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A QUEUEING/INVENTORY AND AN INSURANCE RISK MODEL

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Abstract

We study an M/G/1-type queueing model with the following additional feature. The server works continuously, at fixed speed, even if there are no service requirements. In the latter case, it is building up inventory, which can be interpreted as negative workload. At random times, with an intensity $\omega(x)$ when the inventory is at level $x > 0$, the present inventory is removed, instantaneously reducing the inventory to 0. We study the steady-state distribution of the (positive and negative) workload levels for the cases $\omega(x)$ is constant and $\omega(x) = ax$. The key tool is the Wiener–Hopf factorization technique. When $\omega(x)$ is constant, no specific assumptions will be made on the service requirement distribution. However, in the linear case, we need some algebraic hypotheses concerning the Laplace–Stieltjes transform of the service requirement distribution. Throughout the paper, we also study a closely related model arising from insurance risk theory.

Keywords: M/G/1 queue; Cramér–Lundberg insurance risk model; workload; inventory; ruin probability; Wiener–Hopf technique

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1. Introduction

In this paper we study two related stochastic models: a queueing/inventory model and an insurance risk model. The insurance risk model is a relaxation of the classical Cramér–Lundberg model. Unlike that model, when the capital of the insurance company becomes negative, the company continues to operate in the same way. However, during periods of negative surplus, the company can go *bankrupt*. It goes bankrupt according to some bankruptcy rate $\omega(x)$ when the negative surplus is equal to x; see Figure 1. This relaxation of the ruin concept was introduced in [4], and studied in [3] for exponential claim sizes and various bankruptcy rates. One of our goals in the present paper is to extend some of the results in [3] to *general claim size distributions*. In particular, we aim to study the bankruptcy probability when starting at x, for both positive *and* negative values of x.

It is well known that the Cramér–Lundberg model is dual to the M/G/1 queueing model with the same arrival rate and with a service time distribution that is equal to the claim size distribution in the Cramér–Lundberg model. More precisely, see [6, p. 52]: the probability of ruin in the Cramér–Lundberg model with initial capital x is equal to the probability that the steady-state virtual waiting time (or workload) in the M/G/1 queue exceeds x. This has led us

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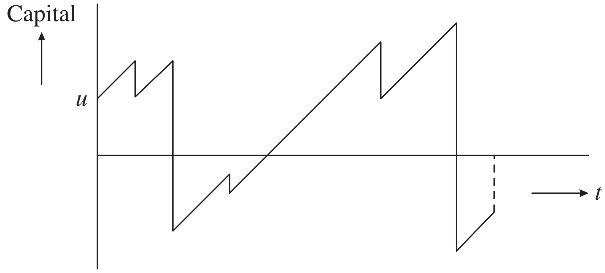

FIGURE 1: The capital of an insurance company.

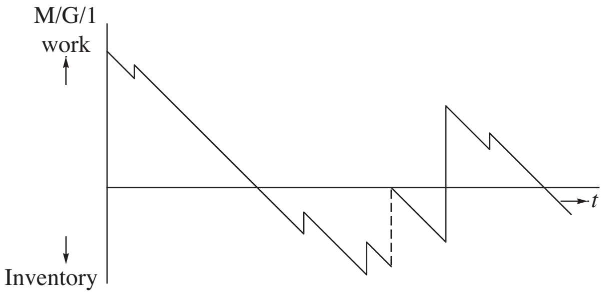

FIGURE 2: Work and inventory in a queueing/inventory model.

to think about queueing models that are relaxations of the M/G/1 queue in a similar way as the Albrecher–Lautscham bankruptcy model is a relaxation of the Cramér–Lundberg model; see also [5].

In this paper we study the following queueing/inventory model. Customers arrive according to a Poisson process, and require independent and identically distributed service times. When there are customers, the server works at unit speed. So far, this is the M/G/1 setting. However, when there are no customers, the server still keeps on working at the same speed. In that way, it is building up inventory.

During periods in which there are no customers, inventory is instantaneously removed according to a Poisson process with rate $\omega(x)$ when the amount of inventory is equal to x. That inventory is, for example, sold. The server just keeps on working; and when a customer arrives and its service request can be satisfied from the inventory, then that is done instantaneously. See Figure 2. In [5] this two sided queueing/inventory model has been analyzed for the case of exponentially distributed service times. Our queueing/inventory model is related to classical M/G/1 and inventory models (see [5] and the references therein). An important inventory model is the basestock model, in which a server produces products until the inventory has reached a certain basestock level, with requests for products arriving according to a Poisson process. A request that cannot immediately be satisfied joins a backorder queue. However, that model has a finite basestock level, and, hence, essentially differs from our model. Two papers which are to some extent related to our paper are [7] and [15]. In these the authors considered a production/inventory model with a so-called sporadic clearing policy. The system is continuously filled at fixed rate, and satisfies demands at Poisson epochs. Under the sporadic clearing policy, clearing of all inventory takes place at a random time (which in [15] is independent of the content process). The authors obtained explicit results for an expected discounted cost functional. The authors in [15] allow the demands to have a general distribution, whereas these are exponentially distributed in [7]. A model that slightly resembles our two-sided model is that of the double-ended queue (for example, persons queueing for a taxi, or taxis queueing for a customer); see [12], [13], and [17].

The main contributions of our paper are

- an exact analysis of the Albrecher–Lautscham bankruptcy model and the queueing/inventory model, with generally distributed claim sizes, respectively generally distributed service requirements, for the $\omega(x) \equiv \omega$ case; and
- a detailed analysis of the queueing/inventory model for the $\omega(x) = ax$ case.

The latter case turns out to lead to an inhomogeneous first-order differential equation with removable singularities, and its analysis gives rise to intricate calculations. A key tool that we are using in the analysis of the two models is Wiener–Hopf factorization. The results of this paper might be used for optimization purposes; for example, one might try to choose ω or a (in the $\omega(x) = ax$ case) such that a particular objective function is optimized.

The paper is organized as follows. The queueing/inventory and insurance risk model are both described in detail in Section 2. Integral equations for the main performance measures (workload and inventory densities in the queueing/inventory model, bankruptcy probability when starting at level x in the insurance risk model) are presented in Section 3. In Section 4 these equations are solved for $\omega(x) \equiv \omega$ and general service requirement distribution, respectively general claim size distribution. The queueing/inventory model is treated in Section 5 for the $\omega(x) = ax$ case. That analysis makes use of Laplace transforms and complex analysis. We assume that the service requirements have a rational Laplace transform. In Section 6 we consider that very same case, under the assumption of exponentially distributed service requirements, without resorting to Laplace transforms. In Section 7 we sketch how to solve the first order differential equation that shows up in the case of linear $\omega(\cdot)$.

2. Model description

2.1. Queueing/inventory model

We study the following model (see Figure 2). Customers arrive according to a Poisson process with rate λ. Their service requirements are independent and identically distributed random variables $B_1, B_2, \ldots$ with common distribution $B(\cdot)$ and Laplace–Stieltjes transform (LST) $\beta(\cdot)$. The server works continuously, at a fixed speed which is normalized to 1—even if there are no service requirements. In the latter case, the server is building up inventory, which can be interpreted as negative workload. At random times, with an intensity $\omega(x)$ when the inventory is at level $x > 0$, the present inventory is removed, instantaneously reducing the inventory to 0 (see the dotted line in Figure 2). Put differently, inventory is removed according to a Poisson process with a rate that depends on the amount of inventory present.

Denote the required work per time unit by $\rho := \lambda \mathbb{E} B$. We assume that $\rho < 1$. This ensures that the steady-state workload distribution exists. Let $V_+(x)$, $x > 0$, denote this steady-state workload distribution, and $v_+(x)$ its density.

During the times in which the inventory level is positive, there is an upward drift $1 - \rho$ of that inventory level; but when $\omega(x) > 0$ for sufficiently large x, the inventory level will always eventually return to 0, and the steady-state inventory distribution will exist. Let $V_-(x)$, $x > 0$, denote this steady-state distribution, and $v_-(x)$ its density.

2.2. Insurance risk model

The problem that we will deal with here was introduced by Albrecher *et al.* [4] and investigated by Albrecher and Lautscham in [3]. We will examine the bankruptcy probability for a surplus process with jumps. Consider a Cramér–Lundberg setup to describe the insurer's surplus C_t at time t as

$$C_t = x + ct - S_t,$$

where $C_0 = x$ is the initial surplus, c is the premium rate, and S_t is the aggregate claim amount up to time t modeled as a compound Poisson process with intensity λ and positive jump sizes $Y_1, Y_2, \dots$ with cumulative distribution function $F_Y(\cdot)$. In order to compare the results for the bankruptcy model with those for the queueing/inventory model, we shall take $F_Y(\cdot) = B(\cdot)$, so its LST is $\beta(s)$. It is assumed here that the insurer may be allowed to continue the business despite a temporary negative surplus. More precisely, consider a suitable locally bounded bankruptcy rate function $\omega(-C_s)$ depending on the size of the negative surplus $C_s < 0$. If no bankruptcy event has occurred yet at time s, then the probability of bankruptcy in the time interval $[s, s + \mathrm{d}t)$ is $\omega(-C_s)\,\mathrm{d}t$. We assume that $\omega(\cdot) \geq 0$ and $\omega(x) \geq \omega(y)$ for $|x| \geq |y|$ to reflect that the likelihood of bankruptcy does not decrease as the surplus becomes more negative. Let τ be the resulting time of bankruptcy, and define the overall probability of bankruptcy as

$$u(x) = \mathbb{E}[\mathbf{1}_{\{\tau<\infty\}} \mid C_0 = x] = \mathbb{P}[\tau < \infty \mid C_0 = x].$$

The idea is that whenever the surplus level becomes negative, there may still be a chance to survive, and survival is less likely the lower such a negative surplus is. For $x > 0$, set $u_+(x) := u(x)$, $\tilde{u}_-(x) := u(-x)$, and $u_-(x) := 1 - \tilde{u}_-(x)$.

