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KESTEN'S BOUND FOR SUBEXPONENTIAL DENSITIES ON THE REAL LINE AND ITS MULTI-DIMENSIONAL ANALOGUES

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Abstract

We study the tail asymptotic of subexponential probability densities on the real line. Namely, we show that the n-fold convolution of a subexponential probability density on the real line is asymptotically equivalent to this density multiplied by n. We prove Kesten's bound, which gives a uniform in n estimate of the n-fold convolution by the tail of the density. We also introduce a class of regular subexponential functions and use it to find an analogue of Kesten's bound for functions on $\mathbb{R}^d$. The results are applied to the study of the fundamental solution to a nonlocal heat equation.

Keywords: Subexponential density; long-tail function; convolution tail; heavy-tailed distribution; tail-equivalence; asymptotic behavior

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1. Introduction

Let F be a probability distribution on $\mathbb{R}$. Denote by $\bar{F}(s) := F((s,\infty))$, $s \in \mathbb{R}$, its tail function. For probability distributions F_1, F_2 on $\mathbb{R}$, their convolution $F_1 * F_2$ has the tail function

$$\overline{F_1 * F_2}(s) = \int_{\mathbb{R}} \bar{F}_1(s-\tau) F_2 \, \mathrm{d}\tau = \int_{\mathbb{R}} \bar{F}_2(s-\tau) F_1 \, \mathrm{d}\tau,$$

where $\bar{F}_1$, $\bar{F}_2$ are the corresponding tail functions of F_1, F_2.

If a probability distribution F is concentrated on $\mathbb{R}_+ := [0,\infty)$ and $\bar{F}(s) > 0$, $s \in \mathbb{R}$, then (see, e.g. [7])

$$\liminf_{s\to\infty} \frac{\overline{F * F}(s)}{\bar{F}(s)} \geq 2. \tag{1.1}$$

If, additionally, F is heavy-tailed, i.e. $\int_{\mathbb{R}} \mathrm{e}^{\lambda s} F \, \mathrm{d}s = \infty$ for all $\lambda > 0$, then the equality in (1.1) holds; see [13]. An important subclass of heavy-tailed distributions concentrated on $\mathbb{R}_+$ are the *subexponential* ones, for which

$$\lim_{s\to\infty} \frac{\overline{F * F}(s)}{\bar{F}(s)} = 2. \tag{1.2}$$

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Any subexponential distribution on $\mathbb{R}_+$ is (right-side) long-tailed on $\mathbb{R}$ (see, e.g. [7]), i.e. (see Definition 2.1 below)

$$\lim_{s\to\infty}\frac{\bar F(s+t)}{\bar F(s)}=1 \quad \text{for each } t>0. \tag{1.3}$$

If distributions F_1, F_2 on $\mathbb{R}$ have probability densities $f_1\geq 0$, $f_2\geq 0$ with $\int_{\mathbb{R}} f_1(s)\,\mathrm{d}s=\int_{\mathbb{R}} f_2(s)\,\mathrm{d}s=1$, then F_1*F_2 has the density

$$(f_1*f_2)(s):=\int_{\mathbb{R}} f_1(s-t)f_2(t)\,\mathrm{d}t, \qquad s\in\mathbb{R}.$$

The density f of a subexponential distribution F concentrated on $\mathbb{R}_+$ (i.e. $f(s)=0$ for $s<0$) is said to be subexponential on $\mathbb{R}_+$ if f is long-tailed, i.e. (1.3) holds with $\bar F$ replaced by f (see also Definition 2.1 below), and (see (1.2))

$$\lim_{s\to\infty}\frac{(f*f)(s)}{f(s)}=2. \tag{1.4}$$

It can be shown (see, e.g. [2], [14], and [15]) that, in this case, for any $n\in\mathbb{N}$,

$$\lim_{s\to\infty}\frac{f^{*n}(s)}{f(s)}=n, \tag{1.5}$$

where $f^{*n}:=f*\cdots*f$ ($n-1$ times). Note that, in general, the density of a subexponential distribution concentrated on $\mathbb{R}_+$ even being long-tailed does not need to be a subexponential one; the corresponding characterization can be found in [2] and [14]; see (2.13) below. Property (1.5) implies, in particular, that, for each $\delta>0$, $n\in\mathbb{N}$, there exists $s_n>0$ such that $f^{*n}(s)\leq(n+\delta)f(s)$ for $s>s_n$. In many situations it is important to have similar inequalities 'uniformly' in n, i.e. on a set independent of n. A possible solution is given by the so-called Kesten's bound (see [2] and [15]): for a bounded subexponential density f on $\mathbb{R}_+$ and for any $\delta>0$, there exist $c_\delta, s_\delta>0$ such that

$$f^{*n}(s)\leq c_\delta(1+\delta)^n f(s), \qquad s>s_\delta,\ n\in\mathbb{N}. \tag{1.6}$$

For the corresponding results for distributions; see [3], [7], [8], and [14]. Kesten's bounds were used to study series of convolutions of distributions on $\mathbb{R}_+$, $\sum_{n=1}^\infty \lambda_n F^{*n}$, and of the corresponding densities, $\sum_{n=1}^\infty \lambda_n f^{*n}$, appeared in different contexts: starting from the renewal theory (the motivation for [7]) to branching age-dependent processes, random walks, queue theory, risk theory and ruin probabilities, compound Poisson processes, and the study of infinitely divisible laws; see, e.g. [2], [4], [10], [14], [15], and [22] and the references therein.

If F is a probability distribution on the whole of $\mathbb{R}$ such that F^+, given by $F^+(B):=F(B\cap\mathbb{R}_+)$ for all Borel $B\subset\mathbb{R}$, is subexponential on $\mathbb{R}_+$, then (see, e.g. [14, Lemma 3.4]) F is long-tailed on $\mathbb{R}$ and (1.2) holds. The distributions on $\mathbb{R}$ and their densities were considered by several authors; see, e.g. [18] and [20]–[22]. In particular, Watanabe [22] reviewed the difficulties associated in the case of the whole of $\mathbb{R}$ and closed several gaps in the preceding results. However, even some basic properties of subexponential densities on the whole $\mathbb{R}$ remained open. Namely, Foss *et al.* [14, Lemma 4.13] showed that if an integrable function f on $\mathbb{R}$

- is (right-side) long-tailed and, being restricted on $\mathbb{R}_+$ and normalized in $L^1(\mathbb{R}_+)$, satisfies (1.4) (we will say then that f is *weekly subexponential* on $\mathbb{R}$; Definition 2.3 below), and if

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- the condition

$$f(s+\tau) \le Kf(s), \qquad s>\rho,\ \tau>0 \tag{1.7}$$

holds for some $K, \rho > 0$ (in particular, if f decays to 0 at ∞; see Definition 2.5),

then (1.4) also holds for the original f on $\mathbb{R}$. We generalize this to an analogue of (1.5) with a general $n \in \mathbb{N}$. In particular, in Theorem 2.1, we prove the following result.

Theorem 1.1. *Let f be an integrable weakly subexponential function on $\mathbb{R}$ such that (1.7) holds. Then f, being normalized in $L^1(\mathbb{R})$, satisfies (1.5) for all $n \in \mathbb{N}$.*

Moreover, in Theorem 2.2, we prove that (1.6) also holds. Namely, we have the following result.

Theorem 1.2. *Let f be a bounded weakly subexponential probability density on $\mathbb{R}$ such that (1.7) holds. Then, for each $\delta > 0$, there exist $c_\delta, s_\delta > 0$ such that (1.6) holds.*

Note that all the 'classical' examples of subexponential functions satisfy the assumptions of Theorems 1.1 and 1.2; see Subsection 3.2.

The multi-dimensional version of the above constructions is much more nontrivial. Currently, there exist at least three different definitions of subexponential distributions on $\mathbb{R}^d$ for $d > 1$; see [9], [17], and [19]. The variety is mainly related to different possibilities to describe the zones in $\mathbb{R}^d$ where an analogue of the equivalence (1.2) takes place. However, there is an absence of any results about subexponential densities in $\mathbb{R}^d$, $d > 1$. Note also that if, e.g. a is radially symmetric, i.e. $a(x) = b(|x|)$, $x \in \mathbb{R}^d$ (here $|x|$ denotes the Euclidean norm on $\mathbb{R}^d$) and b, being normalized, is a subexponential density on $\mathbb{R}_+$, then $(a * a)(x) := \int_{\mathbb{R}^d} a(x-y)a(y)\,\mathrm{d}y = p(|x|)$, $x \in \mathbb{R}^d$, for some $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ (i.e. $a * a$ is also radially symmetric); however, the asymptotic behavior of b and p at ∞ are difficult to compare. Leaving this as an open problem, we focus in this paper on an analogue of Kesten's bound (1.6) in the multi-dimensional case.

To this end, we introduce a special class $\tilde{\mathcal{S}}_{\mathrm{reg},d}$ of regular subexponential functions on $\mathbb{R}_+$ (see Definitions 3.1 and 4.2). Functions from this class are either inverse polynomials (i.e. (4.5) holds) or decay at ∞ faster than any polynomial (i.e. (4.15) holds), but slower than any exponential function, with the fastest allowed asymptotic $\exp(-s(\log s)^{-q})$ with $q > 1$; see Remark 3.5. Then, in Corollary 4.1, we prove the following result.

Theorem 1.3. *Let $a = a(x)$ be a probability density on $\mathbb{R}^d$ such that $a(x) = b(|x|)$, $x \in \mathbb{R}^d$, for some $b \in \tilde{\mathcal{S}}_{\mathrm{reg},d}$. Then, for each $\delta > 0$ and for each $\alpha < 1$ close enough to 1,*

$$a^{*n}(x) \le c_\alpha(1+\delta)^n a(x)^\alpha, \qquad |x| > s_\alpha,\ n \in \mathbb{N} \tag{1.8}$$

for some $c_\alpha = c_\alpha(\delta) > 0$ and $s_\alpha = s_\alpha(\delta) > 0$.

Clearly, $a(x) = o(a(x)^\alpha)$, $|x| \to \infty$, for any $\alpha \in (0,1)$, hence, (1.8) is weaker than (1.6) for the $d = 1$ case.

The result of Corollary 4.1 is based on the more general Theorem 4.1, which says that if, for some $b \in \tilde{\mathcal{S}}_{\mathrm{reg},d}$ and decreasing on $\mathbb{R}_+$ function p,

$$a(x) \le p(|x|), \quad x \in \mathbb{R}^d, \qquad \log p(s) \sim \log b(s), \quad s \to \infty,$$

then (1.8) holds with $a(x)$ replaced by $b(|x|)$ in the right-hand side.

The paper is organized as follows. In Section 2 we consider properties of general subexponential functions on the real line and prove the results leading to the proofs of Theorems 1.1 and 1.2. In Section 3 we define and study properties of regular subexponential functions on the real line and consider the corresponding examples. In Section 4 we prove Theorem 1.3 and its generalizations. Finally, in Appendix A, we apply the obtained results to the study of a nonlocal heat equation.

2. Subexponential functions and Kesten's bound on the real line

Definition 2.1. A function $b\colon \mathbb{R} \to \mathbb{R}_+$ is said to be *(right-side) long-tailed* if there exists $\rho \ge 0$ such that $b(s) > 0$, $s \ge \rho$, and, for any $\tau \ge 0$,

$$\lim_{s\to\infty} \frac{b(s+\tau)}{b(s)} = 1. \tag{2.1}$$

Remark 2.1. By [14, Equation (2.18)], the convergence in (2.1) is equivalent to the locally uniform in τ convergence, namely, (2.1) can be replaced by the assumption that, for all $h > 0$,

$$\lim_{s\to\infty} \sup_{|\tau|\le h} \left| \frac{b(s+\tau)}{b(s)} - 1 \right| = 0. \tag{2.2}$$

A long-tailed function has to have a 'heavier' tail than any exponential function; namely, the following statement holds.

