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Renormalization in the golden-mean semi-Siegel Hénon family: universality and non-rigidity

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Abstract. It was recently shown in Gaidashev and Yampolsky [Golden mean Siegel disk universality and renormalization. *Preprint*, 2016, arXiv:1604.00717] that appropriately defined renormalizations of a sufficiently dissipative golden-mean semi-Siegel Hénon map converge super-exponentially fast to a one-dimensional renormalization fixed point. In this paper, we show that the asymptotic two-dimensional form of these renormalizations is *universal* and is parameterized by the average Jacobian. This is similar to the limit behavior of period-doubling renormalizations in the Hénon family considered in de Carvalho *et al* [Renormalization in the Hénon family, I: universality but non-rigidity. *J. Stat. Phys.* **121** (5/6) (2006), 611–669]. As an application of our result, we prove that the boundary of the golden-mean Siegel disk of a dissipative Hénon map is non-smoothly rigid.

Key words: low-dimensional dynamics, smooth dynamics, tiling
2010 Mathematics Subject Classification: 37F25 (Primary); 37E20, 32H50 (Secondary)

1. *Introduction*

The archetypical class of examples in holomorphic dynamics is given by the *quadratic family*

$$f_c(z) = z^2 + c \quad \text{for } c \in \mathbb{C}.$$

Despite its apparent simplicity, the dynamics of this family is incredibly rich, and exhibits many of the key features that are observed in the general case. In the dynamics of several complex variables, the role of the quadratic family is assumed by its two-dimensional extension

$$H_{c,b}(x, y) = (x^2 + c - by, x) \quad \text{for } c \in \mathbb{C} \text{ and } b \in \mathbb{C} \setminus \{0\},$$

which is called the *(complex quadratic) Hénon family*.

Since

$$H_{c,b}^{-1}(x, y) = \left(y, \frac{y^2 + c - x}{b}\right),$$

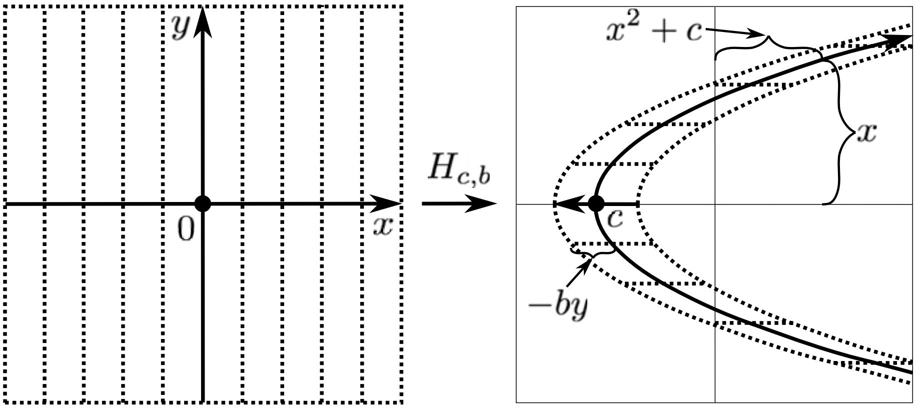

FIGURE 1. A Hénon map $H_{c,b}$. Note that vertical lines are scaled uniformly by $-b$ and then are mapped to horizontal lines.

a Hénon map $H_{c,b}$ is a polynomial automorphism of $\mathbb{C}^2$. Moreover, it is easy to see that $H_{c,b}$ has constant Jacobian: i.e.,

$$\operatorname{Jac} H_{c,b} \equiv b.$$

Note that for $b = 0$, the map $H_{c,b}$ degenerates to the embedding of f_c given by

$$(x, y) \mapsto (f_c(x), x).$$

Hence, the parameter b determines how far $H_{c,b}$ is from being a degenerate one-dimensional system. In this paper, we will always assume that $H_{c,b}$ is a dissipative map (i.e. $|b| < 1$).

A Hénon map $H_{c,b}$ is determined uniquely by the multipliers μ and ν at a fixed point $\mathbf{p}$. In particular,

$$b = \mu\nu,$$

and

$$c = (1 + \mu\nu)\left(\frac{\mu}{2} + \frac{\nu}{2}\right) - \left(\frac{\mu}{2} + \frac{\nu}{2}\right)^2.$$

When convenient, we will write $H_{\mu,\nu}$ instead of $H_{c,b}$ to denote a Hénon map.

Suppose that one of the multipliers, say, μ, is irrationally indifferent, so that

$$\mu = e^{2\pi i\theta} \quad \text{for some } \theta \in (0, 1) \setminus \mathbb{Q}.$$

Then

$$|b| = |\nu|.$$

In this case, the Hénon map $H_{\mu,\nu}$ is said to be *semi-Siegel* if there exist neighborhoods N of $(0, 0)$ and $\mathcal{N}$ of $\mathbf{p}$, and a biholomorphic change of coordinates

$$\phi : (N, (0, 0)) \to (\mathcal{N}, \mathbf{p})$$

such that

$$H_{\mu,\nu} \circ \phi = \phi \circ L,$$

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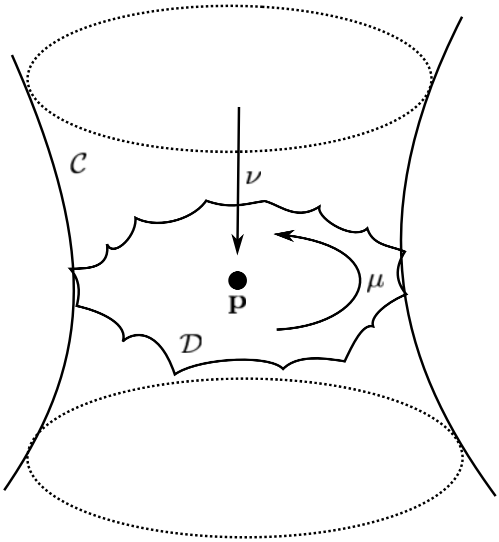

FIGURE 2. The Siegel cylinder $\mathcal{C}$ and the Siegel disk $\mathcal{D}$ of $H_{\mu,\nu}$.

where $L(x, y) := (\mu x, \nu y)$. A classic theorem of Siegel states, in particular, that $H_{\mu,\nu}$ is semi-Siegel whenever θ is *Diophantine*. That is, for some constants C and d,

$$q_{n+1} < C q_n^d,$$

where p_n/q_n are the continued fraction convergents of θ (see §2). In this case, the linearizing map ϕ can be biholomorphically extended to

$$\phi : (\mathbb{D} \times \mathbb{C}, (0, 0)) \to (\mathcal{C}, \mathbf{p})$$

so that its image $\mathcal{C} := \phi(\mathbb{D} \times \mathbb{C})$ is *maximal* (see [**MNTU**]). We call $\mathcal{C}$ the *Siegel cylinder* of $H_{\mu,\nu}$. In the interior of $\mathcal{C}$, the dynamics of $H_{\mu,\nu}$ is conjugate to rotation by θ in one direction and compression by ν in the other direction. Clearly, the orbit of every point in $\mathcal{C}$ converges to the analytic disk $\mathcal{D} := \phi(\mathbb{D} \times \{0\})$ at height 0. We call $\mathcal{D}$ the *Siegel disk* of $H_{\mu,\nu}$.

The geometry of Siegel disks in one dimension is a challenging and important topic that has been studied by numerous authors, including Herman [**He**], McMullen [**Mc**], Petersen [**P**], Inou and Shishikura [**ISh**], Yampolsky [**Ya**], and others. In the two-dimensional Hénon family, the corresponding problems have been wide open until a very recent work of Gaidashev, Radu, and Yampolsky [**GaRYa**], who proved the following theorem.

THEOREM 1.1. (Gaidashev, Radu and Yampolsky) *Let $\theta_* = (\sqrt{5} - 1)/2$ be the inverse golden-mean, and let $\mu_* = e^{2\pi i \theta_*}$. Then there exists $\epsilon > 0$ such that if $|\nu| < \epsilon$, then the boundary of the Siegel disk $\mathcal{D}$ of $H_{\mu_*,\nu}$ is a homeomorphic image of the circle. In fact, the linearizing map*

$$\phi : \mathbb{D} \times \{0\} \to \mathcal{D}$$

extends continuously and injectively (but not smoothly) to the boundary.

In the author's joint paper with Yampolsky [**YaY**], we have obtained the first geometric result about Siegel disks in the Hénon family.

THEOREM 1.2. (Yampolsky) *Let μ_* and $\epsilon > 0$ be as in Theorem 1.1. Then for $|\nu| < \epsilon$ the boundary of the Siegel disk $\mathcal{D}$ of $H_{\mu_*,\nu}$ is not C^1-smooth.*

The proofs of Theorem 1.1 and 1.2 are based on the renormalization theory developed by Gaidashev and Yampolsky in [**GaYa**]. Generally speaking, a renormalization of a dynamical system is defined as a rescaled first return map on an appropriately chosen subset of the phase space. In their paper, Gaidashev and Yampolsky considered the semi-Siegel Hénon maps within the context of a Banach space $\mathcal{B}_2$ of dynamical systems called *almost commuting pairs*. They then formulated renormalization as an operator $\mathbf{R}_{\text{GY}}$ from $\mathcal{B}_2$ to itself. They were able to show that this operator is analytic, and that it has a hyperbolic fixed point $\Sigma_* \in \mathcal{B}_2$. In [**GaRYa**], they went on to prove that the stable manifold of Σ_* does indeed contain the almost commuting pairs that correspond to sufficiently dissipative semi-Siegel Hénon maps of the golden-mean type.

It is important to note that the fixed point Σ_* for $\mathbf{R}_{\text{GY}}$ is a degenerate one-dimensional system. Hence, when the renormalization sequence of an almost commuting pair Σ converges to Σ_*, it loses its dependence on the second variable along the way. In fact, Gaidashev and Yampolsky showed that this must happen at a super-exponential rate.

In this paper, we describe the behavior of almost commuting pairs as they approach the space of degenerate one-dimensional systems under renormalization. For this purpose, we adopt a new renormalization operator $\mathbf{R}$ that we obtain by modifying the construction of $\mathbf{R}_{\text{GY}}$. The main difference between these two operators is that while $\mathbf{R}_{\text{GY}}$ is based on a diagonal embedding of the pairs of one-dimensional maps,

$$(\eta, \xi) \mapsto \left(\begin{bmatrix} \eta \\ \eta \end{bmatrix}, \begin{bmatrix} \xi \\ \xi \end{bmatrix} \right),$$

the operator $\mathbf{R}$ is based on a Hénon-like embedding (see (6)). Although the former embedding has the benefit of being more symmetric, the latter embedding allows us to track two-dimensional deviations from its image more precisely and more explicitly. However, it should be noted that $\mathbf{R}$ and $\mathbf{R}_{\text{GY}}$ are still related closely enough that a number of proofs given in [**GaRYa**] can be directly transferred to our setting, *mutatis mutandis* (in particular, see Theorems 5.7 and 5.8).

The central result of this paper is that in the limit of renormalization, the almost commuting pairs take on a *universal* two-dimensional shape as they flatten into degenerate one-dimensional systems. This statement is formulated explicitly in Theorem 7.3. The proof relies on an analysis of the average Jacobian of almost commuting pairs on their invariant renormalization arcs. A similar approach was taken by de Carvalho, Lyubich, and Martens in [**dCLM**] to study the limits of period-doubling renormalization in the Hénon family.

The universality phenomenon described in Theorem 7.3 has deep consequences on the geometry of the golden-mean Siegel disk of dissipative Hénon maps. In [**dCLM**], de Carvalho, Lyubich, and Martens used universality to show, in particular, that the invariant Cantor set for period-doubling renormalization is non-smoothly rigid. In this paper, we are able to obtain the following analogous result.

NON-RIGIDITY THEOREM. *Let μ_* and $\epsilon > 0$ be as in Theorem 1.1. If $|\nu_1|, |\nu_2| < \epsilon$ and $|\nu_1| \neq |\nu_2|$, then the two semi-Siegel Hénon maps H_{μ_*,ν_1} and H_{μ_*,ν_2} cannot be C^1 conjugate on the boundary of their respective Siegel disks.*

Non-rigidity is the first known property of Siegel disks of Hénon maps that is unique to higher dimensions. In the one-dimensional case, McMullen showed that two quadratic-like maps with a Siegel disk of the same bounded type rotation number are $C^{1+\alpha}$ conjugate on their Siegel boundary (see [**Mc**]).

2. *Motivation*

Let $\theta \in \mathbb{R}/\mathbb{Z}$ be an irrational rotation number. Then θ is represented by an infinite continued fraction: i.e.,

$$\theta = [a_0, a_1, \ldots] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}}.$$

The *nth partial convergent of* θ is the rational number

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n].$$

The denominator q_n is called the nth *closest return moment*. The sequence $\{q_n\}_{n=0}^{\infty}$ satisfies the inductive relation

$$q_0 = 1, \quad q_1 = a_0 \quad \text{and} \quad q_{n+1} = a_n q_n + q_{n-1} \quad \text{for } n \geq 1. \tag{1}$$

We say that θ is of *bounded type* if a_n are uniformly bounded. The simplest example of a bounded type rotation number is the inverse golden-mean

$$\theta_* = \frac{\sqrt{5}-1}{2} = [1, 1, \ldots].$$

For the remainder of this section, we assume that θ is of bounded type.

2.1. *One-dimensional renormalization.*

Consider the quadratic polynomial

$$f_{c_0}(z) = z^2 + c_0$$

that has a fixed Siegel disc $\mathcal{D}_0 \subset \mathbb{C}$ with rotation number θ. If θ is equal to the inverse golden-mean θ_*, then we denote f_{c_0} as simply f_*.

We are interested in understanding the small-scale behavior of the dynamics of f_{c_0} near its Siegel boundary $\partial\mathcal{D}_0$. The following theorem is due to Douady, Ghys, Herman, and Shishikura (see [**He2**]).

THEOREM 2.1. *The quadratic polynomial f_{c_0} has its critical point 0 on its Siegel boundary $\partial\mathcal{D}_0$, and the restriction $f_{c_0}|_{\partial\mathcal{D}_0} : \partial\mathcal{D}_0 \to \partial\mathcal{D}_0$ is quasi-symmetrically conjugate to the rigid rotation of the unit circle $\partial\mathbb{D}$ by angle θ.*

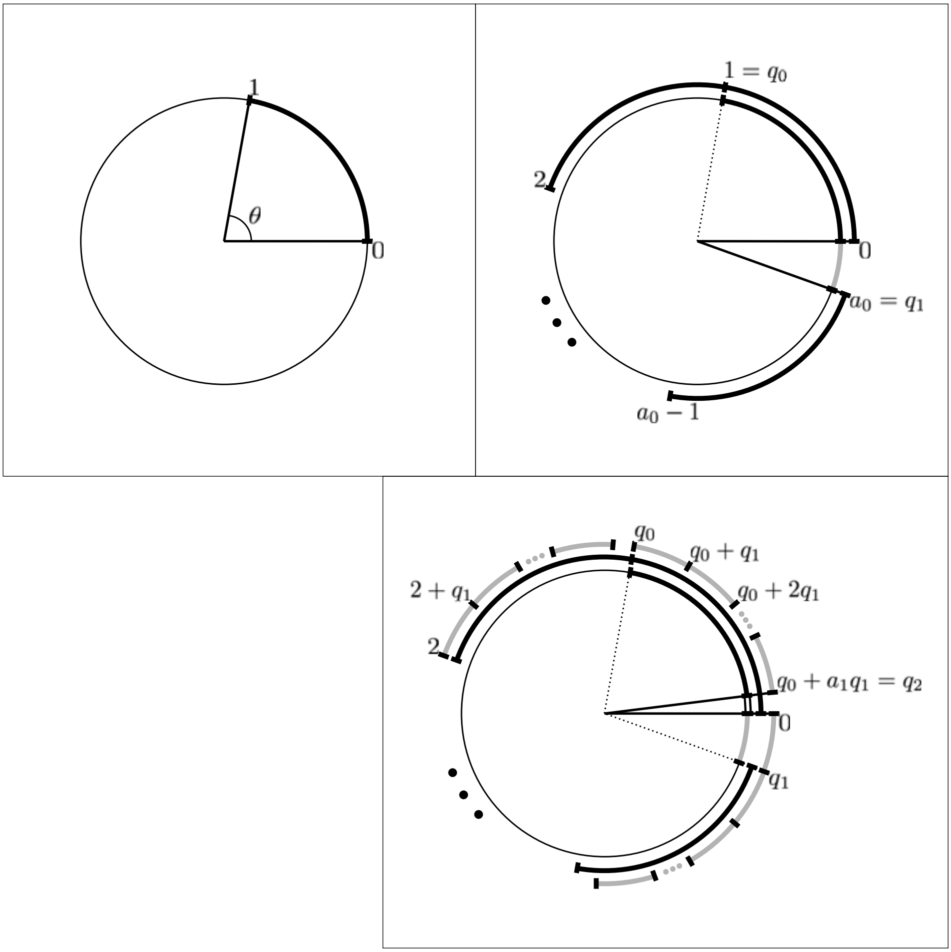

FIGURE 3. The combinatorics of the closest return moments $\{q_n\}_{n=0}^{\infty}$ for $\theta = [a_0, a_1, \ldots]$ illustrated on the circle.

Consider the orbit of the critical point 0 under f_{c_0} given by

$$\mathcal{O}(0) := \{0,\ f_{c_0}(0),\ f_{c_0}^2(0), \ldots\} \subset \partial\mathcal{D}_0.$$

Define the nth *critical arc* $\hat{\Delta}_n \subset \partial\mathcal{D}_0$ as the closed arc containing the critical point 0 whose end points are $f_{c_0}^{q_{2n}}(0)$ and $f_{c_0}^{q_{2n+1}}(0)$. The critical arc $\hat{\Delta}_n$ can be expressed as the union of two closed subarcs E_{2n} and E_{2n+1}, where E_i has its end points at 0 and $f_{c_0}^{q_i}(0)$. Observe that:

(i) $E_{2n} \cap E_{2n+1} = \{0\}$;

(ii) $E_{2n} \supset E_{2n+2}$;

(iii) $f_{c_0}^k(E_{2n}) \cap \hat{\Delta}_n = \varnothing$ for $0 < k < q_{2n+1}$ and $f_{c_0}^{q_{2n+1}}(E_{2n}) \subset \hat{\Delta}_n$; and

(iv) $f_{c_0}^k(E_{2n+1}) \cap \hat{\Delta}_n = \varnothing$ for $0 < k < q_{2n}$ and $f_{c_0}^{q_{2n}}(E_{2n+1}) \subset E_{2n} \subset \hat{\Delta}_n$.

The subarc E_i is called the ith *closest return arc*.

