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CENTRAL IDEMPOTENT MEASURES ON UNITARY GROUPS

DANIEL RIDER†

1. Introduction. Let G be a locally compact group and $M(G)$ the space of finite regular Borel measures on G. If μ and ν are in $M(G)$, their convolution is defined by

$$\mu * \nu(E) = \int \mu(Ex^{-1})\, d\nu(x).$$

Thus, if f is a continuous bounded function on G,

$$\int f(x)\, d\mu * \nu(x) = \iint f(xy)\, d\mu(x)\, d\nu(y).$$

μ is *central* if $\mu(Ex) = \mu(xE)$ for all $x \in G$ and all measurable sets E. μ is *idempotent* if $\mu * \mu = \mu$.

The idempotent measures for abelian groups have been classified by Cohen [**1**]. In this paper we will show that for a certain class of compact groups, containing the unitary groups, the *central* idempotents can be characterized. The method consists of showing that, in these cases, the central idempotents arise from idempotents on abelian groups and applying Cohen's result.

In § 2 we show the existence of central idempotent measures in terms of the hypercoset ring of the space of representations of G and state the main result of the paper. In § 3 we first extend the class of central idempotent measures to the class of sums of such measures. It is then shown that under a suitable condition on G such measures decompose into measures supported on the centre of G (which can be handled by Cohen's theorem) and measures whose Fourier series utilize only representations of bounded degree. These measures are characterized in § 4, at least when G does not have too many representations of the same degree. Finally, in § 5, the unitary groups are shown to satisfy the conditions of §§ 3 and 4.

The referee has pointed out that one special case of the results appearing here is already known. *Positive* idempotent measures have been characterized for locally compact groups by Kelley [**4**] and for complete separable metric groups by Parthasarathy [**5**]. The proofs use the positivity assumption whereas our results are valid for complex-valued central idempotent measures.

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2. The hypercoset ring. Henceforth, G will be a compact group. Γ will denote the set of equivalence classes of irreducible unitary representations of G. For $\alpha \in \Gamma$, T_α is a member of the class α, ψ_α is the character of the class, and $d(\alpha)$ the degree. Γ has a hypergroup structure (cf. [**2**]) in the following sense. If α, $\beta \in \Gamma$, then $T_\alpha \otimes T_\beta$ has a decomposition into irreducible unitary components. If $\mu_{\alpha,\beta}(\gamma)$ is the number of times T_γ appears in this decomposition, then

$$\psi_\alpha(x)\psi_\beta(x) = \sum \mu_{\alpha,\beta}(\gamma)\psi_\gamma(x) \qquad (x \in G).$$

A subset $\mathscr{H}$ of Γ is called a *subhypergroup* if α, $\beta \in \mathscr{H}$ and $\mu_{\alpha,\beta}(\gamma) \neq 0$ imply $\gamma \in \mathscr{H}$. A subhypergroup is *normal* if $\alpha \in \mathscr{H}$ implies $\bar{\alpha} \in \mathscr{H}$, where $T_{\bar{\alpha}}$ is the representation conjugate to T_α.

If H is a closed normal subgroup of G, let $H^\perp$ be the set of $\alpha \in \Gamma$ such that $T_\alpha(x) = E$, the identity transformation, for all $x \in H$. If $\mathscr{H}$ is a normal subhypergroup of Γ, let $\mathscr{H}^\perp$ be the set of $x \in G$ such that $T_\alpha(x) = E$ for all $\alpha \in \mathscr{H}$. Helgason [**1**] has shown that $H^\perp$ is a normal subhypergroup and $\mathscr{H}^\perp$ is a closed normal subgroup. Also $H^{\perp\perp} = H$ and $\mathscr{H}^{\perp\perp} = \mathscr{H}$.

