Macroeconomic Dynamics, **25**, 2021, 1779–1809. Printed in the United States of America.
doi:10.1017/S1365100519000865

RISK SHOCKS, RISK MANAGEMENT, AND INVESTMENT

JONATHAN GOLDBERG
Federal Reserve Board

This paper studies the macroeconomic effects of shocks to idiosyncratic business risk in an economy with endogenously incomplete markets. I develop a model in which firms face idiosyncratic risk and obtain insurance from intermediaries through contracts akin to credit lines. Insurance is imperfect due to limited commitment in financial contracts. Although steady-state capital is higher than if firms were constrained to issue only standard equity, a rise in uncertainty about idiosyncratic business outcomes leads to an endogenous reduction in risk sharing. This deterioration in risk sharing results from a general-equilibrium shortage of pledgeable assets and implies that the economy's response to an increase in idiosyncratic business risk can be amplified by financial contracting rather than dampened. In a parametrized version of the model, a rise in idiosyncratic business risk generates a large increase in uncertainty about aggregate investment.

Keywords: Endogenously Incomplete Markets, Idiosyncratic Risk, Aggregate Fluctuations

1. INTRODUCTION

In the presence of financial frictions, fluctuations in idiosyncratic business risk can affect macroeconomic outcomes. In this paper, I develop a model in which firms can partly insure against idiosyncratic shocks using financial contracts such as credit lines. I study how shocks to uncertainty about idiosyncratic business outcomes affect the provision of insurance and how endogenous changes in the availability of insurance affect the macroeconomic response to uncertainty shocks.

Firms use financial contracts to manage risk because their owners or managers are risk-averse [Stulz (1984), Meh and Quadrini (2006), and Chen et al. (2010)] or because financial frictions render firms effectively risk-averse [Froot et al. (1993)

I am very grateful to two anonymous referees, Marios Angeletos, Nick Bloom, Ricardo Caballero, Giovanni Favara, Guido Lorenzoni, Anna Orlik, Vasia Panousi, Rebecca Wasyk, and Iván Werning for their helpful suggestions. I also would like to thank the seminar participants at ESEM 2017, Boston College, Boston University, the University of California-Berkeley (Haas), MIT, the University of Maryland (Smith), the University of Michigan, the World Bank, the Federal Reserve Banks of Kansas City, New York and Richmond, and conferences. I gratefully acknowledge the financial support from the Kauffman Foundation. The views expressed here are those of the author and need not represent the views of the Federal Reserve Board or its staff. Address correspondence to: Jonathan Goldberg, Federal Reserve Board, Mailstop K1-164, 20th Street and Constitution Avenue NW, Washington, DC 20551, USA. e-mail: jonathan.goldberg@frb.gov.

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and Rampini et al. (2014)]. Although financial contracts such as credit lines can provide insurance against idiosyncratic shocks, this insurance can only be provided by financial intermediaries if they are able to credibly commit to make payments. The ability to make such commitments depends on the availability of pledgeable assets, which may be scarce due to the same frictions that lead firms to seek insurance, as in Holmström and Tirole (1998).

A large literature shows that, with financial frictions, a rise in uncertainty about idiosyncratic business outcomes can generate a drop in investment [e.g., Christiano et al. (2014)]. However, this literature has placed significant constraints on the ability of firms to manage risk through financial contracts. For example, in Bewley-style models with risk-averse entrepreneurs, markets are exogenously incomplete and contracts contingent on idiosyncratic shocks are ruled out by assumption [e.g., Angeletos (2007)].

This paper develops a model with entrepreneurial risk aversion and limited commitment in financial contracts. The entrepreneurs experience idiosyncratic shocks when converting purchased capital into effective capital; I refer to a rise in uncertainty about the idiosyncratic shock as a risk shock.

First, I study the response to a risk shock in a competitive equilibrium with optimal contracting under limited commitment. In this setting, each entrepreneur contracts with a financial intermediary; the contract can specify payments that are contingent on an entrepreneur's idiosyncratic shock but the contract must satisfy limited commitment constraints. In particular, the entrepreneur can commit to pay only up to a fraction of its post-production assets. Moreover, promised payments from an intermediary must be backed by credible claims on other entrepreneurs. The equilibrium contract can be implemented using credit lines.[1]

Second, I study the response to a risk shock in an economy that is identical except that only equity claims are allowed. In this setting, each entrepreneur issues equity against a fraction of its assets and holds the market portfolio; as a result, each entrepreneur's return to investment is linear in its idiosyncratic shock. I compare the responses to a risk shock with and without optimal contracting, both qualitatively and quantitatively.

The role of optimal contracting in propagating a risk shock reflects two opposing forces. On the one hand, optimal contracting reduces entrepreneurs' exposure to idiosyncratic risk and thus dampens the macroeconomic effects of risk shocks. On the other hand, optimal contracting amplifies a risk shock through an endogenous decrease in the supply of insurance. Which of these forces dominates following a risk shock depends on the initial level of risk and the magnitude of the risk shock. With sufficiently low uncertainty about the idiosyncratic shock, optimal contracting provides almost complete insurance against idiosyncratic shocks. Thus, starting from low uncertainty about the idiosyncratic shock, optimal contracting dampens the macroeconomic effects of a marginal increase in risk. However, if there is a sufficiently large increase in idiosyncratic uncertainty, steady-state investment under optimal contracting declines more than steady-state investment under equity only. Dynamically, if risk is sufficiently high prior to the risk shock, aggregate investment is more sensitive to a rise in risk under optimal contracting than in the equity-only setting, conditional on the same level of aggregate capital prior to the risk shock.

The possibility that optimal contracting can either amplify or dampen the effects of a risk shock stands in contrast to the implications of optimal contracting for the levels of steady-state capital and investment, which are unambiguously higher with optimal contracting. Optimal contracting allows for more insurance than issuing equity and holding the market portfolio: for any distribution of the idiosyncratic shock, idiosyncratic investment returns after taking into account optimal contracting are less risky (in the sense of second-order stochastic dominance). Thus, while investment is lower than with complete markets in both the optimal-contracting and equity-only settings, there is less under-investment in steady state when optimal contracting is permitted. This steady-state result is consistent with the literature on optimal contracting and idiosyncratic business risk [Meh and Quadrini (2006)].

Risk sharing can endogenously deteriorate following a risk shock because entrepreneurs promise (through intermediaries) that if they are lucky, they will make payments to unlucky entrepreneurs. Due to limited commitment, the entrepreneurs can commit to pay only part of any unexpected gains; thus, unlucky entrepreneurs can be only partially compensated for their unexpected losses. As idiosyncratic uncertainty rises, more entrepreneurs find themselves with unexpected non-pledgeable gains and more entrepreneurs have unlucky shocks that require insurance; thus, an entrepreneur facing a given unexpected loss will receive less compensation from intermediaries. In contrast, in the equity-only setting, the distribution of the idiosyncratic shock does not affect the mapping from an entrepreneur's idiosyncratic shock to its return after paying equity holders and receiving payments from its equity holdings.

I parametrize the model to understand the potential quantitative importance of optimal contracting for the economy's response to a risk shock. I study a risk shock that occurs when risk is already high, as in the intensification of a crisis. In this scenario, investment growth falls by 1.5 percentage points more under optimal contracting than under equity only. A risk shock also leads to an endogenous increase in uncertainty about future aggregate investment, consistent with the observation made by Bloom (2009) and others that idiosyncratic uncertainty and aggregate uncertainty display positive co-movement. Here, aggregate investment uncertainty rises endogenously with a rise in idiosyncratic risk, and can rise more under optimal contracting than under equity-only contracting, depending on the initial level of idiosyncratic risk.