Lemma 2.1. (See [14, Lemma 2.17].) *Let $b\colon \mathbb{R} \to \mathbb{R}_+$ be a long-tailed function. Then, for any $k > 0$, $\lim_{s\to\infty} \mathrm{e}^{ks} b(s) = \infty$.*

The constant h in (2.2) may be arbitrarily large. It is quite natural to ask what will happen if h increases to ∞ consistently with s.

Lemma 2.2. (See [14, Lemma 2.19, Proposition 2.20].) *Let $b\colon \mathbb{R} \to \mathbb{R}_+$ be a long-tailed function. Then there exists a function $h\colon (0,\infty) \to (0,\infty)$ with $h(s) < s/2$ and $\lim_{s\to\infty} h(s) = \infty$ such that (see (2.2))*

$$\lim_{s\to\infty} \sup_{|\tau|\le h(s)} \left| \frac{b(s+\tau)}{b(s)} - 1 \right| = 0. \tag{2.3}$$

Following [14], we will say that b is h-insensitive. Of course, for a given long-tailed function b, the function h that fulfills (2.3) is not unique; see also [14, Proposition 2.20].

The convergence in (2.1) will not, in general, be monotone in s. To obtain this monotonicity, we consider the following class of functions.

Definition 2.2. A function $b\colon \mathbb{R} \to \mathbb{R}_+$ is said to be *(right-side) tail-log-convex* if there exists $\rho > 0$ such that $b(s) > 0$, $s \ge \rho$, and the function $\log b$ is convex on $[\rho, \infty)$.

Remark 2.2. It is well known that any function which is convex on an open interval is continuous there. Therefore, a tail-log-convex function $b = \exp(\log b)$ is also continuous on (ρ_b, ∞).

Lemma 2.3. *Let $b\colon \mathbb{R} \to \mathbb{R}_+$ be tail-log-convex with $\rho = \rho_b$. Then, for any $\tau > 0$, the function $b(s+\tau)/b(s)$ is nondecreasing in $s \in [\rho, \infty)$.*

Proof. Take any $s_1 > s_2 \ge \rho$. Set $B(s) := \log b(s) \le 0$, $s \in [\rho, \infty)$. Then the desired inequality $b(s_1+\tau)/b(s_1) \ge b(s_2+\tau)/b(s_2)$ is equivalent to $B(s_1+\tau) + B(s_2) \ge B(s_2+\tau) + B(s_1)$. Since B is convex, we have, for $\lambda = \tau/(s_1 - s_2 + \tau) \in (0,1)$,

$$\begin{aligned} B(s_1) &= B(\lambda s_2 + (1-\lambda)(s_1+\tau)) \le \lambda B(s_2) + (1-\lambda) B(s_1+\tau), \\ B(s_2+\tau) &= B((1-\lambda)s_2 + \lambda(s_1+\tau)) \le (1-\lambda) B(s_2) + \lambda B(s_1+\tau), \end{aligned}$$

which implies the required inequality. □

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Due to the terminology mentioned in the introduction, we will use the following definition.

Definition 2.3. We will say that a function $b\colon \mathbb{R} \to \mathbb{R}_+$ is *weakly (right-side) subexponential on* $\mathbb{R}$ if b is long-tailed, $b \in L^1(\mathbb{R}_+)$, and the function (with $\mathbf{1}$ denoting the indicator function)

$$b_+(s) := \mathbf{1}_{\mathbb{R}_+}(s)\biggl(\int_{\mathbb{R}_+} b(\tau)\,\mathrm{d}\tau\biggr)^{-1} b(s), \qquad s \in \mathbb{R}, \tag{2.4}$$

satisfies the following asymptotic relation (as $s \to \infty$):

$$(b_+ * b_+)(s) = \int_{\mathbb{R}} b_+(s-\tau)b_+(\tau)\,\mathrm{d}\tau = \int_0^s b_+(s-\tau)b_+(\tau)\,\mathrm{d}\tau \sim 2b_+(s). \tag{2.5}$$

In the next statement we show that a long-tailed tail-log-convex function is weakly subexponential on $\mathbb{R}$ provided that it decays at ∞ fast enough.

Lemma 2.4. (See [14, Theorem 4.15].) *Let $b\colon \mathbb{R} \to \mathbb{R}_+$ be a long-tailed tail-log-convex function such that $b \in L^1(\mathbb{R}_+)$. Suppose that, for a function $h\colon (0,\infty) \to (0,\infty)$ with $h(s) < s/2$ and $\lim_{s\to\infty} h(s) = \infty$, the asymptotic (2.3) holds, and*

$$\lim_{s\to\infty} sb(h(s)) = 0. \tag{2.6}$$

Then b is weakly subexponential on $\mathbb{R}$.

Remark 2.3. Let $b\colon \mathbb{R} \to \mathbb{R}_+$ be a weakly subexponential function on $\mathbb{R}$. Then, by (2.4) and (2.5), we have

$$\int_0^s b(s-\tau)b(\tau)\,\mathrm{d}\tau \sim 2\biggl(\int_{\mathbb{R}_+} b(\tau)\,\mathrm{d}\tau\biggr)b(s), \qquad s \to \infty. \tag{2.7}$$

Definition 2.4. We say that a function $b\colon \mathbb{R} \to \mathbb{R}_+$ is *(right-side) subexponential on* $\mathbb{R}$ if b is long-tailed, $b \in L^1(\mathbb{R})$, and the following asymptotic relation holds (see (2.5) and (2.7)):

$$(b * b)(s) = \int_{\mathbb{R}} b(s-\tau)b(\tau)\,\mathrm{d}\tau \sim 2\biggl(\int_{\mathbb{R}} b(\tau)\,\mathrm{d}\tau\biggr)b(s), \qquad s \to \infty. \tag{2.8}$$

Remark 2.4. By [14, Lemma 4.12], a subexponential function on $\mathbb{R}$ is weakly subexponential there. In the following lemma we present a sufficient condition for us to obtain the converse.

Lemma 2.5. (See [14, Lemma 4.13].) *Let $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ be a weakly subexponential function on $\mathbb{R}$. Suppose that there exist $\rho = \rho_b > 0$ and $K = K_b > 0$ such that*

$$b(s+\tau) \le Kb(s), \qquad s > \rho,\ \tau > 0. \tag{2.9}$$

Then (2.8) holds, i.e. b is subexponential on $\mathbb{R}$.

Remark 2.5. For $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$, condition (2.9) yields that $\sup_{t\ge s} b(t) \to 0$, $s \to \infty$. In particular, $b(s) \to 0$ as $s \to \infty$.

An evident sufficient condition which ensures (2.9) is that b is decreasing on $[\rho,\infty)$. Consider the corresponding definition.

Definition 2.5. A function $b\colon \mathbb{R} \to \mathbb{R}_+$ is said to be *(right-side) tail-decreasing* if there exists a number $\rho = \rho_b \ge 0$ such that $b = b(s)$ is strictly decreasing on $[\rho,\infty)$ to 0. In particular, $b(s) > 0$, $s \ge \rho$.

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The proof of the following useful statement is straightforward.

Proposition 2.1. *Let $b\colon \mathbb{R} \to \mathbb{R}_+$ be a tail-decreasing function. Let $h\colon (0,\infty) \to (0,\infty)$ with $h(s) < s/2$ and $\lim_{s\to\infty} h(s) = \infty$. Then (2.3) holds if and only if*

$$\lim_{s\to\infty} \frac{b(s \pm h(s))}{b(s)} = 1. \tag{2.10}$$

The next statement and its proof follow the ideas of [14, Lemma 4.9, Lemma 4.13].

Proposition 2.2. *Let $b\colon \mathbb{R} \to \mathbb{R}_+$ be a weakly subexponential function on $\mathbb{R}$ such that (2.9) holds. Let $b_1, b_2 \in L^1(\mathbb{R} \to \mathbb{R}_+)$ and there exist $c_1, c_2 \geq 0$ such that*

$$\lim_{s\to\infty} \frac{b_j(s)}{b(s)} = c_j, \qquad j = 1, 2. \tag{2.11}$$

Then

$$\lim_{s\to\infty} \frac{(b_1 * b_2)(s)}{b(s)} = c_1 \int_{\mathbb{R}} b_2(\tau)\,d\tau + c_2 \int_{\mathbb{R}} b_1(\tau)\,d\tau. \tag{2.12}$$

Proof. Since b_+, given by (2.4), is long-tailed, and (2.5) holds, we have, by [14, Theorem 4.7], that there exists an increasing function $h\colon (0,\infty) \to (0,\infty)$ such that $h(s) < s/2$, $\lim_{s\to\infty} h(s) = \infty$, and

$$\int_{h(s)}^{s-h(s)} b_+(s-\tau) b_+(\tau)\,d\tau = o(b_+(s)), \qquad s \to \infty, \tag{2.13}$$

and, clearly, we can replace b_+ by b in (2.13).

Next, for any $b_1, b_2 \in L^1(\mathbb{R} \to \mathbb{R}_+)$, we can easily obtain

$$(b_1 * b_2)(s) = \int_{-\infty}^{h(s)} (b_1(s-\tau) b_2(\tau) + b_1(\tau) b_2(s-\tau))\,d\tau + \int_{h(s)}^{s-h(s)} b_1(s-\tau) b_2(\tau)\,d\tau.$$

Take an arbitrary $\delta \in (0,1)$. By (2.3), (2.11), and (2.13) (the latter with b_+ replaced by b), there exist $K, \rho > 0$ such that (2.9) holds and, for all $s \geq h(\rho)$,

$$|b_j(s) - c_j b(s)| + \sup_{|\tau| \leq h(s)} |b(s+\tau) - b(s)| + \int_{h(s)}^{s-h(s)} b(s-\tau) b(\tau)\,d\tau \leq \delta b(s), \tag{2.14}$$

$$\int_{-\infty}^{-h(s)} b_j(\tau)\,d\tau + \int_{h(s)}^{\infty} b_j(\tau)\,d\tau \leq \delta, \qquad j = 1, 2. \tag{2.15}$$

Then, by (2.9), (2.14), and (2.15), for $s \geq \rho > h(\rho)$,

$$\int_{-\infty}^{-h(s)} (b_1(s-\tau) b_2(\tau) + b_1(\tau) b_2(s-\tau))\,d\tau \leq \delta K (c_1 + c_2 + 2\delta) b(s).$$

Set $B_j := \int_{\mathbb{R}} b_j(s)\,ds$, $j = 1, 2$. By (2.14), for $s \geq \rho$,

$$\left| \int_{-h(s)}^{h(s)} b_1(s-\tau) b_2(\tau)\,d\tau - c_1 b(s) \int_{-h(s)}^{h(s)} b_2(\tau)\,d\tau \right| \leq \delta B_2 (1 + c_1) b(s),$$

$$\int_{h(s)}^{s-h(s)} b_1(s-\tau) b_2(\tau)\,d\tau \leq \delta (c_1 + \delta)(c_2 + \delta) b(s).$$

From these estimates, it is straightforward to obtain, for some $M > 0$, $|(b_1 * b_2)(s) - (c_1 B_2 + c_2 B_1) b(s)| \leq \delta M b(s)$ for $s \geq \rho$. The latter implies (2.12). □

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Corollary 2.1. *For the property of an integrable function on $\mathbb{R}$ to be weekly subexponential on $\mathbb{R}$ depends on its tail property only. Namely, for a weakly subexponential function on $\mathbb{R}$, $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ and, for any $c \in L^1(\mathbb{R} \to \mathbb{R}_+)$ and $s_0 \in \mathbb{R}$, the function $\tilde{b}(s) = \mathbf{1}_{(-\infty,s_0)}(s)c(s) + \mathbf{1}_{[s_0,\infty)}(s)b(s)$ is weakly subexponential on $\mathbb{R}$; see Theorem 3.1 below.*

We now have a generalization of Lemma 2.5.