If $\mathscr{H} \subset \Gamma$ and $\beta \in \Gamma$, then define

$$\beta\mathscr{H} = \{\gamma\colon \mu_{\alpha,\beta}(\gamma) \neq 0 \text{ for some } \alpha \in \mathscr{H}\}.$$

If $\mathscr{H}$ is a normal subhypergroup, then $\beta\mathscr{H}$ is called a *hypercoset.*

If μ is a central measure on G, then μ has a Fourier-Stieltjes series of the form

$$\mu \sim \sum \hat{\mu}(\alpha)\, d(\alpha)\psi_\alpha(x),$$

where

$$\hat{\mu}(\alpha) = \frac{1}{d(\alpha)} \int \bar{\psi}_\alpha(x)\, d\mu(x).$$

μ is idempotent if $\hat{\mu}(\alpha)$ is always 0 or 1. If μ is idempotent, let

$$S(\mu) = \{\alpha \in \Gamma\colon \hat{\mu}(\alpha) = 1\}.$$

The family Ω of all sets $S(\mu)$, for central idempotent μ, is clearly a ring of sets. That is, it is closed under the formation of unions, intersections, and complements. The *hypercoset ring* of Γ will be the smallest ring containing all the hypercosets.

THEOREM 1. (a) *Ω contains the hypercoset ring.*

(b) *Let H be a closed normal subgroup of G with Haar measure m. Let $\beta \in \Gamma$ and*

$$\frac{1}{c} = \int_H |\psi_\beta(h)|^2\, dm(h).$$

Then $d\mu(x) = cd(\beta)\psi_\beta(x)dm(x)$ is a central idempotent measure on G and $S(\mu) = \beta H^\perp$.

Proof. (a) follows from (b) since by (b) every hypercoset is in the ring Ω. The measure μ is clearly central, and so it remains to show that $\hat{\mu}$ is the characteristic function of $\beta H^{\perp}$. Now if $\gamma \in \Gamma$, then

$$\int_H \bar{\psi}_\gamma(h)\psi_\beta(h)\,dm(h) = 0 \qquad \text{or} \qquad \left.\frac{\psi_\gamma}{d(\gamma)}\right|_H = \left.\frac{\psi_\beta}{d(\beta)}\right|_H .$$

Thus

$$\hat{\mu}(\gamma) = c\,\frac{d(\beta)}{d(\gamma)}\int_H \bar{\psi}_\gamma(h)\psi_\beta(h)\,dm(h) = 1 \text{ or } 0$$

so that μ is idempotent. Now $S(\mu)$ consists precisely of those γ for which $\int \bar{\psi}_\gamma\psi_\beta\,dm \neq 0$. Since $\int \psi_\alpha\,dm = 0$ unless $\alpha \in H^{\perp}$, it follows that

$$\int \bar{\psi}_\gamma\psi_\beta\,dm = \sum_\alpha \mu_{\bar{\gamma},\beta}(\alpha)\int \psi_\alpha\,dm \neq 0$$

exactly when $\mu_{\bar{\gamma},\beta}(\alpha) \neq 0$ for some $\alpha \in H^{\perp}$. Since $\mu_{\bar{\gamma},\beta}(\alpha) = \mu_{\bar{\alpha},\beta}(\gamma)$, this holds for $\gamma \in \beta H^{\perp}$.

We can now state the main result for unitary groups.

THEOREM 2. *Let G be the group of unitary $n \times n$ matrices (for some integer n). A subset E of Γ is $S(\mu)$ for some central idempotent measure μ if and only if E belongs to the hypercoset ring of Γ.*

This is analogous to Cohen's theorem as it appears in [**4**; Theorem 3.1.3]. Theorem 2 will follow directly from Corollary 8 of § 4 and the remarks of § 5.

3. A reduction to measures of bounded representation type. It is convenient, as in the abelian case, to enlarge the class of central idempotent measures. Let $F(G)$ be the set of central measures μ on G for which $\hat{\mu}$ is integral-valued. Let $S(\mu) = \{\alpha\colon \hat{\mu}(\alpha) \neq 0\}$. It is clear that if $\mu \in F(G)$, then $\mu = \sum n_i\mu_i$, where the n_i are integers and the μ_i are mutually orthogonal central idempotents. We will say that a central measure μ is of *bounded representation type* (b.r.t.) if there is an integer M such that $\hat{\mu}(\alpha) = 0$ whenever $d(\alpha) > M$.

Let Z be the centre of G. If $F(Z)$ is considered as a subset of $M(G)$, then $F(Z) \subset F(G)$. A useful form of Cohen's characterization of $F(Z)$ is given in [**3**]. In particular, it follows that if $\mu \in F(Z)$, then $S(\mu)$ is in the hypercoset ring.