There is empirical support for the key assumption of entrepreneurial risk aversion. Managers of US public firms and owners of private firms are exposed to idiosyncratic business risk, and this exposure affects firm behavior [Panousi and Papanikolaou (2012); Kartashova (2014), and Glover and Levine (2015)].[2]

This paper is related to several strands of the literature on investment in the presence of idiosyncratic risk and financial frictions. One strand of that literature focuses on contracting subject to limited commitment. I build in particular on Holmström and Tirole (1998) and Meh and Quadrini (2006). Holmström and Tirole (1998) study a static model and Meh and Quadrini (2006) study a dynamic economy without aggregate shocks. In contrast, this paper embeds the contracting problem into a stochastic real business cycle model and studies the dynamic response to a risk shock. In addition, this paper provides qualitative and quantitative analyses of how the macroeconomic implications of optimal contracting depend on the level of risk and the size of a risk shock.

Risk-averse entrepreneurs and idiosyncratic investment risk have been studied extensively in Bewley-style models. In these models, the idiosyncratic risk of investment in physical capital is exogenous and there is no scope for risk management [e.g., Angeletos (2007)].[3] The results in this paper point to the importance of the contracting environment in determining the effects of risk shocks.

This paper also contributes to the literature on risk shocks by providing theoretical results characterizing the economy's response to a risk shock. Angeletos (2007) and others have shown that steady-state capital with idiosyncratic investment risk can be lower than with complete markets. I generalize this result by characterizing the dynamics of a stochastic model with endogenously incomplete markets. In particular, I show that macroeconomic quantities are exactly identical to those implied by a model with perfect risk sharing, but with preference shocks (i.e., shocks to the Euler equation). The results in this paper also suggest that risk management may be important for assessing the effects of risk shocks in settings without risk-averse entrepreneurs, but with firms that are effectively risk averse because negative idiosyncratic shocks are especially costly for other reasons.[4]

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 studies aggregation and the properties of the equilibrium contract. Section 4 analyzes the macroeconomic implications of risk shocks. Section 5 compares outcomes under optimal contracting and in the equity-only setting. Section 6 conducts a quantitative analysis using a parametrized version of the model. Section 7 concludes.

2. THE MODEL

Time is discrete, indexed by $t \in \{0, 1, \ldots, \infty\}$. The economy is populated by entrepreneurs, intermediaries, and workers that only supply labor.

2.1. Entrepreneurs

There is a continuum of risk-averse entrepreneurs with Epstein–Zin preferences. The total measure of entrepreneurs is normalized to one. I describe the entrepreneur's problem in terms of an entrepreneur with net worth n in period t.

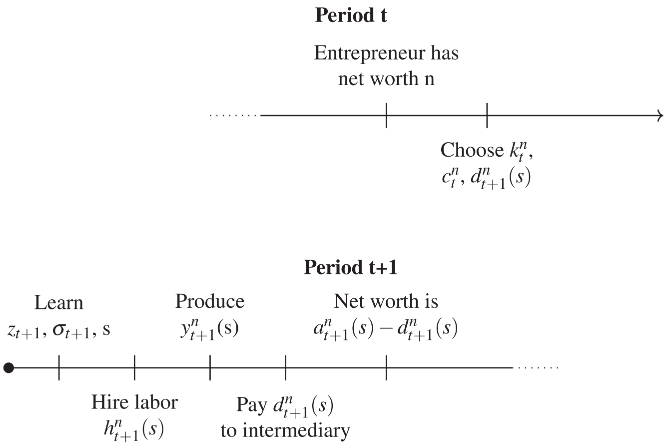

FIGURE 1. Timing.

Each entrepreneur purchases k_t^n units of physical capital and c_t^n consumption goods using its net worth and any proceeds from contracting with an intermediary to make a post-production payment in period $t+1$. The post-production payment, $d_{t+1}^n(s)$, can be contingent on aggregate and idiosyncratic shocks, but limited commitment restricts the payment to be less than a fraction of the firm's post-production assets.[5]

Before production, capital is converted into effective capital; this conversion is subject to idiosyncratic risk. In period $t+1$, each n-type entrepreneur hires labor and produces. The entrepreneur's net worth in period $t+1$ is equal to the resulting profits and undepreciated capital less the payment $d_{t+1}^n(s)$. The timing is summarized in Figure 1.

2.1.1. Production. Each n-type entrepreneur begins period $t+1$ with effective capital sk_t^n, where s is an idiosyncratic shock. s has a unit-mean log normal distribution and is drawn independently across time and entrepreneurs. The standard deviation of log s is denoted by σ_t and follows a first-order univariate autoregression,

$$\log(\sigma_t) = \log(\bar{\sigma})(1-\rho_\sigma) + \rho_\sigma \log(\sigma_{t-1}) + \varepsilon_t^\sigma, \quad \varepsilon_t^\sigma \sim N\left(0, \sigma_\sigma^2\right). \tag{1}$$

An n-type entrepreneur with idiosyncratic shock s hires labor $h_{t+1}^n(s)$ in a competitive market at wage ω_{t+1}. The entrepreneur then produces $y_{t+1}^n(s)$ final goods, where

$$y_{t+1}^n(s) = F\left(sk_t^n, h_{t+1}^n(s), z_{t+1}\right) \tag{2}$$

and z_{t+1} is productivity. The law of motion for productivity is

$$\log(z_t) = \rho_z \log(z_{t-1}) + \varepsilon_{z,t}, \quad \varepsilon_{z,t} \sim N\left(0, \sigma_z^2\right). \tag{3}$$

The expectation of a random variable $x_{t+1}(\sigma_{t+1}, z_{t+1}, s)$ conditional on σ_t and z_t is denoted $E_t[x_{t+1}]$.

The production technology F is neoclassical: it exhibits constant returns to scale in effective capital and labor, has positive and strictly diminishing marginal products, and satisfies the standard Inada conditions. The resulting profits are given by

$$\pi^n_{t+1}(s) = y^n_{t+1}(s) - \omega_{t+1} h^n_{t+1}(s). \tag{4}$$

The n-type entrepreneur's assets $a^n_{t+1}(s)$ are defined as profits and the value of undepreciated capital. Post-production assets $a^n_{t+1}(s)$ are inclusive of profits $\pi^n_{t+1}(s)$. The depreciation rate is δ. Thus,

$$a^n_{t+1}(s) = \pi^n_{t+1}(s) + (1-\delta) s k^n_t. \tag{5}$$

2.1.2. Budget constraint. The budget constraint of an n-type entrepreneur is

$$c^n_t + k^n_t \le n + E_t\left[q_{t+1} d^n_{t+1}(s)\right], \tag{6}$$

where q_{t+1} is the state-price density.[6]

2.1.3. Financial contracting and limited commitment. Each n-type entrepreneur enters into a contract with a financial intermediary. The contract specifies a payment to be made by the entrepreneur after production in period $t+1$. The payment $d^n_{t+1}(s)$ can depend on productivity z_{t+1} and risk σ_{t+1} as well as the idiosyncratic shock s. In return, the intermediary pays the entrepreneur $E_t\left[q_{t+1} d^n_{t+1}(s)\right]$, where q_{t+1} is the state-price density.

The set of possible contracts is constrained by limited commitment. In particular, for any realization of productivity z_{t+1}, risk σ_{t+1}, and the idiosyncratic shock, the payment $d^n_{t+1}(s)$ must be less than θ share of post-production assets,

$$d^n_{t+1}(s) \le \theta a^n_{t+1}(s), \tag{7}$$

with $\theta < 1$. One can motivate this constraint by assuming that if the entrepreneur fails to pay $d^n_{t+1}(s)$, it gets to make a take-it-or-leave-it offer regarding payment. If the intermediary rejects the offer, the firm is liquidated and a fraction $(1-\theta)$ of the firm's assets is lost.[7] The payment promised by the entrepreneur can be negative (i.e., $d^n_{t+1}(s) < 0$), representing a payment from the intermediary to the entrepreneur.

