} = M_i$, as required.
2. Statement 2 follows from statement 1. □

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The next theorem gives an explicit description of the sets $\widehat{\mathcal{A}}(\mathbb{K}[E]\text{-torsion}, \lambda)$ for all $\lambda \in \mathbb{K}^*$.

THEOREM 6.4. *Let $\lambda \in \mathbb{K}^*$. Then $\widehat{\mathcal{A}}(\mathbb{K}[E]\text{-torsion}, \lambda) = \{[\mathbb{K}[H] \otimes M] \mid [M] \in \widehat{\mathcal{R}}\}$ and the simple $\mathcal{A}$-modules $\mathbb{K}[H] \otimes M$ and $\mathbb{K}[H] \otimes M'$ (where $[M], [M'] \in \widehat{\mathcal{R}}$) are isomorphic iff the $\mathcal{R}$-modules M and M' are isomorphic.*

Proof. By Proposition 6.3.(2), every simple module in $\widehat{\mathcal{A}}(\mathbb{K}[E]\text{-torsion}, \lambda)$, where $\lambda \neq 0$, is isomorphic to the factor module

$$\frac{U(\lambda)}{\mathbb{K}[H] \otimes I\bar{1}} \simeq \frac{\mathbb{K}[H] \otimes \mathcal{R}\bar{1}}{\mathbb{K}[H] \otimes I\bar{1}} \simeq \mathbb{K}[H] \otimes \mathcal{R}/I \simeq \mathbb{K}[H] \otimes M,$$

where I is a maximal left ideal of $\mathcal{R}$ and $M := \mathcal{R}/I$ is a simple $\mathcal{R}$-module. If simple $\mathcal{A}$-modules $\mathbb{K}[H] \otimes M$ and $\mathbb{K}[H] \otimes M'$ are isomorphic (where M and M' are simple $\mathcal{R}$-modules), then the $\mathcal{R}$-modules $\ker(E - \lambda)_{\mathbb{K}[H] \otimes M} = M$ and $\ker(E - \lambda)_{\mathbb{K}[H] \otimes M'} = M'$ are isomorphic (see (6.7)). □

The set $\widehat{\mathcal{A}}(\mathbb{K}[E]$-torsion, 0, faithful). The element X is a normal element of $\mathcal{A}$. By Lemma 5.5, for every $[M] \in \widehat{\mathcal{A}}(\mathbb{K}[E]\text{-torsion}, 0, \text{faithful})$, the map $X_M : M \to M$, $m \mapsto Xm$, is a bijection. Therefore, the $\mathcal{A}$-module M coincides with its localisation M_X at $\{X^i \mid i \geqslant 0\}$; that is, $M = M_X$ is an $\mathcal{A}_X$-module. By (6.1), $\mathcal{A}_X = R' \otimes A'_1$ is the tensor product of algebras. Recall that $E \in A'_1$ and $C_{\mathcal{A}_X}(E) = R' \otimes \mathbb{K}[E]$ (Lemma 6.1.(1)). The A'_1-module $V' := A'_1/A'_1E = \mathbb{K}[YX^{-1}]\bar{1} = \bigoplus_{i \geqslant 0} \mathbb{K}(YX^{-1})^i\bar{1}$ is a simple A'_1-module, where $\bar{1} = 1 + A'_1E$. The $\mathcal{A}_X$-module $\mathcal{V}' := \mathcal{A}_X/\mathcal{A}_XE \simeq R' \otimes V'$ is a tensor product of the R'-module R' and the A'_1-module V'. The next proposition describes all the submodules of the $\mathcal{A}_X$-module $\mathcal{V}'$ and all its simple factor modules.

PROPOSITION 6.5. *Let $\mathcal{I}_l(R')$ be the set of all left ideals of the algebra R'. Then*

1. *$\{I \otimes V' \mid I \in \mathcal{I}_l(R')\}$ is the set of distinct submodules of the $\mathcal{A}_X$-module $\mathcal{V}'$.*
2. *$\{M \otimes V' \mid [M] \in \widehat{R'}\}$ is the set of all simple factor modules of the $\mathcal{A}_X$-module $\mathcal{V}'$. The $\mathcal{A}_X$-modules $M \otimes V'$ and $M' \otimes V'$ are isomorphic (where $[M], [M'] \in \widehat{R'}$) iff the R'-modules M and M' are isomorphic.*

Proof.