3. Main equations

In this section we present integral equations for the main performance measures (workload and inventory densities in the queueing/inventory model, bankruptcy probability when starting at level x in the insurance risk model).

3.1. Queueing model

The level crossing technique [9] yields the following integral equations for the workload and inventory densities, by equating the rates at which level x is downcrossed and upcrossed in steady state, i.e.

$$v_+(x) = \lambda \int_0^x \mathbb{P}[B > x - y] v_+(y)\,\mathrm{d}y + \lambda \int_0^{+\infty} \mathbb{P}[B > x + y] v_-(y)\,\mathrm{d}y, \qquad x > 0, \quad (1)$$

$$v_-(x) = \lambda \int_x^{+\infty} \mathbb{P}[B > y - x] v_-(y)\,\mathrm{d}y + \int_x^{+\infty} \omega(y) v_-(y)\,\mathrm{d}y, \qquad x > 0. \quad (2)$$

We introduce the Laplace transforms

$$\phi_+(s) := \int_0^{+\infty} \mathrm{e}^{-sx} v_+(x)\,\mathrm{d}x \quad \text{and} \quad \phi_-(s) := \int_0^{+\infty} \mathrm{e}^{-sx} v_-(x)\,\mathrm{d}x \quad \text{for } \operatorname{Re} s \geq 0.$$

Multiplying both sides of (1) with e^{-sx} for $\operatorname{Re} s \geq 0$ and both sides of (2) with e^{sx} for $\operatorname{Re} s \leq 0$, integrating and adding both equations, after some calculations, we obtain

$$\left[1 - \lambda \frac{1 - \beta(s)}{s}\right][\phi_+(s) + \phi_-(-s)] = \frac{1}{s} \int_0^{+\infty} (\mathrm{e}^{sy} - 1)\omega(y) v_-(y)\,\mathrm{d}y \quad \text{for } \operatorname{Re} s = 0. \quad (3)$$

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3.2. Insurance model

According to [3], one can write

$$0 = cu'_+(x) - \lambda u_+(x) + \lambda\bigg(\int_0^x u_+(x-y)\,\mathrm{d}B(y) + \int_x^{+\infty} u_-(y-x)\,\mathrm{d}B(y)\bigg), \qquad x > 0, \tag{4}$$

$$0 = -c\tilde{u}'_-(x) - (\lambda + \omega(-x))\tilde{u}_-(x) + \omega(-x) + \lambda\int_0^{+\infty} \tilde{u}_-(x+y)\,\mathrm{d}B(y), \qquad x > 0. \tag{5}$$

Adding and subtracting 1, (5) is equivalent to

$$0 = -cu'_-(x) + \lambda u_-(x) + \omega(-x)u_-(x) - \lambda\int_0^{+\infty} u_-(x+y)\,\mathrm{d}B(y), \qquad x > 0. \tag{6}$$

One can dominate the function u_+ by the classic ruin function and under the assumptions on the existence of the second moment of $B(\cdot)$, the classic ruin function is integrable. Hence, the function u_+ is integrable. Similarly, one can argue that the function u_- is integrable. For $\operatorname{Re} s \geq 0$, introduce the Laplace transforms $\Psi_+(s) := \int_0^{+\infty} \mathrm{e}^{-sx} u_+(x)\,\mathrm{d}x$, $\Psi_-(s) := \int_0^{+\infty} \mathrm{e}^{-sx} u_-(x)\,\mathrm{d}x$, and $\beta(s) := \int_0^{+\infty} \mathrm{e}^{-sy}\,\mathrm{d}B(y)$.

Multiply both sides of (4) with e^{-sx} for $\operatorname{Re} s \geq 0$ and both sides of (6) with e^{sx} for $\operatorname{Re} s \leq 0$; integrate and add both equations for $\operatorname{Re} s = 0$. After some calculations and using the fact that the continuity of the function u in 0 implies that $u_+(0) = 1 - u_-(0)$, we obtain

$$\begin{aligned}(\lambda\beta(s) + cs - \lambda)\Psi_+(s) &+ \frac{\lambda}{s}(1 - \beta(s)) - c \\ &= (\lambda\beta(s) + cs - \lambda)\Psi_-(-s) - \int_0^{+\infty} \omega(-x)u_-(x)\mathrm{e}^{sx}\,\mathrm{d}x, \qquad \operatorname{Re} s = 0.\end{aligned} \tag{7}$$

In the next section we restrict ourselves to the case where the function $\omega(\cdot)$ is constant.

4. Analysis for $\omega(\cdot)$ constant

4.1. Queueing model

In this section we assume that the function $\omega(\cdot)$ introduced in Subsection 2.1 is constant, i.e. there exist $\omega > 0$ such that for all $x \geq 0$, we have $\omega(x) = \omega$. Equation (3) can be expressed as

$$[s - \lambda(1 - \beta(s))]\phi_+(s) = [\omega - (s - \lambda(1 - \beta(s)))]\phi_-(-s) - \omega\phi_-(0) \quad \text{for } \operatorname{Re} s = 0. \tag{8}$$

We are going to determine both unknown functions $\phi_+(s)$ and $\phi_-(-s)$ for $\operatorname{Re} s \geq 0$ by formulating and solving a Wiener–Hopf problem (see [10]). A key step in this procedure is to write (8) such that the left-hand side is analytic on $\operatorname{Re} s > 0$ and the right-hand side is analytic on $\operatorname{Re} s < 0$. Liouville's theorem can subsequently be used to identify the left-hand and right-hand sides.

Set $f_{\omega,\lambda}\colon s \mapsto \lambda\beta(s) + s - \lambda - \omega$, $s \geq 0$, and $f_{0,\lambda}\colon s \mapsto \lambda\beta(s) + s - \lambda$, $s \geq 0$. According to [11, p. 548], the constant ω being positive, the function $f_{\omega,\lambda}$ has only one zero $s = \delta(\omega, \lambda)$ and this zero is simple satisfying $\operatorname{Re}\delta(\omega, \lambda) > 0$. In fact, as ω is real in our case, a plot of $\omega + \lambda(1 - \beta(s))$ versus s immediately shows that this zero $\delta(\omega, \lambda)$ is real. Also, the function $f_{0,\lambda}$ has $s = 0$ as its only zero and this zero is simple. In particular, the functions $g_{\omega,\lambda}\colon s \mapsto f_{\omega,\lambda}(s)/(s - \delta(\omega, \lambda))$ and $g_{0,\lambda}\colon s \mapsto f_{0,\lambda}(s)/s$ are analytic on $\operatorname{Re} s > 0$, continuous on $\operatorname{Re} s \geq 0$, and take nonzero values on $\operatorname{Re} s \geq 0$.

We can write (8) for $\operatorname{Re} s = 0$ as

$$\frac{(s-\delta(\omega,\lambda))[s-\lambda(1-\beta(s))]\phi_+(s)}{s-\lambda(1-\beta(s))-\omega} + \frac{\omega\phi_-(0)(s-\delta(\omega,\lambda))}{s-\lambda(1-\beta(s))-\omega} = -(s-\delta(\omega,\lambda))\phi_-(-s). \tag{9}$$

We now use the Wiener–Hopf factorization technique. The left-hand side of (9) is analytic on $\operatorname{Re} s > 0$ and continuous on $\operatorname{Re} s \geq 0$; the right-hand side is analytic on $\operatorname{Re} s < 0$ and continuous on $\operatorname{Re} s \leq 0$. In addition, both sides coincide on $\operatorname{Re} s = 0$. Then, by Liouville's theorem (see [18, p. 85]), there exist $n \geq 0$ and a polynomial $R_n(s)$ of degree n such that

$$-(s-\delta(\omega,\lambda))\phi_-(-s) = R_n(s) \quad \text{for } \operatorname{Re} s \leq 0, \tag{10}$$

$$\frac{(s-\delta(\omega,\lambda))[s-\lambda(1-\beta(s))]\phi_+(s)}{s-\lambda(1-\beta(s))-\omega} + \frac{\omega\phi_-(0)(s-\delta(\omega,\lambda))}{s-\lambda(1-\beta(s))-\omega} = R_n(s) \quad \text{for } \operatorname{Re} s \geq 0. \tag{11}$$

Using (10) and the fact that $\lim_{s\to-\infty}\phi_-(-s) = 0$, we have $\deg(R_n(s)) = 0$; say $R_n(s) = A$, where $A \in \mathbb{C}$. We obtain

$$\phi_-(-s) = \frac{A}{\delta(\omega,\lambda)-s} \quad \text{for } \operatorname{Re} s \leq 0,$$

in particular,

$$\phi_-(0) = \frac{A}{\delta(\omega,\lambda)}. \tag{12}$$

On the other hand, (11) yields

$$\phi_+(s) = \frac{A(s-\lambda(1-\beta(s))-\omega)}{(s-\lambda(1-\beta(s)))(s-\delta(\omega,\lambda))} - \frac{\omega\phi_-(0)}{s-\lambda(1-\beta(s))}.$$

After some calculations and using the notation introduced above and (12), we obtain

$$\phi_+(s) = \frac{A(g_{\omega,\lambda}(s)-\omega/\delta(\omega,\lambda))}{s g_{0,\lambda}(s)}. \tag{13}$$

We now calculate the unknown constant A, and through (12) and (13) we determine the functions ϕ_+ and ϕ_-. Note that $g_{\omega,\lambda}(0) = -f_{\omega,\lambda}(0)/\delta(\omega,\lambda) = \omega/\delta(\omega,\lambda)$; therefore, we can write, for $\operatorname{Re} s \geq 0$,

$$\phi_+(s) = \frac{A}{g_{0,\lambda}(s)} \frac{g_{\omega,\lambda}(s)-g_{\omega,\lambda}(0)}{s}, \tag{14}$$

the function $s \mapsto (g_{\omega,\lambda}(s)-g_{\omega,\lambda}(0))/s$ clearly being analytic for $\operatorname{Re} s > 0$ and continuous for $\operatorname{Re} s \geq 0$.