2.1.4. Preferences. Entrepreneurs have Epstein–Zin preferences with constant elasticity of intertemporal substitution (EIS) ε and constant relative risk aversion γ. The utility $v_t(n)$ of an n-type entrepreneur satisfies the following recursion:

$$v_t(n) = U^{-1}\left[U(c^n_t) + \beta U\left(\mathbb{CE}_t\left[v_{t+1}(a^n_{t+1}(s) - d^n_{t+1}(s))\right]\right)\right], \tag{8}$$

where the discount factor satisfies $\beta < 1$ and $\mathbb{CE}_t(x) = \Upsilon^{-1}\left(E_t\left[\Upsilon(x)\right]\right)$ denotes the certainty equivalent of a random variable x conditional on σ_t and z_t. Υ and U

https://doi.org/10.1017/S1365100519000865 Published online by Cambridge University Press

are given by:

$$\Upsilon(c) = c^{1-\gamma} \text{ and } U(c) = c^{1-\frac{1}{\varepsilon}}, \tag{9}$$

with $\varepsilon > 0$ and $\gamma > 0$.

Epstein–Zin preferences help to clarify how the macroeconomic response to a risk shock depends *qualitatively* on the EIS and relative risk aversion. Standard expected utility is nested by allowing the EIS to equal the inverse of relative risk aversion (i.e., $\varepsilon = \frac{1}{\gamma}$).

In summary, an n-type entrepreneur chooses c_t^n, k_t^n, and $d_{t+1}^n(s)$, subject to the budget constraint (6) and the limited commitment constraint (7), to maximize utility (8).

2.1.5. Aggregation (definitions). The distribution of entrepreneur net worth in period t is denoted by $f_t(n)$. Define aggregate net worth $N_t = \int_0^\infty n\, f_t(n)\, dn$, aggregate capital $K_t = \int_0^\infty k_t^n\, f_t(n)\, dn$, aggregate consumption $C_t = \int_0^\infty c_t^n\, f_t(n)\, dn$, and aggregate profits $\Pi_{t+1} = \int_0^\infty \pi_{t+1}^n(s)\, f_t(n) p(s;\sigma_t)\ \ ds\, dn$, where $p(s;\sigma_t)$ is the probability density function for s conditional on risk σ_t.

2.2. Intermediaries

The intermediary sector is perfectly competitive. Apart from their contracts with entrepreneurs, intermediaries in period t do not have access to period $t+1$ markets. Intermediaries entering in period-t have zero net wealth and are restricted to contracts in which promised payments ($d_{t+1}^n(s) < 0$) are backed by credible promises from entrepreneurs.

2.3. Workers

The model requires workers in order to generate decreasing returns to scale in aggregate capital together with linear returns for each entrepreneur in its own capital. I assume that workers are hand-to-mouth, each period consuming the entirety of their labor income. Each worker supplies ω_t^ς units of labor, with $\varsigma > 0$. The measure of workers is H, implying that aggregate labor supply is $H_t = H\omega_t^\varsigma$.

The assumption of hand-to-mouth workers is made to maintain tractability. However, hand-to-mouth behavior might arise endogenously if workers face their own limited commitment problem that prevents them from borrowing from entrepreneurs. In addition, there is also some empirical justification for assuming hand-to-mouth workers; a significant fraction of households participates only minimally in financial markets and has consumption that responds strongly to even temporary changes in income [e.g., Johnson et al. (2006)]. Hand-to-mouth workers are also present in other models of liquidity and investment, including Angeletos (2007) and Bigio (2015).

2.4. Equilibrium

The market-clearing conditions are, for the labor market,

$$H_t = \int_0^\infty \int_0^\infty h_t^n(s) p(s;\sigma_{t-1}) f_{t-1}(n)\, ds\, dn, \tag{10}$$

and, for consumption goods,

$$C_t + K_{t+1} = \Pi_t + (1-\delta)K_t, \tag{11}$$

and, for state-contingent promised payments,

$$\int_0^\infty \int_0^\infty d_{t+1}^n(s) p(s;\sigma_t) f_t(n)\, ds\, dn = 0. \tag{12}$$

Equation (12) must hold in each period $t+1$ state of nature (i.e., for each realization of period-$t+1$ productivity z_{t+1} and risk σ_{t+1}).

Initial conditions. The initial condition of the economy is given by productivity z_0, risk σ_0, and a distribution over entrepreneurs' net worth $f_0(n)$.

Remarks. In the model, a shortage of pledgeable assets and risk aversion are both required for risk shocks to affect macroeconomic outcomes. If θ was equal to one, then entrepreneurs would choose full insurance against idiosyncratic shocks; a risk shock would have no effect on aggregate quantities or welfare. If relative risk aversion was equal to zero, a risk shock would likewise have no effect on aggregate quantities or welfare.[8]

3. EQUILIBRIUM CHARACTERIZATION

3.1. Individual Behavior

Here, I analyze the entrepreneur's problem for given prices (q_{t+1}, ω_t) and show that optimal consumption, investment, and financial contracts are linear in net worth.[9] This result facilitates aggregation and is used later in analyzing the equilibrium contract (Sections 3.2 and 3.3) and the macroeconomic implications of risk shocks (Section 4).

First, I describe how an entrepreneur's post-production assets depend on the entrepreneur's capital. This problem is simplified by two assumptions: labor is hired after the idiosyncratic shock is realized and F is constant returns to scale in effective capital and labor. These assumptions imply that post-production assets are linear in capital. Each entrepreneur, regardless of net worth, receives a return on capital sR_{t+1}, where R_{t+1} is a function of productivity z_{t+1} and the period-$t+1$ prices of capital and labor.

LEMMA 1. *Given prices, an n-type entrepreneur's assets and labor demand are linear in the entrepreneur's capital, with*

$$\frac{a_{t+1}^n(s)}{k_t^n} = sR_{t+1} \ \ \textit{and} \ \ \frac{h_{t+1}^n(s)}{k_t^n} = sh_{t+1}, \tag{13}$$

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where $h_{t+1} = \arg\max_h (F(1, h, z_{t+1}) - \omega_{t+1} h)$ *and* $R_{t+1} = F(1, h_{t+1}, z_{t+1}) - \omega_{t+1} h_{t+1} + (1 - \delta)$.

Second, I show that utility and equilibrium policies for an entrepreneur are linear in its net worth n. Linearity follows from a combination of homothetic preferences (constant relative risk aversion and constant EIS), the linearity of the limited commitment constraint (7), and the result in Lemma 1 that post-production assets $a^n_{t+1}(s)$ are linear in the capital of each n-type entrepreneur.

LEMMA 2. *Given prices, utility* $v_t(n)$ *is linear in net worth:*

$$v_t(n) = \psi_t n, \tag{14}$$

where

$$\psi_t = \max_{c_t, k_t, d_{t+1}(s)} U^{-1}\left[U(c_t) + \beta U\left(\mathbb{CE}_t\left[\psi_{t+1}(a_{t+1}(s) - d_{t+1}(s))\right]\right)\right] \tag{15}$$

subject to

$$c_t + k_t \leq 1 + E_t\left[q_{t+1} d_{t+1}(s)\right], \tag{16}$$

$$d_{t+1}(s) \leq \theta a_{t+1}(s), \tag{17}$$

and

$$a_{t+1}(s) = s R_{t+1} k_t. \tag{18}$$

Lemma 2 implies that the equilibrium policies for an n-type entrepreneur are linear in n. That is,

$$c^n_t = c_t n, \tag{19}$$

$$k^n_t = k_t n, \tag{20}$$

$$d^n_{t+1}(s) = d_{t+1}(s) n, \tag{21}$$

$$a^n_{t+1}(s) = s R_{t+1} k_t n, \tag{22}$$

where $(c_t, k_t, d_{t+1}(s))$ are solutions to (15)–(18). This homogeneity simplifies aggregation: the evolution of aggregate variables such as the capital stock depends on aggregate net worth, but it does not depend on the distribution of net worth. For example, in the case of aggregate capital, $K_t = \int_0^\infty k^n_t f_t(n)\, dn = k_t N_t$, where aggregate net worth $N_t = \int_0^\infty n f_t(n)\, dn$.