Definition. G is said to satisfy condition I *provided that*

$$\lim_{d(\alpha)\to\infty} \frac{\psi_\alpha(x)}{d(\alpha)} = 0$$

for all $x \notin Z$.

If G satisfies this condition, we have the following singular decomposition of measures in $F(G)$.

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THEOREM 3. *Let G satisfy condition* I *and* $\mu \in F(G)$. *Then* $\mu = \nu + \lambda$, *where* ν *and* λ *are singular*, $\nu \in F(Z)$, *and* λ *is of bounded representation type.*

Proof. If μ itself is not of b.r.t., there is a sequence α_i such that $d(\alpha_i) \to \infty$ and $\hat{\mu}(\alpha_i) \neq 0$. Since $\mu \in F(G)$ and $\hat{\mu}$ is bounded, we can assume that $\hat{\mu}(\alpha_i) = n \neq 0$. Let

$$\gamma_i = \left.\frac{\psi_{\alpha_i}}{d(\alpha_i)}\right|_Z.$$

Then $\gamma_i \in \Gamma(Z)$; i.e. γ_i is a character on Z of degree 1. Furthermore, every character of Z is obtained by such a restriction. Let $\nu = \mu|_Z$ be considered as a measure on Z. Then, since $\psi_{\alpha_i}/d_{\alpha_i} \to 0$ boundedly off Z,

$$n = \int \frac{\bar{\psi}_{\alpha_i}(x)}{d_{\alpha_i}}\, d\mu(x) = \int_Z \bar{\gamma}_i\, d\nu + o(1) \quad \text{as } i \to \infty.$$

Thus

$$\text{(1)} \qquad \lim \hat{\nu}(\gamma_i) = n.$$

Let $\nu_i = \bar{\gamma}_i \nu$. Then a subsequence of ν_i, say ν_i itself, converges weakly to a measure $\sigma \in M(Z)$. $\sigma \neq 0$ since

$$\int d\sigma = \lim \int \bar{\gamma}_i\, d\nu = \lim \hat{\nu}(\gamma_i) = n.$$

We will show that $\sigma \in F(Z)$. Let $\gamma \in \Gamma(Z)$ and fix $\alpha \in \Gamma$ such that

$$\gamma = \left.\frac{\psi_\alpha}{d(\alpha)}\right|_Z.$$

For $\beta \in \Gamma$ let

$$\gamma_\beta = \left.\frac{\psi_\beta}{d(\beta)}\right|_Z.$$

Now

$$\text{(2)} \qquad d(\alpha)d(\alpha_i)\gamma\gamma_i = \sum_\beta \mu_{\alpha,\alpha_i}(\beta)\gamma_\beta\, d(\beta).$$

Since $\sum_\beta \mu_{\alpha,\alpha_i}(\beta)\, d(\beta) = d(\alpha)\, d(\alpha_i)$, it follows from (2) that

$$\text{(3)} \qquad \gamma\gamma_i = \gamma_\beta \quad \text{whenever } \mu_{\alpha,\alpha_i}(\beta) \neq 0.$$

If $\mu_{\alpha,\alpha_i}(\beta) \neq 0$, then $\mu_{\alpha,\bar{\beta}}(\bar{\alpha}_i) \neq 0$ so that $T_{\bar{\alpha}_i}$ appears in the decomposition of $T_\alpha \otimes T_{\bar{\beta}}$. This then implies that

$$d(\alpha)\, d(\beta) \geqq d(\alpha_i).$$

Hence if the β_i are chosen so that $\mu_{\alpha,\alpha_i}(\beta_i) \neq 0$, then $d(\beta_i) \to \infty$ and, by (3), $\gamma\gamma_i = \gamma_{\beta_i}$. Thus, as in (1), it follows that

$$\hat{\nu}(\gamma\gamma_i) = \hat{\nu}(\gamma_{\beta_i}) = \hat{\mu}(\beta_i) + o(1) \quad \text{as } i \to \infty.$$

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But

$$\hat{\sigma}(\gamma) = \lim_i \hat{\nu}(\gamma\gamma_i) = \lim_i \hat{\mu}(\beta_i) = \text{integer}$$

so that $\sigma \in F(Z)$.