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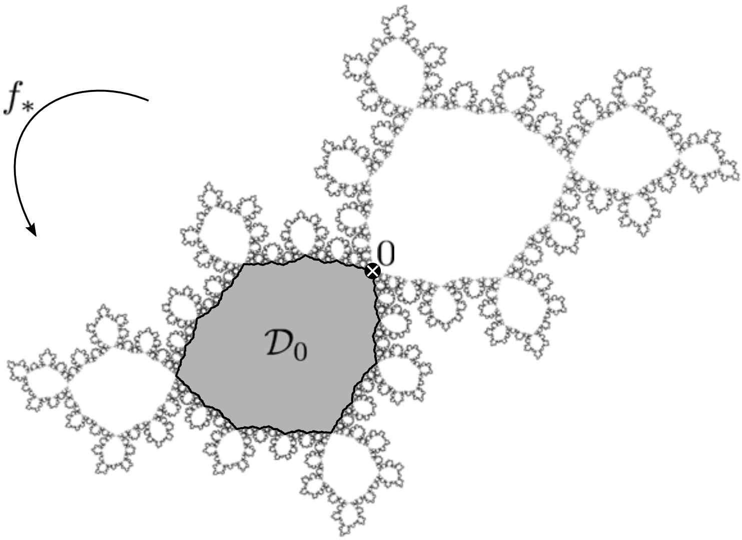

FIGURE 4. The Siegel disk $\mathcal{D}_0$ of the golden-mean Siegel quadratic polynomial f_*. The critical point 0 is on $\partial\mathcal{D}_0$, and the restriction $f_*|_{\partial\mathcal{D}_0} : \partial\mathcal{D}_0 \to \partial\mathcal{D}_0$ is quasi-symmetrically conjugate to the rigid rotation of the unit circle $\partial\mathbb{D}$ by the angle $\theta_* = (\sqrt{5}-1)/2$.

The critical arcs $\hat{\Delta}_n$ form a nested neighborhood of 0 in $\partial\mathcal{D}_0$, and by Theorem 2.1, we see that

$$\bigcap_{n=0}^{\infty} \hat{\Delta}_n = \{0\}. \tag{2}$$

Define the nth *pre-renormalization* $p\mathcal{R}^n(f_{c_0}) : \hat{\Delta}_n \to \hat{\Delta}_n$ of f_{c_0} as the *first return map* on $\hat{\Delta}_n = E_{2n} \cup E_{2n+1}$ under iterates of f_{c_0}. It is not hard to see that

$$p\mathcal{R}^n(f_{c_0})(x) = \begin{cases} f_{c_0}^{q_{2n+1}}(x) & \text{if } x \in E_{2n}, \\ f_{c_0}^{q_{2n}}(x) & \text{if } x \in E_{2n+1}. \end{cases}$$

Hence, we can consider $p\mathcal{R}^n(f_{c_0})$ as a pair of maps

$$\hat{\zeta}_n = (\hat{\eta}_n, \hat{\xi}_n) := p\mathcal{R}^n(f_{c_0}) = (f_{c_0}^{q_{2n+1}}|_{E_{2n}}, f_{c_0}^{q_{2n}}|_{E_{2n+1}}) \tag{3}$$

acting on $\hat{\Delta}_n$. Letting $n = 0$, we obtain a pair representation of f_{c_0},

$$\hat{\zeta}_{f_{c_0}} := \hat{\zeta}_0 = p\mathcal{R}^0(f_{c_0}) = (f_{c_0}^{a_0}|_{E_0}, f_{c_0}|_{E_1}).$$

Intuitively, $p\mathcal{R}^n(f_{c_0})$ captures the dynamics of f_{c_0} on the Siegel boundary $\partial\mathcal{D}_0$ that occurs at the scale of $\hat{\Delta}_n$.

Note that we can obtain the $(n+1)$th pre-renormalization $\hat{\zeta}_{n+1}$ by taking the first return map on $\hat{\Delta}_{n+1} \Subset \hat{\Delta}_n$ under iterates of the nth pre-renormalization $\hat{\zeta}_n$: i.e.,

$$\hat{\zeta}_{n+1} = p\mathcal{R}(\hat{\zeta}_n).$$

For the inverse golden-mean θ_*, this corresponds to taking the iterate of $\hat{\zeta}_n$ given by

$$\hat{\zeta}_{n+1} = p\mathcal{R}(\hat{\zeta}_n) = p\mathcal{R}((\hat{\eta}_n, \hat{\xi}_n)) = (\hat{\eta}_n \circ \hat{\xi}_n \circ \hat{\eta}_n|_{E_{2(n+1)}}, \hat{\eta}_n \circ \hat{\xi}_n|_{E_{2(n+1)+1}}).$$

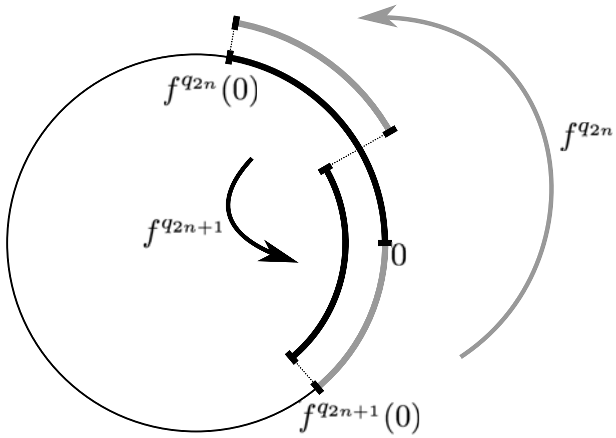

FIGURE 5. The first return map $\hat{\zeta}_n := p\mathcal{R}^n(f_{c_0})$ on $\hat{\Delta}_n = E_{2n} \cup E_{2n+1}$ under iterates of f_{c_0}. The Siegel boundary $\partial\mathcal{D}_0$ is represented as a round circle.

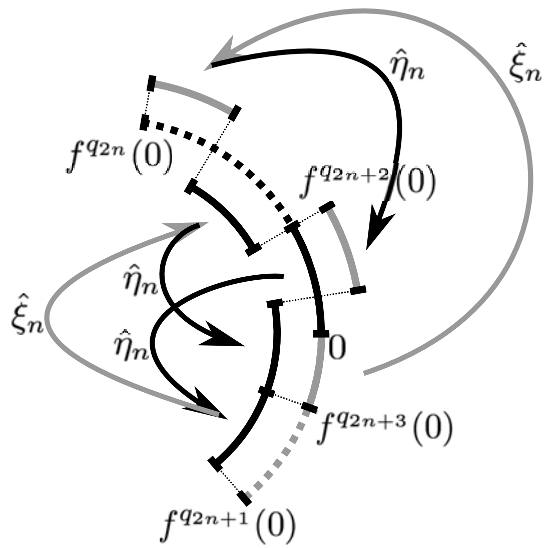

FIGURE 6. The $(n+1)$th pre-renormalization $\hat{\zeta}_{n+1}$ as the first return map on $\hat{\Delta}_{n+1} \Subset \hat{\Delta}_n$ under iterates of the nth pre-renormalization $\hat{\zeta}_n$ for the inverse golden-mean rotation number θ_*. Refer to Figure 5 for an illustration of $\hat{\zeta}_n$ acting on $\hat{\Delta}_n$.

These observations suggest that the sequence of pre-renormalizations of f_{c_0} can be realized as the orbit of $\hat{\zeta}_0 = \hat{\zeta}_{f_{c_0}}$ under the action of some *pre-renormalization operator* $p\mathcal{R}$ defined on a space of pairs of maps.

By (2), we see that $p\mathcal{R}^n(\hat{\zeta}_{f_{c_0}})$ degenerates as $n \to \infty$ to a pair of maps acting on a single point (namely, 0). To obtain a more meaningful asymptotic behavior, we need to magnify the dynamics of $p\mathcal{R}^n(\hat{\zeta}_{f_{c_0}})$ and bring it to some fixed scale. The simplest way to do this is to conjugate

$$p\mathcal{R}^n(\hat{\zeta}_{f_{c_0}}) = \hat{\zeta}_n = (\hat{\eta}_n, \hat{\xi}_n)$$

by a linear map that sends the critical value $\hat{\xi}_n(0)$ to 1. The resulting rescaled dynamical system

$$\mathcal{R}^n(\zeta_{f_{c_0}}) = \zeta_n = (\eta_n, \xi_n) \quad \text{with } \xi_n(0) = 1$$

is called the nth *renormalization* of f_{c_0}. If we denote the rescaling operator on pairs by Λ, we can define the *renormalization operator* as

$$\mathcal{R} := \Lambda \circ p\mathcal{R}.$$

Note that ζ_n acts on an arc Δ_n that is a linear rescaling of the critical arc $\hat{\Delta}_n$. Since Δ_n contains 0 and 1, it does not degenerate to a single point as $n \to \infty$.

Similarly to $p\mathcal{R}$, the renormalization operator $\mathcal{R}$ acts on the space of certain pairs of maps. If $\zeta = (\eta, \xi)$ belongs to this space, then it should satisfy the following properties.

(i) The maps η and ξ each have a unique simple critical point at 0.

(ii) The scale of ζ is normalized, so that the critical value $\xi(0)$ is at 1.

(iii) The maps η and ξ extend to holomorphic maps on some neighborhoods Z and W of 0 in $\mathbb{C}$.

(iv) Where η and ξ are both defined, these maps should commute: i.e.,

$$\eta \circ \xi = \xi \circ \eta.$$

Observe that commutativity clearly holds for $\zeta = \zeta_n = \mathcal{R}^n(f_{c_0})$ since, in this case, $\eta = \eta_n$ and $\xi = \xi_n$ represent different iterates of the same map f_{c_0}.

In [**MN**], Manton and Nauenberg observed numerically that the Siegel boundary $\partial\mathcal{D}_0$ for the golden-mean Siegel quadratic polynomial f_* exhibits a self-similar universal scaling property near the critical point 0. More precisely, they observed that:

(i) the scaling constants

$$\lambda_n := \frac{\operatorname{diam}(\hat{\Delta}_n)}{\operatorname{diam}(\hat{\Delta}_{n-1})}$$

converge to some universal constant λ_*; and

(ii) the rescaled critical arcs Δ_n converge to some universal arc Δ_*.

To explain this phenomenon, Widom [**Wi**] introduced the renormalization scheme that defines the operator $\mathcal{R}$. Based on numerical evidence, he made the following two conjectures.

(i) The renormalization sequence $\zeta_n = \mathcal{R}^n(f_*)$ of the golden-mean Siegel quadratic polynomial f_* converges to some universal limit ζ_*.

(ii) In some suitable function space, this limit ζ_* is a hyperbolic fixed point for $\mathcal{R}$, and the differential $D_{\zeta_*}\mathcal{R}$ is repelling in one-direction and attracting in all other directions.

The first partial result of Widom's conjecture was obtained by Stirnemann [**Stir**], who gave a computer-assisted proof of the existence of a fixed point ζ_* for $\mathcal{R}$ in the golden-mean case. In [**Mc**], McMullen proved (without computer assistance) the existence and uniqueness of ζ_*, and showed that the convergence of $\zeta_n = \mathcal{R}^n(f_*)$ to ζ_* occurs geometrically fast. The hyperbolicity part of Widom's conjecture was left open for a long time, until it was finally resolved by Gaidashev and Yampolsky in their recent work [**GaYa2**]. A detailed statement of their result is given in Theorem 3.7.

It should be noted that in [**MP**], MacKay and Persival expanded Widom's conjecture to include other rotation numbers, and they postulated the existence of a hyperbolic

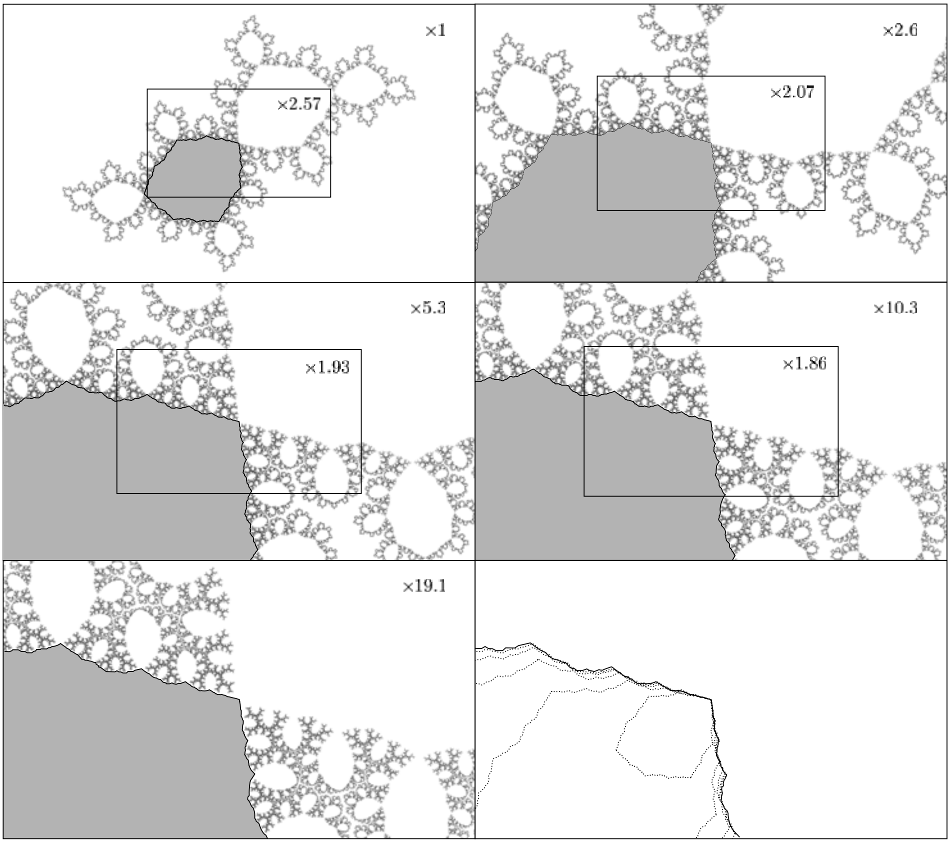

FIGURE 7. Self-similar universal scaling property of the Siegel boundary $\partial\mathcal{D}_0$ for f_* near the critical point 0. Under magnification by scaling constants λ_n^{-1} that converge to some universal constant $\lambda_*^{-1} \approx 1.82$, the rescaled critical arcs Δ_n converge to some universal arc Δ_*.

horseshoe for $\mathcal{R}$ that is analogous to Lanford's horseshoe for critical circle maps (see [**Lan1**], [**Lan2**]). For results in this direction, see [**Mc**] and [**DuLSe**] (for bounded type rotation numbers) and [**ISh**] and [**Ya**] (for high type rotation numbers).

2.2. *Two-dimensional renormalization.* The main goal of this paper is to extend the theory of Siegel renormalization to a higher dimensional setting. To this end, consider a quadratic Hénon map

$$H_{c_b,b}(x, y) = (x^2 + c_b - by, x)$$

that has a semi-Siegel fixed point $\mathbf{p}_b$ with multipliers $\mu = e^{2\pi i\theta}$ and $\nu \in \mathbb{D} \setminus \{0\}$. Such Hénon maps are parameterized by their Jacobian: i.e.,

$$b = \nu/\mu \equiv \text{Jac}\, H_{c_b,b}.$$

As $b \to 0$, the semi-Siegel Hénon map $H_{c_b,b}$ degenerates to the two-dimensional embedding of the Siegel quadratic polynomial f_{c_0} given by

$$(x, y) \mapsto (f_{c_0}(x), x).$$

Hence, for $|b| \ll 1$, the dynamics of $H_{c_b,b}$ can be considered as a small perturbation of the dynamics of f_{c_0}.

Let $\mathcal{D}_b$ be the two-dimensional Siegel disk of $H_{c_b,b}$. *A priori*, we do not have an analog of Theorem 2.1 that characterizes the dynamics of $H_{c_b,b}$ on $\partial\mathcal{D}_b$. However, we can still define the nth *pre-renormalization* $p\mathbf{R}^n(H_{c_b,b})$ of $H_{c_b,b}$ by taking the same iterates as in (3): i.e.,

$$p\mathbf{R}^n(H_{c_b,b}) = \hat{\Sigma}_n = (\hat{A}_n, \hat{B}_n) := (H_{c_b,b}^{q_{2n+1}}|_{\Omega_n}, H_{c_b,b}^{q_{2n}}|_{\Gamma_n}). \tag{4}$$

In (4), the sets Ω_n and Γ_n are chosen to be some suitable domains in $\mathbb{C}^2$ which intersect $\partial\mathcal{D}_b$. By letting $n = 0$, we obtain a pair representation of $H_{c_b,b}$ given by

$$\hat{\Sigma}_{H_{c_b,b}} := \hat{\Sigma}_0 = p\mathbf{R}^0(H_{c_b,b}) = (H_{c_b,b}^{a_0}|_{\Omega_0}, H_{c_b,b}|_{\Gamma_0}).$$

Analogously to the one-dimensional case, the sequence of pre-renormalizations of $H_{c_b,b}$ can be realized as the orbit of $\hat{\Sigma}_{H_{c_b,b}}$ under the action of some *pre-renormalization operator* $p\mathbf{R}$ defined on a space of pairs of two-dimensional maps. To transform $p\mathbf{R}$ into a proper *renormalization operator* $\mathbf{R}$, we need to compose $p\mathbf{R}$ with some suitable *rescaling operator* $\boldsymbol{\Lambda}$. However, this turns out to be a more intricate problem in two-dimensions than in the one-dimensional case. To ensure tractable asymptotic behavior under iterations of $\mathbf{R}$, it is not only important to fix the scale of the dynamical systems, but we must also bring them back to Hénon-like form after each renormalization. To achieve this, we incorporate a non-linear change of coordinates to the definition of $\boldsymbol{\Lambda}$. Further details are provided in §3.

Suppose that the renormalizations of $H_{c_b,b}$ are given by

$$\mathbf{R}^n(\Sigma_{H_{c_b,b}}) = \Sigma_n = (A_n, B_n),$$

where A_n and B_n are defined on some fixed neighborhoods Ω and Γ of $(0, 0)$ in $\mathbb{C}^2$. Recall that A_n and B_n represent rescalings of the q_{2n+1} and q_{2n} iterates of $H_{c_b,b}$, respectively. If $H_{c_b,b}$ is sufficiently dissipative, so that $|b| < \epsilon$ for some $\epsilon < 1$, then by the chain rule, the Jacobians of A_n and B_n are on the order of $\epsilon^{q_{2n+1}}$ and $\epsilon^{q_{2n}}$, respectively. Hence, if the renormalization sequence $\{\Sigma_n\}_{n=0}^{\infty}$ converges to some limit $\Sigma_* = (A_*, B_*)$, then

$$\operatorname{Jac} A_* = \lim_{n\to\infty} O(\epsilon^{q_{2n+1}}) = 0 \quad \text{and} \quad \operatorname{Jac} B_* = \lim_{n\to\infty} O(\epsilon^{q_{2n}}) = 0.$$

Thus, we see that the limit Σ_* of the renormalizations of $H_{c_b,b}$ must be a degenerate one-dimensional system.

3. *Renormalization of almost commuting pairs*

In this section, we formalize the ideas discussed in §2. While previously, we considered any rotation number θ of bounded type, we will henceforth restrict our work to the case of the inverse golden-mean

$$\theta_* = \frac{\sqrt{5}-1}{2} = [1, 1, \ldots].$$

3.1. *One-dimensional renormalization.*

For a domain $Z \subset \mathbb{C}$, we denote by $\mathcal{A}(Z)$ the Banach space of bounded analytic functions $f : Z \to \mathbb{C}$, equipped with the norm

$$\|f\| = \sup_{x\in Z} |f(x)|.$$

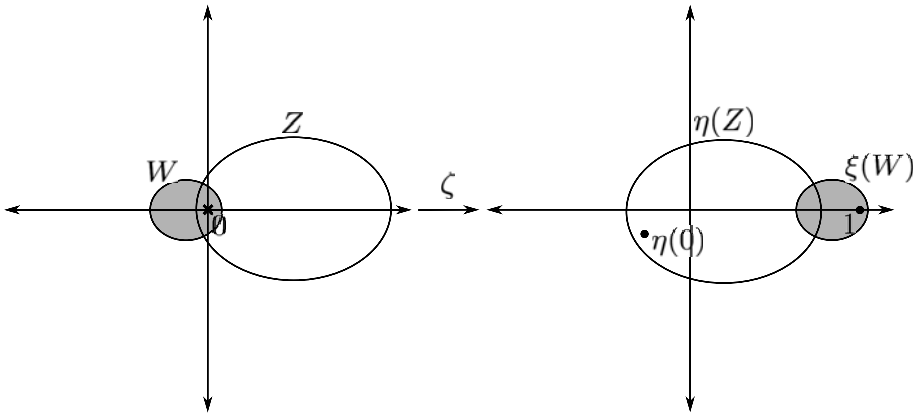

FIGURE 8. A critical pair $\zeta = (\eta, \xi) \in \mathcal{C}(Z, W)$.