3.2. Contracting in Partial Equilibrium

This section analyzes the optimal contract $d_{t+1}(s)$ taking prices as given. The entrepreneur's desire for insurance implies that whenever the limited commitment constraint is not binding, the return on net worth

$$a_{t+1}(s) - d_{t+1}(s) \tag{23}$$

must not depend on the idiosyncratic shock s. As a result, $d_{t+1}(s)$ will rise one-for-one with the post-production return $a_{t+1}(s)$ except insofar as the limited

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commitment constraint binds. This intuition is formalized in the next lemma. For any c_t and k_t, the entrepreneur chooses $d_{t+1}(s)$ to maximize the certainty equivalent of period-$t+1$ utility, subject to the requirement of satisfying the budget constraint and the limited commitment constraint. Since utility is linear in net worth (Lemma 2), maximizing the certainty equivalent of period-$t+1$ utility implies seeking insurance against idiosyncratic shocks.

LEMMA 3. *The equilibrium contract is*

$$d_{t+1}(s) = \min\{-l_{t+1} + s, \theta s\} R_{t+1} k_t, \tag{24}$$

for some $l_{t+1} > 0$.

I refer to l_{t+1} as liquidity, because in the event of an arbitrarily bad shock s, an n-type entrepreneur will receive a transfer about equal to $l_{t+1} R_{t+1} k_t^n$; in particular, $\lim_{s \to 0^+} d_{t+1}^n(s) = -l_{t+1} R_{t+1} k_t^n$. Correspondingly, for any realization of z_{t+1} and σ_{t+1}, the growth of an entrepreneur's net worth between t and $t+1$ will be:

$$a_{t+1}(s) - d_{t+1}(s) = \max\{l_{t+1}, (1-\theta)s\} R_{t+1} k_t. \tag{25}$$

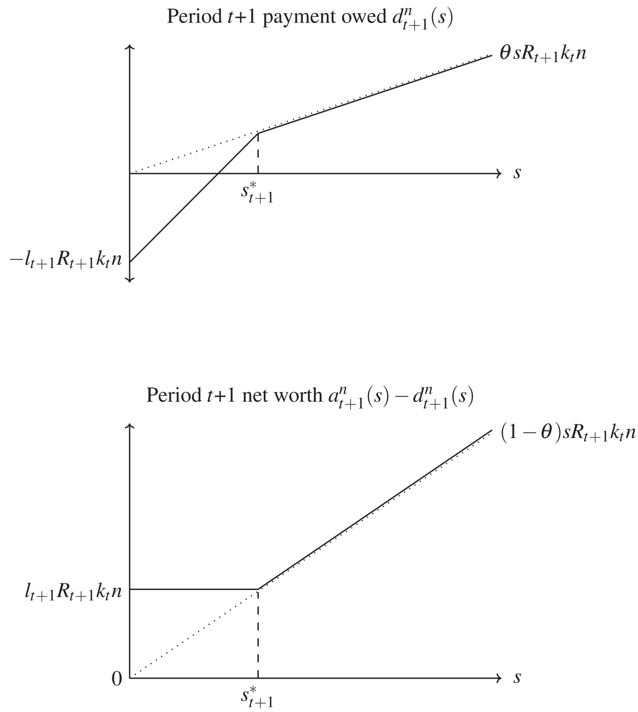
Figure 2 presents a graphical representation of the equilibrium contract and the evolution of net worth. The equilibrium payment $d_{t+1}(s)$ is positive if and only if $s > l_{t+1}$. Thus, each entrepreneur contracts to make a positive payment if its idiosyncratic shock is high ($s > l_{t+1}$), but to receive a transfer if the shock is low. In addition, the limited commitment constraint binds if and only if the idiosyncratic shock exceeds a threshold, $s_{t+1}^* = \frac{l_{t+1}}{1-\theta}$. The entrepreneur would like to reallocate wealth from high-idiosyncratic-return states to low-idiosyncratic-return states, but this would conflict with the limited commitment constraint.

Due to limited commitment and the desire for insurance, the equilibrium payment $d_{t+1}(s)$ is concave in the idiosyncratic shock; correspondingly, the equilibrium return on net worth $a_{t+1}(s) - d_{t+1}(s)$ is convex. Since the equilibrium payment $d_{t+1}(s)$ is concave in the idiosyncratic shock, an increase in the dispersion of s reduces the expected payment by the entrepreneur, all else equal. Thus, the concavity of the equilibrium payment will play an important role in how the economy responds to a risk shock.

3.3. Contracting in General Equilibrium

In general equilibrium, for each realization of period-$t+1$ productivity z_{t+1} and risk σ_{t+1}, total payments by entrepreneurs to intermediaries must sum to zero, as shown in (12). Linearity of equilibrium policies implies that total payments by n-type entrepreneurs in period $t+1$ sum to zero for each n. That is, for any z_{t+1} and σ_{t+1},

$$\int_0^\infty d_{t+1}(s) p(s; \sigma_t)\, ds = 0. \tag{26}$$



Note: The top panel shows the period-$t+1$ payment owed, $d^n_{t+1}(s)$, as a function of the idiosyncratic shock s, conditional on a given realization of σ_{t+1} and z_{t+1}. The limited-commitment constraint requires that the period-$t+1$ payment owed, $d^n_{t+1}(s)$, is at or below the dotted line in the top panel. The bottom panel shows period-$t+1$ net worth, $a^n_{t+1}(s) - d^n_{t+1}(s)$. The limited-commitment constraint requires that period-$t+1$ net worth is at or above the dotted line in the bottom panel. The constraint is binding for all $s > s^*_{t+1}$.

FIGURE 2. Graphical representation of the equilibrium contract.

Substituting from (24) into (26), one obtains:

$$\int_0^\infty \min\{-l_{t+1} + s, \theta s\}\, p(s; \sigma_t)\, ds = 0. \qquad \textbf{(27)}$$

This immediately implies the next result:

PROPOSITION 1. *(a) For any values of period $t+1$ productivity z_{t+1} and risk σ_{t+1}, liquidity l_{t+1} is the unique solution to (27).*
(b) Liquidity is scarce: $l_{t+1} < 1$.
(c) Liquidity is strictly decreasing in risk σ_t.
(d) $l_{t+1} > \theta$.

To see why liquidity is scarce, suppose instead that there were perfect risk sharing ($l_{t+1} = 1$). This would violate the constraint (27), since there would be a

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strictly positive probability of a shock $s > \frac{1}{1-\theta}$ for which the limited commitment constraint is binding.

