Can. J. Math.Vol. 44 (1), 1992 pp. 167–179

THE CONDUCTOR OF POINTS HAVING THE HILBERT FUNCTION OF A COMPLETE INTERSECTION IN P^2

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ABSTRACT. Let A be the coordinate ring of a set of s points in $P^n(k)$. After examining what the Hilbert function of A tells us about the conductor of A, we then determine the possible conductors for those coordinate rings which have the Hilbert function of a complete intersection in $P^2(k)$.

Let A be the coordinate ring of a set of s points $X = \{P_1, \ldots, P_s\}$ in $P^n(k)$ (k an algebraically closed field). The integral closure of A in its total ring of quotients is of the form $\bar{A} = \prod_{i=1}^{s} k[t_i]$ (where $k[t_i]$ is isomorphic to the coordinate ring of P_i) and Orecchia [9] has shown that as an ideal of $\bar{A}$, the conductor of A, $C = \{a \in \bar{A} \mid a\bar{A} \subset A\}$, is of the form

$$C = \prod_{i=1}^{s} t_i^{d_i} k[t_i]$$

where d_i is the least degree of any hypersurface which passes through all of X except for P_i.

This description of the conductor allows one to determine C solely from knowledge of the Hilbert function (of the coordinate ring) of $X \backslash \{P_i\}$ for each P_i (see [6], § 4).

In this paper we consider the following question:

"What can one say about C given the Hilbert function of A?"

After giving some general results about Hilbert functions and the conductor, we completely determine the possible conductors for those sets of points whose coordinate ring has the Hilbert function of a complete intersection in P^2.

1. **Preliminaries.** Throughout, k will denote an algebraically closed field and $R = k[X_0, \ldots, X_n]$ ($n \geq 2$) will denote the homogenous coordinate ring of $P^n = P^n(k)$. We usually write $R = \oplus_{i \geq 0} R_i$, where R_i denotes the $\binom{i+n}{n}$-dimensional k-space of forms of degree i in R, to emphasize that R is a (naturally) graded k-algebra. By "I is an ideal in R" we mean "$I = \oplus_{i>0} I_i$ is a homogeneous ideal in R". For any algebraic subset $V \subset P^n$, $I(V) \subset R$ will denote the ideal of V; that is $I(V)$ is the largest ideal defining V as a subscheme of P^n.

If $A = \oplus_{i \geq 0} A_i$ is a graded k-algebra of finite type, then $\dim_k A_t < \infty$ $\forall t \in N$. The *Hilbert Function* of $A, H(A, \_) = \{H(A,t)\}_{t \geq 0}$, is defined by $H(A,t) = \dim_k A_t$ and the

Received by the editors July 25, 1990; revised May 24, 1991 .
AMS subject classification: Primary: 14M10; secondary: 14A05, 13H15.

https://doi.org/10.4153/CJM-1992-010-6 Published online by Cambridge University Press

difference function of A, $\Delta H(A, \_) = \{\Delta H(A, t)\}_{t \geq 0}$, is given by

$$\Delta H(A, t) = H(A, t) - H(A, t-1)$$

(where $H(A, -1) = 0$). We adopt the convention that $H(A, i) = \Delta H(A, i) = 0$ if i is a negative integer. Also, if A is the coordinate ring of a set of points $X \subset P^n$, then we sometimes write $H(X, \_)$ and $\Delta H(X, \_)$ for $H(A, \_)$ and $\Delta H(A, \_)$.

For any ideal $I \subset R$, we set

$$\alpha(I) = \min\{t \mid I_t \neq \emptyset\}$$

and

$$\beta(I) = \min\{t \mid \text{height } J_t \geq 2\}$$

where J_t is the ideal generated by $\bigcup_{i=1}^{t} I_i$. If I is the ideal of a set of points $X \subset P^n$, then we write $\alpha(X)$ for $\alpha(I)$ and $\beta(X)$ for $\beta(I)$.

We also set $\sigma(X) = \min\{t \mid \Delta H(X, t) = 0\}$.