Denote by $\mathcal{A}(Z, W)$ the Banach space of bounded pairs of analytic functions $\zeta = (\eta, \xi)$ from domains $Z \subset \mathbb{C}$ and $W \subset \mathbb{C}$, respectively, to $\mathbb{C}$, equipped with the norm

$$\|\zeta\| = \tfrac{1}{2}(\|\eta\| + \|\xi\|).$$

Henceforth, we assume that the domains Z and W are topological disks containing 0.

Define the *rescaling map* Λ as

$$\Lambda(\zeta) := (s_\zeta^{-1} \circ \eta \circ s_\zeta, s_\zeta^{-1} \circ \xi \circ s_\zeta) \quad \text{for } \zeta = (\eta, \xi) \in \mathcal{A}(Z, W),$$

where

$$s_\zeta(x) := \lambda_\zeta x \quad \text{and} \quad \lambda_\zeta := \xi(0).$$

We say that $\zeta = (\eta, \xi) \in \mathcal{A}(Z, W)$ is *normalized* if $\xi(0) = 1$. Note that the space of all normalized pairs is equal to $\Lambda(\mathcal{A}(Z, W))$.

Definition 3.1. A normalized pair $\zeta = (\eta, \xi) \in \Lambda(\mathcal{A}(Z, W))$ is said to be a *critical pair* if η and ξ each have a simple unique critical point at 0. The space of critical pairs in $\Lambda(\mathcal{A}(Z, W))$ is denoted by $\mathcal{C}(Z, W)$.

Definition 3.2. We say that $\zeta = (\eta, \xi) \in \mathcal{A}(Z, W)$ is a *commuting pair* if

$$\eta \circ \xi = \xi \circ \eta.$$

It turns out that requiring strict commutativity is too restrictive in the category of analytic functions. Hence, we work with the following less restrictive condition.

Definition 3.3. We say that $\zeta = (\eta, \xi) \in \mathcal{C}(Z, W)$ is an *almost commuting pair* (cf. [**Bur**, **Stir**]) if

$$\frac{d^i[\eta, \xi]}{dx^i}(0) := \frac{d^i(\eta \circ \xi - \xi \circ \eta)}{dx^i}(0) = 0 \quad \text{for } i = 0, 2.$$

The space of almost commuting pairs in $\mathcal{C}(Z, W)$ is denoted by $\mathcal{B}(Z, W)$.

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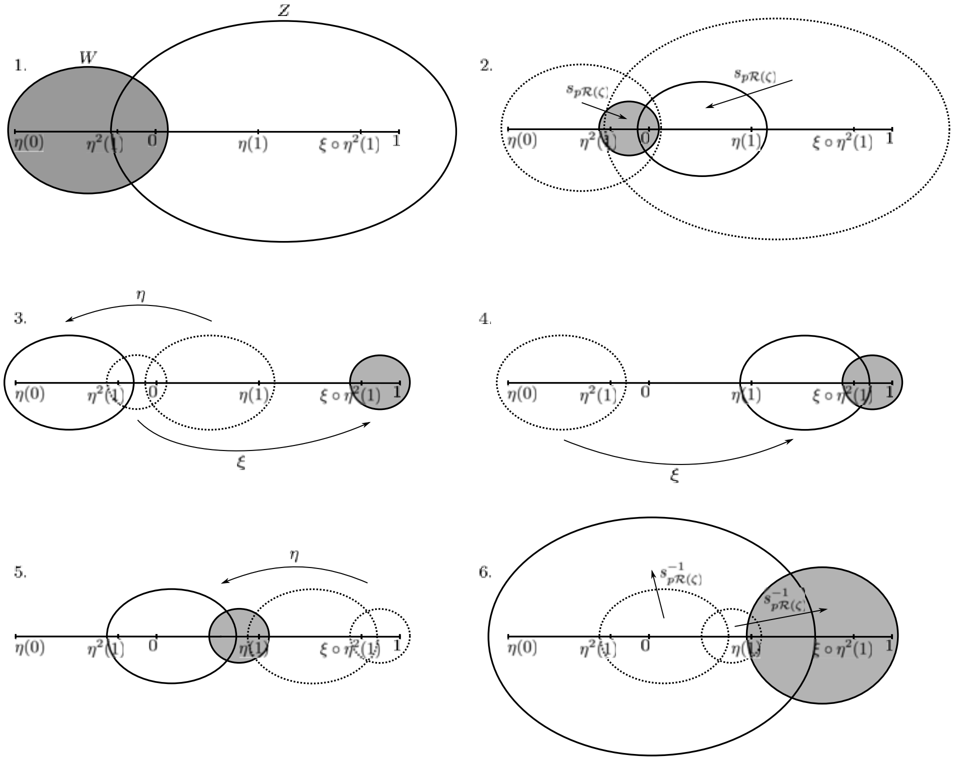

FIGURE 9. The one-dimensional renormalization $\mathcal{R}(\zeta) := \Lambda(p\mathcal{R}(\zeta)) := \Lambda((\eta\circ\xi\circ\eta, \eta\circ\xi))$.

Note that if $\zeta = (\eta, \xi)$ is a critical pair, then the first-order commuting relation is automatically satisfied: i.e.,

$$\frac{d[\eta, \xi]}{dx}(0) = \eta'(1)\xi'(0) - \xi'(\eta(0))\eta'(0) = 0.$$

It is easy to see that the following statement holds.

PROPOSITION 3.4. *The spaces $\Lambda(\mathcal{A}(Z, W))$, $\mathcal{C}(Z, W)$ and $\mathcal{B}(Z, W)$ have the structure of an immersed Banach submanifold of $\mathcal{A}(Z, W)$ of codimension one, three and five, respectively.*

Definition 3.5. Let $\zeta = (\eta, \xi) \in \mathcal{B}(Z, W)$. The *pre-renormalization* of ζ is defined as

$$p\mathcal{R}(\zeta) := (\eta\circ\xi\circ\eta, \eta\circ\xi).$$

The *renormalization* of ζ is defined as

$$\mathcal{R}(\zeta) := \Lambda(p\mathcal{R}(\zeta)).$$

We say that ζ is *renormalizable* if there exists $Z' \subset Z$ and $W' \subset W$ such that $0 \in Z' \cap W'$ and $\mathcal{R}(\zeta) \in \mathcal{B}(Z', W')$. The space of all renormalizable pairs in $\mathcal{B}(Z, W)$ is denoted by $\mathcal{D}(Z, W)$.

PROPOSITION 3.6. *The space $\mathcal{D}(Z, W)$ is an open subset of $\mathcal{B}(Z, W)$.*

Proof. Let $\zeta \in \mathcal{D}(Z, W)$, so that $\mathcal{R}(\zeta) \in \mathcal{B}(Z', W')$ for some $Z' \subset Z$ and $W' \subset W$ with $0 \in Z' \cap W'$. For any $\epsilon > 0$, it is easy to see that there exists $\delta > 0$ such that, for any $\tilde{\zeta} = (\tilde{\eta}, \tilde{\xi})$ contained in a δ-neighborhood of ζ in $\mathcal{B}(Z, W)$, we have $\mathcal{R}(\tilde{\zeta}) \in \mathcal{C}((1-\epsilon)Z', (1-\epsilon)W')$. To check almost commutativity, let

$$(\tilde{\eta}_1, \tilde{\xi}_1) := (\tilde{\eta} \circ \tilde{\xi} \circ \tilde{\eta}, \tilde{\eta} \circ \tilde{\xi}).$$

Observe that

$$[\tilde{\eta}_1, \tilde{\xi}_1](x) = \tilde{\eta}_1 \circ [\tilde{\eta}, \tilde{\xi}](x),$$

$$\frac{d[\tilde{\eta}_1, \tilde{\xi}_1]}{dx}(x) = \tilde{\eta}_1'([\tilde{\eta}, \tilde{\xi}](x)) \cdot \frac{d[\tilde{\eta}, \tilde{\xi}]}{dx}(x)$$

and

$$\frac{d^2[\tilde{\eta}_1, \tilde{\xi}_1]}{dx^2}(x) = \tilde{\eta}_1''([\tilde{\eta}, \tilde{\xi}](x)) \cdot \left(\frac{d[\tilde{\eta}, \tilde{\xi}]}{dx}(x)\right)^2 + \tilde{\eta}_1'([\tilde{\eta}, \tilde{\xi}](x)) \cdot \frac{d^2[\tilde{\eta}, \tilde{\xi}]}{dx^2}(x).$$

The result follows. □

The following theorem is shown in [**GaYa2**].

THEOREM 3.7. (One-dimensional renormalization hyperbolicity) *There exists a commuting pair $\zeta_* = (\eta_*, \xi_*) \in \mathcal{D}(Z, W)$ such that the following statements hold.*

(i) *There exist a neighborhood $\mathcal{N}$ of ζ_* in $\mathcal{D}(Z, W)$ and topological disks $Z' \Supset Z$ and $W' \Supset W$ such that $\mathcal{R} : \mathcal{N} \to \mathcal{B}(Z', W')$ is a well-defined analytic operator.*

(ii) *The pair ζ_* is the unique fixed point of $\mathcal{R}$ in $\mathcal{N}$. In particular,*

$$\lambda_*^{-1} \eta_* \circ \xi_* \circ \eta_*(\lambda_* x) = \eta_*(x) \quad \textit{and} \quad \lambda_*^{-1} \eta_* \circ \xi_*(\lambda_* x) = \xi_*(x),$$

where

$$\lambda_* := \eta_* \circ \xi_*(0)$$

is a universal scaling factor.

(iii) *The differential $D_{\zeta_*}\mathcal{R}$ is a compact linear operator. Moreover, $D_{\zeta_*}\mathcal{R}$ has a single, simple eigenvalue with modulus greater than one. The rest of its spectrum lies inside the open unit disk $\mathbb{D}$ (and hence is compactly contained in $\mathbb{D}$ by the spectral theory of compact operators).*

Let

$$f_*(z) = z^2 + c_*$$

be the quadratic polynomial with a Siegel fixed point of multiplier $\mu_* = e^{2\pi i \theta_*}$, where $\theta_* = (\sqrt{5} - 1)/2$ is the inverse golden-mean rotation number. For c sufficiently close to c_*, we can identify the quadratic polynomial f_c as a pair in $\mathcal{D}(Z, W)$ as

$$\zeta_{f_c} := \Lambda(f_c^2|_{Z_c}, f_c|_{W_c}), \tag{5}$$

where

$$Z_c := s_{f_c}(Z) = f_c(0) \cdot Z \quad \text{and} \quad W_c := s_{f_c}(W) = f_c(0) \cdot W.$$

The following theorem is shown in [**GaRYa**].

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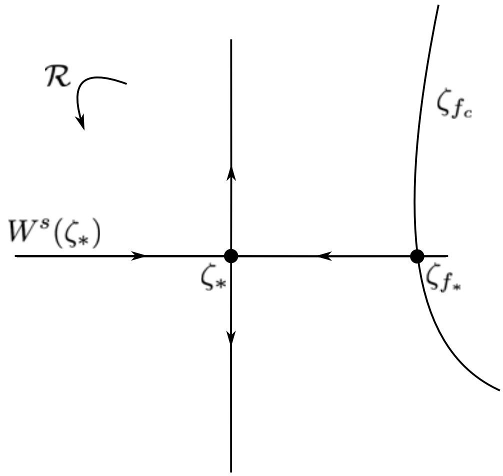

FIGURE 10. The stable manifold $W^s(\zeta_*)$ of the fixed point ζ_* for the one-dimensional renormalization operator $\mathcal{R}$. The family of quadratic polynomials ζ_{f_c} intersect $W^s(\zeta_*)$ transversely at the golden-mean Siegel quadratic polynomial ζ_{f_*}.

THEOREM 3.8. *The one-parameter family $\{\zeta_{f_c}\}_c$ intersects the stable manifold $W^s(\zeta_*) \subset \mathcal{D}(Z, W)$ of the fixed point ζ_* for the one-dimensional renormalization operator $\mathcal{R}$. Moreover, this intersection is transversal, and occurs at ζ_{f_*}.*

3.2. *Two-dimensional renormalization.* For a domain $\Omega \subset \mathbb{C}^2$, we denote by $\mathcal{A}_2(\Omega)$ the Banach space of bounded analytic functions $F : \Omega \to \mathbb{C}^2$, equipped with the norm

$$\|F\| = \sup_{(x,y)\in\Omega} \|F(x, y)\|.$$

Define

$$\|F\|_x := \sup_{(x,y)\in\Omega} \|\partial_x F(x, y)\| \quad \text{and} \quad \|F\|_y := \sup_{(x,y)\in\Omega} \|\partial_y F(x, y)\|.$$

Denote by $\mathcal{A}_2(\Omega, \Gamma)$ the Banach space of bounded pairs of analytic functions $\Sigma = (A, B)$ from domains $\Omega \subset \mathbb{C}^2$ and $\Gamma \subset \mathbb{C}^2$, respectively, to $\mathbb{C}^2$, equipped with the norm

$$\|\Sigma\| = \tfrac{1}{2}(\|A\| + \|B\|).$$

Define

$$\|\Sigma\|_x := \tfrac{1}{2}(\|A\|_x + \|B\|_x) \quad \text{and} \quad \|\Sigma\|_y := \tfrac{1}{2}(\|A\|_y + \|B\|_y).$$

Henceforth, we assume that

$$\Omega = Z \times U \quad \text{and} \quad \Gamma = W \times V,$$

where Z, U, W and V are topological disks in $\mathbb{C}$ containing 0.

Define the *projection map* π_1 as

$$\pi_1 F(x) := f_1(x, 0) \quad \text{for } F(x, y) := \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} \in \mathcal{A}_2(\Omega) \text{ or } \mathcal{A}_2(\Gamma)$$

and

$$\pi_1 \Sigma := (\pi_1 A, \pi_1 B) \quad \text{for } \Sigma = (A, B) \in \mathcal{A}_2(\Omega, \Gamma).$$

Define the *rescaling map* Λ as

$$\Lambda(\Sigma) := (s_\Sigma^{-1} \circ A \circ s_\Sigma, s_\Sigma^{-1} \circ B \circ s_\Sigma) \quad \text{for } \Sigma = (A, B) \in \mathcal{A}_2(\Omega, \Gamma),$$

where

$$s_\Sigma(x, y) := (\lambda_\Sigma x, \lambda_\Sigma y) \quad \text{and} \quad \lambda_\Sigma := \pi_1 B(0).$$

We say that $\Sigma = (A, B) \in \mathcal{A}_2(\Omega, \Gamma)$ is *normalized* if $\pi_1 B(0) = 1$. Note that the space of all normalized pairs is equal to $\Lambda(\mathcal{A}_2(\Omega, \Gamma))$.

For $\epsilon > 0$, define $\mathcal{A}_2(\Omega, \Gamma, \epsilon)$ to be the set of all pairs Σ such that

$$\|\Sigma\|_y < \min\{\epsilon, \epsilon\|\Sigma\|_x\}.$$

Clearly, $\mathcal{A}_2(\Omega, \Gamma, \epsilon)$ is an open subset of $\mathcal{A}_2(\Omega, \Gamma)$. Define $\mathcal{A}_2(\Omega, \Gamma, 0)$ to be the set of all pairs $\Sigma = (A, B)$ such that

$$\|\Sigma\|_y = 0.$$

Note that, in this case,

$$A(x, y) = \begin{bmatrix} a(x) \\ h(x) \end{bmatrix} \quad \text{and} \quad B(x, y) = \begin{bmatrix} b(x) \\ x \end{bmatrix}$$

for some $a, h \in \mathcal{A}(Z)$ and $b \in \mathcal{A}(W)$.

The following definitions are analogs of Definitions 3.1–3.3.

Definition 3.9. For $\epsilon \geq 0$, a normalized pair $\Sigma = (A, B)$ in $\Lambda(\mathcal{A}_2(\Omega, \Gamma, \epsilon))$ is said to be an *ϵ-critical pair* if $\pi_1 A$ and $\pi_1 B$ each have a simple unique critical point contained in the ϵ-neighborhood of 0 (the ϵ-neighborhood is interpreted to be $\{0\}$ if $\epsilon = 0$). The space of ϵ-critical pairs in $\Lambda(\mathcal{A}_2(\Omega, \Gamma, \epsilon))$ is denoted by $\mathcal{C}_2(\Omega, \Gamma, \epsilon)$.

Definition 3.10. We say that $\Sigma = (A, B) \in \mathcal{A}_2(\Omega, \Gamma)$ is a *commuting pair* if

$$A \circ B = B \circ A.$$

Definition 3.11. For $\epsilon > 0$, we say that $\Sigma = (A, B) \in \mathcal{C}_2(\Omega, \Gamma, \epsilon)$ is an *ϵ-almost commuting pair* if

$$\left| \frac{d^i \pi_1 [A, B]}{dx^i}(0) \right| := \left| \frac{d^i \pi_1 (A \circ B - B \circ A)}{dx^i}(0) \right| < \epsilon \quad \text{for } i = 0, 2.$$

The space of ϵ-almost commuting pairs in $\mathcal{C}_2(\Omega, \Gamma, \epsilon)$ is denoted by $\mathcal{B}_2(\Omega, \Gamma, \epsilon)$. Define $\mathcal{B}_2(\Omega, \Gamma, 0) \subset \mathcal{C}_2(\Omega, \Gamma, 0)$ by replacing '$< \epsilon$' in the above inequality with '$= 0$'.

It is easy to see that the following statement holds.

PROPOSITION 3.12. *The space $\Lambda(\mathcal{A}_2(\Omega, \Gamma))$ has the structure of an immersed Banach submanifold of $\mathcal{A}_2(\Omega, \Gamma)$ of codimension one. For $\epsilon > 0$, the spaces $\Lambda(\mathcal{A}_2(\Omega, \Gamma, \epsilon))$, $\mathcal{C}_2(\Omega, \Gamma, \epsilon)$ and $\mathcal{B}_2(\Omega, \Gamma, \epsilon)$ are open subsets of $\Lambda(\mathcal{A}_2(\Omega, \Gamma))$.*

Let $\mathcal{D}(Z, W) \subset \mathcal{B}(Z, W)$ be the set of one-dimensional renormalizable pairs. We define an embedding ι of $\mathcal{D}(Z, W)$ into the space of two-dimensional-almost commuting pairs as follows. Let $\zeta = (\eta, \xi) \in \mathcal{D}(Z, W)$, so that $\mathcal{R}(\zeta) \in \mathcal{B}(Z', W')$ for some $Z' \subset Z$ and $W' \subset W$ with $0 \in Z' \cap W'$. Define

$$\iota(\zeta) := \Lambda((A_\zeta, B_\zeta)),$$

where

$$A_\zeta(x, y) := \begin{bmatrix} \eta \circ \xi \circ \eta(x) \\ \eta(x) \end{bmatrix} \quad \text{and} \quad B_\zeta(x, y) := \begin{bmatrix} \eta \circ \xi(x) \\ x \end{bmatrix}. \tag{6}$$

Observe that

$$\pi_1 \circ \iota(\zeta) = \mathcal{R}(\zeta). \tag{7}$$

Hence, $\iota(\zeta) \in \mathcal{B}_2(Z' \times \mathbb{C}, W' \times \mathbb{C}, 0)$.