Since $\sigma \in F(Z)$ and $\sigma \neq 0$, it follows from Cohen's theorem (cf. [**3**]) that there is a closed subgroup $Z_1 \subset Z$ such that $\sigma|Z_1 \in F(Z)$ and is not singular to the Haar measure of Z_1. Since $\sigma|Z_1$ is the weak limit of $\bar{\gamma}_i\nu|Z_1$, it follows from Helson's translation lemma [**4**, Theorem 3.5.1] that, for some i, $\sigma|Z_1 = \bar{\gamma}_i\nu|Z_1$. Hence

$$\nu_1 = \mu|Z_1 = \nu|Z_1 \in F(Z) \subset F(G).$$

ν_1 and $\mu - \nu_1$ are singular so that $||\mu - \nu_1|| \leqq ||\mu|| - 1$. If $\mu - \nu_1$ is of b.r.t., our proof is complete. Otherwise, we can apply the same argument and since the norm decreases by at least 1, we will finally obtain the desired decomposition.

COROLLARY 4. *Let G satisfy condition* I *and assume that G has only finitely many representations of any fixed degree. If $\mu \in F(G)$, then $\mu = \nu + \lambda$, where $\nu \in F(Z)$ and λ is absolutely continuous. That is,*

$$d\lambda = \sum n_i d(\alpha_i)\psi_{\alpha_i}\, dx,$$

where dx is a Haar measure on G, the n_i are integers, and the sum is finite.

4. Characterization of measures of b.r.t. Let Γ_1 consist of those α with $d(\alpha) = 1$. That is, Γ_1 is the group of continuous complex homomorphisms of G. For $\alpha \in \Gamma_1$ we can identify α, T_α, and ψ_α. If G' is the closure of the commutator subgroup of G, then $\Gamma_1 = (G')^\perp$ and Γ_1 is the dual group of G/G'. If $\alpha \in \Gamma_1$ and $\beta \in \Gamma$, then $\alpha\beta$ is the irreducible representation with character $\alpha\psi_\beta$.

LEMMA 5. *If $\mu \in F(G)$ and $\hat{\mu}(\alpha) = 0$ whenever $\alpha \notin \Gamma_1$, then $S(\mu)$ is in the coset ring of Γ_1.*

Proof. If m is the Haar measure of G', then $\mu = \mu * m$ so that μ can be considered as a measure on the abelian group G/G'. That is, if π is the natural projection of G onto G/G' and

$$\int_{G/G'} f\, d\pi\mu = \int_G f(\pi(x))\, d\mu(x),$$

then the Fourier series for μ and $\pi\mu$ have the same form:

$$\mu \sim \sum_{\alpha\in\Gamma_1} \hat{\mu}(\alpha)\alpha(x); \qquad \pi\mu \sim \sum_{\alpha\in\Gamma_1} \hat{\mu}(\alpha)\alpha(xG').$$

By Cohen's theorem [**4**, Theorem 3.1.3], $S(\mu) = S(\pi\mu)$ is in the coset ring of Γ_1.

The coset ring of Γ_1 is contained in the hypercoset ring of Γ. Furthermore, if E is in the coset ring of Γ_1 and $\beta \in \Gamma$, then βE is in the hypercoset ring.

Definition. *G is said to satisfy condition* II *provided that for each positive integer t there are finitely many irreducible representations* $\beta_1, \ldots, \beta_s$ *of degree t such that if* $d(\beta) = t$ *then* $\beta = \alpha\beta_i$ *for some i and some* $\alpha \in \Gamma_1$.

Condition II is equivalent to saying that all representations of a fixed degree are contained in finitely many hypercosets of Γ_1.

THEOREM 6. *If G satisfies condition* II, $\mu \in F(G)$, *and* μ *is of b.r.t., then* $S(\mu)$ *is in the hypercoset ring.*

The next two corollaries follow immediately from Theorems 1, 3, and 6.

COROLLARY 7. *If G satisfies conditions* I *and* II *and* $\mu \in F(G)$, *then* $S(\mu)$ *is in the hypercoset ring.*

COROLLARY 8. *Let G satisfy conditions* I *and* II. *A subset E of* Γ *is* $S(\mu)$ *for some central idempotent measure* μ *if and only if E belongs to the hypercoset ring of* Γ.