Theorem 2.1. *Let $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ be a weakly subexponential function on $\mathbb{R}$ such that (2.9) holds (e.g. let b be tail-decreasing). Then*

$$\lim_{s\to\infty} \frac{b^{*n}(s)}{b(s)} = n\biggl(\int_{\mathbb{R}} b(\tau)\,\mathrm{d}\tau\biggr)^{n-1}, \qquad n \geq 2, \tag{2.16}$$

where

$$b^{*n}(s) = (\underbrace{b * \cdots * b}_{n})(s), \qquad s \in \mathbb{R}.$$

Proof. Set, in Proposition 2.2, $b_1 = b_2 = b$, i.e. $c_1 = c_2 = 1$. Then, for $B := \int_{\mathbb{R}} b(\tau)\,\mathrm{d}\tau$, we obtain $b^{*2}(s) \sim 2Bb(s),\ s \to \infty$. Proving by induction, assume that

$$b^{*(n-1)}(s) \sim (n-1)B^{n-2}b(s), \qquad s \to \infty,\ n \geq 3.$$

Set, in Proposition 2.2, $b_1 = b$, $b_2 = b^{*(n-1)}$, $c_1 = 1$, and $c_2 = (n-1)B^{n-2}$. Then, since $\int_{\mathbb{R}} b^{*(n-1)}(\tau)\,\mathrm{d}\tau = B^{n-1}$, we obtain

$$\lim_{s\to\infty} \frac{b^{*n}(s)}{b(s)} = B^{n-1} + B(n-1)B^{(n-2)} = nB^{(n-1)}. \qquad \square$$

We now consider some general statements in the Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$. Fix the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. In what follows, all functions on $\mathbb{R}^d$ are assumed to be $\mathcal{B}(\mathbb{R}^d)$-measurable. Let $0 \leq a \in L^1(\mathbb{R}^d)$ be a fixed probability density on $\mathbb{R}^d$, i.e.

$$\int_{\mathbb{R}^d} a(x)\,\mathrm{d}x = 1. \tag{2.17}$$

Let $f : \mathbb{R} \to \mathbb{R}$; we will say that the convolution

$$(a * f)(x) := \int_{\mathbb{R}^d} a(x-y)f(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^d,$$

is well defined if the function $y \mapsto a(x-y)f(y)$ belongs to $L^1(\mathbb{R}^d)$ for almost all $x \in \mathbb{R}^d$. In particular, this holds if $f \in L^\infty(\mathbb{R}^d)$. Next, for a function $\phi \colon \mathbb{R}^d \to (0, +\infty)$, we define, for any $f : \mathbb{R}^d \to \mathbb{R}$,

$$\|f\|_\phi := \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{\phi(x)} \in [0, \infty].$$

Proposition 2.3. *Let a function $\phi \colon \mathbb{R}^d \to (0, +\infty)$ be such that $a * \phi$ is well defined, $\|a\|_\phi < \infty$, and, for some $\gamma \in (0, \infty)$,*

$$\frac{(a * \phi)(x)}{\phi(x)} \leq \gamma, \qquad x \in \mathbb{R}^d. \tag{2.18}$$

Then $a^{*n}(x) \leq \gamma^{n-1}\|a\|_\phi \phi(x),\ x \in \mathbb{R}^d$.

Proof. For any $f\colon \mathbb{R}^d \to \mathbb{R}$ with $\|f\|_\phi < \infty$, we have, for $x \in \mathbb{R}^d$,

$$\left|\frac{(a * f)(x)}{\phi(x)}\right| \le \int_{\mathbb{R}^d} \frac{a(y)\phi(x-y)}{\phi(x)} \frac{|f(x-y)|}{\phi(x-y)}\,\mathrm{d}y \le \frac{a * \phi(x)}{\phi(x)} \|f\|_\phi \le \gamma \|f\|_\phi.$$

In particular, since $\|a\|_\phi < \infty$, we obtain $\|a * a\|_\phi \le \gamma\|a\|_\phi < \infty$. Proceeding inductively, we have $\|a^{*n}\|_\phi \le \gamma\|a * a^{n-1}\|_\phi \le \gamma^{n-1}\|a\|_\phi < \infty$, which yields the statement. □

Proposition 2.4. *Let a function* $\omega\colon \mathbb{R}^d \to (0, +\infty)$ *be such that, for any* $\lambda > 0$,

$$\Omega_\lambda := \Omega_\lambda(\omega) := \{x \in \mathbb{R}^d : \omega(x) < \lambda\} \ne \varnothing. \tag{2.19}$$

Further suppose that

$$\eta := \limsup_{\lambda\to 0+} \sup_{x\in\Omega_\lambda} \frac{(a * \omega)(x)}{\omega(x)} \in (0, \infty). \tag{2.20}$$

Then, for any $\delta \in (0, 1)$, *there exists* $\lambda = \lambda(\delta, \omega) \in (0, 1)$ *such that (2.18) holds with*

$$\phi(x) := \omega_\lambda(x) := \min\{\lambda, \omega(x)\}, \qquad x \in \mathbb{R}^d, \tag{2.21}$$

and $\gamma := \max\{1, (1 + \delta)\eta\}$.

Proof. By (2.21), for an arbitrary $\lambda > 0$, we have $\omega_\lambda(x) \le \lambda$, $x \in \mathbb{R}^d$; then we also have $(a * \omega_\lambda)(x) \le \lambda$, $x \in \mathbb{R}^d$. In particular (see (2.21)),

$$(a * \omega_\lambda)(x) \le \omega_\lambda(x), \qquad x \in \mathbb{R}^d \setminus \Omega_\lambda. \tag{2.22}$$

Next, by (2.20), for any $\delta > 0$ there exists $\lambda = \lambda(\delta) \in (0, 1)$ such that

$$\sup_{x\in\Omega_\lambda} \frac{(a * \omega)(x)}{\omega(x)} - \eta \le \delta\eta,$$

in particular, $(a * \omega)(x) \le (1 + \delta)\eta\omega(x) = (1 + \delta)\eta\omega_\lambda(x)$, $x \in \Omega_\lambda$. Therefore,

$$(a * \omega_\lambda)(x) = (a * \omega)(x) - (a * (\omega - \omega_\lambda))(x) \le (1 + \delta)\eta\omega_\lambda(x) \quad \text{for all } x \in \Omega_\lambda; \tag{2.23}$$

here, we used the fact that $\omega \ge \omega_\lambda$. Then (2.22) and (2.23) yield the statement. □

We are now ready to prove Kesten's bound on $\mathbb{R}$.

Theorem 2.2. *Let* $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ *be a bounded weakly subexponential function on* $\mathbb{R}$ *with* $\int_{\mathbb{R}} b(s)\,\mathrm{d}s = 1$ *such that (2.9) holds. Then, for any* $\delta \in (0, 1)$, *there exist* $C_\delta, s_\delta > 0$ *such that*

$$b^{*n}(s) \le C_\delta(1 + \delta)^n b(s), \qquad s > s_\delta,\ n \in \mathbb{N}. \tag{2.24}$$

Proof. Fix $\delta \in (0, 1)$ and $\varepsilon \in (0, \delta]$ with $(1 + \varepsilon)^3 \le 1 + \delta/2$. Let $s_1, \lambda_1 > 0$ satisfy

$$\int_{-\infty}^{-s_1} b(s)\,\mathrm{d}s + \int_{s_1}^{\infty} b(s)\,\mathrm{d}s \le \frac{\varepsilon}{2}, \qquad 4s_1\lambda_1 \le \varepsilon. \tag{2.25}$$

Define the functions, for $s \in \mathbb{R}$,

$$\begin{aligned} \tilde{b}(s) &:= \mathbf{1}_{(-\infty,-s_1)}(s)b(-s) + \mathbf{1}_{[-s_1,s_1]}(s)\max\{\lambda_1, b(s)\} + \mathbf{1}_{(s_1,\infty)}(s)b(s), \\ b_1(s) &:= \mathbf{1}_{(-\infty,-s_1)}(s)b(s), \qquad a(s) := \frac{\tilde{b}(s)}{\|\tilde{b}\|_1}. \end{aligned} \tag{2.26}$$

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Here and below, $\|\cdot\|_1$ denotes the norm in $L^1(\mathbb{R})$. Then, by (2.25) and (2.26), $\|b-\tilde{b}\|_1 \le \varepsilon$ and, hence,

$$\|\tilde{b}\|_1 \le 1+\varepsilon. \tag{2.27}$$

By Corollary 2.1, both functions $s \mapsto \tilde{b}(s)$ and $s \mapsto \tilde{b}(-s)$ are weakly subexponential on $\mathbb{R}$. Hence, by Theorem 2.1,

$$\lim_{s\to\pm\infty} \frac{\tilde{b}^{*m}(s)}{\tilde{b}^{*k}(s)} = \frac{m}{k}\|\tilde{b}\|_1^{m-k} \quad \text{for any } k, m \in \mathbb{N}.$$

In particular, there exist $m_0 \in \mathbb{N}$ and $s_2 \ge s_1$ such that, for $\omega := \tilde{b}^{*m_0}$ and $|s| \ge s_2$,

$$\frac{\tilde{b} * \omega(s)}{\omega(s)} \le (1+\varepsilon)\|\tilde{b}\|_1, \tag{2.28}$$

$$m_0(1-\varepsilon)\|\tilde{b}\|_1^{m_0-1} \le \frac{\omega(s)}{\tilde{b}(s)} \le m_0(1+\varepsilon)\|\tilde{b}\|_1^{m_0-1}. \tag{2.29}$$

For an $r>0$, $\operatorname{ess\,inf}_{s\in[-r,r]} \tilde{b}(s) > 0$. It is straightforward to check that by induction,

$$\operatorname*{ess\,inf}_{-r\le s\le r} \omega(s) = \operatorname*{ess\,inf}_{-r\le s\le r} \tilde{b}^{*m_0}(s) > 0, \qquad r>0. \tag{2.30}$$

However, by Remark 2.5 and (2.29), we have

$$\lim_{s\to\pm\infty} \omega(s) = \lim_{s\to\pm\infty} \tilde{b}(s) = 0. \tag{2.31}$$

Hence, there exists $\lambda_2 \in (0, \lambda_1]$ such that, for any $\lambda \in (0, \lambda_2)$, the set Ω_λ defined by (2.19) will be a nonempty subset of $(-\infty, -s_\lambda) \cup (s_\lambda, \infty)$, where $s_\lambda \to \infty$ as $\lambda \searrow 0$. Therefore, by (2.28), condition (2.20) holds with a given by (2.26) and $\eta \le 1+\varepsilon$. Then, by Proposition 2.4, there exists $\lambda_3 \in (0, \min\{\lambda_2, 1\})$ such that (2.18) holds with

$$\phi(s) := \min\{\lambda_3, \omega(s)\}, \quad s \in \mathbb{R}, \qquad \gamma := (1+\varepsilon)^2.$$

By (2.29) and (2.30), and since $\tilde{b}$ is bounded, we have $\|\tilde{b}\|_\phi < \infty$. Hence, by Proposition 2.3 and using the fact that $\gamma\|\tilde{b}\|_1 \le 1+\delta/2$ due to (2.27) and the choice of ε, we easily obtain

$$\tilde{b}^{*n}(s) \le \left(1+\frac{\delta}{2}\right)^{n-1} \|\tilde{b}\|_\phi \min\{\lambda_3, \omega(s)\}, \qquad s \in \mathbb{R},\ n \in \mathbb{N}.$$