We have

$$\phi_+(0) = \lim_{s\to0}\phi_+(s) = \lim_{s\to0}\frac{A}{g_{0,\lambda}(s)}\frac{g_{\omega,\lambda}(s)-g_{\omega,\lambda}(0)}{s}.$$

However,

$$g_{0,\lambda}(0) = \lim_{s\to0}\frac{f_{0,\lambda}(s)}{s} = \lim_{s\to0} 1 + \lambda\frac{\beta(s)-1}{s} = 1-\lambda\mathbb{E}(B) = 1-\rho.$$

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We also have

$$\lim_{s\to 0}\frac{g_{\omega,\lambda}(s)-g_{\omega,\lambda}(0)}{s}=g'_{\omega,\lambda}(0)=\frac{\omega-(1-\rho)\delta(\omega,\lambda)}{\delta^2(\omega,\lambda)}.$$

Then,

$$\phi_+(o)=A\frac{\omega-(1-\rho)\delta(\omega,\lambda)}{(1-\rho)\delta^2(\omega,\lambda)}. \tag{15}$$

Using the relation $\phi_-(0)+\phi_+(0)=1$, and (12) and (15), we obtain

$$A\bigg[\frac{1}{\delta(\omega,\lambda)}+\frac{\omega-(1-\rho)\delta(\omega,\lambda)}{(1-\rho)\delta^2(\omega,\lambda)}\bigg]=1,$$

which implies that

$$A=\frac{(1-\rho)\delta^2(\omega,\lambda)}{\omega}. \tag{16}$$

Finally, we obtain the following expressions for $\phi_-(s)$ for $\operatorname{Re} s\le 0$, and for $\phi_+(s)$ for $\operatorname{Re} s\ge 0$:

$$\phi_-(s)=\frac{(1-\rho)\delta^2(\omega,\lambda)}{\omega}\frac{1}{\delta(\omega,\lambda)-s},\qquad \operatorname{Re} s\le 0,$$

$$\phi_+(s)=\frac{(1-\rho)\delta(\omega,\lambda)}{\omega}\bigg[\frac{(\delta(\omega,\lambda)-\omega)s-\lambda\delta(\omega,\lambda)(1-\beta(s))}{(s-\lambda(1-\beta(s)))(s-\delta(\omega,\lambda))}\bigg],\qquad \operatorname{Re} s\ge 0. \tag{17}$$

We immediately see that the density $v_-(x)$ is exponential, i.e.

$$v_-(x)=\frac{(1-\rho)\delta^2(\omega,\lambda)}{\omega}\mathrm{e}^{-\delta(\omega,\lambda)x}\quad \text{for } x>0. \tag{18}$$

We can write the expression for $\phi_+(s)$ in the following form:

$$\phi_+(s)=\bigg[(1-\rho)\lambda\frac{(1-\beta(s))}{s-\lambda(1-\beta(s))}\bigg]\times\bigg[\frac{\delta(\omega,\lambda)}{\omega}\frac{((\delta(\omega,\lambda)-\omega)/\lambda)(s/(1-\beta(s)))-\delta(\omega,\lambda)}{s-\delta(\omega,\lambda)}\bigg].$$

Remark 1. The previous equation expresses the function $\phi_+(s)$ as a product of the Laplace transform of the density of the M/G/1 workload; namely,

$$\frac{(1-\rho)\lambda(1-\beta(s))}{s-\lambda(1-\beta(s))}$$

and a second factor. That fact has led us to the observation that the queue behaves like an M/G/1 queue with different first service time in a busy period (see Welch [19] or Wolff [20, pp. 392–394, 401]). Let us explain this in some detail.

When restricting ourselves to the time intervals with a positive workload in the queue, $v_+(x)$ behaves like the workload density in an M/G/1 queue with Poisson(λ) arrival process and with independent and identically distributed service times $B_1, B_2, \ldots$ with distribution $B(\cdot)$ and LST $\beta(\cdot)$, but with the first service time $\hat{B}$ of each busy period having a *different* distribution $\hat{B}(\cdot)$ with LST $\hat{\beta}(\cdot)$. Indeed, $\hat{B}$ is distributed like the overshoot above 0 of a service time B, when starting from some negative value $-V_-=-x$ which, by Poisson arrivals see time averages

(PASTA), has steady-state density $v_-(x)$. In fact, it is easy to determine $\hat{\beta}(\cdot)$, since $v_-(x)$ is exponentially distributed with parameter $\delta(\omega, \lambda)$, as seen in (18). For simplicity of notation, we write $\delta := \delta(\omega, \lambda)$. Then

$$\begin{aligned}\hat{\beta}(s) &= \int_{0+}^{\infty} \mathrm{e}^{-sx}\, \mathrm{d}_x \mathbb{P}[B - V_- < x \mid B - V_- > 0] \\ &= \frac{\int_{0+}^{\infty} \mathrm{e}^{-sx}\, \mathrm{d}_x \mathbb{P}[B - V_- < x]}{\mathbb{P}[B - V_- > 0]} \\ &= \frac{\int_{x=0+}^{\infty} \mathrm{e}^{-sx} \int_{y=0}^{\infty} \delta \mathrm{e}^{-\delta y}\, \mathrm{d}_x \mathbb{P}[B < x + y]\, \mathrm{d}y}{1 - \beta(\delta)} \\ &= \frac{\delta}{s - \delta} \frac{\beta(\delta) - \beta(s)}{1 - \beta(\delta)},\end{aligned}$$

and

$$1 - \hat{\beta}(s) = \frac{s}{s - \delta} - \frac{\delta}{s - \delta} \frac{1 - \beta(s)}{1 - \beta(\delta)}. \tag{19}$$

From [19] or [20, p. 401] it is seen that the LST of the steady-state workload (and waiting time) distribution in this queueing model with exceptional first service is given by

$$\mathbb{E}[\mathrm{e}^{-sW}] = \pi_0 \lambda \frac{\beta(s) - \hat{\beta}(s)(1 - s/\lambda)}{s - \lambda(1 - \beta(s))},$$

with π_0 the probability of an empty system, and, hence,

$$\mathbb{E}[\mathrm{e}^{-sW} \mid W > 0] = \frac{\mathbb{E}[\mathrm{e}^{-sW}] - \pi_0}{1 - \pi_0} = \frac{\pi_0 \lambda}{1 - \pi_0} \frac{1 - \hat{\beta}(s)}{s - \lambda(1 - \beta(s))}. \tag{20}$$

A balance argument, or the observation that the expressions in (20) should equal 1 for $s = 0$, and that the LSTs of the residual ordinary service time $(1 - \beta(s))/s\mathbb{E}B$ and of the residual special service time $(1 - \hat{\beta}(s))/s\mathbb{E}\hat{B}$ are equal to 1 for $s = 0$, readily yields

$$\pi_0 = \frac{1 - \rho}{1 - \rho + \lambda \mathbb{E}\hat{B}}.$$

It readily follows from (19) that here

$$\mathbb{E}\hat{B} = \frac{\mathbb{E}B}{1 - \beta(\delta)} - \frac{1}{\delta}.$$

Now compare (20) and (17). We claim they agree up to a multiplicative constant, which is $\int_0^{\infty} v_+(x)\, \mathrm{d}x$. Indeed, one can write the term between square brackets in (17) as (replace $\delta - \omega$ by $\lambda(1 - \beta(\delta))$, using the fact that $1 - \beta(\delta) = (\delta - \omega)/\lambda$, which follows from the definition of δ as the zero of $s - \lambda(1 - \beta(s)) = \omega$),

$$(1 - \beta(\delta)) \left[\frac{1}{s - \lambda(1 - \beta(s))} \left(\frac{s}{s - \delta} - \frac{\delta}{1 - \beta(\delta)} \frac{1 - \beta(s)}{s - \delta} \right) \right]. \tag{21}$$

Now use (19) to see that the factor between square brackets in (21) is equal to the factor $(1 - \hat{\beta}(s))/(s - \lambda(1 - \beta(s)))$ in (20).