Let $\Sigma = (A, B) \in \mathcal{B}_2(\Omega, \Gamma, \epsilon)$. We define the *renormalization* $\mathbf{R}(\Sigma)$ of Σ as follows. To avoid introducing too much new notation in our discussion, we will use $C > 0$ to represent any constant which only depends on $\|\Sigma\|$ (and, in particular, does not depend on ϵ).

First, define the *pre-renormalization* of Σ as

$$p\mathbf{R}(\Sigma) = (A_1, B_1) := (B \circ A^2, B \circ A). \tag{8}$$

Next, we denote

$$a_y(x) := a(x, y),$$

and consider the non-linear changes of coordinates

$$H(x, y) := \begin{bmatrix} a_y^{-1}(x) \\ y \end{bmatrix}. \tag{9}$$

Note that this is completely analogous to the non-linear changes of coordinates used in the definition of period-doubling renormalization in [**dCLM**]. Define

$$p\tilde{\mathbf{R}}(\Sigma) = (A_2, B_2) := (H^{-1} \circ A_1 \circ H, H^{-1} \circ B_1 \circ H)$$

and

$$\tilde{\mathbf{R}}(\Sigma) = \Lambda(p\tilde{\mathbf{R}}).$$

Let

$$\zeta = (\eta, \xi) := \pi_1 \Sigma.$$

It is not hard to check that we have the estimates

$$\|\tilde{\mathbf{R}}(\Sigma)\|_y < C\epsilon^2 \quad \text{and} \quad \|\tilde{\mathbf{R}}(\Sigma) - \iota(\zeta)\| < C\epsilon. \tag{10}$$

Thus, we see that under $\tilde{\mathbf{R}}$, the y-dependence of pairs shrinks super-exponentially, and when restricted to pairs with no y-dependence, the action of $\tilde{\mathbf{R}}$ is equivalent to the action of the one-dimensional renormalization operator $\mathcal{R}$ (see (7)).

To complete the definition of the two-dimensional renormalization operator $\mathbf{R}$, we need the following two lemmas. It should be noted that similar results are proved in [**GaYa2**].

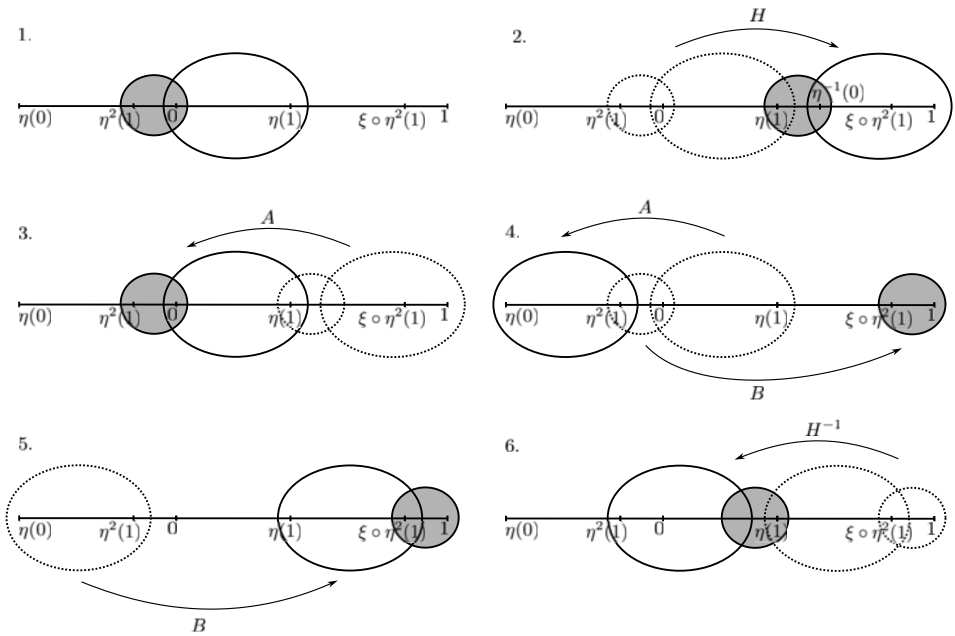

FIGURE 11. The two-dimensional pre-renormalization $p\tilde{\mathbf{R}}(\Sigma) := (H^{-1} \circ B \circ A^2 \circ H,\ H^{-1} \circ B \circ A \circ H)$.

LEMMA 3.13. *There exists an analytic projection operator Π_{crit} such that, for any ϵ-almost commuting pair $\Sigma = (A, B)$, the following statements hold.*

(i) *We have $\|\Pi_{\text{crit}}(\Sigma) - \Sigma\| < C\epsilon$.*

(ii) *If $\|\Sigma\|_y < \delta \ll \epsilon$, then $\Pi_{\text{crit}}(\Sigma)$ is a $C\delta$-critical pair.*

(iii) *Let c_{ba} be the unique simple critical point for $\pi_1(B \circ A)$. If Σ is a commuting pair, then*

$$\Pi_{\text{crit}}(\Sigma) = \Lambda((T_{ba}^{-1} \circ A \circ T_{ba},\ T_{ba}^{-1} \circ B \circ T_{ba})),$$

where

$$T_{ba}(x, y) := (x + c_{ba}, y).$$

Proof. By the argument principle, we see that $\pi_1(B \circ A) = \pi_1 B \circ \pi_1 A + O(\epsilon)$ has a simple unique critical point c_{ba} in the $C\epsilon$-neighborhood of 0. Set

$$T_{ba}(x, y) := (x + c_{ba}, y), \tag{11}$$

and let

$$\Sigma_1 = (A_1, B_1) := (T_{ba}^{-1} \circ A \circ T_{ba},\ T_{ba}^{-1} \circ B \circ T_{ba}).$$

Again by the argument principle, we see that $\pi_1(A_1 \circ B_1) = \pi_1 A \circ \pi_1 B + O(\epsilon)$ has a simple unique critical point c_{ab} in the $C\epsilon$-neighborhood of 0. Set

$$T_{ab}(x, y) := (x + c_{ab}, y),$$

and let

$$\Sigma_2 = (A_2, B_2) := (T_{ab}^{-1} \circ A_1,\ B_1 \circ T_{ab}).$$

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Define

$$\Pi_{\mathrm{crit}}(\Sigma) := \Lambda(\Sigma_2).$$

Clearly,

$$\|\Pi_{\mathrm{crit}}(\Sigma) - \Sigma\| < C\epsilon.$$

Suppose that $\|\Sigma\|_y < \delta \ll \epsilon$. Observe that

$$0 = (\pi_1(B \circ A))'(c_{ba}) = (\pi_1(B_2 \circ A_2))'(0) = \|B_2\|(\pi_1 A_2)'(0) + O(\delta)$$

and

$$0 = (\pi_1(A_1 \circ B_1))'(c_{ab}) = (\pi_1(A_2 \circ B_2))'(0) = \|A_2\|(\pi_1 B_2)'(0) + O(\delta).$$

It follows that $\Pi_{\mathrm{crit}}(\Sigma)$ is a $C\delta$-critical pair.

Lastly, assume that $\Sigma = (A, B)$ is a commuting pair. Then A_1 and B_1 would also commute. In this case, we would have

$$(\pi_1(A_1 \circ B_1))'(0) = (\pi_1(B_1 \circ A_1))'(0) = (\pi_1(B \circ A))'(c_{ba}) = 0.$$

Hence, $c_{ab} = 0$ and $\Sigma_2 = \Sigma_1$. □

LEMMA 3.14. *There exists an analytic projection operator Π_{ac} such that, for any ϵ-almost commuting pair $\Sigma = (A, B)$, the following statements hold.*

(i) *We have $\|\Pi_{\mathrm{ac}}(\Sigma) - \Sigma\| < C\epsilon$.*

(ii) *If Σ is a δ-critical pair for some $0 \le \delta \ll \epsilon$, then $\Pi_{\mathrm{ac}}(\Sigma)$ is a $C\delta$-almost commuting pair.*

(iii) *If Σ is a commuting pair, then $\Pi_{\mathrm{ac}}(\Sigma) = \Sigma$.*

Proof. Write

$$A(x, y) = \begin{bmatrix} a(x, y) \\ h(x, y) \end{bmatrix} \quad \text{and} \quad B(x, y) = \begin{bmatrix} b(x, y) \\ x \end{bmatrix}.$$

Let

$$B_1(x, y) = \begin{bmatrix} b(x, y) + cx^2 + dx^3 \\ x \end{bmatrix},$$

where c and d are constants to be determined later. Define

$$\Pi_{\mathrm{ac}}(\Sigma) := (A, B_1).$$

Observe that

$$\pi_1 B_1(0) = \pi_1 B(0) = 1.$$

We compute

$$\pi_1[A, B_1](0) = \pi_1[A, B](0) - ca(0, 0)^2 - da(0, 0)^3, \tag{12}$$

$$\frac{d\pi_1[A, B_1]}{dx}(0) = \frac{d\pi_1[A, B]}{dx}(0) - (2ca(0, 0) + 3da(0, 0)^2)(\pi_1 A)'(0) \tag{13}$$

and

$$\begin{aligned}\frac{d^2\pi_1[A, B_1]}{dx^2}(0) = \frac{d^2\pi_1[A, B]}{dx^2}&(0) - (2c + 6da(0, 0))(\pi_1 A)'(0)^2 \\ &- (2ca(0, 0) + 3da(0, 0)^2)(\pi_1 A)''(0).\end{aligned} \tag{14}$$

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Choose the constants c and d so that

$$ca(0,0)^2 + da(0,0)^3 = 0$$

and

$$\frac{d^2\pi_1[A,B]}{dx^2}(0) - (2ca(0,0) + 3da(0,0)^2)(\pi_1 A)''(0) = 0.$$

Then we have

$$|c|,\ |d| < C\left|\frac{d^2\pi_1[A,B]}{dx^2}(0)\right|.$$

In particular, $|c|, |d| < C\epsilon$, and if Σ is a commuting pair, then $c = d = 0$ and $\Pi_{\mathrm{ac}}(\Sigma) = \Sigma$.

Lastly, if Σ is a δ-critical pair, then it follows from (12), (13) and (14) that $\Pi_{\mathrm{ac}}(\Sigma)$ is a $C\delta$-almost commuting pair. □

Definition 3.15. For $\epsilon \geq 0$, let $\Sigma = (A, B) \in \mathcal{B}_2(\Omega, \Gamma, \epsilon)$. The *renormalization* of Σ is defined as

$$\mathbf{R}(\Sigma) := \Pi_{\mathrm{ac}} \circ \Pi_{\mathrm{crit}} \circ \tilde{\mathbf{R}}(\Sigma).$$

We say that Σ is *renormalizable* if there exists $\Omega' \subset \Omega$ and $\Gamma' \subset \Gamma$ such that $(0,0) \in \Omega' \cap \Gamma'$ and $\mathbf{R}(\Sigma) \in \mathcal{B}_2(\Omega', \Gamma', \delta)$ for some $\delta > 0$. The space of all renormalizable pairs in $\mathcal{B}_2(\Omega, \Gamma, \epsilon)$ is denoted by $\mathcal{D}_2(\Omega, \Gamma, \epsilon)$.

PROPOSITION 3.16. *The space $\mathcal{D}_2(\Omega, \Gamma, \epsilon)$ is an open subset of $\mathcal{B}_2(\Omega, \Gamma, \epsilon)$.*

Proof. Let $\Sigma \in \mathcal{D}_2(\Omega, \Gamma, \epsilon)$, so that $\mathcal{R}(\Sigma) \in \mathcal{B}_2(\Omega', \Gamma', \delta)$ for some constant $\delta > 0$ and some $\Omega' \subset \Omega$ and $\Gamma' \subset \Gamma$ with $(0,0) \in \Omega' \cap \Gamma'$. For any $e > 0$, it is easy to see that there exists $d > 0$ such that, for any $\tilde{\Sigma}$ contained in a d-neighborhood of Σ in $\mathcal{B}_2(\Omega, \Gamma, \epsilon)$, we have $\mathcal{R}(\tilde{\Sigma}) \in \mathcal{B}_2((1-e)\Omega', (1-e)\Gamma', (1+e)d)$. □

We now generalize Theorem 3.7 to the two-dimensional setting.

THEOREM 3.17. (Two-dimensional renormalization hyperbolicity) *Consider the one-dimensional renormalization fixed point $\zeta_* \in \mathcal{D}(Z, W)$. Let $\iota(\zeta_*) =: (A_*, B_*)$ be the embedding of ζ_* into $\mathcal{D}_2(\Omega, \Gamma, 0)$. For $\epsilon > 0$ sufficiently small, the following statements hold.*

(i) *There exists a neighborhood $\mathbf{N}$ of $\iota(\zeta_*)$ in $\mathcal{D}_2(\Omega, \Gamma, \epsilon)$, topological bidisks $\Omega' \Supset \Omega$ and $\Gamma' \Supset \Gamma$ and a constant $C \ll 1/\epsilon$ such that $\mathbf{R} : \mathbf{N} \to \mathcal{B}_2(\Omega', \Gamma', C\epsilon^2)$ is a well-defined analytic operator.*

(ii) *For $\Sigma \in \mathbf{N}$, we have*

$$\|\mathbf{R}(\Sigma) - \iota \circ \pi_1(\Sigma)\| < C\epsilon.$$

Consequently,

$$\|\pi_1 \circ \mathbf{R}(\Sigma) - \mathcal{R} \circ \pi_1(\Sigma)\| < C\epsilon.$$

(iii) *The pair $\iota(\zeta_*)$ is the unique fixed point of $\mathbf{R}$ in $\mathbf{N}$.*

(iv) *The differential $D_{\iota(\zeta_*)}\mathbf{R}$ is a compact linear operator whose spectrum coincides with that of $D_{\zeta_*}\mathcal{R}$. More precisely, let $\mathcal{N} \subset \mathcal{D}(Z, W)$ be a sufficiently small neighborhood of ζ_*. Then, in the spectral decomposition of $D_{\iota(\zeta_*)}\mathbf{R}$, the complement to the tangent space of $\iota(N)$ corresponds to the zero eigenvalue.*

Proof. Statement (ii) follows immediately from (10) and Lemmas 3.13 and 3.14.

Write

$$A_*(x, y) := \begin{bmatrix} \eta_*(x) \\ \lambda_*^{-1}\eta_*(\lambda_* x) \end{bmatrix} \quad \text{and} \quad B_*(x, y) := \begin{bmatrix} \xi_*(x) \\ x \end{bmatrix}.$$

It is easy to see that $\iota(\zeta_*)$ is a commuting pair that is fixed under $\mathbf{R}$. Moreover, $\mathbf{R}(\iota(\zeta_*))$ is defined as the restriction of some iterate of $\iota(\zeta_*)$ on topological bidisks $\lambda_*\Omega \Subset \Omega$ and $\lambda_*\Gamma \Subset \Gamma$. Hence, $\mathbf{R}(\iota(\zeta_*))$ extends to some larger topological bidisks $\Omega' \Supset \Omega$ and $\Gamma' \Supset \Gamma$. By continuity, we can assume that the same is true for nearby pairs Σ in a small neighborhood $\mathbf{N}$ of $\iota(\zeta_*)$.

The fact that the image of pairs in $\mathbf{N}$ under $\mathbf{R}$ is $C\epsilon^2$-almost commuting is given by (10) and Lemmas 3.13 and 3.14. If $\Sigma_0 \in \mathbf{N}$ is a fixed point for $\mathbf{R}$, then it follows that $\|\Sigma_0\|_y = 0$. Since $\mathbf{R}$ restricted to pairs with no y-dependence is equivalent to $\mathcal{R}$, it follows that $\Sigma_0 = \iota(\zeta_*)$ by the uniqueness of the fixed point ζ_* for $\mathcal{R}$ in $\mathcal{N}$.

Finally, let E be the quotient of the tangent space of $\mathbf{N}$ by the tangent space of $\iota(\mathcal{N})$. Let $M : E \to E$ be the operator induced by $D_{\iota(\zeta_*)}\mathbf{R}$. Then $\|M^n\| = O(\epsilon^{2^n})$, and hence the spectrum of M is equal to $\{0\}$. □

Let $H_{\mu_*,\nu}$ be the Hénon map with a semi-Siegel fixed point $\mathbf{p}$ of multipliers $\mu_* = e^{2\pi i\theta_*}$ and ν, where $\theta_* = (\sqrt{5}-1)/2$ is the inverse golden-mean rotation number and $|\nu| < \epsilon$. For μ sufficiently close to μ_*, we can identify the Hénon map $H_{\mu,\nu}$ as a pair in $\mathcal{D}_2(\Omega, \Gamma, \epsilon)$ as

$$\Sigma_{H_{\mu,\nu}} := \Lambda(H^2_{\mu,\nu}|_{\Omega_{\mu,\nu}}, H_{\mu,\nu}|_{\Gamma_{\mu,\nu}}), \tag{15}$$

where

$$\Omega_{\mu,\nu} := s_{H_{\mu,\nu}}(\Omega) = \pi_1 H_{\mu,\nu}(0)\cdot\Omega \quad \text{and} \quad \Gamma_{\mu,\nu} := s_{H_{\mu,\nu}}(\Gamma) = \pi_1 H_{\mu,\nu}(0)\cdot\Gamma.$$

The following corollary is a consequence of Theorems 3.8 and 3.17.

COROLLARY 3.18. *The two parameter family* $\{\Sigma_{H_{\mu,\nu}}\}_{\mu,\nu}$ *intersects the stable manifold* $W^s(\iota(\zeta_*)) \subset \mathcal{D}_2(\Omega, \Gamma, \epsilon)$ *of the fixed point* $\iota(\zeta_*)$ *for* $\mathbf{R}$.

4. *The combinatorics of golden-mean rotation*

In this section, we study the combinatorics of the two-dimensional renormalization defined in §3. To simplify our analysis, we model the dynamics of almost commuting pairs by rigid interval exchange maps of the inverse golden-mean rotation type.