By (2.31) and (2.29), $\|\tilde{b}\|_\phi \min\{\lambda_3, \omega(s)\} \le C_\delta \tilde{b}(s) = C_\delta b(s)$ for $s > s_3$ with some $s_3 \ge s_2$ and $C_\delta = C_\delta(m_0) > 0$. As a result,

$$\tilde{b}^{*n}(s) \le C_\delta\left(1+\frac{\delta}{2}\right)^n b(s), \qquad s > s_3,\ n \in \mathbb{N}. \tag{2.32}$$

By (2.26), $b \le b_1 + \tilde{b}$ and, hence, $b^{*n} \le \sum_{k=0}^n \binom{n}{k} b_1^{*k} * \tilde{b}^{*(n-k)}$, $n \in \mathbb{N}$, pointwise. Since $b_1^{*k}(s) = 0$ for all $s \ge -s_1$, $k \in \mathbb{N}$, then, by (2.32), we obtain, for $s \ge s_3$,

$$\begin{aligned}
b_1^{*k} * \tilde{b}^{*(n-k)}(s) &= \int_{s+s_1}^{\infty} b_1^{*k}(s-y)\tilde{b}^{*(n-k)}(y)\,\mathrm{d}y \\
&\le C_\delta\left(1+\frac{\delta}{2}\right)^{n-k} \int_{s+s_1}^{\infty} b_1^{*k}(s-y)\tilde{b}(y)\,\mathrm{d}y \\
&= C_\delta\left(1+\frac{\delta}{2}\right)^{n-k} \int_{-\infty}^{-s_1} b_1^{*k}(y)\tilde{b}(s-y)\,\mathrm{d}y,
\end{aligned}$$

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and, by (2.9), (2.25), and (2.27), we can continue, for $s \ge s_\delta := \max\{s_3, \rho\}$,

$$b_1^{*k} * \tilde{b}^{*(n-k)}(s) \le C_\delta \left(1 + \frac{\delta}{2}\right)^{n-k} \left(\frac{\varepsilon}{2}\right)^k K\tilde{b}(s) \le C_\delta \left(1 + \frac{\delta}{2}\right)^{n-k} \left(\frac{\delta}{2}\right)^k b(s),$$

which yields (2.24). □

Remark 2.6. Following the scheme of the proof of [2, Proposition 8], we see that Kesten's bound (2.24) holds under the weaker assumption that $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ is a bounded function on $\mathbb{R}$ with $\int_{\mathbb{R}} b(s)\,\mathrm{d}s = 1$ such that (2.16) holds, i.e.

$$b^{*n}(x) \sim nb(x), \qquad x \to \infty,\ n \ge 2$$

(recall that, in contrast to (1.5), b is not necessary concentrated on $\mathbb{R}_+$). By Theorem 2.1, a sufficient condition for the latter asymptotic relation is that b is weakly subexponential (i.e. its normalized restriction b_+ on $\mathbb{R}_+$ is subexponential; see (2.4) and (2.5)) and inequality (2.9) holds.

3. Regular subexponential densities on $\mathbb{R}$

We are going to obtain an analogue of Kesten's bound on $\mathbb{R}^d$ with $d > 1$, at least for radially symmetric functions. Our technique will require us to deal with functions $b(|x|)^\alpha$, $x \in \mathbb{R}^d$, where b is a subexponential function on $\mathbb{R}_+$ and $\alpha < 1$ is close enough to 1; in particular, we have to be sure that b^α is also subexponential on $\mathbb{R}_+$. Moreover, to weaken the condition of radial symmetry, we will allow double-sided estimates by functions of the form $p(|x|)b(|x|)$ for appropriate p on $\mathbb{R}_+$ (say, polynomial). Again, we will need to check whether the functions pb are also subexponential on $\mathbb{R}_+$. To check such a stability of the class of subexponential functions on $\mathbb{R}_+$ with respect to power and multiplicative perturbations, we have to reduce the class to appropriately regular subexponential functions. Then the mentioned stability takes place; see Theorem 3.1 and Proposition 3.3. Examples of regular subexponential functions can be found in Subsection 3.2. Analogues of Kesten's bound on $\mathbb{R}^d$ are considered in Section 4.

3.1. Main properties

Definition 3.1. Let $\mathcal{S}_{\mathrm{reg}}$ be the set of all functions $b\colon \mathbb{R} \to \mathbb{R}_+$ such that

(i) $b \in L^1(\mathbb{R}_+)$ and b is bounded on $\mathbb{R}$;

(ii) there exists $\rho = \rho_b > 1$ such that b is log-convex and strictly decreasing to 0 on $[\rho, \infty)$ (i.e. b is simultaneously tail-decreasing and tail-log-convex), and $b(\rho) \le 1$ (without loss of generality);

(iii) there exist $\delta = \delta_b \in (0, 1)$ and an increasing function $h = h_b\colon (0, \infty) \to (0, \infty)$ with $h(s) < s/2$ and $\lim_{s\to\infty} h(s) = \infty$ such that asymptotic (2.10) holds, and (see (2.6))

$$\lim_{s\to\infty} b(h(s))s^{1+\delta} = 0. \tag{3.1}$$

For any $n \in \mathbb{N}$, we denote by $\mathcal{S}_{\mathrm{reg},n}$ the subclass of functions b from $\mathcal{S}_{\mathrm{reg}}$ such that

$$\int_{-\infty}^{\rho} b(s)\,\mathrm{d}s + \int_{\rho}^{\infty} b(s)s^{n-1}\,\mathrm{d}s < \infty. \tag{3.2}$$

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Remark 3.1. It is worth noting again that, for a tail-decreasing function, (2.10) implies that b is long-tailed.

Remark 3.2. By Lemma 2.4, any function $b \in \mathcal{S}_{\mathrm{reg}}$ is weakly subexponential on $\mathbb{R}$. Moreover, by Lemma 2.5, any function $b \in \mathcal{S}_{\mathrm{reg},1}$ is subexponential on $\mathbb{R}$.

Remark 3.3. Let $b \in \mathcal{S}_{\mathrm{reg}}$ and $s_0 > 0$ be such that $h(2s_0) > \rho$. Then the monotonicity of b and h implies that $b(s) \le b(h(2s))$, $s > s_0$; hence, due to (3.1), for $B := 2^{-1-\delta}$, there exists $s_1 \ge s_0$ such that

$$b(s) \le Bs^{-1-\delta}, \qquad s \ge s_1. \tag{3.3}$$

Below we will show that $\mathcal{S}$ and $\mathcal{S}_{\mathrm{reg},n}$, $n \in \mathbb{N}$, are closed under some simple transformations of functions. For an arbitrary function $b \in \mathcal{S}_{\mathrm{reg}}$, we consider the following transformed functions:

- for fixed $p > 0$, $q > 0$, $r \in \mathbb{R}$, we set

$$\tilde{b}(s) := pb(qs + r), \qquad s \in \mathbb{R}; \tag{3.4}$$

- for a fixed $s_0 > 0$ and a fixed bounded function $c\colon \mathbb{R} \to \mathbb{R}_+$, we set

$$\breve{b}(s) := \mathbf{1}_{(-\infty,s_0)}(s)c(s) + \mathbf{1}_{[s_0,\infty)}(s)b(s), \qquad s \in \mathbb{R}; \tag{3.5}$$

- for any $\alpha \in (0, 1]$, we denote

$$b_\alpha(s) := (b(s))^\alpha, \qquad s \in \mathbb{R}.$$

Theorem 3.1. (i) *Let $b \in \mathcal{S}_{\mathrm{reg}}$. Then the functions $\tilde{b}$ and $\breve{b}$ defined in (3.4) and (3.5) also belong to $\mathcal{S}_{\mathrm{reg}}$ for all admissible values of their parameters. If, additionally, there exists $\alpha' \in (0, 1)$ such that $b_{\alpha'} \in L^1(\mathbb{R}_+)$, then there exists $\alpha_0 \in (\alpha', 1)$ such that $b_\alpha \in \mathcal{S}_{\mathrm{reg}}$ for all $\alpha \in [\alpha_0, 1]$.*

(ii) *Let $b \in \mathcal{S}_{\mathrm{reg},n}$ for some $n \in \mathbb{N}$. Then $\tilde{b} \in \mathcal{S}_{\mathrm{reg},n}$. If, additionally, the function c in (3.5) is integrable on $(-\infty, s_0)$ then $\breve{b} \in \mathcal{S}_{\mathrm{reg},n}$. Finally, if there exists $\alpha' \in (0, 1)$ such that (3.2) holds for $b = b_{\alpha'}$ then there exists $\alpha_0 \in (\alpha', 1)$ such that $b_\alpha \in \mathcal{S}_{\mathrm{reg},n}$ for all $\alpha \in [\alpha_0, 1]$. Moreover, in the latter case, there exist $B_0 > 0$ and $\rho_0 > 0$ such that, for all $\alpha \in (\alpha_0, 1]$,*

$$\int_{\mathbb{R}} (b(s-\tau))^\alpha (b(\tau))^\alpha \,\mathrm{d}\tau \le B_0 (b(s))^\alpha, \qquad s \ge \rho_0. \tag{3.6}$$

Proof. It is straightforward to check that if b is long-tailed, tail-decreasing, and tail-log-convex, then $\tilde{b}$, $\breve{b}$, b_α also have these properties for all admissible values of their parameters. Let $h\colon (0,\infty) \to (0,\infty)$ be such that $h(s) < s/2$, $\lim_{s\to\infty} h(s) = \infty$ and (2.10) hold. Also let also (3.1) hold for some $\delta > 0$. We deal with the proof in four steps.

Step 1. Evidently, both (2.10) and (3.1) hold with b replaced by $\breve{b}$. Next, $\breve{b} \in L^1(\mathbb{R}_+)$ and $\breve{b}$ is bounded. Hence, $\breve{b} \in \mathcal{S}_{\mathrm{reg}}$. If $b \in \mathcal{S}_{\mathrm{reg},n}$ and c is integrable on $(-\infty, s_0)$, then (3.2) holds for b replaced by $\breve{b}$. See also Corollary 2.1.

Step 2. Set, for the given $q > 0$, $r \in \mathbb{R}$,

$$\tilde{h}(s) := \frac{1}{q} h(qs + r) - \frac{r}{2q} \mathbf{1}_{\mathbb{R}_+}(r), \qquad s \in [s_1, \infty),$$

where $s_1 > 0$ is such that $qs_1 + r > 0$ and $h(qs + r) > r/2q$ for all $s \ge s_1$, and $\tilde{h}$ is increasing on $(0, s_1)$ such that $\tilde{h}(s) < \min\{s/2, \tilde{h}(s_1)\}$, $s \in (0, s_1)$. By Proposition 2.1, (2.10) is equivalent

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to (2.3). Then, by (3.4), we have, $\sup_{|\tau|\le\tilde{h}(s)}|\tilde{b}(s+\tau)/\tilde{b}(s)-1|\to 0$ as $s\to\infty$. Therefore, again by Proposition 2.1, (2.10) holds for b replaced by $\tilde{b}$. Next, from (2.1) and (3.1), it is straightforward to obtain $\tilde{b}(\tilde{h}(s))s^{1+\delta}\to 0$ as $s\to\infty$. Therefore, $\tilde{b}\in\mathcal{S}_{\mathrm{reg}}$. Finally, $b\in\mathcal{S}_{\mathrm{reg},n}$ for some $n\in\mathbb{N}$, trivially implies that $\tilde{b}\in\mathcal{S}_{\mathrm{reg},n}$.