If $X = \{P_1, \ldots, P_s\}$ is a set of points with coordinate ring A, then the conductor of A, considered as an ideal of $\bar{A}$, has the form

$$C_X = \prod_{i=1}^{s} t_i^{d_i} k[t_i] \subset \prod_{i=1}^{s} k[t_i] = \prod_{i=1}^{s} R/I(P_i)$$

where d_i is the least degree of any hypersurface which passes through all of X except for P_i ([9], 4.3). Accordingly we call d_i the *degree of conductor* of P_i in X and write $\deg_X(P_i)$ for d_i. Also, we refer to C_X as the conductor of X.

By relabelling if necessary, we assume $d_1 \leq d_2 \leq \cdots \leq d_s$ and we write $\langle d_1, \ldots, d_s \rangle$ as a short form for

$$\prod_{i=1}^{s} t_i^{d_i} k[t_i].$$

Finally, if $S = \{b_i\}_{i \geq 0}$ is the Hilbert function of some set of points $X \subset P^{b_1}$, then we set $C(S) = \{\langle d_1, \ldots, d_s \rangle \mid \prod_{i=1}^{s} t_i^{d_i} k[t_i]$ is the conductor of some sets of points with Hilbert function $S\}$.

2. We begin by reviewing some basic facts about Hilbert functions and the conductor of points in P^n, referring the reader to [2],[3],[5] or [6] for a more complete discussion. Throughout this section X will denote a set of s points in P^n.

PROPOSITION 1.

1. $H(X, t+1) \geq H(X, t) \geq 1, \quad \forall t \in N$

2. $H(X, t) = H(X, t+1) \Rightarrow H(X, t+2) = H(X, t)$.

3. $H(X, t) = s$ *for* $t \gg 0$.

If $X \subset P^2$ *then we also have:*

4. $\Delta H(X, i) = i + 1$ *for* $0 \leq i < \alpha(X)$

5. $\Delta H(X, i) \geq \Delta H(X, i+1)$ *for* $i \geq \alpha(X)$

6. $\Delta H(X,i) > \Delta H(X,i+1)$ *for* $\beta(X) \leq i < \sigma(X)$.

PROOF. For (1) and (2) see ([5], 1.1) and (3) is a well-known fact from multiplicity theory (see, for example, [8] I.7). For (4)–(6) see ([2], 3.9).

We note that (3) says that $\Delta H(X,i) = 0$ for some i (i.e. $\sigma(X) < \infty$) and (1) says that $\Delta H(X,i) \geq 0$, $\forall i \in N$.

It is not hard to show that if $Y \subset X$, then $\Delta H(Y,i) \leq \Delta H(X,i)$, $\forall i \in N$. Using this it is not hard to establish the following result.

PROPOSITION 2. *For any* $P \in X$,

$$\Delta H(X\backslash\{P\},i) = \begin{cases} \Delta H(X,i) & i \neq \deg_X(P) \\ \Delta H(X,i) - 1 & i = \deg_X(P) \end{cases}$$

PROOF. See ([6], 2.3).

Since $\Delta H(X,i) \geq \Delta H(X\backslash\{P\},i) \geq 0$, $\forall i \in N$, Proposition 2 tells us that $\deg_X(P) \leq \sigma(X) - 1$, $\forall P \in X$. The following result allows us to say more.

PROPOSITION 3. *Let* $P, Q \in X$ *and suppose* $\deg_X(P) \neq \deg_X(Q)$. *Then* $\deg_{X\backslash\{P\}}(Q) = \deg_X(Q)$.

PROOF. Let $S = \{a_i\}_{i\geq 0}$ be the Hilbert function of X and set $q = \deg_X(Q)$ and $p = \deg_X(P)$.

Clearly $q \geq \deg_{X\backslash\{P\}}(Q)$, so suppose $q > \deg_{X\backslash\{P\}}(Q)$.

Case 1. $q > p$.

Since we are assuming $q > \deg_{X\backslash\{P\}}(Q)$, we have by Proposition 2 that

$$H\big((X\backslash\{P\})\backslash\{Q\}, q-1\big) = H(X\backslash\{P\}, q-1) - 1.$$

Also $q > p \Rightarrow q - 1 \geq p$ and so

$$H(X\backslash\{P\}, q-1) = H(X, q-1) - 1.$$

Consequently,

$$H\big((X\backslash\{P\})\backslash\{Q\}, q-1\big) = a_{q-1} - 2.$$

Now putting P back gives

$$H(X\backslash\{Q\}, q-1) = a_{q-1} - 1.$$

But this means that $q - 1 \geq \deg_X(Q) = q$ which is absurd. So if $q > p$ then $\deg_{X\backslash\{P\}}(Q) = \deg_X(Q)$.