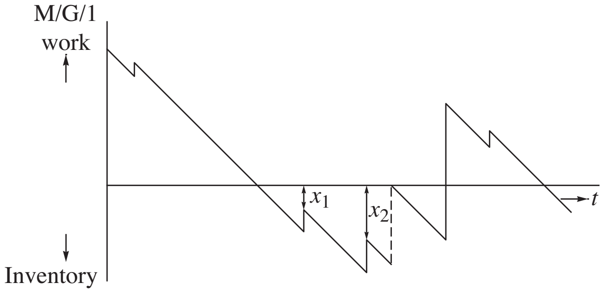

FIGURE 3: The workload and inventory process.

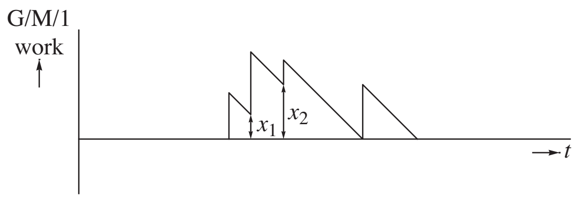

FIGURE 4: A G/M/1 busy period ('process 3').

Remark 2. It may at first sight seem surprising that $v_-(x)$ is an exponential density. The fact that $v_-(x)$ is exponential may be explained as follows. Consider the inventory process, so look at Figure 2 upside down ('process 1'). Next, consider this figure (now reproduced as Figure 3) by looking from right to left ('process 2'). Subsequently, replace the line segments that go up at an angle of 45 degrees by upward jumps equal to the increase along the line segment; and replace the jumps downward by line segments that go down at an angle of 45 degrees, by an amount equal to the jump ('process 3'; see Figure 4). We now have the representation of the workload process in a busy period of a G/M/1 queue. Indeed, the jumps upward (service times) are exp (λ) distributed, and the intervals between jumps have distribution $B(\cdot)$. Note that, in particular, the waiting times in the G/M/1 queue are identical to the heights after jumps in process 2. By PASTA, these heights have the same distribution as the steady-state workload distribution in process 2, and, hence, also in process 1. Finally, use the fact that the waiting time in the G/M/1 queue is exponentially distributed.

Example 1. (*Exponential service requirements in the queueing/inventory model.*) We will retain the same notation as previously. In this case, we have

$$\mathbb{P}[B > x] = \mathrm{e}^{-\mu x} \quad \text{with } \mu > 0$$

and, for $\mathrm{Re}\, s \geq 0$,

$$\beta(s) = \frac{\mu}{\mu + s}, \qquad \mathbb{E}[B] = \frac{1}{\mu}, \qquad \rho = \frac{\lambda}{\mu}.$$

The functions $f_{\omega,\lambda}$ and $f_{0,\lambda}$ in this case are given by

$$f_{\omega,\lambda}(s) = \frac{s^2 + (\mu - \lambda - \omega)s - \omega\mu}{\mu + s} \quad \text{and} \quad f_{0,\lambda}(s) = \frac{s(s + \mu - \lambda)}{s + \mu}.$$

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The function $f_{\omega,\lambda}$ has two zeros, i.e.

$$\delta(\omega,\lambda)=\frac{\sqrt{(\mu-\lambda-\omega)^2+4\omega\mu}-(\mu-\lambda-\omega)}{2}>0,$$
$$\eta(\omega,\lambda)=\frac{-\sqrt{(\mu-\lambda-\omega)^2+4\omega\mu}-(\mu-\lambda-\omega)}{2}<0.$$

The functions $g_{\omega,\lambda}$ and $g_{0,\lambda}$ are then given by

$$g_{\omega,\lambda}(s)=\frac{s-\eta(\omega,\lambda)}{s+\mu}\quad\text{and}\quad g_{0,\lambda}(s)=\frac{s+\mu-\lambda}{s+\mu}. \tag{22}$$

Also A is given by (16). The function $v_-(x)$ for $x>0$ is given by (18), therefore,

$$v_-(x)=\frac{\mu-\lambda}{2\omega}\Big[(\mu-\lambda-\omega)^2-2\omega\mu-(\mu-\lambda-\omega)\sqrt{(\mu-\lambda-\omega)^2+4\omega\mu}\Big]$$
$$\times\,\mathrm{e}^{-x\left(\left(\sqrt{(\mu-\lambda-\omega)^2+4\omega\mu}-(\mu-\lambda-\omega)\right)/2\right)}.$$

Using (14) and (22), and after some calculations, we obtain, for $\operatorname{Re} s\geq 0$,

$$\phi_+(s)=\frac{(\mu-\lambda)(\mu-\eta(\omega,\lambda))\delta^2(\omega,\lambda)}{\mu^2\omega}\frac{1}{s+\mu-\lambda}.$$

Consequently, we can deduce, for $x>0$,

$$v_+(x)=\frac{(\mu-\lambda)(\mu-\eta(\omega,\lambda))\delta^2(\omega,\lambda)}{\mu^2\omega}\mathrm{e}^{-(\mu-\lambda)x}.$$

Figures 5 and 6 represent, respectively, the steady-state inventory density v_- and the steady-state workload density v_+, in the exponential service requirements case for the particular values $\mu=2,\ \lambda=1$, and $\omega=2$.

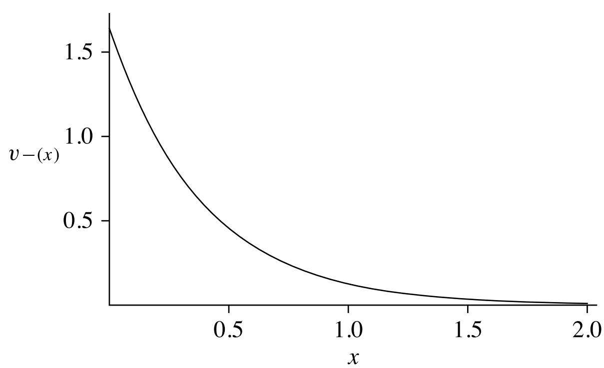

FIGURE 5: Steady state inventory density ($\mu=2,\ \lambda=1$, and $\omega=2$).

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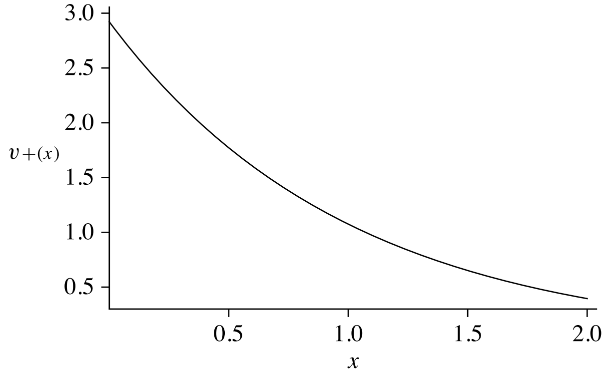

FIGURE 6: Steady state workload density ($\mu = 2,\ \lambda = 1$, and $\omega = 2$).

4.2. Insurance model

We now turn to the insurance risk model with bankruptcy. Equation (7) can be written as, for $\operatorname{Re} s = 0$,

$$(cs - \lambda(1 - \beta(s)))\Psi_+(s) + \frac{\lambda(1 - \beta(s)) - cs}{s} = (cs - \lambda(1 - \beta(s)) - \omega)\Psi_-(-s). \tag{23}$$

In particular, when $s = 0$ in (23), we have $\lambda \lim_{s\to 0}((1 - \beta(s))/s) - c = -\omega\Psi_-(0)$, which implies that

$$\Psi_-(0) = \frac{c - \lambda\mathbb{E}[Y]}{\omega}.$$

We follow the same procedure as in the queueing/inventory model. We reformulate (23) into a Wiener–Hopf problem. Set $f_{\omega,\lambda,c}\colon s \mapsto cs - \lambda(1 - \beta(s)) - \omega$, $\operatorname{Re} s \geq 0$, and $f_{0,\lambda,c}\colon s \mapsto cs-\lambda(1-\beta(s))$, $\operatorname{Re} s \geq 0$. According to [11, p. 548], the constant ω being positive, the function $f_{\omega,\lambda,c}$ has one zero $s = \delta(\omega, \lambda, c)$ and this zero is simple satisfying $\operatorname{Re}\delta(\omega, \lambda, c) > 0$. Also, the function $f_{0,\lambda,c}$ has $s = 0$ as its only zero and this zero is simple. In particular, the functions $g_{\omega,\lambda,c}\colon s \mapsto f_{\omega,\lambda,c}(s)/(s - \delta(\omega, \lambda, c))$ and $g_{0,\lambda,c}\colon s \mapsto f_{0,\lambda,c}(s)/s$ are analytic for $\operatorname{Re} s > 0$, continuous for $\operatorname{Re} s \geq 0$, and take nonzero values on $\operatorname{Re} s \geq 0$. Dividing by $cs-\lambda(1-\beta(s))-\omega$ and multiplying by $s - \delta(\omega, \lambda, c)$ in (23), we obtain, for $\operatorname{Re} s = 0$,

$$\frac{cs - \lambda(1 - \beta(s))}{cs - \lambda(1 - \beta(s)) - \omega}(s - \delta(\omega, \lambda, c))\Psi_+(s) + \frac{\lambda(1 - \beta(s)) - cs}{s}\frac{s - \delta(\omega, \lambda, c)}{cs - \lambda(1 - \beta(s)) - \omega}$$
$$= \Psi_-(-s)(s - \delta(\omega, \lambda, c)). \tag{24}$$