Step 3. Clearly, (2.10) holds for b replaced by b_α, with the same h and for any $\alpha\in(0,1)$. Next, let $\alpha'\in(0,1)$ be such that $b_{\alpha'}\in L^1(\mathbb{R}_+)$. By the well-known log-convexity of L^p-norms (for $p>0$), for any $\alpha\in(\alpha',1)$ and for $\beta:=\alpha-\alpha'/\alpha(1-\alpha')\in(0,1)$, we have $1/\alpha=(1-\beta)/\alpha'+\beta$ and

$$\|b\|_{L^\alpha(\mathbb{R}_+)}\le\|b\|^{1-\beta}_{L^{\alpha'}(\mathbb{R}_+)}\|b\|^{\beta}_{L^1(\mathbb{R}_+)}<\infty, \tag{3.7}$$

i.e. $b_\alpha\in L^1(\mathbb{R}_+)$ for all $\alpha\in(\alpha',1)$. Take and fix an $\alpha_0\in(\max\{\alpha',1/(1+\delta)\},1)$. Then, for any $\alpha\in[\alpha_0,1]$, we have $\delta':=\alpha(1+\delta)-1\in(0,\delta]$ and, hence, by (3.1),

$$\lim_{s\to\infty}b_\alpha(h(s))s^{1+\delta'}=\lim_{s\to\infty}(b(h(s))s^{1+\delta})^\alpha=0.$$

Therefore, $b_\alpha\in\mathcal{S}_{\mathrm{reg}}$, $\alpha\in[\alpha_0,1]$.

Let, additionally, (3.2) hold for both b and $b_{\alpha'}$ (i.e. in particular, $b\in\mathcal{S}_{\mathrm{reg},n}$) and for some $n\in\mathbb{N}$. Then we can again use the log-convexity of L^p-norms, now for $L^p((\rho,\infty),s^n\,\mathrm{d}s)$ spaces, to deduce that $b_\alpha\in\mathcal{S}_{\mathrm{reg},n}$, $\alpha\in[\alpha_0,1]$.

Step 4. Finally, $b,b_{\alpha_0}\in\mathcal{S}_{\mathrm{reg},n}$, $n\in\mathbb{N}$, implies that $b,b_{\alpha_0}\in\mathcal{S}_{\mathrm{reg},1}$ and, hence, (see Remark 3.2) b and b_{α_0} are subexponential on $\mathbb{R}$, i.e. (2.8) holds for both b and b_{α_0}. Therefore, for an arbitrary $\varepsilon\in(0,1)$, there exists $\rho_0=\rho_0(\varepsilon,b,b_{\alpha_0})>\rho$ (where ρ is from Definition 3.1) and $B_0:=2(1+\varepsilon)\max\{\int_{\mathbb{R}}b(s)\,\mathrm{d}s,\int_{\mathbb{R}}b_{\alpha_0}(s)\,\mathrm{d}s\}>0$ such that, for all $s\ge\rho_0$,

$$\int_{\mathbb{R}}b(s-\tau)b(\tau)\,\mathrm{d}\tau\le B_0b(s),\qquad \int_{\mathbb{R}}b_{\alpha_0}(s-\tau)b_{\alpha_0}(\tau)\,\mathrm{d}\tau\le B_0b_{\alpha_0}(s). \tag{3.8}$$

Then, applying again the norm log-convexity arguments (see (3.7)), we obtain, for any fixed $s\ge\rho_0$ and for all $\alpha\in(\alpha_0,1)$,

$$\int_{\mathbb{R}}(b(s-\tau)b(\tau))^\alpha\,\mathrm{d}\tau\le\left(\int_{\mathbb{R}}(b(s-\tau)b(\tau))^{\alpha_0}\,\mathrm{d}\tau\right)^{(1-\beta)\alpha/\alpha_0}\left(\int_{\mathbb{R}}b(s-\tau)b(\tau)\,\mathrm{d}\tau\right)^{\beta\alpha},$$

where $\beta=(\alpha-\alpha_0)/\alpha(1-\alpha_0)\in(0,1)$. Combining the latter inequality with (3.8), we obtain

$$\int_{\mathbb{R}}(b(s-\tau)b(\tau))^\alpha\,\mathrm{d}\tau\le(B_0(b(s))^{\alpha_0})^{(1-\beta)\alpha/\alpha_0}(B_0b(s))^{\beta\alpha}=B_0(b(s))^\alpha,$$

completing the proof. □

It is natural to expect that asymptotically small changes of the tail properties preserve the subexponential property of a function. Namely, consider the following definition.

Definition 3.2. Two functions $b_1,b_2\colon\mathbb{R}\to\mathbb{R}_+$ are said to be *weakly tail-equivalent* if

$$0<\liminf_{s\to\infty}\frac{b_1(s)}{b_2(s)}\le\limsup_{s\to\infty}\frac{b_1(s)}{b_2(s)}<\infty$$

or, in other words, if there exist $\rho>0$ and $C_2\ge C_1>0$ such that

$$C_1b_1(s)\le b_2(s)\le C_2b_1(s),\qquad s\ge\rho.$$

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The proof of the following statement is a straightforward consequence of [14, Theorem 4.8].

Proposition 3.1. *Let $b_1 : \mathbb{R} \to \mathbb{R}_+$ be a weakly subexponential function on $\mathbb{R}$. Let $b_2 : \mathbb{R} \to \mathbb{R}_+$ be a long-tailed function which is weakly tail-equivalent to b_1. Then b_2 is also weakly subexponential on $\mathbb{R}$. If, additionally, (2.9) holds for $b = b_1$ then b_2 is subexponential on $\mathbb{R}$.*

Proposition 3.2. *Let $b_1 \in \mathcal{S}_{\text{reg}}$ and let $b_2 : \mathbb{R} \to \mathbb{R}_+$ be a bounded, tail-decreasing, and tail-log-convex function such that*

$$\lim_{s\to\infty} \frac{b_2(s)}{b_1(s)} = C \in (0, \infty). \tag{3.9}$$

Then $b_2 \in \mathcal{S}_{\text{reg}}$.

Proof. Clearly, b_2 is long-tailed as b_1 is long-tailed. Let $\delta \in (0, 1)$ and h be an increasing function with $h(s) < s/2$ and $\lim_{s\to\infty} h(s) = \infty$ such that (2.10) and (3.1) hold for $b = b_1$. Let $\varepsilon \in (0, \min\{1, C\})$ and choose $\rho > 1$ such that b_2 is decreasing and log-convex on $[\rho, \infty)$, and $b_2(\rho) \le 1$. By (3.9) and (2.10) (for $b = b_1$), there exists $\rho_1 \ge \rho$ such that

$$0 < (C - \varepsilon)b_1(s) \le b_2(s) \le (C + \varepsilon)b_1(s), \qquad \left|\frac{b_1(s \pm h(s))}{b_1(s)} - 1\right| < \varepsilon \tag{3.10}$$

for all $s \ge \rho_1$. Since b_2 is bounded and $b_1 \in L^1(\mathbb{R}_+)$, from (3.10), we have $b_2 \in L^1(\mathbb{R}_+)$. By (3.10), for any $s \ge 2\rho_1$,

$$\left|\frac{b_2(s \pm h(s))}{b_2(s)} - 1\right| < \max\left\{\varepsilon\frac{C+\varepsilon}{C-\varepsilon} + \frac{C+\varepsilon}{C-\varepsilon} - 1, \varepsilon\frac{C-\varepsilon}{C+\varepsilon} + 1 - \frac{C-\varepsilon}{C+\varepsilon}\right\}.$$

Since the latter expression may be arbitrary small, by an appropriate choice of ε, we see that (2.10) holds for $b = b_2$. Finally, (3.1) with $b = b_1$ and (3.9) imply that (3.1) holds with $b = b_2$ and the same δ and h. □

Remark 3.4. In the assumptions of the previous theorem, if, additionally, $b_1 \in \mathcal{S}_{\text{reg},n}$ for some $n \in \mathbb{N}$, and b_2 is integrable on $(-\infty, -\rho_2)$ for some $\rho_2 > 0$, then $b_2 \in \mathcal{S}_{\text{reg},n}$ (due to (3.10) and the boundedness of b_2).

On the other hand, if we can check that both functions b_1 and b_2 satisfy (2.10) with the same function $h(s)$, then the sufficient condition to verify (3.1) for $b = b_2$, provided that it holds for $b = b_1$, is much weaker than (3.9). To present the corresponding statement, consider the following definition.

Definition 3.3. Let $b_1, b_2 : \mathbb{R} \to \mathbb{R}_+$ and, for some $\rho \ge 0$, $b_i(s) > 0$ for all $s \in [\rho, \infty)$, $i = 1, 2$. The functions b_1 and b_2 are said to be *(asymptotically) log-equivalent* if

$$\log b_1(s) \sim \log b_2(s), \qquad s \to \infty. \tag{3.11}$$

Proposition 3.3. *Let $b_1 \in \mathcal{S}_{\text{reg}}$ and let h be the function corresponding to Definition 3.1 with $b = b_1$. Let $b_2 : \mathbb{R} \to \mathbb{R}_+$ be a bounded tail-decreasing and tail-log-convex function such that (2.10) holds with $b = b_2$ and the same h. Suppose that b_1 and b_2 are log-equivalent. Then $b_2 \in \mathcal{S}_{\text{reg}}$. If, additionally, there exists $\alpha' \in (0, 1)$ such that (3.2) holds with $b = (b_1)^{\alpha'}$ and b_2 is integrable on $(-\infty, \rho)$, then $b_2 \in \mathcal{S}_{\text{reg},n}$.*

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Proof. Let $\delta \in (0, 1)$ be such that (3.1) holds for b replaced by b_1. Take an arbitrary $\varepsilon \in (0, \delta/(1+\delta))$. By (3.11), there exists $\rho_\varepsilon > 0$ such that $b_i(s) < 1$, $s > \rho_\varepsilon$, $i = 1, 2$, and

$$-(1-\varepsilon)\log b_1(s) \le -\log b_2(s) \le -(1+\varepsilon)\log b_1(s), \qquad s > \rho_\varepsilon,$$
$$b_1(s)^{1+\varepsilon} \le b_2(s) \le b_1(s)^{1-\varepsilon}, \qquad s > \rho_\varepsilon. \tag{3.12}$$

Since $h(s) \to \infty$, $s \to \infty$, there exists $\rho_0 > \rho_\varepsilon$ such that $h(s) > \rho_\varepsilon$ for any $s > \rho_0$. Then, by (3.12), we have, for all $s > \rho_0$,

$$b_2(h(s))s^{(1+\delta)(1-\varepsilon)} < b_1(h(s))^{1-\varepsilon}s^{(1+\delta)(1-\varepsilon)} = (b_1(h(s))s^{1+\delta})^{1-\varepsilon}$$

and, therefore, (3.1) holds with $b = b_2$ and δ replaced by $(1+\delta)(1-\varepsilon)-1 = \delta-\varepsilon(1+\delta) \in (0, 1)$, which proves the first statement. To prove the second statement, assume, additionally, that $\varepsilon < 1-\alpha'$. Then, by (3.12), we have $b_2(s)s^{n-1} \le b_1(s)^{1-\varepsilon}s^{n-1} < b_1(s)^{\alpha'}s^{n-1}$ for all $s > \rho_\varepsilon$, since $b_1(s) < 1$ here. □

3.2. Examples

We consider examples of functions for $b \in \mathcal{S}_{\mathrm{reg}}$. Due to Proposition 3.3, we will classify these functions 'up to log-equivalence', i.e. by the tail properties of the function

$$l(s) := -\log b(s).$$

Taking into account the result of Theorem 3.1 concerning the function $\check{b}$, it will be enough to define b on some (s_0, ∞), $s_0 > 0$ only. Next, by Lemma 2.4, the function b_+ defined by (2.4) is a subexponential density on $\mathbb{R}_+$. Therefore, we can use the classical examples of such densities; see, e.g. [14]. However, using the result of Theorem 3.1 concerning the function $\tilde{b}$, we can consider that examples in their 'simplest' forms (ignoring any shifts of the argument or scales of the argument or the function itself).