Roughly speaking, we have proved that removal of a point with small degree of conductor does not affect the points in X having strictly higher degree.

Case 2. $p > q$.

Since we are assuming that $q > \deg_{X\setminus\{P\}}(Q)$, $p > q \Rightarrow H(X\setminus\{P,Q\}, q-1) = a_{q-1} - 1$. Now

$$H(X\setminus\{Q\}, q-1) = a_{q-1},$$

so we must have

$$\deg_{X\setminus\{Q\}}(P) \leq q - 1.$$

But by Case 1 (reversing the roles of P and Q) we have $\deg_{X\setminus\{Q\}}(P) = p$ which contradicts the fact that $p > q$. So if $p > q$ then we again have $\deg_{X\setminus\{P\}}(Q) = \deg_X(Q)$.

As an immediate consequence we have:

COROLLARY 4. *For each $\zeta \in N$, set*

$$Y_\zeta = \{P \in X \mid \deg_X(P) < \zeta\}.$$

Then either $Y_\zeta = X$ or $\deg_X(P) = \deg_{X\setminus Y_\zeta}(P)$, $\forall P \in X\setminus Y_\zeta$.

COROLLARY 5. *Let $Y = \{P \in X \mid \deg_X(P) < \sigma(X) - 1\}$. Then*

$$|X\setminus Y| > \sigma(X).$$

That is, X contains at least $\sigma(X)$ points with degree of conductor $= \sigma(X) - 1$.

PROOF. By the definition of Y, we have

$$\Delta H(X\setminus Y, \sigma(X) - 1) = \Delta H\big(X, \sigma(X) - 1\big) \neq 0.$$

But $X\setminus Y$ is a set of distinct points in P^n, so $\Delta H(X\setminus Y, i) \neq 0$ for $0 \leq i \leq \sigma(X) - 1$. Thus

$$|X\setminus Y| \geq \sigma(X)$$

as required.

In the following example, we compute $C(S)$ for a particular Hilbert function S.

EXAMPLE 6. Let S be the sequence

$$1 \quad 3 \quad 4 \quad 5 \quad \longrightarrow$$

and suppose that $Y \subset P^2$ is any set of points with Hilbert function S. They Y lies on 2 independent conics, F_1 and F_2. Since $|Y| = 5$, by Bezout's Theorem, $F_1 = LL_1$ and $F_2 = LL_2$ for some distinct lines L_1, L_2 and L. L necessarily contains 4 points of Y. If $P \in Y$ lies on L then $\deg_Y(P) = 3$; otherwise $\deg_Y(P) = 1$. So

$$C_Y = \prod_{i=1}^{4} t_i^3 k[t_i] \times t_5 k[t_5]$$

for any set of points Y having Hilbert function S. Since S is the Hilbert function of 5 points in P^2, 4 on a line and one off the line, $C(S)$ is the singleton set

$$C(S) = \{\langle 1, 3, 3, 3, 3\rangle\}.$$

Building on earlier work of Macaulay, Geramita, *et al.*, have given a simple combinatorical characterization of those sequences $S = \{b_i\}_{i\geq 0}$ which are the Hilbert function of some set of points in P^{b_1} ([6], 3.3). They have shown that $S = \{b_i\}_{i\geq 0}$ is the Hilbert function of a set of points in P^{b_1} if and only if S is a zero-dimensional differentiable 0-sequence (a zdd-sequence for short). When $b_1 = 2$ or 3 (the cases which will be of interest to us in this paper) the zdd's can be described quite simply. We do this in the next proposition. The reader wanting more information (or proofs) can refer to [4],[7] or [11].

PROPOSITION 7. *Let $S = \{b_i\}_{i\geq 0}$ be a sequence of integers with $b_0 = 1$, $b_1 = 2$ or 3 and set $\Delta b_i = b_i - b_{i-1}$. Then S is a zdd-sequence if and only if $\exists \alpha, \sigma \in N$ such that $\Delta b_i = i+1$ for $0 \leq i < \alpha$, $\Delta b_\alpha \geq \cdots \geq \Delta b_{\sigma-1} > 0$ and $\Delta b_\sigma = 0$, $\forall i \geq \sigma$.*

PROOF. See ([7], § 2).