4.1. *Pre-renormalization operator for golden-mean rotation.*

Consider $s \in (0, \theta_*]$ and $t \in (0, 1]$ such that $s/t = \theta_* = (\sqrt{5}-1)/2$. Let

$$I = [1-t-s, 1-t] \quad \text{and} \quad J = [1-t, 1].$$

Note that we have

$$|I| = s < t = |J|.$$

Define the maps $S : J \to I \cup J$ and $T : I \to J$ as

$$S(x) := x - s \quad \text{and} \quad T(x) := x + t.$$

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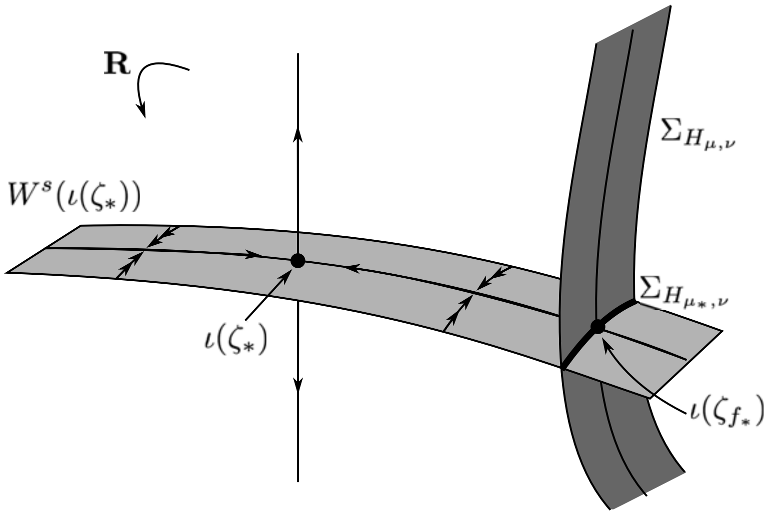

FIGURE 12. The stable manifold $W^s(\iota(\zeta_*))$ of the fixed point $\iota(\zeta_*)$ for the two-dimensional renormalization operator **R**. Every pair in $W^s(\iota(\zeta_*))$ converges super-exponentially fast to the space of degenerate one-dimensional pairs. Every degenerate one-dimensional pair in $W^s(\iota(\zeta_*))$ converges exponentially fast to $\iota(\zeta_*)$, at a rate given by Theorem 3.7. By Theorem 5.8, the family of Hénon maps $\{\Sigma_{H_{\mu,\nu}}\}_{\mu,\nu}$ intersects $W^s(\zeta_*)$ at the golden-mean semi-Siegel Hénon maps $\{\Sigma_{H_{\mu*,\nu}}\}_\nu$. Note that $\Sigma_{H_{\mu*,0}} = \iota(\zeta_{f_*})$, where ζ_{f_*} is the pair representation of the golden-mean Siegel quadratic polynomial f_*. Compare with Figure 10.

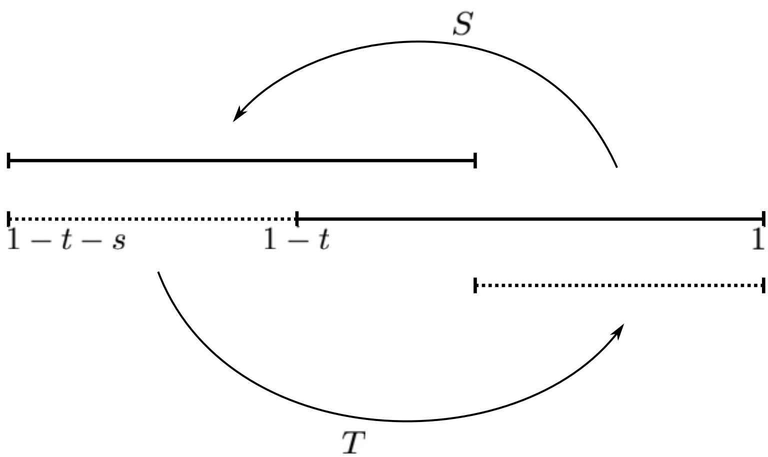

FIGURE 13. A rigid rotation pair $R = (S|_J, T|_I)$, where $I = [1-t-s, 1-t]$ and $J = [1-t, 1]$.

The action of the pair of maps $R = (S|_J, T|_I)$ on the interval $I \cup J$ represents the rigid rotation of the circle by the angle θ_*.

We define the *pre-renormalization* $p\mathcal{R}(R)$ of R as follows. Let

$$s' := 2s - t \in (0, s) \quad \text{and} \quad t' := t - s \in (0, t).$$

Then define

$$p\mathcal{R}(R) = (T \circ S^2|_{J'}, T \circ S|_{I'}),$$

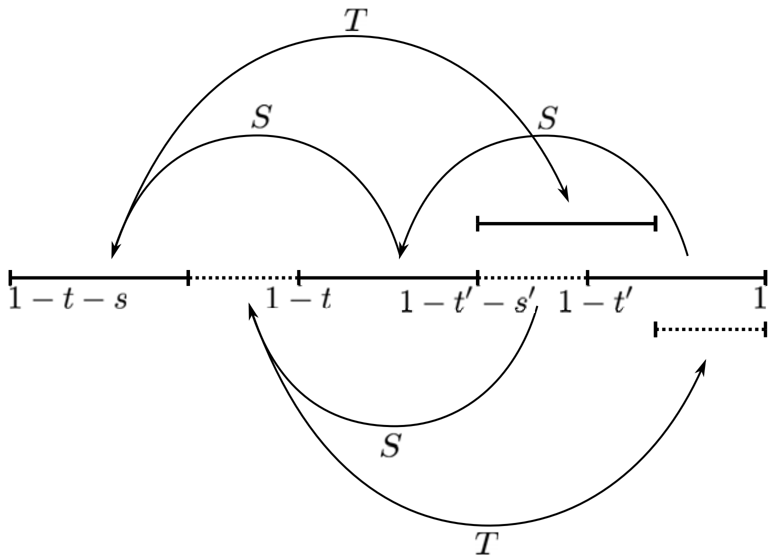

FIGURE 14. The pre-renormalization $p\mathcal{R}(R) = (T \circ S^2|_{J'}, T \circ S|_{I'})$, where $I' = [1 - t' - s', 1 - t']$ and $J' = [1 - t', 1]$.

where $I' = [1 - t' - s', 1 - t']$ and $J' = [1 - t', 1]$. Similarly to before, we have $s'/t' = \theta_*$ and

$$|I'| = s' < t' = |J'|.$$

Hence, the action of $p\mathcal{R}(R)$ on the interval $I' \cup J'$ represents the rigid rotation of the circle by the angle θ_*

4.2. *Dynamical partitions.* Set

$$s_0 := \theta_*, \quad t_0 := 1, \quad I_0 := [-\theta_*, 0] \quad \text{and} \quad J_0 := [0, 1].$$

Define

$$S_0(x) := x - \theta_*, \quad T_0(x) := x + 1, \tag{16}$$

and consider the pair $R_0 = (S_0|_{J_0}, T_0|_{I_0})$ acting on the interval $[-\theta_*, 1]$.

For $n \in \mathbb{N}$, denote the nth pre-renormalization of R_0 by

$$R_n = (S_n|_{J_n}, T_n|_{I_n}) := p\mathcal{R}^n(R_0),$$

where

$$I_n = [1 - t_n - s_n, 1 - t_n] \quad \text{and} \quad J_n = [1 - t_n, 1], \tag{17}$$

and

$$S_n(x) := x - s_n \quad \text{and} \quad T_n(x) := x + t_n.$$

Then

$$\frac{s_n}{t_n} = \theta_*. \tag{18}$$

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Notation 4.1. For $n \in \mathbb{N}$, consider an n-tuple

$$\overline{\omega} = (\alpha_{n-1}, \ldots, \alpha_0)$$

constructed inductively from $i = n - 1$ to $i = 0$ as follows.

(i) Choose $\alpha_{n-1} \in \{0, 1, 2\}$.

(ii) If $\alpha_{i+1} = 2$, then choose $\alpha_i \in \{0, 1\}$.

(iii) If α_{i+1} was chosen from $\{0, 1\}$, and $\alpha_{i+1} = 1$, then choose $\alpha_i \in \{0, 1\}$.

(iv) Otherwise, choose $\alpha_i \in \{0, 1, 2\}$.

Denote the set of all n-tuples constructed as above by $\mathcal{J}_n$. For $n = 0$, we define $\mathcal{J}_0 := \{(0)\}$.

We also denote by $\mathcal{I}_n$ the set of all n-tuples

$$\overline{\gamma} = (\beta_{n-1}, \ldots, \beta_0)$$

constructed identically as for $\mathcal{J}_n$, except that step (i) is replaced by

(i′) Choose $\beta_{n-1} \in \{0, 1\}$.

LEMMA 4.2. *Let*

$$\overline{\omega} = (\alpha_{n-1}, \ldots, \alpha_0) \in \mathcal{J}_n \quad \textit{and} \quad \overline{\gamma} = (\beta_{n-1}, \ldots, \beta_0) \in \mathcal{I}_n.$$

Denote

$$R_0^{\overline{\omega}} := S_0^{\alpha_0}|_{J_0} \circ \cdots \circ S_{n-1}^{\alpha_{n-1}}|_{J_{n-1}} \quad \textit{and} \quad R_0^{\overline{\gamma}} := S_0^{\beta_0}|_{J_0} \circ \cdots \circ S_{n-1}^{\beta_{n-1}}|_{J_{n-1}}.$$

Then $R_0^{\overline{\omega}}$ and $R_0^{\overline{\gamma}}$ are well defined on J_n and I_n, respectively.

LEMMA 4.3. *Let*

$$\overline{\omega}_n^{\max} := (2, 1, 1, \ldots, 1) \in \mathcal{J}_n \quad \textit{and} \quad \overline{\gamma}_n^{\max} := (1, 1, \ldots, 1) \in \mathcal{I}_n.$$

Then

$$R_n = p\mathcal{R}^n(R_0) = (T_0 \circ R_0^{\overline{\omega}_n^{\max}}|_{J_n}, T_0 \circ R_0^{\overline{\gamma}_n^{\max}}|_{I_n}).$$

LEMMA 4.4. *Define*

$$\mathcal{P}_n := \{R_0^{\overline{\omega}}(J_n) \mid \overline{\omega} \in \mathcal{J}_n\}$$

and

$$\mathcal{Q}_n := \{R_0^{\overline{\gamma}}(I_n) \mid \overline{\gamma} \in \mathcal{I}_n\}.$$

*Then $\mathcal{P}_n \cup \mathcal{Q}_n$ forms a cover of $[-\theta_*, 1]$ such that its members are disjoint except at the endpoints. The collection $\mathcal{P}_n \cup \mathcal{Q}_n$ is called the n*th dynamical partition *of* $[-\theta_*, 1]$.

LEMMA 4.5. *For $n \geq 0$, let $U \in \mathcal{P}_n$. Listing in order from left to right, the element U consists of one element in $\mathcal{P}_{n+1}$, one element in $\mathcal{Q}_{n+1}$ and another element in $\mathcal{P}_{n+1}$.*

Similarly, let $V \in \mathcal{Q}_n$. Listing in order from left to right, the element V consists of one element in $\mathcal{P}_{n+1}$ and one element in $\mathcal{Q}_{n+1}$.

LEMMA 4.6. *Let $\{q_n\}_{n=0}^{\infty} \subset \mathbb{N}$ be the* Fibonacci sequence *defined by the inductive relation*

$$q_0 = 1, \quad q_1 = 1 \quad \textit{and} \quad q_{n+1} = q_n + q_{n-1} \quad \textit{for } n \geq 1.$$

Then $q_{2n+1} = |\mathcal{J}_n|$ and $q_{2n} = |\mathcal{I}_n|$.

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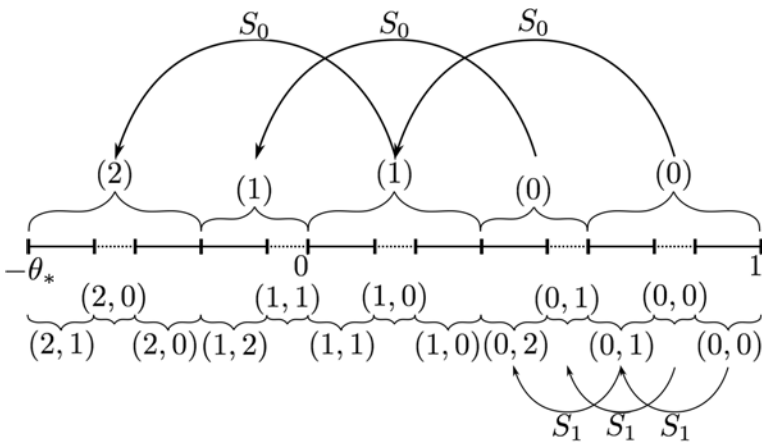

FIGURE 15. The elements of the 1st and 2nd dynamic partitions $\mathcal{P}_1 \cup \mathcal{Q}_1$ and $\mathcal{P}_2 \cup \mathcal{Q}_2$.

Define

$$Q_n := \bigcup_{V \in \mathcal{Q}_n} V.$$

By Lemmas 4.4 and 4.6, the set Q_n is a union of q_{2n} intervals of length s_n. The following result shows that these intervals are well distributed over $[-\theta_*, 1]$ in the sense that the average of any sufficiently well-behaved function on $[-\theta_*, 1]$ is approximately equal to its average on Q_n. Moreover, the error is of the same order of magnitude as s_n.

PROPOSITION 4.7. *Let $f : [-\theta_*, 1] \to \mathbb{C}$ be a piecewise-smooth function with finitely many discontinuities, whose derivative is bounded by M. Then*

$$\frac{1}{q_{2n} s_n} \int_{Q_n} f(x)\, dx = \frac{1}{1+\theta_*} \int_{-\theta_*}^{1} f(x)\, dx + O(M s_n).$$

Proof. Denote

$$m_n := \frac{1+\theta_*}{q_{2n} s_n} > 1,$$

and let $u_n : Q_n \to [-\theta_*, 1]$ be the unique surjective map satisfying the following two properties.

(i) The restriction of u_n to any element $V \in \mathcal{Q}_n$ is an affine map of the form

$$u_n|_V(x) = m_n x + b_V$$

for some $b_V \in \mathbb{R}$.

(ii) For any $x, y \in Q_n$, if $x < y$, then $u_n(x) \le u_n(y)$.

Then

$$\int_{-\theta_*}^{1} f(u_n^{-1}(x))\, dx = m_n \int_{Q_n} f(x)\, dx.$$

Write

$$m_n \int_{Q_n} f(x)\, dx = \int_{-\theta_*}^{1} f(x)\, dx + E_n,$$

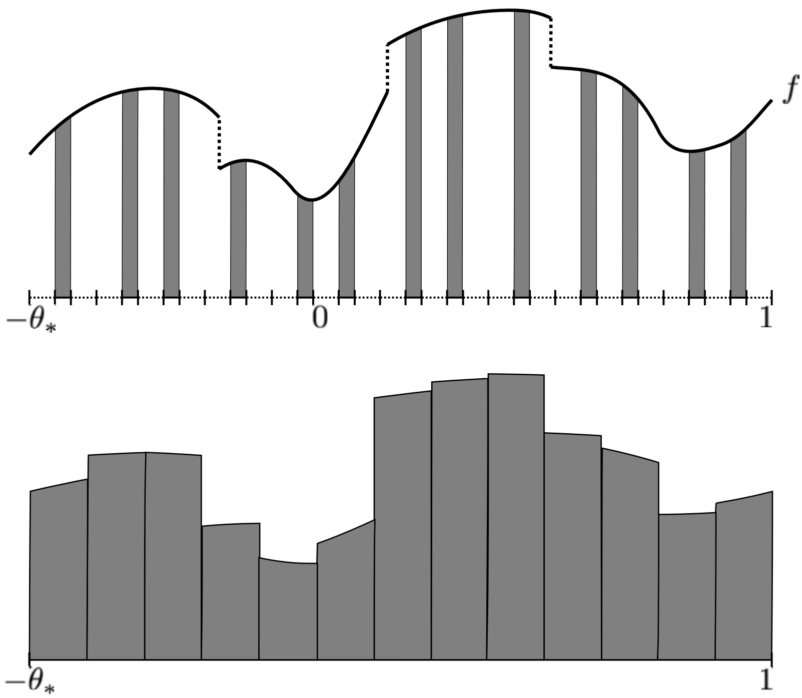

FIGURE 16. The integrals $\int_{-\theta_*}^{1} f(x)\,dx$ (top), $\int_{Q_n} f(x)\,dx$ (top, in grey) and $\int_{-\theta_*}^{1} f(u_n^{-1}(x))\,dx = m_n \int_{Q_n} f(x)\,dx$ (bottom).

where

$$E_n := \int_{-\theta_*}^{1} f(u_n^{-1}(x))\,dx - \int_{-\theta_*}^{1} f(x)\,dx.$$

Observe that

$$\begin{aligned} |E_n| &\leq \int_{-\theta_*}^{1} |f(u_n^{-1}(x)) - f(x)|dx \\ &\leq M \int_{-\theta_*}^{1} |u_n^{-1}(x) - x|dx. \end{aligned} \tag{19}$$

To estimate (19), we need to find a bound on the displacement of points under u_n.

Consider the kth dynamic partition $\mathcal{P}_k \cup \mathcal{Q}_k$ for $0 \leq k \leq n-1$. The map u_n acts by eliminating the elements that belong to $\mathcal{P}_n$ and stretching the elements that belong to $\mathcal{Q}_n$ by a factor of m_n. Denote the change in size under u_n of each element in $\mathcal{P}_k$ and $\mathcal{Q}_k$ by τ_n^k and σ_n^k, respectively, where

$$\tau_n^k := |u_n(U_k)| - |U_k| \quad \text{for any } U_k \in \mathcal{P}_k,$$

and

$$\sigma_n^k := |u_n(V_k)| - |V_k| \quad \text{for any } V_k \in \mathcal{Q}_k.$$

See Figure 17.

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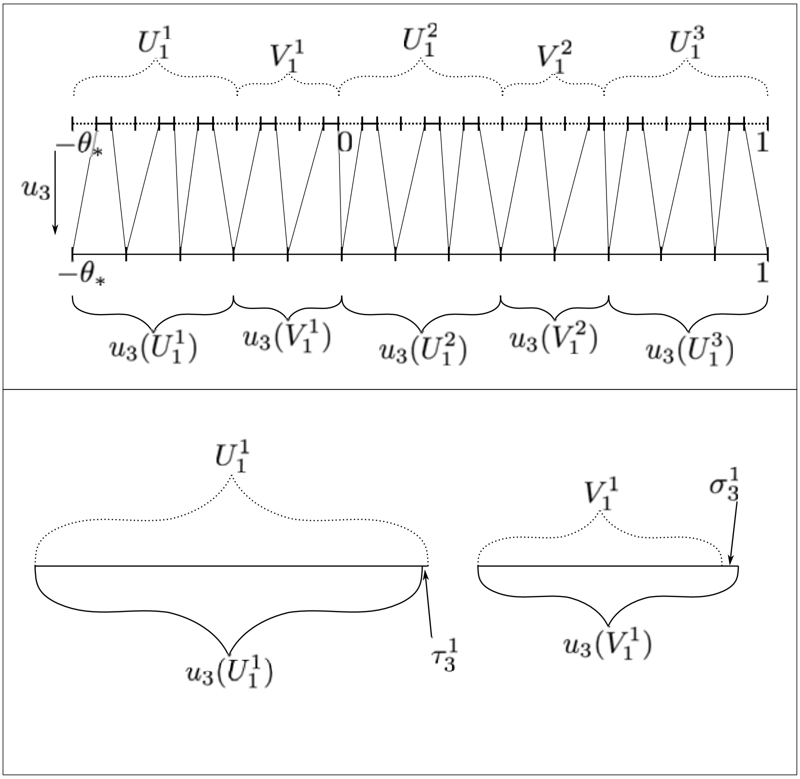

FIGURE 17. The images of the elements $U_1^1, U_1^2, U_1^3 \in \mathcal{P}_1$ and $V_1^1, V_1^2 \in \mathcal{Q}_1$ under the piece-wise affine map u_3. Under u_3, the elements of $\mathcal{P}_3$ are discarded, and the elements of $\mathcal{Q}_3$ are stretched by a factor of m_3. As a result, the elements in $\mathcal{P}_1$ shrink by τ_3^1, while the elements in $\mathcal{Q}_1$ expand by σ_3^1.