Now we consider different asymptotics of the function $l(s) = -\log b(s)$. In all particular examples below, it is straightforward to check that each particular bounded function b is such that $b'(s) < 0$ and $(\log b(s))'' > 0$ for all large enough values of s, i.e. b is tail-decreasing and tail-log-convex.

3.2.1. *Class 1:* $l(s) \sim D\log s$, $s \to \infty$, $D > 1$. (i) *Polynomial decay.* Let $b\colon \mathbb{R} \to \mathbb{R}_+$ be a bounded tail-decreasing and tail-log-convex function such that $b(s) \sim qs^{-D}$, $s \to \infty$, $D > 1$, $q > 0$. By Proposition 3.2, in order to show that $b \in \mathcal{S}_{\mathrm{reg}}$, it is enough to prove it for

$$b(s) = \mathbf{1}_{\mathbb{R}_+}(s)(1+s)^{-D}, \qquad s \in \mathbb{R}.$$

For an arbitrary $\gamma \in (1/D, 1)$, consider $h(s) = s^\gamma$, $s > 0$. Then it is straightforward to check that (2.10) and (3.1) hold, provided that $\delta \in (0, \gamma D - 1) \subset (0, 1)$. As a result, $b \in \mathcal{S}_{\mathrm{reg}}$. Clearly, $b \in \mathcal{S}_{\mathrm{reg},n}$ for $D > n$; see Remark 3.4.

Classical examples of the polynomially decaying probability densities in [14] can be described by the following functions.

- *Student's t-function*: $\mathcal{T}(s) = (1+s^2)^{-p}$, $s > 0$, $p > \frac{1}{2}$. Note that $\mathcal{T} \in \mathcal{S}_{\mathrm{reg},n}$, $n \in \mathbb{N}$, if only $p > n/2$. The $p = 1$ case is referred to as the Cauchy distribution, the corresponding function belongs to $\mathcal{S}_{\mathrm{reg},n}$ for $n = 1$ only.
- *The Lévy function*: $\mathcal{L}(s) = s^{-3/2}\exp(-c/s)$, $s > 0$, $c > 0$.

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- *The Burr function*: $\mathfrak{B}(s) = s^{c-1}(1+s^c)^{-k-1}$, $s > 0$, $c > 0$, $k > 0$. Note that the $c = 1$ case is related to the Pareto distribution; the latter has the density $kp^k\mathfrak{B}(s-1)\mathbf{1}_{[p,\infty)}(s)$ for any $p > 0$.

(ii) *Logarithmic perturbation of the polynomial decay.* Let $D > 1$, $\nu \in \mathbb{R}$, and

$$b(s) = \mathbf{1}_{(1,\infty)}(s)(\log s)^{\nu} s^{-D}, \qquad s \in \mathbb{R}.$$

We apply Proposition 3.3 with $b_1(s) = s^{-D}$ and $b_2(s) = (\log s)^{\nu} s^{-D}$. Indeed, (3.11) clearly holds. It remains to check that (2.10) holds for both b_1 and b_2 with the same $h(s) = s^{\gamma}$, $\gamma \in (0, 1)$. We then have $\log(s \pm s^{\gamma})/\log s \to 1$ as $s \to \infty$, as required.

3.2.2. *Class 2:* $l(s) \sim D(\log s)^q$, $s \to \infty$, $q > 1$, $D > 0$. Consider the function

$$N(s) := \mathbf{1}_{\mathbb{R}_+}(s)\exp(-D(\log s)^q), \qquad s \in \mathbb{R}.$$

Take $h(s) = \mathbf{1}_{[\rho,\infty)}(s)s^{1/q}$, where $\rho > 1$ is chosen such that $h(s) < s/2$ for $s \geq \rho$. Since $q > 1$, we have

$$\begin{aligned} \frac{N(s \pm h(s))}{N(s)} &= \exp\biggl\{D(\log s)^q\biggl(1 - \biggl(1 + \frac{\log(1 \pm s^{1/q-1})}{\log s}\biggr)^q\biggr)\biggr\} \\ &\sim \exp\biggl\{D(\log s)^q\biggl(\mp q\frac{s^{1/q-1}}{\log s}\biggr)\biggr\} \sim 1, \qquad s \to \infty, \end{aligned}$$

which proves (2.10). Next, for any $\delta \in \mathbb{R}$,

$$N(s^{1/q})s^{1+\delta} = \exp(-Dq^{-q}(\log s)^q + (1+\delta)\log s) \to 0, \qquad s \to \infty.$$

As a result, $N \in \mathcal{S}_{\mathrm{reg}}$. Moreover, clearly, $N \in \mathcal{S}_{\mathrm{reg},n}$ for any $n \in \mathbb{N}$.

We may also consider Proposition 3.3 for $b_1 = b$ and $b_2 = pb$, where b_2 is a tail-decreasing and tail-log-convex function such that $\log p = o(\log b)$ (i.e. equivalent to $\log b_1 \sim \log b_2$) and p satisfies (2.10) with $h(s) = s^{1/q}$. According to the results above, a natural example of such a $p(s)$ might be s^D, $D \in \mathbb{R}$. It is straightforward to verify that, for any $D \in \mathbb{R}$, $b_2 = pb_1$ is tail-decreasing and tail-log-convex. As a result, $b_2 \in \mathcal{S}_{\mathrm{reg},n}$, $n \in \mathbb{N}$.

The classical log-normal distribution has a density described by the function

$$\mathcal{N}(s) = \frac{1}{s}\exp\biggl(-\frac{(\log s)^2}{2\gamma^2}\biggr), \qquad s > 0,\ \gamma > 0,$$

which is an example of the function b_2 above.

3.2.3. *Class 3:* $l(s) \sim s^{\alpha}$, $\alpha \in (0, 1)$. Consider, for any $\alpha \in (0, 1)$, the so-called *fractional exponent*

$$w(s) = \mathbf{1}_{\mathbb{R}_+}(s)\exp(-s^{\alpha}), \qquad s \in \mathbb{R}. \tag{3.13}$$

Set $h(s) = \mathbf{1}_{[\rho,\infty)}(s)(\log s)^{2/\alpha}$, where $\rho > 0$ is chosen such that $h(s) < s/2$ for $s \geq \rho$. Then $h(s) = o(s)$, $s \to \infty$, and, hence,

$$\frac{w(s \pm h(s))}{w(s)} = \exp\biggl\{-s^{\alpha}\biggl(\biggl(1 \pm \frac{h(s)}{s}\biggr)^{\alpha} - 1\biggr)\biggr\} \sim \exp\biggl\{-s^{\alpha}\biggl(\pm\alpha\frac{h(s)}{s}\biggr)\biggr\} \sim 1 \quad \text{as } s \to \infty,$$

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which proves (2.10). Next, for any $\delta \in \mathbb{R}$,

$$w(h(s))s^{1+\delta} = \exp(-(\log s)^2 + (1+\delta)\log s) \to 0, \qquad s \to \infty.$$

As a result, $w \in \mathcal{S}_{\mathrm{reg}}$. It is also clear that $w \in \mathcal{S}_{\mathrm{reg},n}$ for all $n \in \mathbb{N}$.

Similarly to the above, we can show that $pw \in \mathcal{S}_{\mathrm{reg}}$, provided that, in particular, $\log p = o(\log w)$ and (2.10) holds for $b = p$ and $h(s) = (\log s)^{2/\alpha}$. Again, we can consider $p(s) = s^D$, $D \in \mathbb{R}$, since it satisfies (2.10) with $h(s) = s^\gamma > (\log s)^{2/\alpha}$, $\alpha, \gamma \in (0, 1)$, and large enough s. As before, the verification that, for any $D \in \mathbb{R}$, $b_2 = pb_1$ is tail-decreasing and tail-log-convex is straightforward.

The probability density of the classical Weibull distribution is described by the function $\mathcal{W}(s) = s^{\alpha-1}\exp(-s^\alpha)$, $s \geq \rho > 0$, $\alpha \in (0, 1)$. By the above, $\mathcal{W} \in \mathcal{S}_{\mathrm{reg},n}$, $n \in \mathbb{N}$. Note that $\int_s^\infty \mathcal{W}(\tau)\,\mathrm{d}\tau = (1/\alpha)w(s)$, where w is given by (3.13).

3.2.4. *Class 4:* $l(s) \sim s/(\log s)^q$, $q > 1$. Consider also a function which decays 'slightly' more slowly than an exponential function. Namely, let, for an arbitrary fixed $q > 1$,

$$g(s) = \mathbf{1}_{\mathbb{R}_+}(s)\exp\biggl(-\frac{s}{(\log s)^q}\biggr), \qquad s \in \mathbb{R}.$$

Take, for an arbitrary $\gamma \in (1, q)$, $h(s) = (\log s)^\gamma$, $s > 0$, and denote, for brevity, $p(s) := h(s)/s \to 0$, $s \to \infty$. Then $\log(s + h(s)) = \log s + \log(1 + p(s))$. Set also $r(s) := \log(1 + p(s))/\log s \to 0$, $s \to \infty$. Then, for any $s > \mathrm{e}^{q+1}$, we have

$$\begin{aligned}\log\frac{g(s+h(s))}{g(s)} &= \frac{s}{(\log s)^q}\biggl(1 - \frac{1+p(s)}{(1+r(s))^q}\biggr)\\ &= \frac{1}{(1+r(s))^q}\biggl(q\frac{(1+r(s))^q - 1}{qr(s)}\frac{\log(1+p(s))}{p(s)}(\log s)^{\gamma-q-1} - (\log s)^{\gamma-q}\biggr)\\ &\to 0 \quad \text{as } s \to \infty,\end{aligned}$$

since $\gamma < q$ and, similarly, $\log(g(s - h(s))/g(s)) \to 0$, $s \to \infty$. Therefore, (2.10) holds for $b = g$. Next,

$$\log(g(h(s))s^{1+\delta}) = -(\log s)\biggl(\frac{(\log s)^{\gamma-1}}{\gamma^q(\log\log s)^q} - (1+\delta)\biggr) \to -\infty, \qquad s \to \infty,$$

which yields (3.1) for $b = g$. As a result, $g \in \mathcal{S}_{\mathrm{reg}}$. Again, clearly, $g \in \mathcal{S}_{\mathrm{reg},n}$, $n \in \mathbb{N}$. The same arguments as before show that, for any $D \in \mathbb{R}$, the function $s^D g(s)$ also belongs to $\mathcal{S}_{\mathrm{reg},n}$.

Remark 3.5. Naturally, $q \in (0, 1]$ yields the behavior of $g(s)$ more 'closely' to the exponential function. Unfortunately, our approach does not cover this case: in the analysis above we show that $h(s)$, to fulfill even (2.6), must grow faster than $\log s$, whereas such a 'large' $h(s)$ would not fulfill (2.10). In general, Lemma 2.4 gives a sufficient condition only, allowing us to obtain a subexponential density on $\mathbb{R}_+$. It can be shown (see, e.g. [10, Example 1.4.3]) that a probability distribution, whose density b on $\mathbb{R}_+$ is such that $\int_s^\infty b(\tau)\,\mathrm{d}\tau \sim g(s)$, $s \to \infty$ with $q > 0$, is a subexponential distribution (for the latter definition, see, e.g. [14, Definition 3.1]). Then we expect that $b(s) \sim -g'(s)$, $s \to \infty$, and it is easy to see that $\log(-g'(s)) \sim \log g(s)$, $s \to \infty$. We stress that, in general, the subexponential property of a distribution does not imply the corresponding property of its density; see [14, Section 4.2]. Therefore, we cannot state that the function b above is a subexponential one for $q \in (0, 1]$.

Combining the results above, we have the following statement.