Sometimes the above criterion allows us to quickly determine $C(S)$ for a given sequence S.

EXAMPLE 8. Let $S = \{b_i\}_{i\geq 0}$ be the sequence

$$1 \quad 3 \quad 5 \quad 7 \quad 9 \quad 9 \quad \longrightarrow$$

By Proposition 7, S is the Hilbert function of some set of nine points in P^2. However, again by Proposition 7, the sequence $S' = \{b'_i\}_{i\geq 0}$ given by

$$b'_i = \begin{cases} b_i & 0 \leq i < d \\ b_i - 1 & i \geq d \end{cases}$$

cannot be the Hilbert function of any set of points unless $d = 4$. So if X is any set of points with Hilbert function S, then $\deg_X(P) = 4$, $\forall P \in X$. Therefore

$$C(S) = \{\langle 4,4,4,4,4,4,4,4,4\rangle\}.$$

Given a zdd-sequence, $S = \{b_i\}_{i\geq 0}$, we say that ζ is a *permissible value* for S if the sequence $S' = \{b'_i\}_{i\geq 0}$ where

$$b'_i = \begin{cases} b_i & 0 \leq i < \zeta \\ b_i - 1 & i \geq \zeta \end{cases}$$

is a zdd-sequence. For any $P \in X$, $\deg_X(P)$ is necessarily a permissible value for $H(X, \_)$.

There is a simple criterion to determine whether ζ is a permissible value for a zdd-sequence.

PROPOSITION 9. *Let $S = \{b_i\}_{i\geq 0}$ be a zdd-sequence with $b_0 = 1$, $b_1 = 2$ or 3 and let $\Delta b_i = b_i - b_{i-1}$. Then ζ is a permissible value for $S \Leftrightarrow \Delta b_\zeta > \Delta b_{\zeta+1}$.*

PROOF. See ([7], p. 35).

EXAMPLE 10. Let $S = \{b_i\}_{i\geq 0}$ be the zdd-sequence

$$1 \quad 3 \quad 6 \quad 8 \quad 9 \quad 9 \quad \longrightarrow$$

and set $\Delta b_i = b_i - b_{i-1}$. Using Proposition 9 we have that the permissible values for S are 2, 3 and 4.

The following sets each have Hilbert function S.

$$(i) \qquad X_1: \quad \begin{array}{ccc} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{array}$$

$$(ii) \qquad X_2: \quad \begin{array}{ccccc} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & & \\ \circ & & & & \end{array}$$

$$(iii) \qquad X_3: \quad \begin{array}{cccc} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & & & \end{array}$$

$$(iv) \qquad X_4: \quad \begin{array}{cccc} \circ & \circ & \circ & \circ \\ \circ & \circ & & \\ \circ & \circ & & \end{array}$$

X_1 is the intersection of two cubics and it is well-known that $C_{X_1} = \prod_{i=1}^{9} t_i^4 k[t_i]$. Using little more than Bezout's Theorem one can show that

$$C_{X_2} = t_1^2 k[t_1] \times \Big(\prod_{i=2}^{4} t_i^3 k[t_i]\Big) \times \Big(\prod_{i=5}^{9} t_i^4 k[t_i]\Big)$$

$$C_{X_3} = t_1^2 k[t_1] \times \Big(\prod_{i=2}^{9} t_i^4 k[t_1]\Big)$$

and

$$C_{X_4} = \Big(\prod_{i=1}^{4} t_i^3 k[t_i]\Big) \times \Big(\prod_{i=5}^{9} t_i^4 k[t_i]\Big).$$

So $|C(S)| \geq 4$. Later on we will show that $|C(S)| = 4$.