The proof that liquidity is *strictly* decreasing in risk σ_t relies on the assumption that s is log-normally distributed. However, the relationship between liquidity and risk can also be characterized for other possible distributions of the idiosyncratic shock s: Because $d_{t+1}(s)$ is concave in s, any mean-preserving spread in s must lead to a *weak* decrease in liquidity l_{t+1} in order for (27) to continue to hold.[10]

Proposition 1 provides a characterization of the general-equilibrium change in risk management following a risk shock. The *ex-post* idiosyncratic return on capital $g_{t+1}(s)$ is given by:

$$g_{t+1}(s) = \max\left\{l_{t+1}, (1-\theta)s\right\}. \tag{28}$$

For all aggregate shocks, $g_{t+1}(s) \neq 1$ almost surely. Moreover, a rise in σ_t leads to a mean-preserving spread in $g_{t+1}(s)$ and, thus, a decrease in its certainty equivalent. A risk shock increases the riskiness of $g_{t+1}(s)$ in two ways: endogenously, as liquidity l_{t+1} decreases; and mechanically, as s is riskier. Note that $g_{t+1}(s)$ is the *ex-post* idiosyncratic return on capital, as well as the change in a given entrepreneur's share of total entrepreneurial wealth and consumption.

Implementation. The equilibrium contract $d_{t+1}(s)$ can be implemented using credit lines. Entrepreneurs could set up a limited liability company with capital k_t^n, a credit line from an intermediary equal to $l_{t+1}R_{t+1}k_t^n$, and promised compensation to the entrepreneur of $a_{t+1}^n(s) - d_{t+1}^n(s)$. The entrepreneur would sell standard equity and debt claims against the remaining assets of the firm, $\max\{d_{t+1}^n(s), 0\}$. An intermediary would issue credit lines in return for fees, which it would use to purchase equity and debt claims on a diversified pool of firms. A credit line would generate losses for the intermediary equal to $\min\{d_{t+1}^n(s), 0\}$.[11] From (26), there would be no net cash flow for the intermediary in any period, so the arrangement would be feasible.

4. MACROECONOMIC IMPLICATIONS OF RISK SHOCKS

This section shows how risk management affects the consumption-savings decision of entrepreneurs and thereby affects aggregate quantities. The main result is that a risk shock generates a decrease in investment if entrepreneurs are sufficiently willing to substitute across time (more specifically, if the EIS is greater than one). This result shows the usefulness of Epstein–Zin preferences, which allow the delinking of risk aversion and the EIS, in understanding the economy's response to a risk shock.

From the first-order conditions of the Bellman equation (15), the Euler equation of an entrepreneur can be written as:

$$U'(c_t) = \beta\rho_t U'(\rho_t(1-c_t)), \tag{29}$$

where ρ_t is the risk-adjusted expected return on investing in physical capital measured in units of the marginal utility of wealth. That is,

$$\rho_t = \mathbb{CE}_t \left[\psi_{t+1} R_{t+1} g_{t+1}(s) \right]. \quad \textbf{(30)}$$

Under the equilibrium contract, characterized by (27), liquidity l_{t+1} and hence the *ex-post* idiosyncratic return $g_{t+1}(s)$ are independent of z_{t+1} and σ_{t+1} conditional on σ_t. Thus, the risk-adjusted expected return to investing in physical capital can be rewritten as:

$$\rho_t = \mathbb{CE}_t \left[\psi_{t+1} R_{t+1} \right] \mathbb{CE} \left[g_{t+1}(s) | \sigma_t \right]. \quad \textbf{(31)}$$

Equation (31) emphasizes that a risk shock affects aggregate quantities through the riskiness of the *ex-post* idiosyncratic return on investment. In addition, (31) shows that risk shocks shift the risk-adjusted expected return on investment. More precisely, a risk shock is observationally equivalent to a preference shock with respect to macroeconomic quantities.

PROPOSITION 2. *The policy functions c_t and k_t are the same as in a model without limited commitment ($\theta = 1$), but where the discount factor β follows an exogenous process given by*

$$\beta(\sigma_t) = \beta U \left(\mathbb{CE}[g_{t+1}(s)|\sigma_t] \right). \quad \textbf{(32)}$$

Moreover, β is inversely related to σ_t if and only if $\varepsilon > 1$.

This result shows that in an environment with limited commitment and risk-averse entrepreneurs, risk shocks can drive business cycles. This result also highlights that both a shortage of pledgeable assets and risk aversion are required for risk shocks to affect macroeconomic outcomes. If θ was equal to one, then entrepreneurs would choose full insurance ($g_{t+1}(s) = 1$), implying that $\mathbb{CE}[g_{t+1}(s)|\sigma_t] = 1$. In this scenario, Proposition 2 shows that a rise in risk σ_t would have no effect on individual or aggregate consumption and investment. If relative risk aversion was equal to zero, entrepreneurs would ignore fluctuations in risk, with $\mathbb{CE}[g_{t+1}(s)|\sigma_t] = E[g_{t+1}(s)|\sigma_t] = 1$. In this scenario, Proposition 2 shows that a rise in risk would similarly have no effect on aggregate consumption and investment. If relative risk aversion was equal to zero, a rise in risk does affect the dispersion of *ex-post* idiosyncratic returns and the distribution of net worth, but these changes would be without consequence for aggregate quantities.

Proposition 2 also permits a characterization of how aggregate quantities are related to risk in steady state (in which productivity and risk are constant, but idiosyncratic shocks continue to occur with standard deviation $\bar{\sigma}$).

PROPOSITION 3. *If ε>1, then the steady-state values for aggregate capital, investment, labor, and consumption are decreasing in $\bar{\sigma}$.*

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Equation (32) implies that the steady-state return on physical capital R_{ss} is given by

$$R_{ss} = \frac{1}{\beta(\bar{\sigma})} = \frac{1}{\beta} \frac{1}{U\left(\mathbb{CE}[g_{t+1}(s)|\bar{\sigma}]\right)}. \quad (33)$$

An increase in steady-state risk $\bar{\sigma}$ reduces the certainty equivalent of the idiosyncratic return on investment. If the EIS is greater than one (implying $U(.)$ is an increasing function), this raises the required return on physical capital, implying lower capital, investment, labor, and consumption.

5. COMPARISON TO OTHER CONTRACTING SETTINGS

A central goal of the paper is to compare the economy with optimal contracting subject to limited commitment to an economy in which entrepreneurs issue only equity.

5.1. Definition of Alternative Contracting Settings

I compare optimal contracting with three alternative settings: equity-only; autarky; and complete markets.

Equity-only. In the equity-only setting, each n-type entrepreneur in period t issues an equity claim to ξ share of its post-production assets a^n_{t+1}(s), subject to the constraint that $\xi \leq \theta$. The entrepreneur uses the proceeds to purchase a market portfolio of equities issued by other entrepreneurs (which are the only assets available). As a result, the problem of the entrepreneur in the equity-only setting is the same as in the optimal contracting setting, (15)–(18), except that the limited commitment constraint is replaced by

$$d_{t+1}(s) = \xi(s-1)a_{t+1}(s) \quad (34)$$

and

$$\xi \leq \theta. \quad (35)$$

The choice of $d_{t+1}(s)$ in the equity-only setting thus becomes a choice of ξ, the fraction of assets pledged to equityholders. In equilibrium, (35) binds, reflecting entrepreneurs' desire to maximize insurance. The *ex-post* idiosyncratic return is given by $g^{eq}_{t+1}(s) = \theta + (1-\theta)s$. Note that any contract $d_{t+1}(s)$ feasible under equity-only is also feasible under optimal contracting.

Autarky. In autarky, entrepreneurs do not have access to financial contracts. The entrepreneur's problem is the same as in the optimal contracting setting, except that the limited commitment constraint (17) is replaced by $d_{t+1}(s) = 0$. Consequently, under autarky, the *ex-post* idiosyncratic return is $g^{aut}_{t+1}(s) = s$.

Complete markets. In the complete markets setting, the entrepreneur's problem is the same as in the optimal-contracting setting except that the limited commitment constraint (17) is eliminated. Consequently, $g^{cm}_{t+1}(s) = 1$.

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5.2. Risk Sharing and Aggregate Quantities

The riskiness of the *ex-post* idiosyncratic return under the four contracting settings can be ranked according to second-order stochastic dominance, as the next Proposition shows.