Corollary 3.1. *Let $b\colon \mathbb{R} \to \mathbb{R}_+$ be a bounded tail-decreasing and tail-log-convex function such that, for some $C > 0$, the function $Cb(s)$ has either of the following asymptotics as $s \to \infty$:*

$$(\log s)^\mu s^{-(n+\delta)}, \qquad (\log s)^\mu s^\nu \exp(-D(\log s)^q),$$
$$(\log s)^\mu s^\nu \exp(-s^\alpha), \qquad (\log s)^\mu s^\nu \exp\biggl(-\frac{s}{(\log s)^q}\biggr),$$

where $D, \delta > 0$, $q > 1$, $\alpha \in (0, 1)$, $\nu, \mu \in \mathbb{R}$. Then $b \in \mathcal{S}_{\mathrm{reg},n}$, $n \in \mathbb{N}$.

4. Analogues of Kesten's bound on $\mathbb{R}^d$

We start with a simple corollary of Propositions 2.3 and 2.4.

Proposition 4.1. *Let a function $\omega\colon \mathbb{R}^d \to (0, +\infty)$ be such that (2.19) holds, and*

$$\limsup_{\lambda\to 0+} \sup_{x\in\Omega_\lambda} \frac{(a * \omega)(x)}{\omega(x)} \leq 1. \tag{4.1}$$

We also let $a \in L^\infty(\mathbb{R}^d)$ and $\|a\|_\omega < \infty$. Then, for any $\delta \in (0, 1)$, there exist $c_\delta > 0$ and $\lambda = \lambda(\delta) \in (0, 1)$ such that

$$a^{*n}(x) \leq c_\delta (1 + \delta)^{n-1} \min\{\lambda, \omega(x)\}, \qquad x \in \mathbb{R}^d. \tag{4.2}$$

Proof. Take any $\delta \in (0, 1)$. By Proposition 2.4, there exists $\lambda = \lambda(\delta, \omega) \in (0, 1)$ such that (2.18) holds with ϕ given by (2.21) and $\gamma = 1 + \delta$. Denote $\|a\|_\infty := \|a\|_{L^\infty(\mathbb{R}^d)}$. We have

$$\frac{a(x)}{\omega_\lambda(x)} \leq \frac{\|a\|_\infty}{\lambda} \mathbf{1}_{\mathbb{R}^d\setminus\Omega_\lambda}(x) + \frac{a(x)}{\omega(x)} \mathbf{1}_{\Omega_\lambda}(x) \leq \frac{\|a\|_\infty}{\lambda} + \|a\|_\omega =: c_\delta < \infty, \tag{4.3}$$

and we can apply Proposition 2.3, which yields the statement. □

Remark 4.1. Note that, for $d = 1$, Proposition 2.2 implies that if only $\omega \in \mathcal{S}_{\mathrm{reg},1}$, $a(s) = o(\omega(s))$, $s \to \infty$, and (2.17) holds, then $(a * \omega)(s) \sim \omega(s)$, $s \to \infty$, and then, in particular, (4.1) holds. Next, in the course of the proof of Theorem 2.2 (still for $d = 1$), we slightly weaken the restriction $a = o(\omega)$ for the case where a itself is subexponential by setting $\omega := a^{*m}$ with large enough $m \in \mathbb{N}$, since then $a/\omega \sim 1/m$ may be chosen arbitrary small. As we mentioned in the introduction, for the multi-dimensional case, we do not have a theory of subexponential densities. Therefore, we consider a more 'rough' candidate for ω to ensure (4.1), namely, $\omega(x) = a(x)^\alpha$ for $\alpha \in (0, 1)$; then, in particular, $a(x) = o(\omega(x))$, $|x| \to \infty$ if, e.g. $a(x) = b(|x|)$, $x \in \mathbb{R}^d$, with a tail-decreasing function b; see Definition 4.1 below. In particular, using the results of this section, following from (4.2), we obtain upper bounds for $a^{*n}(x)$ with the right-hand side heavier than $a(x)$ at ∞.

Definition 4.1. Let $\mathcal{D}_d$ be the set of all bounded functions $b\colon \mathbb{R} \to (0, \infty)$ such that b is tail-decreasing (see Definition 2.5) and $\int_0^\infty b(s)s^{d-1}\,\mathrm{d}s < \infty$. Let $\tilde{\mathcal{D}}_d \subset \mathcal{D}_d$ denote the subset of all functions from $\mathcal{D}_d$ which are (strictly) decreasing to 0 on $\mathbb{R}_+$.

Remark 4.2. It is easy to see that if $b^{\alpha_0} \in \tilde{\mathcal{D}}_d$ for some $\alpha_0 \in (0, 1)$ then $b^\alpha \in \tilde{\mathcal{D}}_d$ for all $\alpha \in [\alpha_0, 1]$.

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We consider an analogue of long-tailed functions on $\mathbb{R}^d$.

Lemma 4.1. *Let $b: \mathbb{R}_+ \to \mathbb{R}_+$ be a long-tailed function (see Definition 2.1) and $c(x) = b(|x|)$, $x \in \mathbb{R}^d$. Then, for any $r > 0$,*

$$\lim_{|x|\to\infty} \sup_{|y|\leq r} \left| \frac{c(x+y)}{c(x)} - 1 \right| = 0. \tag{4.4}$$

Proof. Clearly, $|y| \leq r$ implies that $h := |x+y| - |x| \in [-r, r]$ for each $x \in \mathbb{R}^d$. Therefore,

$$\sup_{|y|\leq r} \left| \frac{b(|x+y|)}{b(|x|)} - 1 \right| \leq \sup_{h\in[-r,r]} \left| \frac{b(|x|+h)}{b(|x|)} - 1 \right| \to 0, \qquad |x| \to \infty,$$

due to (2.2). □

We will now assume that a is bounded by a radially symmetric function:

$$\text{there exists } b^+ \in \tilde{\mathfrak{D}}_d \text{ such that } a(x) \leq b^+(|x|) \text{ for almost all } x \in \mathbb{R}^d. \tag{4.5}$$

We start with the following sufficient condition.

Proposition 4.2. *Let (4.5) hold with $b^+ \in \tilde{\mathfrak{D}}_d$, which is log-equivalent (see Definition 3.3) to the function b given by*

$$b(s) := \mathbf{1}_{\mathbb{R}_+}(s) \frac{M}{(1+s)^{d+\mu}}, \qquad s \in \mathbb{R}, \tag{4.6}$$

for some $\mu, M > 0$. Then there exists $\alpha_0 \in (0,1)$ such that, for all $\alpha \in (\alpha_0, 1)$, the function $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$, satisfies (4.1).

Proof. Set $\alpha_0 := (d+\mu/2)/(d+\mu) \in (0,1)$. Take arbitrary $\alpha \in (\alpha_0, 1)$ and $\varepsilon \in (0, 1-\alpha)$. Take also an arbitrary $\delta \in (0,1)$ and define $h(s) = s^\delta$, $s > 0$. By (3.12) applied to $b_1 = b$ and $b_2 = b^+$, there exists $s_\delta > 2r$ such that, for all $s > s_\delta$,

$$h(s) < \frac{s}{2}, \qquad b^+(s) \leq (b(s))^{1-\varepsilon}. \tag{4.7}$$

For an arbitrary $x \in \mathbb{R}^d$ with $|x| > s_\delta$, we have a disjoint expansion $\mathbb{R}^d = D_1(x) \sqcup D_2(x) \sqcup D_3(x)$, where

$$D_1(x) := \{|y| \leq h(|x|)\}, \qquad D_2(x) := \left\{ h(|x|) < |y| \leq \frac{|x|}{2} \right\}, \qquad D_3(x) = \left\{ |y| \geq \frac{|x|}{2} \right\}.$$

Then $(a * \omega)(x)/\omega(x) = I_1(x) + I_2(x) + I_3(x)$, where

$$I_j(x) := \int_{D_j(x)} a(y) \frac{(1+|x|)^{(d+\mu)\alpha}}{(1+|x-y|)^{(d+\mu)\alpha}} \, dy, \qquad j = 1, 2, 3.$$

Using the inequality $|x-y| \geq ||x| - |y||$, $x, y \in \mathbb{R}^d$, we have $|x-y| \geq |x| - |y| \geq |x| - |x|^\delta$ for $y \in D_1(x)$, $|x| > s_\delta$. Then

$$I_1(x) \leq \left(\frac{1+|x|}{1+|x|-|x|^\delta} \right)^{(d+\mu)\alpha} \int_{D_1(x)} a(y) \, dy \to 1, \qquad |x| \to \infty.$$

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Next, we have, for any $|y| < |x|/2$, $1 + |x - y| \geq 1 + |x| - |y| \geq \frac{1}{2}(1 + |x|)$; therefore,

$$I_2(x) \leq 2^{(d+\mu)\alpha} \int_{\{|y|>|x|^\delta\}} a(y)\,dy \to 0, \qquad |x| \to \infty.$$

Finally, by (4.5) and (4.7), the inclusions $y \in D_3(x)$ and $|x| > s_\delta$ imply that $a(y) \leq b^+(|y|) \leq b(|y|)^{1-\varepsilon} \leq b(|x|/2)^{1-\varepsilon}$, and, therefore,

$$\begin{aligned} I_3(x) &\leq M \frac{(1+|x|)^{(d+\mu)\alpha}}{(1+|x|/2)^{(d+\mu)(1-\varepsilon)}} \int_{D_3(x)} \frac{1}{(1+|x-y|)^{(d+\mu)\alpha_0}}\,dy \\ &\leq M \frac{(1+|x|)^{(d+\mu)\alpha}}{(1+|x|/2)^{(d+\mu)(1-\varepsilon)}} \int_{\mathbb{R}^d} \frac{1}{(1+|y|)^{d+\mu/2}}\,dy \\ &\to 0, \qquad |x| \to \infty, \end{aligned}$$

as $1 - \varepsilon > \alpha$. Since b is decreasing on $\mathbb{R}_+$, it follows that, by (2.19) for any $\lambda > 0$, there exists $\rho_\lambda > 0$ such that $\Omega_\lambda = \{x \in \mathbb{R}^d : |x| > \rho_\lambda\}$. This yields (4.1). □

Lemma 4.2. *Let $b \in L^1(\mathbb{R})$ be an even, positive, decreasing to 0 on the whole $\mathbb{R}_+$, and long-tailed function. Suppose that there exist $B, r_b, \rho_b > 0$ such that*

$$\int_{r_b}^{\infty} b(s-\tau)b(\tau)\,d\tau \leq Bb(s), \qquad s > \rho_b. \tag{4.8}$$

Suppose also that

$$\lim_{|x|\to\infty} \frac{a(x)|x|^{d-1}}{b(|x|)} = 0. \tag{4.9}$$

Then inequality (4.1) holds for $\omega(x) := b(|x|)$, $x \in \mathbb{R}^d$.