By Lemma 4.5, we have

$$\tau_n^{n-1} := (m_n - 1)s_n - 2t_n < 0 \quad \text{and} \quad \sigma_n^{n-1} := (m_n - 1)s_n - t_n > 0. \tag{20}$$

It follows that

$$|\tau_n^{n-1}| < t_n \quad \text{and} \quad |\sigma_n^{n-1}| < t_n.$$

Likewise, for $0 \le k < n-1$,

$$\tau_n^k := \sigma_n^{k+1} + 2\tau_n^{k+1} \quad \text{and} \quad \sigma_n^k := \sigma_n^{k+1} + \tau_n^{k+1}. \tag{21}$$

Note that the pairs $\{\tau_n^{k+1}, \sigma_n^{k+1}\}$ and $\{\tau_n^k, \sigma_n^k\}$ each have opposite signs. Hence, (21) implies that the pairs $\{\tau_n^k, \tau_n^{k+1}\}$ and $\{\sigma_n^k, \sigma_n^{k+1}\}$ each have the same sign, and

$$|\sigma_n^k| < |\sigma_n^{k+1}| \quad \text{and} \quad |\tau_n^k| < |\tau_n^{k+1}|.$$

Thus, by (20),

$$\sigma_n^k > 0 \quad \text{and} \quad \tau_n^k < 0 \quad \text{for all } 0 \le k \le n-1.$$

Let $\hat{x} \in Q_n$ be a point of maximum displacement under u_n: i.e.,

$$\max_{-\theta_* \le x \le 1} |u_n^{-1}(x) - x| = |\hat{x} - u_n(\hat{x})|.$$

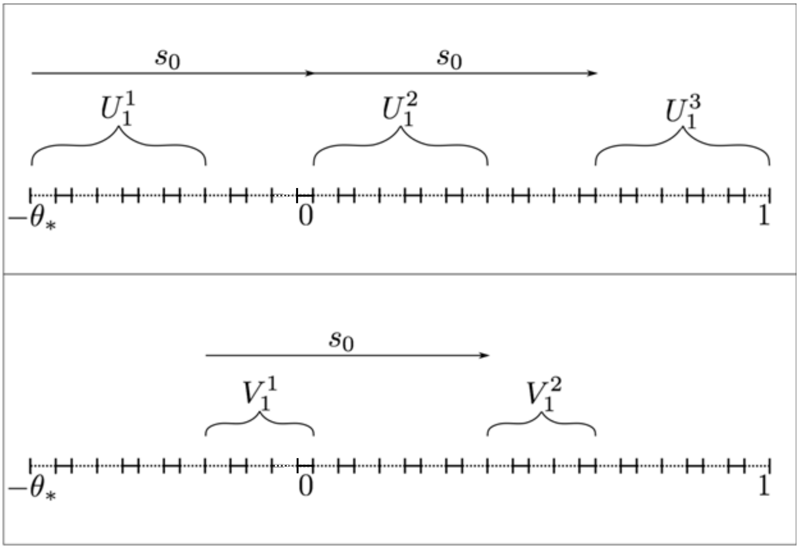

FIGURE 18. Illustration of (22) (top) and (23) (bottom).

To obtain the desired estimate on (19), we will find a bound on the displacement of $\hat{x}$ under u_n.

The interval $[-\theta_*, 0]$ is occupied by the element I_0 in $\mathcal{Q}_0$, and the interval $[0, 1]$ is occupied by the element J_0 in $\mathcal{P}_0$. By Lemma 4.5, listing from left to right, the first dynamic partition $\mathcal{P}_1 \cup \mathcal{Q}_1$ consists of $U_1^1 \in \mathcal{P}_1$, $V_1^1 \in \mathcal{Q}_1$, $U_1^2 \in \mathcal{P}_1$, $V_1^2 \in \mathcal{Q}_1$ and $U_1^3 \in \mathcal{P}_1$. Note that, for $x_1 \in U_1^1 \cap Q_n$, we have $x_1 + s_0 \in U_1^2 \cap Q_n$ and $x_1 + 2s_0 \in U_1^3 \cap Q_n$, and

$$u_n(x_1 + ls_0) = u_n(x_1) + l\sigma_n^0 \quad \text{for } l = 0, 1, 2. \tag{22}$$

Likewise, for $y_1 \in V_1^1 \cap Q_n$, we have $y_1 + s_0 \in V_1^2 \cap Q_n$, and

$$u_n(y_1 + s_0) = u_n(y_1) + \sigma_n^0. \tag{23}$$

See Figures 18 and 19. Since $\sigma_n^0 > 0$, we have the following two possibilities:

(i) $u_n(\hat{x}) - \hat{x} < 0$, and $\hat{x}$ is contained in $U_1^1 \cup V_1^1$; or

(ii) $u_n(\hat{x}) - \hat{x} > 0$, and $\hat{x}$ is contained in $V_1^2 \cup U_1^3$.

Assume case (i). Listing from left to right, the element U_1^1 consists of $U_2^1 \in \mathcal{P}_2$, $V_2 \in \mathcal{Q}_2$ and $U_2^2 \in \mathcal{P}_2$. For $x_2 \in (U_2^1 \cup V_2) \cap Q_n$, we have $x_2 + t_1 \in V_1^1 \cap Q_n$, and

$$u_n(x_2 + t_1) = u_n(x_2) + \tau_n^1. \tag{24}$$

Moreover, for $y_2 \in U_2^2 \cap Q_n$, we have $y_2 + t_2 \in V_1^1 \cap Q_n$, and

$$u_n(y_2 + t_2) = u_n(y_2) + \tau_n^2. \tag{25}$$

Since $\tau_n^1, \tau_n^2 < 0$, it follows that if $u_n(\hat{x}) - \hat{x} < 0$, then $\hat{x} \in V_1^1$. Using a similar argument and proceeding inductively, we see that $\hat{x}$ is contained in $V_{n-1} = [s_{n-1}, 0] \in \mathcal{Q}_{n-1}$. In fact, $\hat{x}$ must be equal to the left endpoint s_n of the unique element of $\mathcal{Q}_n$ contained in V_{n-1}. Thus

$$|u_n(\hat{x}) - \hat{x}| = |\sigma_n^{n-1} + t_n - \sigma_n^0| < 2t_n = 2\theta_*^{-1} s_n.$$

The desired estimate follows. See Figure 20.

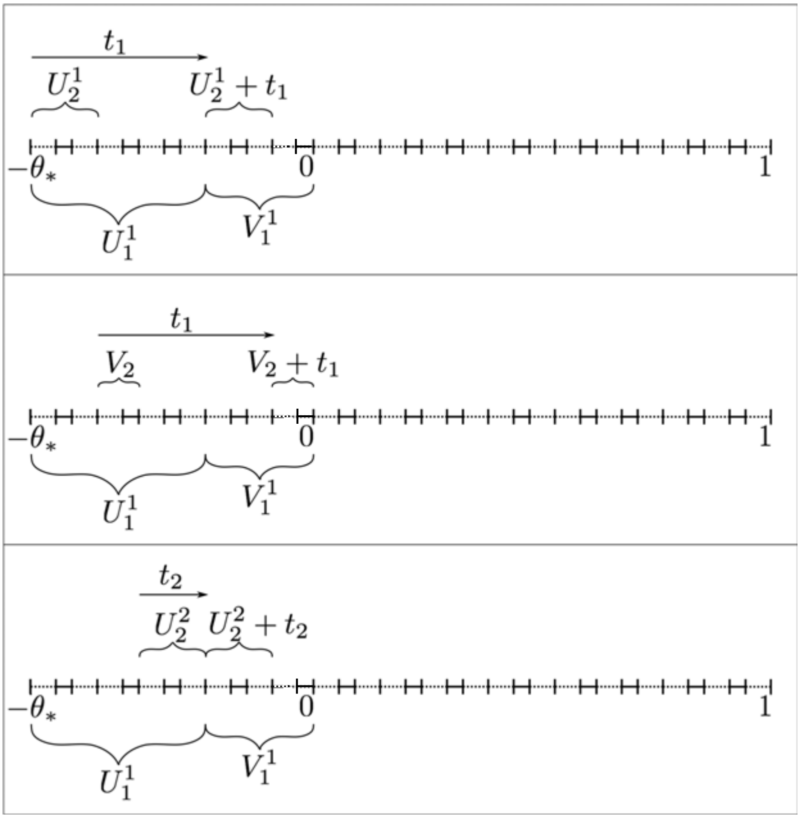

FIGURE 19. Illustration of (24) (top and middle) and (25) (bottom).

Now, assume case (ii). Arguing similarly to above and proceeding inductively, we see that $\hat{x}$ is contained in $J_{n-1} = [1 - t_{n-1}, 1] \in \mathcal{P}_{n-1}$. In fact, $\hat{x}$ must be equal to the right endpoint $1 - t_n$ of the unique element I_n of $\mathcal{Q}_n$ contained in J_{n-1}. Thus

$$|u_n(\hat{x}) - \hat{x}| = |1 - (1 - t_n)| = t_n = \theta_*^{-1} s_n.$$

The desired estimate follows. See Figure 20. □

5. *The renormalization arc*

By Theorem 3.7, the one-dimensional renormalization operator $\mathcal{R}$ has a hyperbolic fixed point $\zeta_* = (\eta_*, \xi_*)$. Consider the embedding $\iota(\zeta_*)$ of ζ_* into the space of two-dimensional-almost commuting pairs given in (6). By Theorem 3.17, $\iota(\zeta_*)$ is a hyperbolic fixed point for the two-dimensional renormalization operator $\mathbf{R}$. Moreover, $\iota(\zeta_*)$ has a codimension one stable manifold $W^s(\iota(\zeta_*))$.

Let $\Sigma = (A, B)$ be a commuting pair contained in $W^s(\iota(\zeta_*))$. Set

$$\Sigma_n = (A_n, B_n) := \mathbf{R}^n(\Sigma),$$

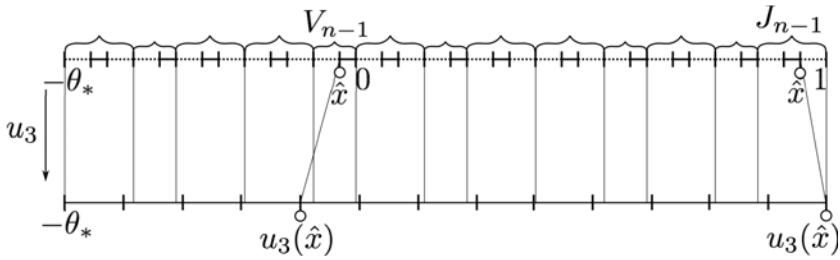

FIGURE 20. Maximal displacements under u_3. The point which moves maximally to the left is in V_{n-1} and the point which moves maximally to the right is in J_{n-1}.

where

$$A_n(x, y) = \begin{bmatrix} a_n(x, y) \\ h_n(x, y) \end{bmatrix} \quad \text{and} \quad B_n(x, y) = \begin{bmatrix} b_n(x, y) \\ x \end{bmatrix}.$$

Let

$$\eta_n(x) := a_n(x, 0), \quad \xi_n(x) := b_n(x, 0) \quad \text{and} \quad \zeta_n := (\eta_n, \xi_n).$$

Then, by Theorem 3.17,

$$\|\Sigma_{n+1} - \iota(\zeta_n)\| < O(\epsilon^{2^n}). \tag{26}$$

Denote

$$(a_n)_y(x) := a_n(x, y),$$

and let

$$H_{n+1}(x, y) := \begin{bmatrix} (a_n)_y^{-1}(x) \\ y \end{bmatrix}$$

be the non-linear changes of coordinates given in (9). If

$$\tilde{B}_{n+1} := H_{n+1}^{-1} \circ B_n \circ A_n \circ H_{n+1} \quad \text{and} \quad \tilde{\beta}_{n+1} := \pi_1 \tilde{B}_{n+1},$$

then by (10), the map $\tilde{\beta}_{n+1}$ has a unique critical point c_{n+1} near 0. Define

$$T_{n+1}(x, y) := (x + c_{n+1}, y),$$

and let

$$s_{n+1}(x, y) := (\lambda_{n+1} x, \lambda_{n+1} y), \quad |\lambda_{n+1}| < 1$$

be the scaling map so that if

$$\Phi_{n+1}(x, y) := H_{n+1} \circ T_{n+1} \circ s_{n+1}(x, y), \tag{27}$$

then we have

$$A_{n+1} = \Phi_{n+1}^{-1} \circ B_n \circ A_n^2 \circ \Phi_{n+1} \quad \text{and} \quad B_{n+1} = \Phi_{n+1}^{-1} \circ B_n \circ A_n \circ \Phi_{n+1}.$$

Note that, by the choice of T_{n+1},

$$(\pi_1 B_{n+1})'(0) = \xi_{n+1}'(0) = 0. \tag{28}$$

The following corollary is a direct consequence of Theorems 3.7 and 3.17.

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COROLLARY 5.1. *As $n \to \infty$, we have the following convergences (each of which occurs at a geometric rate):*
(i) $\zeta_n = (\eta_n, \xi_n) \to \zeta_* = (\eta_*, \xi_*)$;
(ii) $\lambda_n \to \lambda_*$, *where λ_* is the universal scaling constant given in Theorem 3.7; and*
(iii) $\Phi_n \to \Phi_*$, *where*

$$\Phi_*(x, y) = \begin{bmatrix} \phi_*(x) \\ \lambda_* y \end{bmatrix} := \begin{bmatrix} \eta_*^{-1}(\lambda_* x) \\ \lambda_* y \end{bmatrix}.$$

For $0 \leq k < l$, define the *kth microscope map of depth l* by

$$\Phi_k^l := \Phi_{k+1} \circ \Phi_{k+2} \circ \cdots \circ \Phi_l. \tag{29}$$

Let

$$\Omega_k^l := \Phi_k^l(\Omega) \quad \text{and} \quad \Gamma_k^l := \Phi_k^l(\Gamma). \tag{30}$$

Observe that $\{\Omega_k^l \cup \Gamma_k^l\}_{l=k}^{\infty}$ is a nested sequence of open sets. Moreover, for $k < m < l$, we have

$$\Phi_k^m(\Omega_m^l) = \Omega_k^l \quad \text{and} \quad \Phi_k^m(\Gamma_m^l) = \Gamma_k^l.$$

PROPOSITION 5.2. *Let λ_* be the universal scaling factor given in Corollary 5.1. Then, for all $0 \leq k < l$,*

$$\operatorname{diam}(\Omega_k^l \cup \Gamma_k^l) = O(\lambda_*^{l-k}).$$

Consequently, there exists a point $\kappa_k \in Z \subset \mathbb{C}$, called the kth cap, *such that*

$$\bigcap_{l=k+1}^{\infty} \Omega_k^l \cup \Gamma_k^l = (\kappa_k, 0).$$

It is not difficult to see that

$$\Phi_n((\kappa_n, 0)) = (\kappa_{n-1}, 0) \quad \text{and} \quad \Phi_k^l((\kappa_l, 0)) = (\kappa_k, 0). \tag{31}$$

Notation 5.3. We denote by

$$p\Sigma_n = (pA_n, pB_n) \quad \text{for } n \in \mathbb{N}$$

the sequence of pairs of iterates of $\Sigma = (A, B)$ defined as:
(i) let $p\Sigma_0 := \Sigma$; and
(ii) for $n \geq 0$, let

$$p\Sigma_{n+1} := (pB_n \circ pA_n^2, pB_n \circ pA_n).$$

Observe that if

$$\Sigma_n = (A_n, B_n) = \mathbf{R}^n(\Sigma)$$

is the nth renormalization of Σ, then

$$A_n = (\Phi_0^n)^{-1} \circ pA_n \circ \Phi_0^n \quad \text{and} \quad B_n = (\Phi_0^n)^{-1} \circ pB_n \circ \Phi_0^n. \tag{32}$$

The following statements are analogs of Lemmas 4.2 and 4.3.

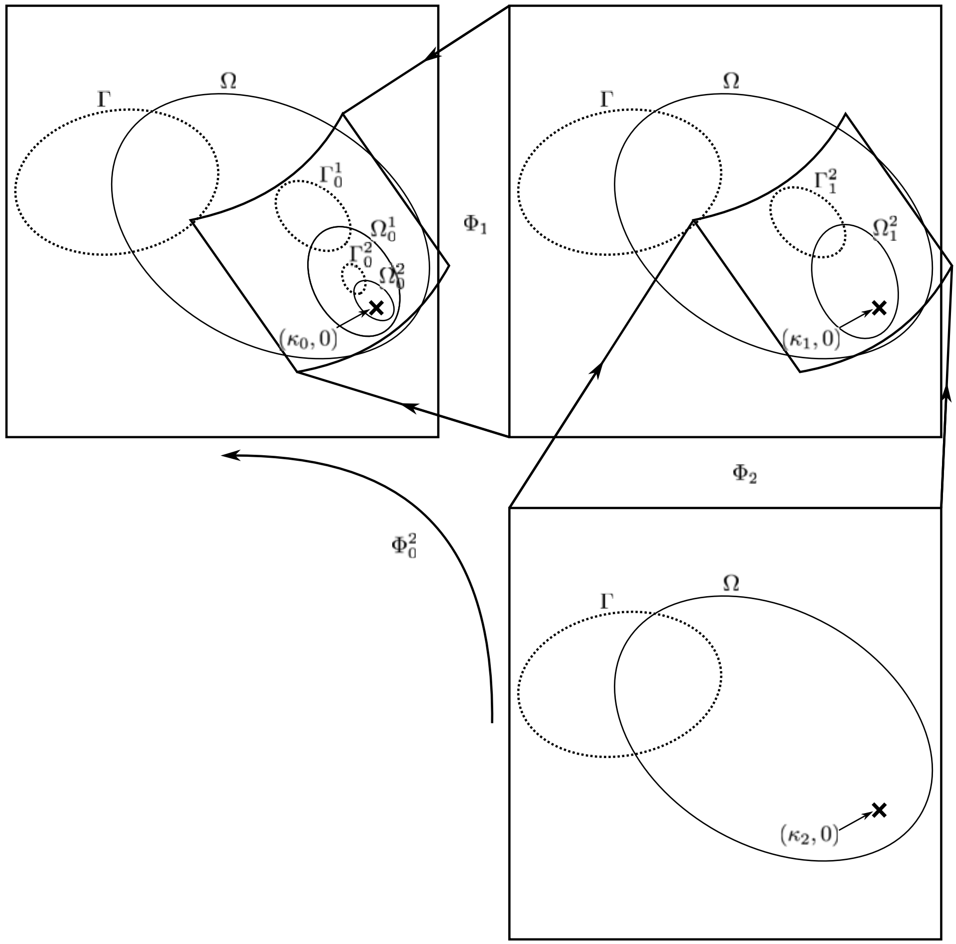

FIGURE 21. The renormalization microscope map Φ_0^2 obtained by composing the non-linear changes of coordinates Φ_1 and Φ_2. We have $\Omega_0^1 = \Phi_1(\Omega)$, $\Gamma_0^1 = \Phi_1(\Gamma)$, $\Omega_0^2 = \Phi_0^2(\Omega)$, $\Gamma_0^2 = \Phi_0^2(\Gamma)$ and $(\kappa_0, 0) = \Phi_1((\kappa_1, 0)) = \Phi_0^2((\kappa_2, 0))$.