PROPOSITION 4. *For any σ_t, the contracting environments can be ranked according to increasing risk of the ex-post idiosyncratic return as follows:*
(i) Complete markets (no risk),
(ii) Optimal contracting,
(iii) Equity-only,
(iv) Autarky (highest risk),
where risk is ranked according to second-order stochastic dominance.

The *ex-post* idiosyncratic return on capital for an entrepreneur with idiosyncratic shock s is lower in the equity-only setting than under optimal contracting if and only if $s < \frac{l_{t+1}-\theta}{1-\theta}$; thus, for any risk σ_t, the *ex-post* idiosyncratic return is riskier with equity-only contracting than with optimal contracting.

Proposition 4 has immediate implications for aggregate quantities and welfare. In particular, note that (32) and (33) still apply in each alternative setting if one replaces the *ex-post* idiosyncratic return under optimal contracting, $g_{t+1}(s)$, with the *ex-post* idiosyncratic return in the alternative setting (i.e., autarky, equity-only, or complete markets). The next result follows immediately.

PROPOSITION 5. *If $\varepsilon>1$, the steady-state values for aggregate capital, investment, labor, and consumption can be ranked from lowest to highest as follows: (i) Autarky (lowest aggregate quantities); (ii) Equity-only; (iii) Optimal contracting; (iv) Complete markets (highest aggregate quantities).*

Proposition 5 shows that steady-state aggregate quantities are unambiguously higher with optimal contracting than with equity-only, if the EIS is greater than one. However, as the next proposition shows, the difference in steady-state aggregate capital becomes vanishingly small as risk $\bar{\sigma}$ becomes arbitrarily small or large.

In order to state this result, it is helpful to define K_{ss}^{cm} as the steady-state capital that obtains under complete markets (i.e., $\theta = 1$), in which case the steady-state required return on capital is $R_{ss}^{cm} = 1/\beta$. It is also helpful to define K_{ss}^{θ}, the steady-state capital that would obtain if the *ex-post* idiosyncratic return on capital was equal to θ, so that the required return on capital would be $R_{ss} = \frac{1}{\beta}\frac{1}{U(\theta)}$. Further, denote steady-state capital with optimal contracting by K_{ss}. Thus, $K_{ss}^{cm} > K_{ss} > K_{ss}^{eq} > K_{ss}^{\theta}$, for any risk $\bar{\sigma}$, if $\varepsilon > 1$.

PROPOSITION 6. *Suppose $\gamma > 1$.*

(a) As steady-state risk $\bar{\sigma}$ approaches zero, steady-state capital under optimal contracting, equity-only, and autarky converge to steady-state capital under complete markets. That is, $\lim_{\bar{\sigma}\to 0} K_{ss} = K_{ss}^{cm}$, $\lim_{\bar{\sigma}\to 0} K_{ss}^{eq} = K_{ss}^{cm}$, and $\lim_{\bar{\sigma}\to 0} K_{ss}^{aut} = K_{ss}^{cm}$.

(b) As steady-state risk $\bar{\sigma}$ approaches infinity, steady-state capital with optimal contracting and with only equity converges to the same value. That is, $\lim_{\bar{\sigma}\to\infty} K_{ss} = K_{ss}^{\theta}$ *and* $\lim_{\bar{\sigma}\to\infty} K_{ss}^{eq} = K_{ss}^{\theta}$.

Thus, for very high or very low risk, aggregate quantities in steady state are similar under optimal contracting and with only equity. This result holds even though the equity-only contract inefficiently allocates resources to lucky entrepreneurs, relative to optimal contracting. In particular, $g_{t+1}(s) - g_{t+1}^{eq}(s) = \theta$ for all $s > \frac{l_{t+1}}{1-\theta}$, for any risk σ_t.

Proposition 6 shows that for a large enough increase in risk, steady-state capital under optimal contracting declines more than steady-state capital under equity-only contracting. In particular, pick a level of steady-state risk $\sigma_A > 0$. Proposition 5 says that steady-state capital under optimal contracting is higher than under equity-only: $K_{ss}(\sigma_A) > K_{ss}^{eq}(\sigma_A)$. Proposition 6 implies that there exists a $\sigma > \sigma_A$ such that $K_{ss}(\sigma_A) - K_{ss}(\sigma_B) > K_{ss}^{eq}(\sigma_A) - K_{ss}^{eq}(\sigma_B)$ for all $\sigma_B > \sigma$. This makes precise the claim that, for a large enough increase in risk, steady-state capital under optimal contracting declines more than steady-state capital under equity-only.

5.3. Welfare

This section studies the welfare implications of state-contingent contracting and the other contracting environments.

The social planner's problem is formulated following Nuño and Moll (2018).[12] Consider a planner that chooses consumption, physical capital, and the financial contract for each entrepreneur. The planner faces the same constraints as the private economy. In particular, the planner is restricted to allocations that satisfy all limited commitment constraints and budget constraints; the planner must also allow the labor market to operate freely under perfect competition. The planner is not allowed to complete the market by transferring goods between lucky and unlucky entrepreneurs, except in a way that satisfies the limited commitment constraints and budget constraints. The social planner has a utilitarian objective function.

In order to specify the social planner's problem, it is necessary to model the hand-to-mouth workers rather than assuming an exogenous labor supply as in Section 2. I assume that hand-to-mouth workers active in period t live only during period t and have indirect utility $U^w(\omega_t)$, where U^w is an increasing function.[13]

The social planner's objective function, W, is the weighted sum of the value function $v_0(n_0)$ of entrepreneurs with initial net worth n_0:

$$W(f_0(n_0), z_0, \sigma_0) = \int v_0(n) f_0(n) dn, \tag{36}$$

where $v_0(n)$ is defined by (8). The social planner maximizes (36) subject to the budget constraint (6), the limited commitment constraint (7), the market-clearing constraints (10)–(12), and optimizing behavior by entrepreneurs in the spot labor

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market (13). The social planner also has to assure that the utility of period-$t+1$ workers is at least $\bar{U}^w_{t+1}$. I assume that $\bar{U}^w_{t+1}$ is such that entrepreneurs get positive utility; if $\bar{U}^w_{t+1}$ is too high, it is possible that the constraint set is empty. The main difference between the entrepreneur's problem and the social planner's problem is that the planner takes into account that the return on physical capital R_{t+1} and the wage ω_{t+1} are endogenous.

Given these constraints, the social planner will choose a financial contract of the same form as entrepreneurs choose in competitive equilibrium, as made precise in Proposition 7. For a given return on physical capital and a given consumption-savings decision (i.e., conditional on R_{t+1} and k_t), the contract chosen in competitive equilibrium delivers as much insurance as possible while respecting the limited commitment constraint, individual budget constraints, and the constraint that total payments by entrepreneurs to intermediaries must sum to zero in every period and for every realization of aggregate shocks. Even if the social planner prefers a different consumption-savings decision than the entrepreneur chooses in the competitive equilibrium, there is no reason for the social planner to deviate from the financial contract that provides the most insurance [i.e., the financial contract satisfying (24) and (27)]; the social planner can directly dictate a different consumption-savings decision than in the competitive equilibrium. Thus, while the equity-only contract is available to the social planner, the social planner does not choose it. The remaining question is whether the social planner will make the same consumption-savings decision as the entrepreneur.

Let the solution to the social planner's problem be denoted by an asterisk.

PROPOSITION 7. *The solution to the social planner's problem can be characterized as follows:*

(a) The policies chosen by the social planner are linear in net worth n.