Clearly, the left-hand side of (24) is analytic for $\operatorname{Re} s > 0$ and continuous for $\operatorname{Re} s \geq 0$; on the other hand, the right-hand side is analytic for $\operatorname{Re} s < 0$ and continuous for $\operatorname{Re} s \leq 0$. In addition, both sides coincide for $\operatorname{Re} s = 0$. By Liouville's theorem, there exist $n \geq 0$ and a polynomial R_n of degree n such that

$$\frac{cs - \lambda(1 - \beta(s))}{cs - \lambda(1 - \beta(s)) - \omega}(s - \delta(\omega, \lambda, c))\Psi_+(s) + \frac{\lambda(1 - \beta(s)) - cs}{s}\frac{s - \delta(\omega, \lambda, c)}{cs - \lambda(1 - \beta(s)) - \omega}$$
$$= R_n(s) \quad \text{for } \operatorname{Re} s \geq 0,$$

$$\Psi_-(-s)(s - \delta(\omega, \lambda, c)) = R_n(s) \quad \text{for } \operatorname{Re} s \leq 0. \tag{25}$$

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Since $\lim_{s\to-\infty}\Psi_-(-s)=0$ and using (25), we can deduce that n must be 0; say, $R_n(s)=Z,\ Z\in\mathbb{C}$. Consequently, we have

$$\Psi_+(s)=Z\frac{cs-\lambda(1-\beta(s))-\omega}{s-\delta(\omega,\lambda,c)}\frac{1}{cs-\lambda(1-\beta(s))}+\frac{1}{s}\quad\text{for }\operatorname{Re}s\geq 0,\tag{26}$$

$$\Psi_-(-s)=\frac{Z}{s-\delta(\omega,\lambda,c)}\quad\text{for }\operatorname{Re}s\leq 0.\tag{27}$$

In particular, (27) implies that

$$\Psi_-(0)=\frac{-Z}{\delta(\omega,\lambda,c)}.\tag{28}$$

Let us now identify the constant Z.

Set $\rho=\lambda\mathbb{E}[Y]/c$. Substituting $s=0$ into (23) and combining it with (28), we obtain

$$Z=-\frac{c\delta(\omega,\lambda,c)}{\omega}(1-\rho);\tag{29}$$

thanks to (27), the latter relation completely determines the function $\Psi_-(-s)$ for $\operatorname{Re}s\leq 0$. In this case, we can immediately deduce $u_-(x)$, which is the survival probability when starting at a negative surplus $-x$, and so also the ruin probability $\tilde{u}_-(x)=1-u_-(x)$. We have, for $x\geq 0$,

$$\tilde{u}_-(x)=1-u_-(x)=1-\frac{c\delta(\omega,\lambda,c)(1-\rho)}{\omega}\mathrm{e}^{-\delta(\omega,\lambda,c)x}.\tag{30}$$

Finally, since the constant Z is known, we can identify the function $\Psi_+(s)$ for $\operatorname{Re}s\geq 0$. Rewriting (26), with this in mind, we have

$$\Psi_+(s)=\frac{1}{g_{0,\lambda,c}(s)}\bigg(\bigg(Zg_{\omega,\lambda,c}(s)-\lambda\frac{1-\beta(s)}{s}+c\bigg)s^{-1}\bigg)\quad\text{for }\operatorname{Re}s\geq 0.\tag{31}$$

Set $h_{\omega,\lambda,c}\colon s\mapsto Zg_{\omega,\lambda,c}(s)-\lambda((1-\beta(s))/s)$, for $\operatorname{Re}s\geq 0$, so $h_{\omega,\lambda,c}(0)=Z\omega/\delta(\omega,\lambda,c)-\lambda\mathbb{E}[Y]$. Equation (29) implies that $h_{\omega,\lambda,c}(0)=-c$. Then, we have

$$\Psi_+(s)=\frac{1}{g_{0,\lambda,c}(s)}\frac{h_{\omega,\lambda,c}(s)-h_{\omega,\lambda,c}(0)}{s};\tag{32}$$

the function $s\mapsto(h_{\omega,\lambda,c}(s)-h_{\omega,\lambda,c}(0))/s$ being analytic for $\operatorname{Re}s>0$ and continuous for $\operatorname{Re}s\geq 0$ with $h_{\omega,\lambda,c}(0)=-c$.

The Laplace transform of the survival probability when starting at a positive surplus x, given by the function $1-u_+(x)$, is equal to

$$\frac{1}{s}-\Psi_+(s)=Z\frac{1}{cs-\lambda(1-\beta(s))}\frac{cs-\lambda(1-\beta(s))-\omega}{\delta(\omega,\lambda,c)-s}.$$

Example 2. (*Exponential claim sizes in the insurance model.*) We will keep the same notation as previously. Fix $\nu>0$. In the exponential claim sizes case, we assume claim size density $\nu\mathrm{e}^{-\nu y}$, and, hence,

$$\beta(s)=\frac{\nu}{s+\nu}\quad\text{for }\operatorname{Re}s\geq 0.$$

In particular, $\rho = \lambda/\nu c$. In this case, we obtain

$$f_{\omega,\lambda,c} = \frac{cs^2 + (c\nu - \omega - \lambda)s - \omega\nu}{s+\nu} \quad \text{for } \operatorname{Re} s \geq 0.$$

This function has two zeros; namely,

$$\eta(\omega,\lambda,c) = \frac{-\sqrt{(c\nu-\omega-\lambda)^2 + 4\omega\nu c} - (c\nu-\omega-\lambda)}{2c} < 0,$$

and

$$\delta(\omega,\lambda,c) = \frac{\sqrt{(c\nu-\omega-\lambda)^2 + 4\omega\nu c} - (c\nu-\omega-\lambda)}{2c} > 0.$$

Therefore,

$$g_{\omega,\lambda,c} = \frac{f_{\omega,\lambda,c}(s)}{s-\delta(\omega,\lambda,c)} = c\frac{s-\eta(\omega,\lambda,c)}{s+\nu} \quad \text{for } \operatorname{Re} s \geq 0.$$

Applying (29), we obtain

$$Z = -\left(\nu - \frac{\lambda}{c}\right)\frac{c}{\nu\omega}\delta(\omega,\lambda,c).$$

Using the relation between the zeros $\delta(\omega,\lambda,c)$ and $\eta(\omega,\lambda,c)$,

$$\delta(\omega,\lambda,c)\eta(\omega,\lambda,c) = -\frac{\omega\nu}{c},$$

we obtain

$$Z = \frac{\nu - \lambda/c}{\eta(\omega,\lambda,c)}.$$

Applying (30), we obtain, for $x \geq 0$,

$$u_-(x) = \frac{\nu-\lambda/c}{-\eta(\omega,\lambda,c)}\mathrm{e}^{-\delta(\omega,\lambda,c)x} \quad \text{and} \quad \tilde{u}_-(x) = 1 - \frac{\nu-\lambda/c}{-\eta(\omega,\lambda,c)}\mathrm{e}^{-\delta(\omega,\lambda,c)x}. \tag{33}$$

These results agree with results in [3, Section 2.1.1]. To explicitly determine the function Ψ_+ as given in (32), and so u_+, we have to make the function $h_{\omega,\lambda,c}$ explicit in this case. After some calculations, we obtain, for $\operatorname{Re} s \geq 0$,

$$h_{\omega,\lambda,c}(s) = \frac{(\nu c - \lambda)s - \nu c\eta(\omega,\lambda,c)}{\eta(\omega,\lambda,c)(s+\nu)}.$$

We now can use (32) and deduce, for $\operatorname{Re} s \geq 0$,

$$\Psi_+(s) = \left(1 - \frac{\nu-\lambda/c}{-\eta(\omega,\lambda,c)}\right)\frac{1}{s+\nu-\lambda/c}.$$

Note that here the condition $\rho < 1$ is equivalent to $\nu - \lambda/c > 0$. Finally, we obtain, for $x \geq 0$,

$$u_+(x) = \left(1 - \frac{\nu-\lambda/c}{-\eta(\omega,\lambda,c)}\right)\mathrm{e}^{-(\nu-\lambda/c)x}. \tag{34}$$

Note that using (33) and (34), we can check immediately that $u_+(0) + u_-(0) = 1$. Furthermore, note that (33) and (34) coincide with [3, Equation (18)].

Figures 7 and 8 represent, respectively, the bankruptcy probability starting from a negative surplus against the initial surplus $-C_0$ and the bankruptcy probability starting from a positive surplus against the initial surplus C_0, in the exponential claim sizes case for the particular values $\nu = 2,\ \lambda = 1,\ \omega = 2$, and $c = 1$.

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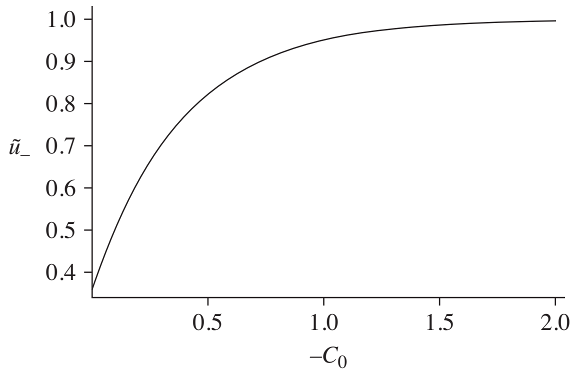

FIGURE 7: Bankruptcy probability starting from a negative surplus against the initial surplus C_0 ($\nu = 2,\ \lambda = 1,\ \omega = 2$, and $c = 1$).