A set of points $X \subset P^2$ is said to be a *complete intersection of type* (a,b) (written $X = C.I.(a,b)$) if X is a set of ab points which is the intersection of a curve of degree a with a curve of degree b. If $X = C.I.(a,b)$, then the ideal of X, $I \subset R = k[X_0, X_1, X_2]$, is of the form $I = (F,G)$ where $F \in R_a$ and $G \in R_b$ and it is not hard to show that the difference function of X is given by

$$\Delta H(X,i) = \begin{cases} i+1 & 0 \leq i < a-1 \\ a & a-1 \leq i \leq b-1 \\ a+b-i-1 & b \leq i \leq a+b-1. \end{cases}$$

In example 10 above, S was the Hilbert function of a $C.I.(3,3)$. We noted that if $X = C.I.(3,3)$ then $\deg_X(P) = 4, \forall P \in X$. More generally, the Cayley-Bacherach Theorem (see [3]) tells us that if $X = C.I.(a,b)$, then $\deg_X(P) = \sigma(X) - 1 = a+b-1, \forall P \in X$.

3. For this section, $I = I(X)$ will denote the ideal of a set of points $X \subset P^n$ having the property that

$$D = g.c.d.\{F \in I \mid F \text{ a form with degree } < \beta(X)\}$$

is a form of degree $d < \alpha(X)$ and we set $Y = \{P \in X \mid D(P) = 0\}$. Our immediate goal is to establish a relationship between C_X, C_Y and $C_{X\setminus Y}$. Central to our discussion is the following decomposition theorem.

PROPOSITION 11.

$$H(X,i) = H(X\setminus Y, i-d) + H\big(R/(D,I), i\big)$$

PROOF. See ([10], Corollary 5).

In general $I(Y) \neq (D,I)$ (see [10] for examples), but if $I(Y) = (D,I)$, then we have the following result.

PROPOSITION 12. *If $I(Y) = (D,I)$, then*

$$\deg_X(P) = \deg_{X\setminus Y}(P) + d$$

$\forall P \in X\setminus Y$.

PROOF. Let $P \in X\setminus Y$. Since $D(Q) = 0$, $\forall Q \in Y$, we have

$$\deg_X(P) \leq \deg_{X\setminus Y}(P) + d.$$

So to prove the result, it remains to show that

$$\deg_{X\setminus Y}(P) \leq \deg_X(P) - d.$$

Let $H \in R$ be a form of least degree which vanishes on all of X except for P. Then $H \in I(Y)$, and since $I(Y) = (D,I)$,

$$H = FD + G$$

for some $F \in R$ and $G \in I$. Now $\forall Q \in (X\setminus\{P\})$, both $G(Q) = 0$ and $H(Q) = 0$. We therefore have $FD(Q) = 0$. But if $Q \notin Y$, then $D(Q) \neq 0$. Consequently,

$$F(Q) = 0 \quad \forall Q \in (X\setminus\{P\})\setminus Y.$$

$G \in I \Rightarrow G(P) = 0$, and by assumption $H(P) \neq 0$; so $FD(P) \neq 0$. This shows that

$$\deg_{X\setminus Y}(P) \leq \deg_X(P) - d.$$

PROPOSITION 13. *If $\Delta H(X,t) = \Delta H\big(R/(D,I),t\big)$ for some $\alpha < t \leq \beta = \beta(X)$ then*
(1) $I(Y) = (D,I)$ and

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(2) $\deg_X(P) = \deg_Y(P)$, $\forall P \in Y$.

PROOF. For the proof that $I(Y) = (D, I)$ see ([10], Theorem 8).

To show that $\deg_X(P) = \deg_Y(P)$ we first show that $\deg_Y(P) \geq t-1$:

Let $P \in Y$ and suppose that $\deg_Y(P) = m < t-1$. Then, $\exists F \in R_m$ that vanishes on all of Y except for P, and we can find two linearly independent linear forms $H_1, H_2 \in R$ such that

$$H_1 F \in I(Y)_{m+1}$$

and

$$H_2 F \in I(Y)_{m+1}.$$

By assumption, $m+1 < t$ and, since $I(Y) = (D, I)$, we have $D \mid H_1 F$ and $D \mid H_2 F$. Since H_1 and H_2 are linear forms and $D \nmid F$, we have that $H_1 \mid D$ and $H_2 \mid D$. Consequently $H_1 H_2 \mid H_1 F$ which implies that $H_2 \mid F$ contradicting the fact that $F(P) \neq 0$. So $\deg_Y(P) \geq t-1$, $\forall P \in Y$. In particular, if $\deg_X(P) = t-1$ (for $P \in Y$), then $\deg_X(P) = \deg_Y(P)$.