1. Let M be an $\mathcal{A}_X$-submodule of $\mathcal{V}' = R' \otimes V' = \bigoplus_{i \geqslant 0} R' \otimes (YX^{-1})^i\bar{1}$. We have to show that $M = I \otimes V'$ for some left ideal of the algebra R'. We may assume that $M \neq 0$. Each element m of M is a unique sum $m = \sum_{i=0}^n a_i \otimes (YX^{-1})^i\bar{1}$ for unique elements $a_i \in R'$. Let I be the left ideal of the algebra R' generated by all the elements a_i for all $m \in M$. Then M is a submodule of the $\mathcal{A}_X$-module $M' := I \otimes V'$. It remains to show that $M' \subseteq M$. To prove this, it suffices to show that for all elements $m = \sum_{i=0}^n a_i \otimes (YX^{-1})^i\bar{1}$ of M, we have $a_i \otimes \bar{1} \in M$ (since then $M \supseteq A'_1a_i \otimes \bar{1} = a_i \otimes A'_1\bar{1} = a_i \otimes V'$, hence $M \supseteq \sum R'a_i \otimes V' = I \otimes V$).
   To prove this statement, we use induction on the degree $n := \deg(m) = \max\{i \mid a_i \neq 0\}$ of the element m. The case $n = 0$ is obvious. So, let $n \geqslant 1$ and we assume that the statement holds for all elements of degree $< n$. The element

$$Em = \sum_{i=1}^n a_i \otimes i(YX^{-1})^{i-1}\bar{1} \in M$$

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has degree $n-1$. Hence, by induction, $a_1 \otimes \bar{1}, \dots, a_n \otimes \bar{1} \in M$. Then $m' := \sum_{i=1}^{n} a_i \otimes (YX^{-1})^i \bar{1} \in M$, and so $a_0 \otimes \bar{1} = m - m' \in M$, as required.

2. Statement 2 follows from statement 1. □

The algebra R' contains the skew polynomial algebra $\Lambda' = \mathbb{K}[H'][X;\sigma]$, where $H' = H + 2YX^{-1}E$ and $\sigma(H') = H' - 1$. The element X is a normal element of Λ' and $\Lambda'_X = R'$. By Lemma 5.5, for a simple Λ'-module M the following conditions are equivalent:

$$\ker(X_M) = 0 \Leftrightarrow XM \neq 0 \Leftrightarrow X_M : M \to M, \;\; m \mapsto Xm \text{ is a bijection} \Leftrightarrow \mathrm{ann}_{\Lambda'}(M) = 0. \qquad (*)$$

Let $\widehat{\Lambda'}\,(*)$ be the set of all simple Λ'-modules that satisfy one of the equivalent conditions (*).

The next theorem gives an explicit description of the set $\widehat{\mathcal{A}}\,(\mathbb{K}[E]\text{-torsion, 0, faithful})$.

Theorem 6.6. *$\widehat{\mathcal{A}}\,(\mathbb{K}[E]\text{-torsion, 0, faithful}) = \{M \otimes V' \mid [M] \in \widehat{\Lambda'}\,(*)\}$ and simple $\mathcal{A}$-modules $M \otimes V'$ and $M' \otimes V'$ are isomorphic (where $[M], [M'] \in \widehat{\Lambda'}\,(*)$) iff the Λ'-modules M and M' are isomorphic.*

Proof. Let $[N] \in \widehat{\mathcal{A}}\,(\mathbb{K}[E]\text{-torsion, 0, faithful})$. The element X is a normal element of $\mathcal{A}$. By Lemma 5.5, $N = N_X$. Hence, N_X is a simple factor module of the $\mathcal{A}_X$-module $\mathcal{V}'$. By Proposition 6.5.(2), $N_X \simeq M \otimes V'$ for some simple R'-module M.

Claim: If M' is a Λ'-submodule of M, then $M' \otimes V'$ is an $\mathcal{A}$-submodule of $M \otimes V'$:

$$XM' \otimes V' = (XM') \otimes V' \subseteq M' \otimes V',$$
$$YM' \otimes V' = XM' \otimes YX^{-1}V' \subseteq M' \otimes V',$$
$$EM' \otimes V' = M' \otimes EV' \subseteq M' \otimes V',$$
$$HM' \otimes V = (H' - 2YX^{-1}E)M' \otimes V' \subseteq H'M' \otimes V' - M' \otimes 2YX^{-1}EV' \subseteq M' \otimes V'.$$

By the Claim and the simplicity of the $\mathcal{A}$-module $M \otimes V'$, we must have $M = M'$, i.e., $M \in \widehat{\Lambda'}\,(*)$.