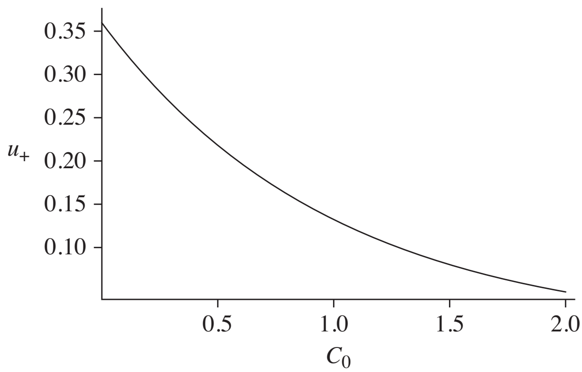

FIGURE 8: Bankruptcy probability starting from a positive surplus against the initial surplus C_0 ($\nu = 2,\ \lambda = 1,\ \omega = 2$, and $c = 1$).

Remark 3. The queueing/inventory and insurance risk models that we treat in this paper are clearly closely related, although they are not dual in the sense discussed in, for example, [6, Section III.2]. The results that we obtain for the densities and the Laplace transforms in [6] are indeed very similar; see (17) and (31), and (18) and (30). It would be interesting to construct a queueing/inventory model that is completely dual to the insurance risk model.

Remark 4. Albrecher and Ivanovs [2] have recently studied exit problems for Lévy processes where the first passage time over a threshold is detected either immediately ('ruin') or at Poisson observation epochs ('bankruptcy'). The authors relate the two exit problems via a nice identity. In the case of the Cramér–Lundberg insurance risk model, their identity is expressed as

$$\hat{s}(x) = \mathbb{E}[s(x + U)], \tag{35}$$

where $s(x)$ is the survival probability in the case of the Cramér–Lundberg model with initial capital x and $\hat{s}(x)$ is the survival probability in the corresponding model with Poisson(ω)-observations (note that $u_+(x) = 1 - \hat{s}(x)$); finally, U is an exp(Φ)-distributed random variable, where Φ is the inverse of the Laplace exponent of the spectrally negative Lévy process corresponding to the Cramér–Lundberg model. In other words, Φ is the zero of $cs - \lambda(1 - \beta(s)) = \omega$. We conclude that $\Phi = \delta(\omega, \lambda, c)$.

From (35), it follows that

$$\hat{s}(x) = \int_{t=0}^{\infty} \delta(\omega, \lambda, c)\, \mathrm{e}^{-\delta(\omega,\lambda,c)t} s(x+t)\, \mathrm{d}t. \tag{36}$$

When $s(\cdot)$ is explicitly known, then we can determine $\hat{s}(x)$ explicitly using (36). In particular, in the case of $\exp(\nu)$ claim sizes, we have (see [6])

$$s(x) = 1 - \frac{\lambda}{\nu c}\mathrm{e}^{-(\nu-\lambda/c)x}, \qquad x > 0,$$

and, hence,

$$\hat{s}(x) = 1 - \frac{\delta(\omega, \lambda, c)}{\delta(\omega, \lambda, c) + \nu c - \lambda} \frac{\lambda}{\nu c}\mathrm{e}^{-(\nu-\lambda/c)x}, \qquad x > 0.$$

Using the definition of $\delta(\omega, \lambda, c)$, it is readily verified that this equation is indeed in agreement with the expression for $u_+(x) = 1 - \hat{s}(x)$ in (34).

5. Analysis for $\omega(\cdot)$ linear

In this section we focus on the queueing/inventory model. In this section we assume that the function ω introduced in Section 2.1 is linear, i.e. there exists a constant $a > 0$ such that, for all $x \geq 0$, we have $\omega(x) = ax$. Equation (3) can be written as

$$\begin{aligned}&[s - \lambda(1 - \beta(s))]\phi_+(s)\\ &\quad = -[s - \lambda(1 - \beta(s))]\phi_-(-s) - a\int_0^{+\infty} y(1 - \mathrm{e}^{sy})v_-(y)\, \mathrm{d}y \quad \text{for } \operatorname{Re} s = 0.\end{aligned}$$

Set $\mathbb{E}I = \int_0^{+\infty} yv_-(y)\, \mathrm{d}y$. After integrating by parts, we obtain

$$\begin{aligned}&[s - \lambda(1 - \beta(s))]\phi_+(s)\\ &\quad = -[s - \lambda(1 - \beta(s))]\phi_-(-s) + a\frac{\mathrm{d}}{\mathrm{d}s}[\phi_-(-s)] - a\mathbb{E}I \quad \text{for } \operatorname{Re} s = 0.\end{aligned} \tag{37}$$

We will now discuss the case where the function β is rational. Suppose that there exist $m \in \mathbb{N}$ and polynomials N and D in $\mathbb{C}[x]$ such that $\deg D = m$, $\deg N \leq m - 1$, $(N \bigvee D) = 1$, i.e they do not have a common factor and $\beta(s) = N(s)/D(s)$ for $\operatorname{Re} s = 0$. In this configuration, necessarily, the polynomial D has no zeros for $\operatorname{Re} s \geq 0$ and m zeros for $\operatorname{Re} s \leq 0$ (counted with multiplicities). Denote them by $-\mu_1, -\mu_2, \dots, -\mu_m$ with $\operatorname{Re} \mu_j \geq 0$. Set

$$N(s) = \sum_{k=0}^{m-1} n_k s^k \quad \text{and} \quad D(s) = \sum_{k=0}^{m} d_k s^k \quad \text{for } s \in \mathbb{C},$$

where $n_0, \dots, n_{m-1}$ and $d_0, \dots, d_m$ are complex numbers. Note that $\beta(0) = 1$ implies that $n_0 = d_0$. Multiplying (37) by $D(s)$, we obtain, for $\operatorname{Re} s = 0$,

$$\begin{aligned}&[(s - \lambda)D(s) + \lambda N(s)]\phi_+(s)\\ &\quad = -[(s - \lambda)D(s) + \lambda N(s)]\phi_-(-s) + aD(s)\frac{\mathrm{d}}{\mathrm{d}s}[\phi_-(-s)] - a\mathbb{E}I\, D(s).\end{aligned}$$

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Using the same techniques as previously, we can deduce that there exists a polynomial R_m such that

$$[(s-\lambda)D(s)+\lambda N(s)]\phi_+(s)=R_m(s) \quad \text{for } \operatorname{Re} s \geq 0, \tag{38}$$

$$\begin{aligned}&-[(s-\lambda)D(s)+\lambda N(s)]\phi_-(-s)+aD(s)\frac{\mathrm{d}}{\mathrm{d}s}[\phi_-(-s)]-a\mathbb{E}ID(s)\\&\quad=R_m(s) \quad \text{for } \operatorname{Re} s \leq 0.\end{aligned} \tag{39}$$

Since $\lim_{s\to+\infty}\phi_+(s)=0$ and using (38), we can deduce that $\deg R_m \leq m$. Equation (38) implies that $R_m(0)=0$. Hence, set $R_m(s)=\sum_{k=1}^{m} r_k s^k$ for $s\in\mathbb{C}$, where $r_1,\dots,r_m\in\mathbb{C}$.

From (38), it follows that

$$\phi_+(s)=\frac{R_m(s)}{(s-\lambda)D(s)+\lambda N(s)}, \qquad \operatorname{Re} s \geq 0. \tag{40}$$

Using (40), we can also express $\phi_+(0)$ in r_1. Indeed,

$$\phi_+(0)=\lim_{s\to 0}\frac{R_m(s)}{(s-\lambda)D(s)+\lambda N(s)}=\frac{r_1}{d_0+\lambda(n_1-d_1)}.$$

Since $\phi_+(0)+\phi_-(0)=1$, we can also deduce that

$$\phi_-(0)=1-\frac{r_1}{d_0+\lambda(n_1-d_1)}. \tag{41}$$

Note that by substituting $-\mu_j$ for $j\in\{1,2,\dots,m\}$ into (39), we obtain the m relations

$$-\lambda N(-\mu_j)\phi_-(\mu_j)=R_m(-\mu_j), \qquad 1\leq j\leq m. \tag{42}$$

For $s\in\mathbb{C}$, introduce the following notations: $\check{N}(s):=N(-s)$, $\check{D}(s):=D(-s)$, and $\check{R}_m(s):=R_m(-s)$. Now, putting $z=-s$ into (39), the function ϕ_- is a solution of the following first-order differential equation:

$$\begin{aligned}&a\check{D}(z)\frac{\mathrm{d}}{\mathrm{d}z}[\phi_-(z)]+[-(z+\lambda)\check{D}(z)+\lambda\check{N}(z))]\phi_-(z)+a\mathbb{E}I\check{D}(z)+\check{R}_m(z)\\&\quad=0 \quad \text{for } \operatorname{Re} z\geq 0.\end{aligned} \tag{43}$$

We can write the previous equation in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}z}[\phi_-(z)]=-\frac{\lambda\check{N}(z)\phi_-(z)+\check{R}_m(z)}{a\check{D}(z)}+\frac{z+\lambda}{a}\phi_-(z)-\mathbb{E}I=0 \quad \text{for } \operatorname{Re} z\geq 0. \tag{44}$$

Equation (42) implies that the function $z\mapsto(\lambda\check{N}(z)\phi_-(z)+\check{R}_m(z))/\check{D}(z)$ is analytic on $\operatorname{Re} z>0$ and continuous on $\operatorname{Re} z\geq 0$. Equation (43) is a first-order algebraic differential equation on the complex half-plane $\{\operatorname{Re} z>0\}$. It has m singularities $\mu_1,\mu_2,\dots,\mu_m$; (44) shows that these singularities are removable.