Now the assumption that $\Delta H(X, t) = \Delta H\big(R/(D, I), t\big)$ means that $\sigma(X \setminus Y) \leq t - d$, which in turn means that $\deg_{X\setminus Y}(Q) \leq t-d-1$, $\forall Q \in X \setminus Y$, and so $\deg_X(Q) \leq t-1$, $\forall Q \in X$. Therefore, by repeated application of Proposition 3, if $\deg_X(P) \geq t$, then $\deg_Y(P) = \deg_X(P)$. This proves the theorem.

REMARK. For a non-reduced analogue of (part of) Proposition 13(1) see Davis ([1], 4.6).

PROPOSITION 14. *Let X be a set of points in P^2 and set $\beta = \beta(X)$. If $\Delta H(X, \beta - 1) = \Delta H(X, \beta) + 1$, then $\Delta H(X, \beta - 1) = \Delta H\big(R/(D, I), \beta - 1\big)$.*

PROOF. We first note that $\beta((D, I)) = \beta$ and $\beta(X \setminus Y) \leq \beta - d - 1$, so by Proposition 1,

$$(*) \qquad \Delta H\big(R/(D, I), \beta - 1\big) > \Delta H\big(R/(D, I), \beta\big)$$

and either

$$\Delta H(X \setminus Y, \beta - d - 1) = 0$$

or

$$\Delta H(X \setminus Y, \beta - d - 1) > \Delta H(X \setminus Y, \beta - d).$$

Suppose $\Delta H(X, \beta - 1) = \Delta H(X, \beta) + 1$. By Proposition 11,

$$\Delta H(X, i) = \Delta H\big(R/(D, I), i\big) + \Delta H(X \setminus Y, i - d),$$

so if $\Delta H(X \setminus Y, \beta - d - 1) > \Delta H(X \setminus Y, \beta - d)$ then $\Delta H\big(R/(D, I), \beta - 1\big) \leq \Delta H\big(R/(D, I), \beta\big)$, contradicting $(*)$. So $\Delta H(X \setminus Y, \beta - d - 1) = 0$ and accordingly $\Delta H\big(R/(D, I), \beta - 1\big) = \Delta H(X, \beta - 1)$.

Combining the last results we have the following:

PROPOSITION 15. *Let X be a set of points with the property that*

$$\Delta H(X, \beta - 1) = \Delta H(X, \beta) + 1 \text{ where } \beta = \beta(X).$$

Then

(1) $I(Y) = (D, I)$
(2) $\deg_X(P) = \deg_Y(P)$, $\forall P \in Y$
(3) $\deg_X(P) = \deg_{X \setminus Y}(P) + \deg D$, $\forall P \in X \setminus Y$.

PROOF. From Proposition 14, we have that $\Delta H(X, \beta - 1) = \Delta H\big(R/(D, I), \beta - 1\big)$ so by Proposition 13 we have (1) and (2). (3) follows from (1) and Proposition 12.

If S is the Hilbert function of a $C.I.(a, b)$, then $\Delta H\big(X, \beta(X) - 1\big) = \Delta H\big(X, \beta(X)\big) + 1$ for any set of points X with Hilbert function S. In the next section we will use this (and Proposition 11) to compute S where S is the Hilbert function of a $C.I.(a, b)$. Before we do this we will first compute $C(S)$ where S is the Hilbert function of $C.I.(3, 3)$.

EXAMPLE 16. Let S be the Hilbert function of a $C.I.(3, 3)$. That is, S is the sequence

$$1 \quad 3 \quad 6 \quad 8 \quad 9 \quad 9 \quad \longrightarrow$$

and let X be a set of points with Hilbert function S. The possible values for $\beta(X)$ are 3, 4 and 5.

If $\beta(X) = 3$, then X is a $C.I.(3, 3)$ and $C_X = \langle 4, 4, 4, 4, 4, 4, 4, 4, 4 \rangle$.