Proof. Assumption (4.9) implies that

$$g(r) := \sup_{|x|\geq r} \frac{a(x)|x|^{d-1}}{\omega(x)} \to 0, \qquad r \to \infty. \tag{4.10}$$

Take an arbitrary $\delta \in (0, 1)$. By (4.10), we can then take $r = r(\delta) > r_b$ such that $g(r) < \delta$. Next, by Lemma 4.1, inequality (4.4) holds for $c = \omega$. Therefore, there exists $\rho = \rho(\delta, r) = \rho(\delta) > \max\{r, \rho_b\}$ such that

$$\sup_{|y|\leq r} \frac{\omega(x-y)}{\omega(x)} < 1 + \delta, \qquad |x| \geq \rho. \tag{4.11}$$

Then, by (4.10) and (4.11), we have

$$\begin{aligned} (a * \omega)(x) &= \omega(x) \int_{\{|y|\leq r\}} a(y) \frac{\omega(x-y)}{\omega(x)}\,dy + \omega(x) \int_{\{|y|\geq r\}} \frac{a(y)|y|^{d-1}}{\omega(y)} \frac{\omega(x-y)\omega(y)}{\omega(x)|y|^{d-1}}\,dy \\ &\leq \omega(x)(1+\delta) \int_{|y|\leq r} a(y)\,dy + g(r)\omega(x) \int_{\{|y|\geq r\}} \frac{b(|x-y|)b(|y|)}{b(|x|)|y|^{d-1}}\,dy, \end{aligned}$$

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and using the fact that b is decreasing on $\mathbb{R}_+$ and the inequality $|x-y| \geq ||x|-|y||$, we obtain (see (2.17) and recall that $g(r) < \delta$)

$$\begin{aligned}(a*\omega)(x) &\leq \omega(x)(1+\delta) + \delta\omega(x) \int_{\{|y|\geq r\}} \frac{b(||x|-|y||)b(|y|)}{b(|x|)|y|^{d-1}}\,\mathrm{d}y \\ &\leq \omega(x)(1+\delta) + \delta\omega(x)\sigma_d \int_r^\infty \frac{b(|x|-p)b(p)}{b(|x|)}\,\mathrm{d}p, \end{aligned} \tag{4.12}$$

where σ_d is the hypersurface area of a unit sphere in $\mathbb{R}^d$ (note that we have omitted an absolute value, as b is even). Finally, using the fact that $r > r_b$ and $\rho > \rho_b$, from (4.8) and (4.12), for any $\delta \in (0,1)$, we obtain

$$(a*\omega)(x) \leq \omega(x)(1+\delta(1+\sigma_d B)), \qquad |x| > \rho(\delta),$$

which implies the statement. □

Lemma 4.3. *Let $b \in \mathcal{S}_{\mathrm{reg},1}$ be an even function. Suppose that there exists $\alpha' \in (0,1)$ such that $b^{\alpha'} \in \tilde{\mathfrak{D}}_d$, i.e.*

$$\int_0^\infty b(s)^{\alpha'} s^{d-1}\,\mathrm{d}s < \infty \tag{4.13}$$

and, for any $\alpha \in (\alpha', 1)$,

$$\lim_{|x|\to\infty} \frac{a(x)}{b(|x|)^\alpha}|x|^{d-1} = 0. \tag{4.14}$$

Then there exists $\alpha_0 \in (\alpha', 1)$ such that (4.1) holds for $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$, for all $\alpha \in (\alpha_0, 1)$.

Proof. We apply Theorem 3.1(ii) for $n = 1$; note that then (4.13) implies (3.2). As a result, for any $\alpha \in (\alpha_0, 1)$, inequality (3.6) holds; in particular, (4.8) holds with b replaced by b^α. The latter together with (4.14) allows us to apply Lemma 4.2 for b replaced by b^α, which fulfills the statement. □

Remark 4.3. Note that, by Remark 4.2, (4.13) implies that $b \in \tilde{\mathfrak{D}}_d$ and, hence (see Definition 3.1), $b \in \mathcal{S}_{\mathrm{reg},d}$.

As a result, we obtain a counterpart of Proposition 4.2 for the case when the function b^+ in (4.5) decays faster than polynomial and $d > 1$.

Proposition 4.3. *Let (4.5) hold for a function $b^+ \in \tilde{\mathfrak{D}}_d$ which is log-equivalent to a function $b \in \mathcal{S}_{\mathrm{reg},1}$. For $d > 1$, we suppose, additionally, that*

$$\lim_{s\to\infty} b(s)s^\nu = 0 \quad \textit{for all } \nu \geq 1. \tag{4.15}$$

Then there exists $\alpha_0 \in (0,1)$ such that, for all $\alpha \in (\alpha_0, 1)$, the function $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$, satisfies (4.1).

Proof. We will use Lemma 4.3. For $d > 1$, from (4.15), it follows that, for any $\nu > 0$, there exists $\rho_\nu \geq 1$ such that $b(s) \leq s^{-\nu}$, $s > \rho_\nu$. In particular, for any $\alpha' \in (0,1)$, we have (4.13). For $d = 1$ and $\sigma = 0$, we instead use the fact that $b \in \mathcal{S}_{\mathrm{reg},1}$ implies (3.3) and, hence, we obtain (4.13) if only $\alpha' \in (1/(1+\delta), 1)$.

Next, for any $d \in \mathbb{N}$, choose an arbitrary $\alpha \in (\alpha', 1)$. Then, by (4.5) and (3.12) applied for $b_1 = b$ and $b_2 = b^+$, it follows that, for any $\varepsilon \in (0, 1-\alpha)$, there exists $\rho_\varepsilon > 0$ such that, for all $|x| > \rho_\varepsilon$,

$$\frac{a(x)}{b(|x|)^\alpha}|x|^{d-1} \le b(|x|)^{1-\varepsilon-\alpha}|x|^{d-1} = (b(|x|)|x|^\nu)^{1-\varepsilon-\alpha}, \tag{4.16}$$

where $\nu = (d-1)/(1-\varepsilon-\alpha) \ge 0$ as $\alpha < 1-\varepsilon$. Clearly, (4.16) together with (4.15), in the $d > 1$ case, imply (4.14), which fulfills the statement. □

Remark 4.4. Note that, in Proposition 4.2, for the function b given by (4.6), we can choose $\alpha' \in (0, 1)$ such that (4.13) holds. This is the same property we checked above for the function b which satisfies the assumptions of Proposition 4.3. As a result, by Remark 4.2, the function $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$, in Proposition 4.3 are integrable for all $\alpha \in (\alpha_0, 1)$.

Definition 4.2. Let the set $\tilde{\mathcal{S}}_{\mathrm{reg},d} \subset \mathcal{S}_{\mathrm{reg},d}$, $d \in \mathbb{N}$, be defined as follows. Let $\tilde{\mathcal{S}}_{\mathrm{reg},1}$ be just the set $\mathcal{S}_{\mathrm{reg},1}$, whereas for $d > 1$, let $\tilde{\mathcal{S}}_{\mathrm{reg},d}$ be the set of all functions $b \in \mathcal{S}_{\mathrm{reg},d}$ such that b is either given by (4.6) for some $M, \mu > 0$ or b satisfies (4.15).

Remark 4.5. All functions in classes 2–4 in Subsection 3.2 clearly satisfy (4.15) and, hence, belong to $\tilde{\mathcal{S}}_{\mathrm{reg},d}$.

Theorem 4.1. *Let (4.5) hold with $b^+ \in \tilde{\mathcal{D}}_d$, which is log-equivalent to a function $b \in \tilde{\mathcal{S}}_{\mathrm{reg},d}$. Then there exists $\alpha_0 \in (0, 1)$ such that, for any $\delta \in (0, 1)$ and $\alpha \in (\alpha_0, 1)$, there exist $c_1 = c_1(\delta, \alpha) > 0$ and $\lambda = \lambda(\delta, \alpha) \in (0, 1)$ such that*

$$a^{*n}(x) \le c_1(1+\delta)^n \min\{\lambda, b(|x|)^\alpha\}, \qquad x \in \mathbb{R}^d.$$

In particular, for some $c_2 = c_2(\delta, \alpha) > 0$ and $s_\alpha = s_\alpha(\delta) > 0$,

$$a^{*n}(x) \le c_2(1+\delta)^n b(|x|)^\alpha, \qquad |x| > s_\alpha, n \in \mathbb{N}. \tag{4.17}$$

Proof. Combining Propositions 4.1–4.3, it follows that, by (4.2) and (4.3), there exist $\tilde{c_\delta} = \tilde{c_\delta}(\omega)$ and $\lambda = \lambda(\delta, \alpha) \in (0, 1)$, where $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$, such that

$$a^{*n}(x) \le \tilde{c_\delta}(1+\delta)^{n-1} \min\{\lambda, b(|x|)^\alpha\}, \qquad x \in \mathbb{R}^d, \tag{4.18}$$

which clearly yields (4.18). Since b is tail-decreasing, we have, for some $s_\alpha > 0$, $b(|x|)^\alpha < \lambda$ for $|x| > s_\alpha$. This implies (4.17). □

Corollary 4.1. *Let $a(x) = b(|x|)$, $x \in \mathbb{R}$, for some $b \in \tilde{\mathcal{S}}_{\mathrm{reg},d}$. Then there exists $\alpha_0 \in (0, 1)$ such that, for any $\delta \in (0, 1)$ and $\alpha \in (\alpha_0, 1)$, there exist $c_{\delta,\alpha} > 0$ and $s_\alpha = s_\alpha(\delta) > 0$ such that*

$$a^{*n}(x) \le c_{\delta,\alpha}(1+\delta)^n a(x)^\alpha, \qquad |x| > s_\alpha,\ n \in \mathbb{N}.$$

Proof. Since, for some $\rho > 0$, b is decreasing on (ρ, ∞) and (3.2) holds, there exists $b^+ \in \tilde{\mathcal{D}}_d$ such that $b^+(s) = b(s)$, $s > \rho$, and $b^+(s) \ge b(s)$, $s \in [0, \rho]$. We then apply Theorem 4.1. □

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Appendix A. Nonlocal heat equation

We apply our results to the study of the regular part of the fundamental solution to the nonlocal heat equation

$$\frac{\partial}{\partial t}u(x,t) = \varkappa \int_{\mathbb{R}^d} a(x-y)(u(y,t)-u(x,t))\,\mathrm{d}y, \qquad x \in \mathbb{R}^d, \tag{A.1}$$

where $\varkappa > 0$ and $0 \le a \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is normalized, i.e. $\int_{\mathbb{R}^d} a(x)\,\mathrm{d}x = 1$; see, e.g. [1], [5], and [16]. Consider an initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{R}^d$, to (A.1) with u_0 from a space E of functions bounded on $\mathbb{R}^d$. Since the operator $Au = \varkappa a * u - \varkappa u$ on the right-hand side of (A.1) is bounded on E, the unique solution to (A.1) is given by

$$u(x,t) = \mathrm{e}^{-\varkappa t}((\delta_0 + \phi_\varkappa(t)) * u_0)(x), \tag{A.2}$$

where δ_0 is the Dirac delta at $0 \in \mathbb{R}^d$ and

$$\phi_\varkappa(x,t) := \sum_{n=1}^{\infty} \frac{\varkappa^n t^n}{n!} a^{*n}(x), \qquad x \in \mathbb{R}^d,\ t \ge 0. \tag{A.3}$$

Note that Chasseigne *et al.* [6, Lemma 2.2] showed that if a is a rapidly decreasing smooth function then ϕ_k is indeed the solution to (A.1) with $u_0 = \delta_0$.

Now, for $d = 1$, suppose that $a(x) = b(x)$, $x \in \mathbb{R}^1$, and b satisfies the conditions of Theorem 2.2. Then, by (2.24), the series in (A.3) converges uniformly on finite-time intervals for each $x > s_0$ and, therefore, by (2.16),

$$\phi_\varkappa(x,t) \sim kt\mathrm{e}^{\varkappa t} a(x), \qquad x \to \infty,\ t > 0.$$

For $d > 1$, let a and b satisfy the conditions of Theorem 4.1. Then, for each $\delta > 0$ and for each $\alpha < 1$ close enough to 1,

$$\phi_\varkappa(x,t) \le c_{\delta,\alpha}(\mathrm{e}^{\varkappa t(1+\delta)} - 1) b(|x|)^\alpha, \qquad |x| > s_\alpha,\ t > 0, \tag{A.4}$$

for some $c_{\delta,\alpha} > 0$ and $s_\alpha = s_\alpha(\delta) > 0$. In particular, if a is radially symmetric and the conditions of Corollary 4.1 hold, then we can replace $b(|x|)$ on $a(x)$ in (A.4).

Moreover, combining (4.18) with (A.2), we also have an estimate for the solution u to (A.1). The further analysis of solutions to (A.1) can be found in [11] and [12].

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