Proof of Theorem 6. Fix an integer t and choose $\beta_1, \ldots, \beta_s$ of degree t such that $\bigcup_1^s \beta_i \Gamma_1$ consists of all representations of degree t. Let m be the Haar measure of G' and

$$\frac{1}{c(i)} = \int_{G'} |\psi_{\beta_i}(h)|^2 \, dm(h).$$

It is easy to see that $1/c(i)$ is the number of $\alpha \in \Gamma_1$ for which $\alpha\beta_i = \beta_i$. These α form a finite subgroup A_i of Γ_1. $A_i^\perp$ is a closed normal subgroup of G containing G'. It is trivial unless ψ_{β_i} is supported on a proper open subgroup of G. In particular, if G/G' is connected, then $c(i) = 1$ for all β_i. By Theorem 1, the measures $\theta_i = c_i t \psi_{\beta_i} m$ are central idempotents and $S(\theta_i) = \beta_i \Gamma_1$. If μ is of b.r.t., then μ is a finite sum of measures $\mu * \theta_i$ (for possibly different t). It thus suffices to prove the theorem when $\mu = \mu * \theta_i$ for some i.

If $\mu = \mu * \theta_i$, let

$$\lambda = \frac{\bar{\psi}_{\beta_i}}{t} \mu * m.$$

Then $\hat{\lambda}(\alpha) = 0$ if $\alpha \notin \Gamma_1$ and a simple calculation yields

$$\hat{\lambda}(\alpha) = \hat{\mu}(\alpha\beta_i) \quad \text{if } \alpha \in \Gamma_1.$$

Thus, by Lemma 5, $S(\lambda)$ is in the coset ring of Γ_1. Since $\hat{\lambda}$ is constant on the cosets of A_i (in Γ_1), it follows that λ is supported on $A_i^\perp$. Another calculation yields

$$\mu = c_i t \psi_{\beta_i} \lambda$$

so that $S(\mu) = \beta_i S(\lambda)$, and hence $S(\mu)$ is in the hypercoset ring.

5. The unitary groups. It is a strong condition to require a group to satisfy both conditions I and II. For example, if H is a closed normal subgroup of such a group, then by considering the Haar measure of H, it follows from Theorem 2 and the proof of Theorem 6 that either $H \subset Z$ or H is an open subgroup of HG'. It is possible, however, to show that the unitary groups satisfy both conditions.

Let U_n be the group of unitary $n \times n$ matrices. Then it is known [**7**, p. 198] that all irreducible characters are obtained in the following way: to each decreasing set of n integers, $m = \{m_1 > m_2 > \ldots > m_n\}$, corresponds an irreducible character ψ_m. If $x \in U_n$, then x is conjugate to a diagonal unitary matrix with entries x_i. When the x_i are all distinct,

$$\psi_m(x) = \frac{\det(x_i^{m_j})}{\det(x_i^{n-j})}. \tag{4}$$

If the x_i are not all distinct, then $\psi_m(x)$ can be evaluated by taking a limit in (4).

The degree of this representation is given by

$$d(m) = \prod_{i<j} \frac{m_i - m_j}{j - i}.$$

This can be seen by evaluating

$$d(m) = \psi_m(e) = \lim_{x \to e} \psi_m(x),$$

where the entries of x are distinct. In the same way it follows that if x is not in the centre of U_n, that is if the entries x_i are not all the same, then $\psi_m(x)/d(m) \to 0$ as $d(m) \to \infty$, so that U_n satisfies condition I.

If ψ_m is a given character, let $m' = \{m_1 - m_n, m_2 - m_n, \ldots, 0\}$ and $\alpha = \{m_n + n - 1, m_n + n - 2, \ldots, m_n\}$. Then $d(m') = d(m)$, $d(\alpha) = 1$, and $\psi_\alpha \psi_{m'} = \psi_m$. Since there are only finitely many characters of the same degree with 0 as the last integer, it follows that U_n satisfies condition II.

It would seem that both conditions should be satisfied by any compact connected Lie group. It also seems reasonable that Theorem 2 should hold for any compact group.

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University of Wisconsin,
Madison, Wisconsin