If $\beta(X) = 4$, then $\deg D = 2$, $Y = C.I.(2, 4)$ and $|X \setminus Y| = 1$. Accordingly, $\deg_{X \setminus Y}(P) = 0$, $\forall P \in X \setminus Y$ and $\deg_Y(P) = 4$, $\forall P \in Y$; so by Proposition 15

$$C_X = \langle 2, 4, 4, 4, 4, 4, 4, 4, 4 \rangle.$$

If $\beta(X) = 5$, then $\deg D = 1$, $Y = C.I.(1, 5)$ and $X \setminus Y$ has the Hilbert function

$$1 \quad 3 \quad 4 \quad 4 \quad \longrightarrow$$

Either $X \setminus Y = C.I.(2, 2)$ (so $C_{X \setminus Y} = \langle 2, 2, 2, 2 \rangle$) or $X \setminus Y$ has 3 collinear points (so $C_{X \setminus Y} = \langle 1, 2, 2, 2 \rangle$). Since $\deg_Y(P) = 4$, $\forall P \in Y$, by Proposition 15 either

$$C_X = \langle 3, 3, 3, 3, 4, 4, 4, 4, 4 \rangle$$

or

$$C_X = \langle 2, 3, 3, 3, 4, 4, 4, 4, 4 \rangle.$$

This shows that $|C(S)| \leq 4$. In Example 10 we showed that $|C(S)| \geq 4$, so we have that $|C(S)| = 4$.

4. In this section we determine $C(S)$ where S is the Hilbert function of a $C.I.(a,b)$. Throughout, X will denote a set of points with the Hilbert function of a $C.I.(a,b)$ and we set $I = I(X)$ and $\beta = \beta(X)$. We have $b \leq \beta \leq \sigma(X)$ and if $\beta > b$ we set

$$D = g.c.d.\{F \in I \mid F \text{ is a form with degree } F < \beta\},$$

and

$$Y = \{P \in X \mid D(P) = 0\}.$$

PROPOSITION 17. *If $\beta > b$ then:*
(1) $d = \Delta H(X, \beta - 1)$.
(2) $I(Y) = (D, I)$.
(3) $\deg_X(P) = \deg_Y(P)$, $\forall P \in Y$.
(4) $\deg_X(P) = \deg_{X\setminus Y}(P) + d$, $\forall P \in X \setminus Y$.
(5) $Y = C.I.(d, \beta)$.
(6) *$X \setminus Y$ has the Hilbert function of a $C.I.(a-d, b-d)$.*

PROOF. Since X has the Hilbert function of a complete intersection, $\Delta H(X, \beta - 1) = \Delta H(X, \beta) + 1$. So we have (1) by Proposition 14 and (2)–(4) by Proposition 15.

From Proposition 11 we conclude that $X \setminus Y$ has the Hilbert function of a $C.I.(a-d, b-d)$, and Proposition 11, (2) and the fact that $I(Y) = (D, I)$ ensures that $Y = C.I.(d, \beta)$.

NOTATION. Let $S = \{b_i\}_{i \geq 0}$ be a zero-dimensional differentiable O-sequence with permissible values

$$\zeta_1 < \zeta_2 < \cdots < \zeta_t$$

and let X be a set of points in P^2 with Hilbert function S.

Suppose X contains precisely q_i points with degree ζ_i, $(1 \leq i \leq t)$, then we write

$$C_X = [q_1, \ldots, q_t]$$

as a short form for

$$C_X = \left(\prod_{j=1}^{t}\left(\prod_{i=1}^{q_j} t_{ij}^{\zeta_j} k[t_{ij}]\right)\right) \subset \left(\prod_{j=1}^{t}\left(\prod_{i=1}^{q_j} k[t_{ij}]\right)\right) \quad q_j \neq 0$$

THEOREM 18. *Let $S = \{b_i\}_{i \geq 0}$ be the Hilbert function of a $C.I.(a,b)$ and let*

$$\zeta_i = b - 2 + i \qquad 1 \leq i \leq a$$

denote the permissible values for S. Set

$$q_i = (b - a) + 2(i-1) + 1 \qquad 1 \leq i \leq a$$

and define $f_0 = q_0 = 0$. Then

$$C(S) = \left\{[f_1, \ldots, f_a] \,\middle|\, f_1 = 0 \text{ or } f_i = q_i - \sum_{j=0}^{i-1}(f_j - q_j)\right.$$
$$\left. 1 \leq i < a \text{ and } f_a = q_a - \sum_{j=0}^{a-1}(f_j - q_j)\right\}$$