(b) The financial contract $d^*_{t+1}(s)$ chosen by the social planner has the same form as in the competitive equilibrium. In particular,

$$d^*_{t+1}(s) = \min\{-l_{t+1} + s, \theta s\} R^*_{t+1} k^*_t, \tag{37}$$

where $l_{t+1} = l^*_{t+1}$.

(c) If $\bar{U}^w_{t+1} = U^{w,CE}_{t+1}$, where $U^{w,CE}_{t+1}$ is the utility of workers in the competitive equilibrium, then the solution to the social planner's problem coincides with the competitive equilibrium. That is, the competitive equilibrium is constrained efficient.

Thus, the competitive equilibrium with state-contingent contracting is constrained efficient, despite the possibility suggested by Proposition 6—and demonstrated in the subsequent quantitative analysis—that the decline in aggregate investment following a risk shock can be larger under optimal contracting than with equity-only contracting. Nonetheless, in a richer environment (e.g., in the presence of a pecuniary externality that exacerbates the welfare consequences of volatility), the competitive equilibrium under state-contingent contracting might no longer be constrained efficient.

The next result orders the contracting environments with respect to welfare.

PROPOSITION 8. *Steady-state welfare in each contracting environment can be ranked as follows: Autarky (lowest); Equity-only; Optimal contracting; and Complete markets (highest).*

5.4. Exploring the Mechanism with a Simple Example

To illustrate the mechanisms that underpin the previous results in Section 5, this section considers a version of the model where the idiosyncratic shock s is binary.

In particular, suppose that s can take two values, $s_L = 0$ and $s_H > 1$; denote the probability of idiosyncratic shock s_H by p_H. The idiosyncratic shock s has unit mean: $s_H p_H = 1$. Idiosyncratic uncertainty is parametrized by s_H, with higher values of s_H corresponding to higher uncertainty in the sense of second-order stochastic dominance. (The variance of s is $s_H - 1$.)

With optimal contracting, if s_H is not too high, full insurance is possible: the entrepreneur contracts to pay $d_{t+1}(s_H) = (s_H - 1)\, R_{t+1}$ in the event of a good idiosyncratic shock and to receive $-d_{t+1}(s_L) = R_{t+1}$ in the event of a bad shock; this is consistent with the limited commitment constraint for s_H if and only if

$$s_H - 1 \leq \theta s_H. \tag{38}$$

Thus, when (38) holds, there is full insurance and the *ex-post* idiosyncratic return does not depend on s. That is, $g_{t+1}(s) = 1$ for $s \in \{s_L, s_H\}$.

Now, consider a rise in idiosyncratic uncertainty in the form of an increase in s_H. If the increase in s_H is small [i.e., equation (38) continues to hold], then full insurance remains possible. However, as s_H increases, the amount that each entrepreneur can pledge to pay in the event of a good shock rises less than one-for-one with period-$t+1$ assets because $\theta < 1$. Thus, for high enough s_H [i.e., equation (38) no longer holds], the limited commitment constraint binds for the payment in the event of a good idiosyncratic shock: $d_{t+1}(s_H) = \theta s_H R_{t+1}$. In that case, $g_{t+1}(s_H) = (1-\theta)s_H > 1$ and $g_{t+1}(s_L) = \frac{s_H \theta}{s_H - 1} < 1$. Note that as s_H rises, fewer entrepreneurs receive the good shock and more receive the bad shock. Thus, as s_H rises, the aggregate payment from the lucky entrepreneurs $p_H \theta s_H R_{t+1} = \theta R_{t+1}$ remains unchanged, but it has to be shared among a greater fraction $1 - p_H$ of unlucky entrepreneurs. Thus, with an increase in s_H, the distribution of the *ex-post* idiosyncratic return $g_{t+1}(s)$ becomes riskier, with more entrepreneurs receiving the bad shock and each entrepreneur with a bad shock receiving a lower *ex-post* return (i.e., $\frac{\partial g_{t+1}(s_L)}{\partial s_H} < 0$).

In summary, with sufficiently low idiosyncratic uncertainty, a marginal rise in uncertainty about the idiosyncratic shock has no effect on the distribution of the *ex-post* idiosyncratic return. However, for sufficiently high idiosyncratic uncertainty, full insurance is not possible and a marginal rise in uncertainty leads to deterioration in risk management.

Comparison to the equity-only setting. Now, I compare how a rise uncertainty about the idiosyncratic shock (i.e., an increase in s_H) affects the distribution of

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ex-post returns under optimal contracting and in the equity-only setting. In the equity-only setting, $g^{eq}_{t+1}(s) = \theta + (1-\theta)s$ for $s \in \{s_L, s_H\}$.

First, for any s_H, the distribution of *ex-post* returns is riskier in the equity-only setting. That is, $g^{eq}_{t+1}(s_L) < g_{t+1}(s_L)$ and thus $g_{t+1}(s)$ second-order stochastically dominates $g^{eq}_{t+1}(s)$. It follows that the certainty equivalent of the *ex-post* idiosyncratic return with optimal contracting is higher than the certainty equivalent with equity only: for any s_H, $\mathbb{CE}(g_{t+1}(s)) > \mathbb{CE}(g^{eq}_{t+1}(s))$.

Second, with optimal contracting, there is full insurance if $s_H \leq \frac{1}{1-\theta}$. In contrast, in the equity-only setting, there is uninsured idiosyncratic risk for any s_H, with $g^{eq}_{t+1}(s) \neq 1$ for $s \in \{s_L, s_H\}$.

Third, under optimal contracting, if $s_H > \frac{1}{1-\theta}$, the entrepreneur's payoff in the event of a bad shock, $g_{t+1}(s_L)$, is strictly declining in s_H. Thus, a rise in uncertainty can cause deterioration in risk sharing. In contrast, in the equity-only setting, the entrepreneur's payoff in the event of a bad shock, $g^{eq}_{t+1}(s_L)$, does not depend on s_H.

Fourth, as uncertainty about the idiosyncratic shock becomes arbitrarily high, the *ex-post* idiosyncratic return on capital under optimal contracting, $g_{t+1}(s)$, converges in probability to θ, as does the *ex-post* idiosyncratic return on capital with equity only, $g^{eq}_{t+1}(s)$.[14] Moreover, for arbitrarily high uncertainty, the certainty equivalent of the *ex-post* idiosyncratic return under optimal contracting converges to the same value as the certainty equivalent with equity only.[15] In this sense, with arbitrarily high risk, optimal contracting has a vanishingly small effect on the distribution of *ex-post* returns. As a result, as uncertainty about the idiosyncratic shock becomes arbitrarily high, steady-state capital with optimal contracting and with only equity converges to the same value. That is, $\lim_{s_H \to \infty} K_{ss} = K^{\theta}_{ss}$ and $\lim_{s_H \to \infty} K^{eq}_{ss} = K^{\theta}_{ss}$.

6. NUMERICAL SIMULATION

Having discussed qualitatively the effects of optimal contracting on the economy's response to a risk shock, I now turn to an evaluation of the potential quantitative importance of optimal contracting. Section 6.1 describes the parameter values used. Section 6.2 studies how aggregate investment and the volatility of investment respond to a risk shock and to a productivity shock.

6.1. Parametrization and Solution Method

The frequency is quarterly. I assume that the production function for consumption goods is Cobb–Douglas, with $F(k, h, z) = zk^{\alpha}h^{1-\alpha}$. I let $\alpha = 0.36$, $\delta = 0.02$, and $\beta = 0.99$. These values are standard in the literature.[16]

As shown in Proposition 2, the EIS is a key parameter governing the economy's response to a risk shock: an EIS greater than one is required if investment, output, and the price of capital are to decrease in response to a risk shock. I set the EIS $\varepsilon = 2$, as in Gourio (2012). There has been considerable debate

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about the value of the EIS. Analyses based on a univariate regression of aggregate consumption growth on the risk-free rate have generated very low estimates (Hall, 1988). However, in the present setting, this approach generates an estimate of EIS that is severely downward biased. Using simulated data from the model with optimal contracting, the estimate of the EIS would be 0.17, far below the parametrized value. The univariate regression underestimates the EIS because it ignores the need for liquidity due to idiosyncratic risk. Moreover, other approaches to estimating the EIS have suggested values considerably higher than in Hall (1988).[17]

The role of relative risk aversion is demonstrated in Proposition 2, which shows how fluctuations in the distribution of *ex-post* idiosyncratic returns affect aggregate quantities. I set $\gamma = 2.5$, about halfway between the values in Gourio (2012) and Christiano et al. (2014).