In principle, one can solve this first-order algebraic differential equation. However, the singularities give rise to several technical difficulties. Below, we consider the $m=1$ case, i.e. $\exp(\mu)$-distributed service requirements, yielding one singularity $z=\mu$. In Subsection 5.1 we sketch its analysis. We formally solve the differential equation (44) only for $\operatorname{Re} z>\mu$, where μ is the singularity. We also determine the two missing constants EI and r_1. In addition, we

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formally invert the Laplace transform $\phi_-(z)$ to find $v_-(x)$, but this inversion is not considered in detail.

In Section 6 we briefly outline a completely different approach to the problem of finding $v_-(x)$ and $v_+(x)$ for the $\omega(x) = ax$ case and $\exp(\mu)$ service requirements. In that section we do not use Laplace transforms, but differentiate (2) twice to obtain a second-order nonlinear differential equation in $v_-(x)$, and we differentiate (1) once to obtain a simple first-order differential equation in $v_+(x)$. The latter equation is easily solved; the solution of the former differential equation is expressed in hypergeometric functions.

Finally, we should add that we do not see how the approach in Section 6 can be extended to the case of an Erlang, hyperexponential, or, more generally, phase-type service requirement distribution, as such distributions will give rise to a higher-order nonlinear differential equation for $v_-(x)$. On the other hand, the approach of Section 7 towards the differential equation (44) seems promising for such service requirement distributions, even though they give rise to multiple singularities $\mu_1, \dots, \mu_m$.

5.1. The Exponential service requirements case

In this case, we have $\mathbb{P}[B > x] = \mathrm{e}^{-\mu x}$ with $\mu > 0$ and for $\operatorname{Re} s \geq 0$, $\beta(s) = \mu/(\mu + s)$ ($N(s) = \mu$ and $D(s) = \mu + s$). Then, $m = 1$ and since $R_1(0) = 0$, we obtain $R_1(s) = r_1 s$, where $r_1 \in \mathbb{C}$. Equation (43) can be written as

$$a(\mu - z)\frac{\mathrm{d}}{\mathrm{d}z}[\phi_-(z)] + z(z + \lambda - \mu)\phi_-(z) + a\mathbb{E}I(\mu - z) - r_1 z = 0 \quad \text{for } \operatorname{Re} z \geq 0. \tag{45}$$

Equation (45) is a first-order algebraic differential equation on the complex half-plane $\{\operatorname{Re} z > 0\}$. It has one singularity which is μ. That makes the study of this differential equation more complicated; therefore, we refer the reader to [8], which is an extended version of the present paper, for a detailed treatment of (44). It exposes a way to handle the singularity $z = \mu$, and holds the promise of allowing an extension to Erlang or hyperexponential service requirement distributions, the latter leading to multiple singularities $\mu_1, \dots, \mu_m$. Remembering that $\phi_-(\mu) = r_1/\lambda$ and writing (45) in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}z}[\phi_-(z)] = \frac{\mu}{\lambda a}\frac{\phi_-(z) - r_1/\lambda}{z - \mu} + \frac{z + \lambda}{a}\phi_-(z) - \mathbb{E}I - \frac{r_1}{a},$$

we deduce that this singularity is regular. Solving (45) on $\{\operatorname{Re} z > \mu\}$ and using the fact that $\lim_{z\to+\infty} \phi_-(z) = 0$, we obtain

$$\phi_-(z) = z\left(\frac{z - \mu}{\mu}\right)^{\lambda\mu/a} \mathrm{e}^{z(z+2\lambda)/2a} \int_1^{+\infty} \left[\mathbb{E}I + \frac{r_1 z t}{a(zt - \mu)}\right]\left(\frac{\mu}{zt - \mu}\right)^{\lambda\mu/a} \mathrm{e}^{-zt(zt+2\lambda)/2a}\,\mathrm{d}t, \tag{46}$$

(note that here, for every $\alpha \in \mathbb{R}$, we consider the principal value of the function $z \mapsto z^\alpha$). For $\operatorname{Re} z > \mu$, we introduce

$$F_{\alpha,\beta}(z) := z\left(\frac{z - \mu}{\mu}\right)^{\alpha} \mathrm{e}^{z(z+2\lambda)/2a} \int_1^{+\infty} \left(\frac{\mu}{zt - \mu}\right)^{\beta} \mathrm{e}^{-zt(zt+2\lambda)/2a}\,\mathrm{d}t.$$

The function $F_{\alpha,\beta}$ is analytic on $\{\operatorname{Re} z > \mu\}$ and one can check by the l'Hopital rule that the function $F_{\alpha,\beta}$ can be analytically continued in μ for $\alpha \geq \beta - 1$. Now fixing $\alpha = \lambda\mu/a$, (46)

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can be written as

$$\phi_-(z) = \left(\mathbb{E}I + \frac{r_1}{a}\right)F_{\alpha,\alpha}(z) + \frac{r_1}{a}F_{\alpha,\alpha+1}(z) \quad \text{for } \operatorname{Re} z > \mu. \tag{47}$$

We will denote by $\mathcal{L}^{-1}$ the inverse Laplace transform, and apply what is commonly known as the Mellin inverse formula or the Bromwich integral. Let γ be any real number such that $\gamma > \mu$, then we have

$$v_-(x) = \frac{1}{2\mathrm{i}\pi}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \phi_-(z)\mathrm{e}^{zx}\,\mathrm{d}z = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_-(\gamma+\mathrm{i}\omega)\mathrm{e}^{(\gamma+\mathrm{i}\omega)x}\,\mathrm{d}\omega \quad \text{for } x > 0.$$

Since the constant γ here is chosen to be larger than μ, one can use the expression of ϕ_- given in (47). Set $f_{\alpha,\beta}(x) = (1/(2\mathrm{i}\pi))\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} F_{\alpha,\beta}(z)\mathrm{e}^{zx}\,\mathrm{d}z$. We obtain

$$v_-(x) = \left(\mathbb{E}I + \frac{r_1}{a}\right)f_{\alpha,\alpha}(x) + \frac{r_1}{a}f_{\alpha,\alpha+1}(x), \qquad x > 0. \tag{48}$$

Integrating the previous equation, we obtain $\phi_-(0) = (\mathbb{E}I + r_1/a)A_{\alpha,\alpha} + (r_1/a)A_{\alpha,\alpha+1}$, where $A_{\alpha,\beta} := \int_0^{+\infty} f_{\alpha,\beta}(x)\,\mathrm{d}x$. According to (41), we have $\phi_-(0) = 1 - r_1/(\mu-\lambda)$, and, hence,

$$1 - \left(\frac{1}{\mu-\lambda} + \frac{A_{\alpha,\alpha} + A_{\alpha,\alpha+1}}{a}\right)r_1 = A_{\alpha,\alpha}\mathbb{E}I. \tag{49}$$

Set $B_{\alpha,\beta} := \int_0^{+\infty} xf_{\alpha,\beta}(x)\,\mathrm{d}x$; multiplying (48) by x, integrating it and remembering the fact that $\mathbb{E}I = \int_0^{+\infty} xv_-(x)\,\mathrm{d}x$, we obtain

$$(1 - B_{\alpha,\alpha})\mathbb{E}I = \frac{B_{\alpha,\alpha} + B_{\alpha,\alpha+1}}{a}r_1. \tag{50}$$

Thanks to (49) and (50), one can deduce the constants r_1 and $\mathbb{E}I$; using (48), one can then completely determine the function v_-.

Finally, using (40), we obtain

$$v_+(x) = r_1\mathrm{e}^{-(\mu-\lambda)x}, \qquad x > 0. \tag{51}$$

It is not surprising that the density of the workload, when positive, is exponentially distributed with the same rate $\mu - \lambda$ as in the corresponding M/M/1 queue (arrival rate λ, service requirements $\exp(\mu)$) *without* inventory. Indeed, every time the workload becomes positive, this occurs because of a customer arrival, and the memoryless property implies that the residual part of the service requirement which makes that workload positive is $\exp(\mu)$-distributed.

6. Direct approach

In this section we use the analysis developed in [5] to state some explicit results when $\omega(x) = ax$ in the exponential service requirements case. In [5], the authors studied directly the functions v_+ and v_- without considering their Laplace transforms. Indeed, differentiating (1) and (2), one can show that the functions v_- and v_+ satisfy some well known differential equations.