Conversely, let $L := M \otimes V'$ for some $M \in \widehat{\Lambda'}\,(*)$. The element X is a normal element of Λ'. By Lemma 5.5, the element X acts bijectively on M. So, $M = M_X$ is a simple R'-module, since $\Lambda'_X = R'$. Hence, L is an $\mathcal{A}_X$-module and $\mathcal{A}$-module. Let us show that the $\mathcal{A}$-module L is simple. It suffices to show that $\mathcal{A}u = L$ for all non-zero elements $u \in L$. Fix a non-zero element $u \in L$. Then $u = \sum_{i=0}^{n} m_i \otimes (YX^{-1})^i \bar{1}$, where $m_i \in M$ and $m_n \neq 0$. Then $E^n u = n! m_n \otimes \bar{1} \in M = M \otimes \bar{1}$. The action of the element H on $M \otimes \bar{1}$ *coincides* with the action of the element $H' = H + 2YX^{-1}E$ since $H'm \otimes \bar{1} = Hm \otimes \bar{1} + 2m \otimes YX^{-1}E\bar{1} = Hm \otimes \bar{1}$ (since $E\bar{1} = 0$), where $m \in M$. Therefore, to say *that $M \otimes \bar{1}$ is a Λ'-module is the same as to say that $M \otimes \bar{1}$ is a $\tilde{\Lambda}$-module, where $\tilde{\Lambda} = \mathbb{K}[H][X;\sigma]$* is a skew polynomial algebra, where $\sigma(H) = H - 1$. Since M is a simple Λ'-module, it is a simple $\tilde{\Lambda}$-module. Hence, $\mathcal{A}u \supseteq \mathcal{A}m_n \otimes \bar{1} \supseteq \tilde{\Lambda}m_n \otimes \bar{1} = M \otimes \bar{1}$. Now, for all $i \geqslant 0$,

$$\mathcal{A}u \supseteq Y^i M \otimes \bar{1} = X^i M \otimes (YX^{-1})^i \bar{1} = M \otimes (YX^{-1})^i \bar{1}.$$

Therefore, $\mathcal{A}u = M \otimes V'$, as required. □

The set $\widehat{\mathcal{A}}\,(\mathbb{K}[E]\text{-torsion, 0, unfaithful})$. Clearly,

$$\widehat{\mathcal{A}}\,(\mathbb{K}[E]\text{-torsion, 0, unfaithful}) = \bigsqcup_{P \in \mathrm{Prim}(\mathcal{A}) \setminus \{(0)\}} \widehat{\mathcal{A}}\,(\mathbb{K}[E]\text{-torsion, 0}, P), \qquad (6.8)$$

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where $\widehat{\mathcal{A}}(\mathbb{K}[E]\text{-torsion}, 0, P) := \{[M] \in \widehat{\mathcal{A}}(\mathbb{K}[E]\text{-torsion}, 0) \mid \text{ann}_{\mathcal{A}}(M) = P\}$. The next proposition gives an explicit description of the set $\widehat{\mathcal{A}}(\mathbb{K}[E]\text{-torsion}, 0, \text{unfaithful})$.

PROPOSITION 6.7. $\widehat{\mathcal{A}}(\mathbb{K}[E]\text{-torsion}, 0, \text{unfaithful}) = \widehat{\mathcal{A}/(E)}\,(\text{faithful}) \bigsqcup_{\mathfrak{p} \in \text{Max}(\mathbb{K}[H])} \mathcal{A}/\widehat{(Y, E, \mathfrak{p})}$. *In more detail,*

(a) $\widehat{\mathcal{A}}(\mathbb{K}[E]\text{-torsion}, 0, (E)) = \widehat{\mathcal{A}/(E)}\,(\text{faithful})$.
(b) For all $\mathfrak{p} \in \text{Max}(\mathbb{K}[H])$, $\widehat{\mathcal{A}}(\mathbb{K}[E]\text{-torsion}, 0, (Y, E, \mathfrak{p})) = \mathcal{A}/\widehat{(Y, E, \mathfrak{p})} = \widehat{\mathbb{K}[H]/\mathfrak{p}}$.
(c) $\widehat{\mathcal{A}}(\mathbb{K}[E]\text{-torsion}, 0, (Y)) = \emptyset$.
(d) $\widehat{\mathcal{A}}(\mathbb{K}[E]\text{-torsion}, 0, (X, \mathfrak{q})) = \emptyset$ *for all* $\mathfrak{q} \in \text{Max}(\mathbb{K}[Z]) \setminus \{(Z)\}$.

Proof. In view of (6.8) and an explicit description of primitive ideals of $\mathcal{A}$ (Proposition 2.8), it suffices to show that statements (a)–(d) hold. The statements (a) and (b) are obvious. The statement (c) follows from the fact that $\mathcal{A}/(Y) \simeq \mathbb{K}[H][E; \sigma] =: L$, where $\sigma(H) = H - 2$ and the annihilator of the $\mathcal{A}$-module L/LE is $(E, Y) \neq (Y)$; see (2.9). (Hence, every simple factor module of the $\mathcal{A}/(Y)$-module L/LE has nozero annihilator in $\mathcal{A}/(Y)$.) Finally, by Theorem 2.5.(3c), for all $\mathfrak{q} \in \text{Max}(\mathbb{K}[Z] \setminus \{(Z)\})$, the element E is a unit in the factor algebra $\mathcal{A}/(X, \mathfrak{q})$, and the statement (d) follows. □

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