LEMMA 5.4. *Consider the sets $\mathcal{J}_n$ and $\mathcal{I}_n$ of ordered n-tuples constructed in Notation 4.1. For*

$$\overline{\omega} = (\alpha_{n-1}, \ldots, \alpha_0) \in \mathcal{J}_n \quad \text{and} \quad \overline{\gamma} = (\beta_{n-1}, \ldots, \beta_0) \in \mathcal{I}_n,$$

denote

$$\Sigma^{\overline{\omega}} := pA_0^{\alpha_0} \circ \cdots \circ pA_{n-1}^{\alpha_{n-1}} \quad \text{and} \quad \Sigma^{\overline{\gamma}} := pA_0^{\beta_0} \circ \cdots \circ pA_{n-1}^{\beta_{n-1}}.$$

Then $\Sigma^{\overline{\omega}}$ and $\Sigma^{\overline{\gamma}}$ are well defined on Ω_0^n and Γ_0^n, respectively.

LEMMA 5.5. *Let*

$$\overline{\omega}_n^{\max} := (2, 1, 1, \ldots, 1) \in \mathcal{J}_n \quad \text{and} \quad \overline{\gamma}_n^{\max} := (1, 1, \ldots, 1) \in \mathcal{I}_n.$$

Then

$$p\Sigma_n = (pA_n, pB_n) = (B_0 \circ \Sigma^{\overline{\omega}_n^{\max}}, B_0 \circ \Sigma^{\overline{\gamma}_n^{\max}}).$$

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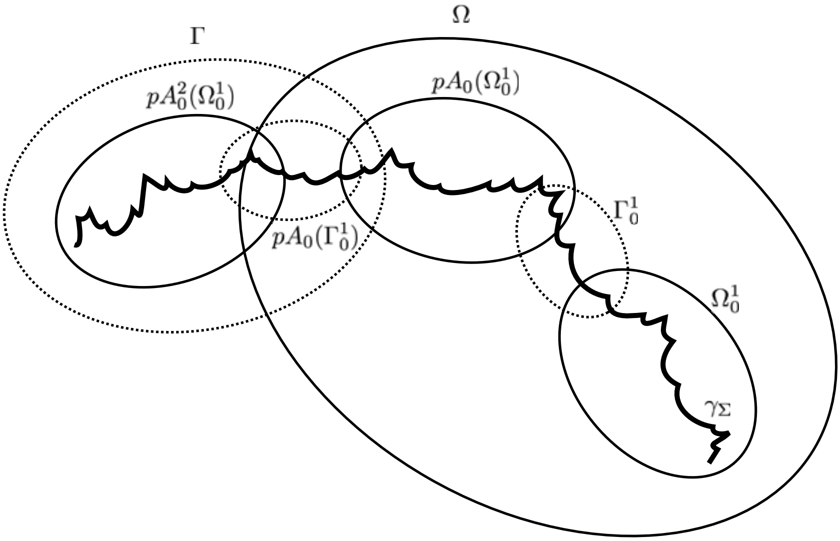

FIGURE 22. The renormalization arc γ_Σ of Σ. The open cover $X_1 \cup Y_1$ is shown.

Definition 5.6. For $n \in \mathbb{N}$, let

$$X_n := \bigcup_{\overline{\omega} \in \mathcal{J}_n} \Sigma^{\overline{\omega}}(\Omega_0^n) \quad \text{and} \quad Y_n := \bigcup_{\overline{\gamma} \in \mathcal{I}_n} \Sigma^{\overline{\gamma}}(\Gamma_0^n).$$

The set

$$\gamma_\Sigma := \bigcap_{n=1}^{\infty} X_n \cup Y_n$$

is called the *renormalization arc* of Σ.

The following theorem justifies the use of the term 'arc' in Definition 5.6. It is the counterpart to [**GaRYa**, Proposition 4.2].

THEOREM 5.7. (Continuity of the Siegel boundary) *Let $R_0 = (S_0|_{J_0}, T_0|_{I_0})$ be the pair representing the rigid rotation of the circle by θ_*, as given by (16). Then there exists a homeomorphism $h : [-\theta_*, 1] \to \gamma_\Sigma$ that conjugates the action of Σ and the action of R_0.*

Proof. The proof is identical, *mutatis mutandis*, to the proof of Proposition 4.2 in [**GaRYa**]. For the reader's convenience, we will outline the main ideas.

The renormalization arc γ_{Σ_k} of the kth renormalization of Σ maps into γ_Σ under the microscope map Φ_0^k. For k sufficiently high, Σ_k is in a small neighborhood of the renormalization fixed point $\iota(\zeta_*)$. For all such pairs, the maps $\Sigma_k^{\overline{\omega}} \circ \Phi_k^{k+n}$ for $\overline{\omega} \in \mathcal{J}_n$ and $\Sigma_k^{\overline{\gamma}} \circ \Phi_k^{k+n}$ for $\overline{\gamma} \in \mathcal{I}_n$ have derivatives bounded above by $C\rho^n$ for some uniform constants $C > 1$ and $\rho < 1$. It readily follows that the theorem holds for γ_{Σ_k}, and hence also for γ_Σ. □

Henceforth, we consider the renormalization arc of Σ as a continuous curve $\gamma_\Sigma = \gamma_\Sigma(t)$ parameterized by the homeomorphism $h : [-\theta_*, 1] \to \gamma_\Sigma$ given in Theorem 5.7.

The following theorem is the counterpart to [**GaRYa**, Proposition 4.6]. The proof is identical, *mutatis mutandis*, and hence it will be omitted.

THEOREM 5.8. *The pair $\Sigma_{H_{\mu_*,\nu}}$ representing the semi-Siegel Hénon map $H_{\mu_*,\nu}$ given in (15) is contained in the stable manifold $W^s(\iota(\zeta_*)) \subset \mathcal{D}_2(\Omega, \Gamma, \epsilon)$ of the fixed point $\iota(\zeta_*)$ for* **R**. *Moreover, a linear rescaling of the renormalization arc $s(\gamma_{\Sigma_{H_{\mu_*,\nu}}})$ is contained in the boundary of the Siegel disc $\mathcal{D}$ of $H_{\mu_*,\nu}$. In fact,*

$$\partial\mathcal{D} = s(\gamma_{\Sigma_{H_{\mu_*,\nu}}}) \cup H_{\mu_*,\nu} \circ s(\gamma_{\Sigma_{H_{\mu_*,\nu}}}).$$

6. *Limit of the microscope maps*

Consider the microscope maps $\Phi_k^l : \Omega \cup \Gamma \to \Omega$ given in (29). By Proposition 5.2, Φ_k^l converges to the constant map $(x, y) \mapsto (\kappa_k, 0)$ as l goes to ∞. In this section, we show that, in the x-coordinate, Φ_k^l behaves asymptotically like the $(l-k)$th iterate of the map $\phi_*(x) := \eta_*^{-1}(\lambda_* x)$ given in Corollary 5.1.

PROPOSITION 6.1. *The map $\phi_* : Z \to Z$ has an attracting fixed point at* 1 *with multiplier λ_*^2.*

Proof. Recall that

$$\lambda_* := \eta_* \circ \xi_*(0) = \eta_*(1).$$

Immediately, we see that the map

$$\phi_*(x) := \eta_*^{-1}(\lambda_* x)$$

fixes the point 1. Moreover, since $\phi_*(Z) \Subset Z$, this fixed point must be attracting.

Since ξ_* has a critical point at 0, we may write

$$\xi_*(x) = 1 + c_* x^2 + O(|x|^3)$$

for some $c_* \in \mathbb{C}$. Thus,

$$\lambda_* \xi_*(x) = \lambda_* + \lambda_* c_* x^2 + O(|x|^3) \quad \text{and} \quad \xi_*(\lambda_* x) = 1 + \lambda_*^2 c_* x^2 + O(|x|^3).$$

Since $\zeta_* = (\eta_*, \xi_*)$ is a renormalization fixed point,

$$\lambda_* \xi(x) = \eta_* \circ \xi_*(\lambda_* x) = \lambda_* + \eta_*'(1)\lambda_*^2 c_* x^2 + O(|x|^3).$$

Therefore

$$\eta_*'(1) = \lambda_*^{-1},$$

and we conclude that

$$\phi_*'(1) = \frac{\lambda_*}{\eta_*'(1)} = \lambda_*^2. \qquad \square$$

Let $t_*(x) := x + 1$ be the translation by 1, and define

$$\check{\phi}_* := t_*^{-1} \circ \phi_* \circ t_*.$$

Since $\check{\phi}_*$ has an attracting fixed point at 0 of multiplier λ_*^2, the sequence $\lambda_*^{-2n}\check{\phi}_*^n$ converges to the linearizing map $u_* : t_*^{-1}(Z) \to \mathbb{C}$ for $\check{\phi}_*$ at 0 as $n \to \infty$.

Let

$$\hat{\phi}_n := \pi_1 \Phi_n.$$

For $0 \leq k < l$, define

$$\hat{\phi}_k^l := \hat{\phi}_{k+1} \circ \hat{\phi}_{k+2} \circ \cdots \circ \hat{\phi}_l.$$

It is not difficult to see that

$$\hat{\phi}_k^l = \pi_1 \Phi_k^l.$$

It follows from (31) that

$$\hat{\phi}_n(\kappa_n) = \kappa_{n-1} \quad \text{and} \quad \hat{\phi}_k^l(\kappa_l) = \kappa_k.$$

Denote

$$d_n := \hat{\phi}_n'(\kappa_n) \quad \text{and} \quad d_k^l := (\hat{\phi}_k^l)'(\kappa_l) = d_{k+1} d_{k+2} \ldots d_l.$$

PROPOSITION 6.2. *As $n \to \infty$, we have the following convergences (each of which occurs at a geometric rate):*

(i) $\hat{\phi}_n \to \phi_*$;

(ii) $\kappa_n \to 1$; *and*

(iii) $d_n \to \lambda_*^2$.

Let $t_n(x) := x + \kappa_n$ be the translation by κ_n, and define

$$\check{\phi}_n := t_{n-1}^{-1} \circ \hat{\phi}_n \circ t_n.$$

Observe that 0 is an attracting fixed point for $\check{\phi}_n$ of multiplier d_n. For $0 \leq k < l$, define

$$\check{\phi}_k^k := \mathrm{Id} \quad \text{and} \quad \check{\phi}_k^l := \check{\phi}_{k+1} \circ \check{\phi}_{k+2} \circ \cdots \circ \check{\phi}_l = t_k^{-1} \circ \hat{\phi}_k^l \circ t_l.$$

PROPOSITION 6.3. *For $k \geq 0$, we have the convergence*

$$(d_k^l)^{-1} \check{\phi}_k^l \to u_* \quad \textit{as } l \to \infty,$$

where u_* is the linearizing map for $\check{\phi}_*$ at 0.

Proof. For $k < m \leq l$, define

$$e_m^l(x) := \lambda_*^{-2(l-m)} d_m^l x.$$

By Proposition 6.2(iii),

$$\lambda_*^{-2(l-m)} d_m^l = 1 + O(\rho_1^{m+1})$$

for some uniform constant $\rho_1 < 1$. It follows that

$$(e_m^l)^{-1} \circ \check{\phi}_m \circ (e_m^l) = \check{\phi}_m + O(\rho_1^{m+1}).$$

By Proposition 6.2(iv), we may write

$$d_m^{-1} (e_m^l)^{-1} \circ \check{\phi}_m \circ (e_m^l) = \lambda_*^{-2} \check{\phi}_* + E_m,$$

where

$$\|E_m\| = O(\rho_2^m)$$

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for some uniform constant $\rho_2 < 1$. Since

$$d_m^{-1}\check{\phi}_m'(0) = 1 = \lambda_*^{-2}\check{\phi}_*'(0),$$

we have

$$E_m'(0) = 0.$$

It follows from Cauchy-estimates that

$$\|E_m(x)\| = O(\rho_2^m|x|^2)$$

for all x such that $|x|$ is sufficiently small.

Let

$$\rho = \max\{|\lambda_*^2|, \rho_2\}.$$

Observe that

$$\begin{aligned}
&(d_k^l)^{-1}\check{\phi}_k^{m-1} \circ (\check{\phi}_m \circ e_m^l) \circ \check{\phi}_*^{l-m} \\
&\quad = (d_k^l)^{-1}\check{\phi}_k^{m-1} \circ (e_m^l \circ (e_m^l)^{-1} \circ \check{\phi}_m \circ e_m^l) \circ \check{\phi}_*^{l-m} \\
&\quad = (d_k^l)^{-1}\check{\phi}_k^{m-1} \circ (e_{m-1}^l \circ \check{\phi}_* + d_m E_m) \circ \check{\phi}_*^{l-m} \\
&\quad = (d_k^l)^{-1}\check{\phi}_k^{m-1} \circ e_{m-1}^l \circ \check{\phi}_*^{l-m+1} + O(\|(d_k^l)^{-1}\check{\phi}_k^{m-1} \circ d_m E_m \circ \check{\phi}_*^{l-m}\|) \\
&\quad = (d_k^l)^{-1}\check{\phi}_k^{m-1} \circ e_{m-1}^l \circ \check{\phi}_*^{l-m+1} + O(\rho^{k-l}\rho^{m-1-k}\rho\rho^m\rho^{2(l-m)}) \\
&\quad = (d_k^l)^{-1}\check{\phi}_k^{m-1} \circ e_{m-1}^l \circ \check{\phi}_*^{l-m+1} + O(\rho^l).
\end{aligned}$$

By induction,

$$(d_k^l)^{-1}\check{\phi}_k^l = \lambda_*^{-2(l-k)}\check{\phi}_*^{l-k} + O((l-k)\rho^l).$$

The result follows. □

For $0 \le k < l$, define

$$\lambda_k^l := \lambda_{k+1}\lambda_{k+2}\cdots\lambda_l.$$

COROLLARY 6.4. *For $k \ge 0$, we have the convergence*

$$(d_k^l\lambda_k^l)^{-1}\,\mathrm{Jac}\,\Phi_k^l(x, y) \xrightarrow{l\to\infty} u_*'(x) \quad \text{for } (x, y) \in \Omega \cup \Gamma.$$

Proof. Write

$$\Phi_n(x, y) = \begin{bmatrix} \phi_n(x, y) \\ \lambda_n y \end{bmatrix},$$

so that $\phi_n(x, 0) = \hat{\phi}_n(x)$. By (26) and the definition of H_n and Φ_n,

$$\|\phi_n\|_y = O(|\lambda_n|\epsilon^{2^n}). \tag{33}$$

Moreover, by Corollary 5.1,

$$\|\partial_x\phi_n\| = O(|\lambda_*^2|). \tag{34}$$

For $k \le m < l$, let $\phi_m^l(x, y)$ be the first coordinate of $\Phi_m^l(x, y)$. Then the following inductive relation holds: i.e.,

$$\phi_m^l(x, y) = \phi_{m+1}(\phi_{m+1}^l(x, y), \lambda_{m+1}^l y). \tag{35}$$

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Hence,

$$\|\phi^l_m\|_y = |\lambda^l_m|\epsilon^{2^{m+1}} + |d_{m+1}|\|\phi^l_{m+1}\|_y.$$

By induction,

$$\begin{aligned}\|\phi^l_m\|_y &= O(|\lambda^l_m|\epsilon^{2^{m+1}} + |d_{m+1}\lambda^l_{m+1}|\epsilon^{2^{m+2}} + \cdots + |d^{l-1}_m\lambda_l|\epsilon^{2^l}) \\ &= O(\rho^{l-m}\epsilon^{2^m})\end{aligned}$$

for some uniform constant ρ such that $|\lambda_*| < \rho < 1$. This means that

$$\phi^l_m(x, y) = \hat{\phi}_m(x) + O(\rho^{l-m}\epsilon^{2^m}). \tag{36}$$

Differentiating (35) with respect to x, we obtain

$$\partial_x\phi^l_m(x, y) = \partial_x\phi_{m+1}(\phi^l_{m+1}(x, y), \lambda^l_{m+1}y) \cdot \partial_x\phi^l_{m+1}(x, y).$$

Note that

$$\begin{aligned}\partial_x\phi_m(\phi^l_m(x, y), \lambda^l_m y) &= \frac{d\hat{\phi}_m}{dx} \circ \phi^l_m(x, y) + O(|\lambda^l_{m-1}|\epsilon^{2^m}) \\ &= \frac{d\hat{\phi}_m}{dx} \circ \hat{\phi}^l_m(x) + O(\rho^{l-m+2}\epsilon^{2^m}) + O(|\lambda^l_{m-1}|\epsilon^{2^m}) \\ &= \frac{d\hat{\phi}_m}{dx} \circ \hat{\phi}^l_m(x) + O(\rho^{l-m}e^{2^m}),\end{aligned}$$

where in the first equality we used (33) and in the second equality we used (34) and (36). By induction,

$$\partial_x\phi^l_k(x, y) = \frac{d\hat{\phi}^l_k}{dx}(x)(1 + O(\rho^{l-k}e^{2^k})).$$

Thus, by Proposition 6.3,

$$(d^l_k)^{-1}\partial_x\phi^l_k(x, y) \xrightarrow{l\to\infty} u'_*(x).$$

The result follows. □

7. *Universality*

Let $\Sigma = (A, B)$ be commuting pair contained in the stable manifold $W^s(\iota(\zeta_*))$ of the two-dimensional renormalization fixed point $\iota(\zeta_*)$. Moreover, assume that there exists a constant δ such that the following estimates hold: i.e.,

$$0 \neq \max_{z\in\gamma_\Sigma} \|\text{Jac } A(z)\| < \delta \quad \text{and} \quad \min_{z\in\gamma_\Sigma} \|\text{Jac } B(z)\| > \delta. \tag{37}$$

Note that these assumptions hold for the pair $\Sigma_{\mu_*,\nu}$ representing the semi-Siegel Hénon map $H_{\mu_*,\nu}$ given in (15).

By (37), we may choose a branch of the logarithm so that the complex-valued function

$$\tau(t) := \begin{cases} \log \text{Jac } A(h(t)), & 0 < t \leq 1, \\ \log \text{Jac } B(h(t)), & -\theta_* \leq t \leq 0, \end{cases}$$

where $h : [-\theta_*, 1] \to \gamma_\Sigma$ is the parameterization of the renormalization arc γ_Σ given in Theorem 5.7 is well defined. We define the *average Jacobian* of Σ to be the complex number

$$b = b_\Sigma := \exp\left(\frac{1}{1 + \theta_*}\int_{-\theta_*}^{1} \tau(t)\,dt\right). \tag{38}$$

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Consider the iterate $p\Sigma_n$ of Σ given in Notation 5.3. Proposition 5.2, Lemmas 5.4 and 5.5 and standard distortion estimates imply the following lemma.

LEMMA 7.1. *There exists a uniform constant* $\rho < 1$ *such that*

$$\frac{\text{Jac } pB_n(z_1)}{\text{Jac } pB_n(z_2)} = 1 + O(\rho^n)$$

for any $z_1, z_2 \in \Gamma_0^n$.