PROOF. *The proof is by induction on a. If $a = 1$, then S is the Hilbert function of a C.I.(a,b), thus the only permissible value for S is $\zeta_1 = b-1$, and any set of points with Hilbert function S is necessarily $q_1 = b$ points on a line. This proves the theorem for $a = 1$. Therefore, assume that $a > 1$ and inductively assume that the theorem is true for each d such that $1 \le d < a$.*

For $i = 1, \ldots, a-1$ recursively define f_i by

$$f_i = 0 \quad \text{or} \quad f_i = q_i - \sum_{j=0}^{i-1}(f_j - q_j)$$

and set

$$f_a = q_a - \sum_{j=1}^{a-1}(f_j - q_j).$$

Now if $f_1 = \cdots = f_{a-1} = 0$, then

$$\begin{aligned} f_a &= \sum_{i=0}^{a} q_i = \sum_{i=1}^{a}\big((b-a) + 2(i-1) + 1\big) \\ &= a(b-a) + 2\frac{a(a-1)}{2} + a \\ &= ab. \end{aligned}$$

So if $X = C.I.(a,b)$ then

$$C_X = [f_1, \ldots, f_{a-1}, f_a]$$

where $f_1 = \cdots = f_{a-1} = 0$. Therefore to prove the theorem we need to show:

(1) If $X \subset P^2$ is a set of points with Hilbert function S, and $X \neq C.I.(a,b)$, then

$$C_X = [f_1, \ldots, f_a]$$

for some $f_1, \ldots, f_a$ as defined above.

(2) For any choice of $f_1, \ldots, f_a$ as defined above we can find a set of points $X \subset P^2$ with Hilbert function S and conductor $C_X = [f_1, \ldots, f_a]$.

PROOF OF (1). If X is a set of points with Hilbert function S and $X \neq C.I.(a,b)$, then by Proposition 17, $\exists d$, $1 \le d < a$, and a set $Y \subset X$ such that

(a) $Y = C.I.(d, a+b-d)$.

and

(b) $X \setminus Y$ has the Hilbert function of a $C.I.(a-d, b-d)$.

The permissible values for $H(X \setminus Y, \_)$ are $\zeta_i' = b - d - 2 + i$ (for $1 \le i \le a-d$) and if we set

$$q_i' = (b-d) - (a-d) + 2(i-1) + 1$$

we have by induction hypothesis that $C_{X \setminus Y}$ is of the form

$$C_{X \setminus Y} = [f_1', \ldots, f_{a-d}']$$

where for $i = 1, \ldots, a-d-1$.

$$f'_i = 0 \quad \text{or} \quad f'_i = q'_i - \sum_{j=0}^{i-1}(f'_j - q'_j)$$

(where $f'_0 = q'_0 = 0$) and

$$f'_{a-d} = q'_{a-d} - \sum_{j=0}^{a-d-1}(f'_j - q'_j).$$

Now $\deg_Y(P) = a+b-2, \forall P \in Y, \deg_{X\setminus Y}(P) \leq a+b-2-2d, \forall P \in X\setminus Y$, so (noting that $q'_i = q_i$) by Proposition 17, (3) and (4),

$$C_X = [f_1, \ldots, f_{a-1}, |Y|]$$

where $f_{a-d+1} = \cdots = f_{a-1} = 0$. Since

$$\begin{aligned} |Y| &= ab - |X\setminus Y| \\ &= \sum_{i=1}^{a} q_i - \sum_{j=0}^{a-1}(f_j - q_j) \end{aligned}$$

(since $f_{a-d} \neq 0$ and $f_i = 0$ for $i = a-d+1, \ldots, a-1$), we have that

$$C_X = [f_1, \ldots, f_{a-1}, f_a]$$

proving (1).

It is not hard to show that for any choice of $f_1, \ldots, f_a$ (as defined above) one can find a subset of $Z = \mathcal{F} \cap \mathcal{G}$ where $\mathcal{F}$ is defined by

$$\prod_{i=1}^{a}(X_1 - iX_0)$$

and $\mathcal{G}$ is defined by

$$\prod_{i=1}^{a+b-1}(X_2 - iX_0)$$

which has the Hilbert function of a $C.I.\,(a,b)$ and conductor $[f_1, \ldots, f_a]$.

We omit the details.

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