Set $C = \int_0^{+\infty}\mathrm{e}^{-\mu x}v_-(x)\,\mathrm{d}x$. Differentiating (1), we obtain

$$v_+(x) = C\lambda\mathrm{e}^{-(\mu-\lambda)x} \quad \text{for all } x \geq 0. \tag{52}$$

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This is in agreement with (51), which we obtained following a Laplace transform approach. On the other hand, differentiating (2), we obtain the following equation for v_-:

$$v_-'(x) + (\lambda + ax)v_-(x) - \lambda\mu \mathrm{e}^{\mu x} \int_x^{+\infty} \mathrm{e}^{-\mu x} v_-(x)\, \mathrm{d}x = 0. \tag{53}$$

Now, differentiating the expression in (53), the function v_- satisfies the following second-order differential equation:

$$v_-''(x) + (\lambda - \mu + ax)v_-'(x) + a(1 - \mu x)v_-(x) = 0. \tag{54}$$

Introduce the function $\theta(x) = v_-(x)\mathrm{e}^{(ax^2/2)+\lambda x}$. The function v_- is a solution of (54) if and only if the function θ is a solution of the following second-order differential equation:

$$\theta''(x) - (\lambda + \mu + ax)\theta'(x) + \lambda\mu\theta(x) = 0. \tag{55}$$

One can check that θ is a solution of (55) if and only if $\theta(x) = \mathcal{J}(\tilde{a}, \tilde{b}, (a/2)(x + (\lambda + \mu)/a)^2)$, where $\tilde{a} = -\lambda\mu/2a$ and $\tilde{b} = \frac{1}{2}$, and $\mathcal{J}(\tilde{a}, \tilde{b}, \cdot)$ is a solution of the degenerate hypergeometric equation

$$zy''(z) + (\tilde{b} - z)y'(z) - \tilde{a}y(z) = 0. \tag{56}$$

According to [14, p. 322] and [16, p. 143], (56) has two standard solutions denoted by $z \mapsto M(\tilde{a}, \tilde{b}, z)$ and $z \mapsto U(\tilde{a}, \tilde{b}, z)$, the so-called Kummer functions. Provided that $\tilde{b} \notin \{-1, -2, \dots\}$, the function $z \mapsto M(\tilde{a}, \tilde{b}, z)$ is given by

$$M(\tilde{a}, \tilde{b}, z) = \sum_0^{+\infty} \frac{(\tilde{a})_s}{(\tilde{b})_s s!} z^s \quad \text{for all } z \in \mathbb{C},$$

where $(c)_s = c(c+1)\cdots(c+s-1)$. The function $U(\tilde{a}, \tilde{b}, z)$ is uniquely determined by the property $U(\tilde{a}, \tilde{b}, z) \sim z^{-a}$ when z goes to $+\infty$. In our case ($\tilde{b} = \frac{1}{2}$), we have

$$U(\tilde{a}, \tilde{b}, z) = \frac{\Gamma(1/2)}{\Gamma(1/2 - \lambda\mu/2a)} M\biggl(-\frac{\lambda\mu}{2a}, \frac{1}{2}, z\biggr) + \Gamma\biggl(-\frac{1}{2}\biggr) z^{\frac{1}{2}} M\biggl(-\frac{\lambda\mu}{2a} + \frac{1}{2}, \frac{3}{2}, z\biggr).$$

The following analysis was developed in detail in [5], using knowledge of the degenerate hypergeometric equation (see, for example, [14, p. 322]). It appears that one needs to distinguish between two cases.

Case 1: $\tilde{a} = -\lambda\mu/2a \notin \mathbb{Z}$, *i.e* $\lambda\mu/2a \notin \mathbb{N}$. Denote $\nu := \lambda\mu/a$. In this case, according to [5], we have

$$v_-(x) = K\mathrm{e}^{-(a/2)x^2 - \lambda x} U\biggl(-\frac{\nu}{2}, \frac{1}{2}, \frac{a}{2}\biggl(x + \frac{\lambda + \mu}{a}\biggr)^2\biggr) \quad \text{for all } x \geq 0, \tag{57}$$

where $K \in \mathbb{R}$. Equations (52) and (57) imply that to determine completely the functions v_+ and v_-, it is enough to determine the constants C and K. According to [5], these constants are given by the following system of equations:

$$C = K(\lambda + \mu)U'\biggl(-\frac{\nu}{2}, \frac{1}{2}, \frac{(\lambda + \mu)^2}{2a}\biggr)(\lambda\mu)^{-1},$$

$$C\frac{\rho}{1 - \rho} + K\int_0^{+\infty} \mathrm{e}^{-(a/2)x^2 - \lambda x} U\biggl(-\frac{\nu}{2}, \frac{1}{2}, \frac{a}{2}\biggl(x + \frac{\lambda + \mu}{a}\biggr)^2\biggr)\, \mathrm{d}x = 1.$$

In particular, if ν is odd, i.e. $\nu = 2n+1$, where $n \in \mathbb{N}$, from [5], we have

$$v_-(x) = 2^{-\nu/2} K \mathrm{e}^{-(a/2)x^2-\lambda x} H_\nu\bigg(\sqrt{a}x + \frac{\lambda+\mu}{\sqrt{a}}\bigg) \quad \text{for all } x \geq 0, \tag{58}$$

where, for $m \in \mathbb{Z}$, $H_m(\cdot)$ is the Hermite polynomial of order m (see [1, p. 775]) given by $H_m(x) = (-1)^m \mathrm{e}^{x^2/2}(\mathrm{d}/\mathrm{d}x)^m[\mathrm{e}^{-x^2/2}]$.

Case 2: $\lambda\mu/2a = n \in \mathbb{N}$, *i.e.* $\nu = 2n$. In this case, according to [5], we have, for all $x \geq 0$,

$$v_-(x) = (-1)^{\nu/2} \frac{(\nu/2)!}{\nu!} 2^{\nu/2} K^* \mathrm{e}^{-ax^2/2-\lambda x} H_\nu\bigg(\sqrt{a}x + \frac{\lambda+\mu}{\sqrt{a}}\bigg), \qquad x \geq 0, \tag{59}$$

where $K^* \in \mathbb{R}$. Furthermore, similarly to case 1, one can find two linear equations involving the unknowns C and K^* and then determine them. One can see that (58) and (59) have the same shape.

7. Analysis of the first-order differential equation

We now sketch a different method to find the unknowns $\mathbb{E}I$ and r_1; full details appear in [8, Section 7]. The solution $\phi_-(z)$ of the differential equation (45) is analytic in $\operatorname{Re} z > 0$ and continuous and bounded in $\operatorname{Re} z \geq 0$, and satisfies the boundary conditions

$$-\mathbb{E}I = \phi_-(0) = 1 - \frac{r_1}{\mu-\lambda}, \qquad \phi_-(z) \to 0, \quad z \to +\infty; \tag{60}$$

see (41) for the first expression. Denote

$$d_k = \frac{\phi_-^{(k)}(\mu)}{k!}, \qquad c_k = (-1)^k d_k, \quad k = 0, 1, \ldots.$$

From (45), there follows the three-term recursion

$$(ka - \mu\lambda)c_k + (\mu+\lambda)c_{k-1} - c_{k-2} = 0, \qquad k = 2, 3, \ldots, \tag{61}$$

with initialization

$$\mu\lambda c_0 = r_1\mu, \qquad (a-\mu\lambda)c_1 + (\mu+\lambda)c_0 = a\mathbb{E}I + r_1. \tag{62}$$

The standard calculus solution

$$y(x) = y_0 \mathrm{e}^{-A(x)} + \mathrm{e}^{-A(x)} \int_{x_0}^{x} Q(t)\mathrm{e}^{A(t)}\,\mathrm{d}t \tag{63}$$

of the boundary value problem

$$y'(x) + P(x)y(x) = Q(x), \qquad y(x_0) = y_0$$

with integrable P and Q and $A(x) = \int_{x_0}^{x} P(t)\,\mathrm{d}t$ cannot be used directly to solve (45) since the latter is singular at $z = \mu$. However, when we let

$$\sigma = \frac{\mu\lambda}{a}, \qquad K = \lceil\sigma\rceil,$$

both

$$\frac{1}{w^\sigma}\left(\phi_-(\mu - w) - \sum_{k=0}^{K-1} c_k w^k\right), \qquad 0 \le w \le \mu \tag{64}$$

and

$$\frac{1}{w^\sigma}\left(\phi_-(\mu + w) - \sum_{k=0}^{K-1} d_k w^k\right), \qquad w \ge 0, \tag{65}$$

do satisfy a regular first-order differential equation. In both (64) and (65), the A that appears in (63) is a quadratic function with known coefficients, while the Q are

$$-(\sigma - K)c_K w^{K-\sigma-1} + \frac{1}{a}c_{K-1}w^{K-\sigma} \quad \text{and} \quad -(\sigma - K)d_K w^{K-\sigma-1} + \frac{1}{a}d_{K-1}w^{K-\sigma}$$

for the two respective cases.

From the second expression in (60), used for (65), we obtain a linear relation between d_{K-1} and d_K and, therefore, a linear relation between c_{K-1} and c_K. Using (61) in a backward direction, we can thus express $c_0, c_1, \ldots, c_{K-1}$ linearly in c_K. Evaluation of the standard calculation solution for (64) at $w = \mu$ then yields $\phi_-(0)$ in terms of $c_0, c_1, \ldots, c_{K-1}, c_K$, i.e. in terms of c_K. Combining the first expression in (60) and (62) then finally gives c_K, and, thus, via (62), we have found r_1 and $\mathbb{E}I$.

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