PROPOSITION 7.2. *Let* $\rho < 1$ *be as in Lemma 7.1. Then*

$$J_n(z) := \text{Jac } pB_n(z) = e^{r_n} b^{q_{2n}} (1 + O(\rho^n)) \quad \textit{for } z \in \Gamma_0^n,$$

where $q_{2n} = |\mathcal{I}_n|$ *is given in Lemma 4.6, and* $r_n \in \mathbb{C}$ *has a uniform upper bound.*

Proof. By Proposition 4.7,

$$\log b = \frac{1}{1+\theta_*} \int_{-\theta_*}^{1} \tau(t)\, dt = \frac{1}{q_{2n} s_n} \int_{Q_n} \tau(t)\, dt + O(s_n).$$

Now, there exists a point x in the interval $I_n := [1 - t_n - s_n, 1 - t_n]$ (see (17)) such that for

$$w := h(x) \in \Gamma_0^n \cap \gamma_\Sigma,$$

we have

$$\int_{Q_n} \tau(t)\, dt = \int_{h(I_n)} \log \text{Jac } pB_n(z)\, dz = s_n \log \text{Jac } pB_n(w).$$

Hence,

$$q_{2n} \log b = \log \text{Jac } pB_n(w) + O(q_{2n} s_n).$$

Observe that

$$q_{2n} s_n < 1 + \theta_*.$$

The result now follows from Lemma 7.1. □

Set

$$\Sigma_n = (A_n, B_n) := \mathbf{R}^n(\Sigma),$$

where

$$A_n(x, y) = \begin{bmatrix} a_n(x, y) \\ h_n(x, y) \end{bmatrix} \quad \text{and} \quad B_n(x, y) = \begin{bmatrix} b_n(x, y) \\ x \end{bmatrix}.$$

Let

$$\eta_n(x) := a_n(x, 0), \quad \xi_n(x) := b_n(x, 0) \quad \text{and} \quad \zeta_n := (\eta_n, \xi_n).$$

By Theorem 3.17, we know that the renormalization sequence Σ_{n+1} approaches the sequence of embeddings $\iota(\zeta_n)$ super-exponentially fast. The following result, which is central to this paper, states that, during this process, the renormalization sequence uniformizes to a certain two-dimensional *universal* form.

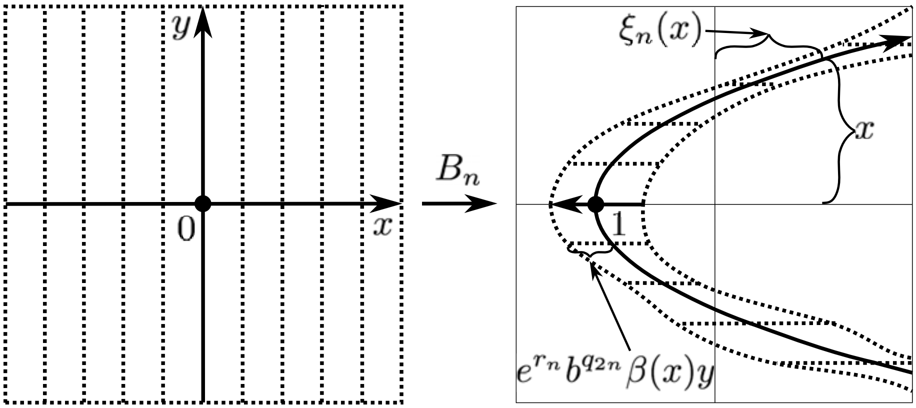

FIGURE 23. The universal two-dimensional form of the nth renormalization $\Sigma_n = (A_n, B_n)$ of the commuting pair $\Sigma = (A, B)$. Unlike in Figure 1, the vertical lines are not scaled by the same constant. However, the scaling factor $e^{r_n} b^{q_{2n}} \beta(x)$ is bounded away from 0 and ∞ and has bounded distortion.

THEOREM 7.3. (Universality) *For some $\rho < 1$,*

$$B_n(x, y) = \begin{bmatrix} \xi_n(x) + e^{r_n} b^{q_{2n}}\ \beta(x)\ y\ (1 + O(\rho^n)) \\ x \end{bmatrix},$$

where b is the average Jacobian, $\{q_i\}_{i=0}^{\infty} \subset \mathbb{N}$ is the Fibonacci sequence, $\{r_i\}_{i=0}^{\infty} \subset \mathbb{C}$ is a uniformly bounded sequence that depends on the pair Σ, and $\beta(x)$ is a universal function that is uniformly bounded away from 0 and ∞ and which has a uniformly bounded derivative and distortion.

Proof. Recall that

$$B_n(x, y) = (\Phi_0^n)^{-1} \circ pB_n \circ \Phi_0^n(x, y).$$

See (32). Hence,

$$\text{Jac } B_n(x, y) = J_n(x, y) \frac{\text{Jac } \Phi_0^n(x, y)}{\text{Jac } \Phi_0^n(B_n(x, y))}, \tag{39}$$

where J_n is the Jacobian of pB_n given in Proposition 7.2. By Corollary 6.4,

$$\frac{\text{Jac } \Phi_0^n(x, y)}{\text{Jac } \Phi_0^n(B_n(x, y))} \to \frac{u_*'(x)}{u_*'(\xi_*(x))} =: \beta(x) \quad \text{as } n \to \infty.$$

Note that the convergence is geometric and that β has the properties claimed in the theorem.

Now write

$$B_n(x, y) = \begin{bmatrix} \xi_n(x) + E_n(x, y) \\ x \end{bmatrix},$$

where E_n is undetermined. Since

$$\partial_y E_n(x, y) = \text{Jac } B_n(x, y),$$

plugging in (39) and integrating both sides, we obtain the desired formula. □

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COROLLARY 7.4. *For some* $\rho < 1$,

$$a_n(x, y) = \eta_n(x) + e^{r_n} b^{q_{2n}} \alpha(x)\, y\, (1 + O(\rho^n)),$$

where a_n *is the first coordinate of* A_n*,* b *is the average Jacobian,* $\{q_i\}_{i=0}^{\infty} \subset \mathbb{N}$ *is the Fibonacci sequence,* $\{r_i\}_{i=0}^{\infty} \subset \mathbb{C}$ *is a uniformly bounded sequence that depends on the pair* Σ*, and* $\alpha(x)$ *is a universal function that is uniformly bounded away from* 0 *and* ∞ *and which has a uniformly bounded derivative and distortion.*

Proof. Recall that

$$A_{n+1} = \Phi_{n+1}^{-1} \circ B_n \circ A_n^2 \circ \Phi_{n+1} \quad \text{and} \quad B_{n+1} = \Phi_{n+1}^{-1} \circ B_n \circ A_n \circ \Phi_{n+1}.$$

Since $\Sigma = (A, B)$ is a commuting pair,

$$A_{n+1} = \Phi_{n+1}^{-1} \circ A_n \circ B_n \circ A_n \circ \Phi_{n+1} = \Phi_{n+1}^{-1} \circ A_n \circ \Phi_{n+1} \circ B_{n+1}.$$

Let

$$H_{n+1}(x, y) = \begin{bmatrix} (a_n)_y^{-1}(x) \\ y \end{bmatrix},$$

and

$$s_{n+1}(x, y) := (\lambda_{n+1}x, \lambda_{n+1}y) \quad \text{and} \quad T_{n+1}(x, y) = (x + c_{n+1}, y),$$

where c_{n+1} is the critical point of $\pi_1(H_{n+1}^{-1} \circ B_n \circ A_n \circ H_{n+1})$ (see (27)). Then

$$\Phi_{n+1} := H_{n+1} \circ T_{n+1} \circ s_{n+1}.$$

Let

$$\tilde{B}_{n+1} := T_{n+1} \circ s_{n+1} \circ B_{n+1}.$$

Then

$$A_{n+1} = (T_{n+1} \circ s_{n+1})^{-1} \circ H_{n+1}^{-1} \circ (A_n \circ H_{n+1}) \circ \tilde{B}_{n+1}.$$

Note that

$$A_n \circ H_{n+1}(x, y) = \begin{bmatrix} x \\ \tilde{h}_n(x, y) \end{bmatrix} := \begin{bmatrix} x \\ h_n((a_n)_y^{-1}(x), y) \end{bmatrix}.$$

Moreover, by Theorem 7.3,

$$\tilde{B}_{n+1}(x, y) = \begin{bmatrix} \tilde{b}_{n+1}(x, y) \\ \lambda_{n+1}x \end{bmatrix},$$

where

$$\tilde{b}_{n+1}(x, y) := \lambda_{n+1}\xi_{n+1}(x) + c_{n+1} + \lambda_{n+1}e^{r_{n+1}} b^{q_{2(n+1)}} \beta(x)\, y\, (1 + O(\rho^{n+1})).$$

Thus,

$$A_{n+1}(x, y) = \begin{bmatrix} a_{n+1}(x, y) \\ h_{n+1}(x, y) \end{bmatrix} = \begin{bmatrix} \lambda_{n+1}^{-1} a_n(\tilde{b}_{n+1}(x, y), \tilde{h}_n \circ \tilde{B}_{n+1}(x, y)) - \lambda_{n+1}^{-1} c_{n+1} \\ \lambda_{n+1}^{-1} \tilde{h}_n \circ \tilde{B}_{n+1}(x, y) \end{bmatrix}.$$

By (26),

$$\|a_n(\tilde{b}_{n+1}(x, 0), \tilde{h}_n \circ \tilde{B}_{n+1}(x, y))\|_y = O(\epsilon^{2^{n-1}} |b^{q_{2(n+1)}}|).$$

Hence,

$$
\begin{aligned}
a_{n+1}(x,\, y) &= \eta_{n+1}(x) + e^{r_{n+1}} b^{q_{2(n+1)}}\ \eta_n'(\tilde{b}_{n+1}(x,\, 0)) \\
&\quad \times\ \beta(x)\ y\ (1 + O(\rho^{n+1})) + O(\epsilon^{2^{n-1}} |b^{q_{2(n+1)}}|) \\
&= \eta_{n+1}(x) + e^{r_{n+1}} b^{q_{2(n+1)}}\ \alpha(x)\ y\ (1 + O(\rho^{n+1})),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha(x) &:= \lim_{n\to\infty} \eta_n'(\tilde{b}_{n+1}(x,\, 0))\ \beta(x) \\
&= \lim_{n\to\infty} \eta_n'(\lambda_{n+1}\xi_{n+1}(x) + c_{n+1})\ \beta(x) \\
&= \eta_*'(\lambda_*\xi_*(x))\beta(x)
\end{aligned}
$$

is universal and has the properties claimed in the corollary. □

Consider the kth cap κ_k given in Proposition 5.2. Denote

$$
D_n := D_{(\kappa_n,0)}\Phi_n \quad \text{and} \quad D_k^l := D_{(\kappa_l,0)}\Phi_k^l.
$$

By (31),

$$
D_k^l = D_{k+1} \cdot D_{k+2} \cdot \cdots \cdot D_l.
$$

COROLLARY 7.5. *Write*

$$
D_n = \begin{bmatrix} 1 & s_n b^{q_{2(n-1)}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_n & 0 \\ 0 & \lambda_n \end{bmatrix},
$$

where b is the average Jacobian of Σ defined in (38). Then there exists a constant $\rho < 1$ such that the following estimates hold for all $n \geq 1$:
(i) $u_n = \lambda_*^2(1 + O(\rho^n))$;
(ii) $\lambda_n = \lambda_*(1 + O(\rho^n))$; *and*
(iii) $|s_n| \asymp 1$.
Consequently, for $0 \leq k < l$,

$$
D_k^l = \begin{bmatrix} 1 & t_k^l b^{q_{2k}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_k^l & 0 \\ 0 & \lambda_k^l \end{bmatrix},
$$

where:
(i) $u_k^l := u_{k+1} \cdot u_{k+2} \cdot \cdots \cdot u_l = \lambda_*^{2(l-k)}(1 + O(\rho^{k+1}))$;
(ii) $\lambda_k^l := \lambda_{k+1} \cdot \lambda_{k+2} \cdot \cdots \cdot \lambda_l = \lambda_*^{l-k}(1 + O(\rho^{k+1}))$; *and*
(iii) $|t_k^l| \asymp 1$.

Proof. By Corollary 5.1 and Proposition 6.2, we have

$$
D_n = \begin{bmatrix} u_n & \sigma_n \\ 0 & \lambda_n \end{bmatrix},
$$

where u_n converges to λ_*^2 geometrically fast in n. It remains to find the desired estimate for σ_n.

Recall that

$$
\Phi_n(x,\, y) := H_n(\lambda_n x + c_n,\, \lambda_n y)
$$

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and

$$H_n^{-1}(x, y) = \begin{bmatrix} a_{n-1}(x, y) \\ y \end{bmatrix},$$

where $a_{n-1}(x, y)$ is the first coordinate of $A_{n-1}(x, y)$. By Corollary 7.4,

$$a_{n-1}(x, y) = \eta_{n-1}(x) + e^{r_{n-1}} b^{q_{2(n-1)}} \alpha(x)\, y\, (1 + O(\rho^{n-1})).$$

Hence,

$$\partial_y a_{n-1}(x, y) = e^{r_{n-1}} b^{q_{2(n-1)}} \alpha(x)(1 + O(\rho^{n-1})).$$

By straightforward computation,

$$\sigma_n = -u_n \lambda_n \partial_y a_{n-1}(\kappa_{n-1}, 0).$$

The result follows. □

8. *Non-rigidity*

As an application of the universality theorem obtained in §7, we show that two commuting pairs cannot be C^1-conjugate on their respective renormalization arcs if their average Jacobians differ in absolute value. Together with Theorem 5.8, this implies the non-rigidity theorem stated in §1. Our proof is similar to the one given in [**dCLM**] that shows non-rigidity of the invariant Cantor set for period-doubling renormalization.

THEOREM 8.1. (Non-rigidity) *Let $\Sigma = (A, B)$ and $\tilde{\Sigma} = (\tilde{A}, \tilde{B})$ be commuting pairs contained in the stable manifold $W^s(\iota(\zeta_*)) \subset \mathcal{D}_2(\Omega, \Gamma, \epsilon)$ of the two-dimensional renormalization fixed point $\iota(\zeta_*)$. Furthermore, assume that Σ and $\tilde{\Sigma}$ both satisfy (37) for some $\delta, \tilde{\delta} > 0$, so that their respective average Jacobians b and $\tilde{b}$ are well defined. Let $f : \gamma_\Sigma \to \gamma_{\tilde{\Sigma}}$ be a homeomorphism which conjugates the action of Σ and $\tilde{\Sigma}$. Then the Hölder exponent of f is at most $\frac{1}{2}(1 + \ln |b| / \ln |\tilde{b}|)$ (and, in particular, cannot be C^1).*

Proof. For brevity, we will only define the notation for Σ. The corresponding objects for $\tilde{\Sigma}$ will be marked with the tilde.

Assume that $|b| \neq |\tilde{b}|$. Then we can choose k sufficiently large so that

$$|\tilde{b}|^{q_{2k}} \ll |b|^{q_{2k}}.$$

Next, choose $n \geq 0$ so that

$$\lambda_*^{n+1} < |\tilde{b}|^{q_{2k}} < \lambda_*^n \ll |b|^{q_{2k}}. \tag{40}$$

For the proof, we work in three different scales: in the scale of $\Sigma = (A, B)$, of $\Sigma_k = (A_k, B_k)$ and of $\Sigma_{k+n} = (A_{k+n}, B_{k+n})$ (see Figure 24). First, in the scale of Σ_{k+n}, let

$$c_{k+n} := B_{k+n}((\kappa_{k+n}, 0)).$$

Then, in the scale of Σ_k, let

$$c_k^{k+n} := \Phi_k^{k+n}(c_{k+n}), \quad z_k^{k+n} := B_k(c_k^{k+n}) \quad \text{and} \quad w_k := B_k((\kappa_k, 0)).$$

Finally, in the scale of Σ, let

$$Z_k^{k+n} := \Phi_0^k(z_k^{k+n}) \quad \text{and} \quad W_k := \Phi_0^k(w_k).$$

Consider the distance between the following pairs of points:

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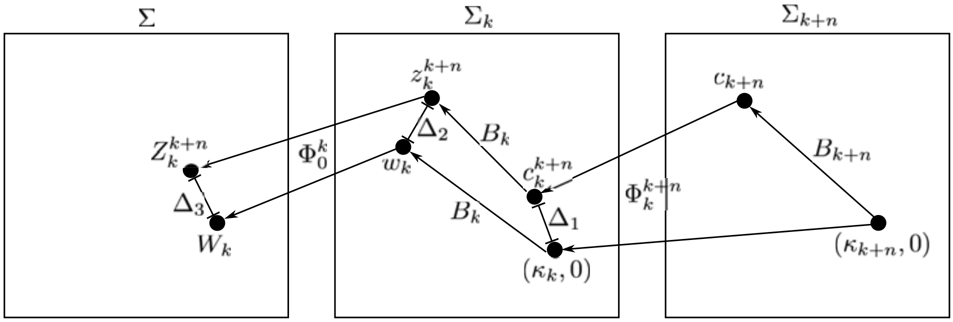

FIGURE 24. The distances Δ_1, Δ_2 and Δ_3.

(1) c_k^{k+n} and $(\kappa_k, 0)$;
(2) z_k^{k+n} and w_k; and
(3) Z_k^{k+n} and W_k.

Let Δ_i^x, Δ_i^y and Δ_i with $i = 1, 2, 3$ denote the horizontal, vertical and Euclidean distance between these pairs of points, respectively.

By Corollary 7.5,

$$\Delta_1^y \asymp |\lambda_*|^n,$$

and for some uniform constant $C > 0$,

$$\Delta_1^x > 2C(|b|^{q_{2k}}|\lambda_*|^n - |\lambda_*|^{2n}) > C|b|^{q_{2k}}|\lambda_*|^n,$$

where, in the last inequality, we used (40). Thus, we see that

$$\Delta_2^y > C|b|^{q_{2k}}|\lambda_*|^n.$$

Again by Corollary 7.5, we arrive at

$$\Delta_3 \geq \Delta_3^y > C|b|^{q_{2k}}|\lambda_*|^{n+k}.$$

Now, consider the corresponding distances for $\tilde{\Sigma}$. Again,

$$\tilde{\Delta}_1^y \asymp |\lambda_*|^n.$$

However, by (40) we see that

$$\tilde{\Delta}_1^x = O(|\tilde{b}|^{q_{2k}}|\lambda_*|^n + |\lambda_*|^{2n}) = O(|\lambda_*|^{2n}).$$

By Theorem 7.3 and (40), we obtain

$$\tilde{\Delta}_2^x = O(\tilde{\Delta}_1^x + |\tilde{b}|^{q_{2k}}\tilde{\Delta}_1^y) = O(|\lambda_*|^{2n}) = \tilde{\Delta}_2^y.$$

Lastly, Corollary 7.5 implies that

$$\tilde{\Delta}_3^x = \tilde{\Delta}_3^y = O(|\lambda_*|^{2n+k}).$$

Hence,

$$\tilde{\Delta}_3 = O(|\lambda_*|^{2n+k}).$$

Observe that any Hölder exponent α for a conjugacy $f : \gamma_\Sigma \to \gamma_{\tilde{\Sigma}}$ between Σ and $\tilde{\Sigma}$ must satisfy

$$\Delta_3 \leq C'(\tilde{\Delta}_3)^\alpha$$

for some uniform constant $C' > 1$. By our estimates above, this means that

$$|b|^{q_{2k}}|\tilde{b}|^{q_{2k}}|\lambda_*|^k < |b|^{q_{2k}}|\lambda_*|^{n+k} \leq C'(|\lambda_*|^{2n+k})^\alpha < C'(|\lambda_*|^{k-2}|\tilde{b}|^{q_{2k}}|\tilde{b}|^{q_{2k}})^\alpha.$$

The theorem follows. □

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