There are two aggregate shocks, productivity and risk. For productivity, I set $\rho_z = 0.95$ and $\sigma_z = .007$ [e.g., Caldara et al. (2012)]. For risk, I set $\bar{\sigma} = 0.26$, $\sigma_\sigma = 0.07$, and $\rho_\sigma = 0.97$. This matches the values in Christiano et al. (2014) for the steady state and autocorrelation of risk as well as the standard deviation of the risk innovation.[18] I set the elasticity of the labor supply to the wage $\varsigma = 0.3$, consistent with estimates from microdata (Chetty et al., 2011), and the measure of workers is $H = 2$.

I set the limited commitment parameter $\theta = 0.4$. The resulting cross-sectional standard deviation of annual idiosyncratic consumption growth is equal to 0.1, on average, as in De Santis (2007) and Herskovic et al. (2015).

A global solution method is necessary since a main point of the exercise is to understand nonlinearities in the responses of key endogenous variables to aggregate shocks (e.g., how the response of aggregate investment to a change in risk σ_t depends on the size of the risk shock, with and without optimal contracting). The theoretical analysis (namely, Propositions 5 and 6, as well as the model with a binary idiosyncratic shock in Section 5.4) demonstrated the existence of such nonlinearities for variables such as steady-state aggregate quantities. I solve for the model's global solution using the spectral method of projection with Chebyshev polynomials as basis functions [e.g., Caldara et al. (2012)], taking advantage of the representation of the economy provided in Proposition 2. For numerical tractability, capital-adjustment costs are added to the model as detailed in Appendix C.

One drawback of the analysis is that I do not allow for government debt, which would make the problem intractable because the value of government debt would affect the distribution of the *ex-post* idiosyncratic return on wealth determined through the optimal contracting problem, which in turn would affect capital accumulation and the value of government debt.

6.2. Investment and Investment Volatility

Figure 3 shows how various outcomes depend on the aggregate shocks in period t, conditional on being at the autarky steady state in period $t-1$. The top panels

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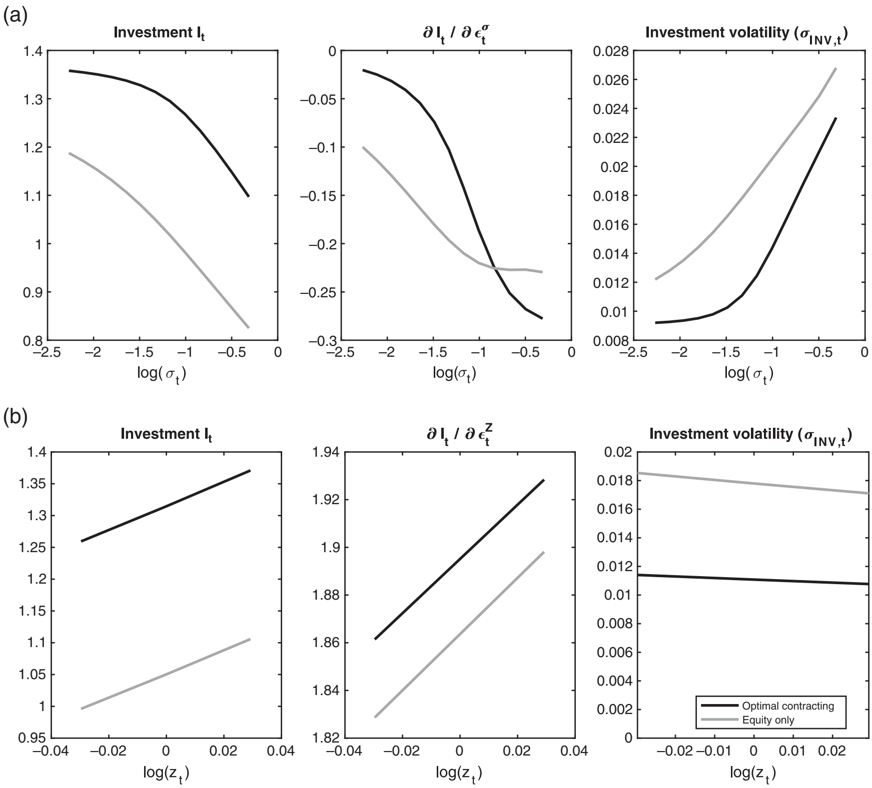
show outcomes as a function of risk σ_t. For all values of risk σ_t, investment is higher with optimal contracting than in the equity-only setting. For low levels of risk, a marginal increase in risk leads to a larger decline in investment in the equity-only setting. However, for high levels of risk, this ordering is reversed: a marginal increase in risk leads to a larger decline in investment in the optimal contracting setting, as shown in the top, middle panel. These results are consistent with the theoretical steady-state results, which showed that the benefits of optimal contracting recede as risk becomes high. Figure 3 also shows that investment volatility is uniformly lower under optimal contracting than under equity only. For low levels of risk, a marginal increase in risk leads to almost no change in investment volatility under optimal contracting. However, for higher levels of risk, a marginal increase in risk is associated with a larger increase in volatility under optimal contracting than under equity only.

The rise in uncertainty about aggregate investment generated by an increase in idiosyncratic risk is consistent with the observation made by Bloom (2009) and others that idiosyncratic uncertainty and aggregate uncertainty display positive co-movement. Here, aggregate investment uncertainty rises endogenously with a rise in idiosyncratic risk.

Suppose that risk σ_t increases from a high level, from 0.5 to 0.65, as in the intensification of a crisis. Investment growth falls by 1.5 percentage points more under optimal contracting than under equity only, showing that optimal contracting can generate economically meaningful amplification of a risk shock. Table D.1 in Appendix D shows that, for a range of alternative parameter values, optimal contracting dampens the response of investment to a risk shock when the initial level of risk is low; however, when the initial level of risk is high, the response of investment to a risk shock under optimal contracting is larger in magnitude than the response under equity only, as in the baseline parametrization.

The bottom panels of Figure 3 describe how outcomes depend on productivity z_t. Investment varies with productivity in similar ways with optimal contracting and equity only. In particular, investment is roughly linear in productivity, while investment volatility varies little with productivity. This contrasts with the nonlinearity of investment and investment volatility in risk σ_t.

It bears noting that risk σ_t does not meaningfully affect period-t output, conditional on productivity z_t and aggregate capital K_{t-1}. The reason is that the initial drop in investment in response to a change in risk is offset by an increase in consumption, since idiosyncratic risk by assumption does not affect the aggregate resource constraint of the economy. This counterfactual conditional negative co-movement between investment and consumption is common in models with risk shocks, including Bachmann and Bayer (2013), Gilchrist et al. (2014), Chugh (2016), and Bloom et al. (2018), again because risk in these models has little or no effect on aggregate output on impact. However, this conditional negative co-movement can be reversed, for example, by assuming that productivity and risk are negatively correlated [e.g., Angeletos (2007), Bachmann and Bayer (2013), and Bloom et al. (2018)]. A risk shock might induce an endogenous decline in



Note: Panel (a) shows aggregate investment (left), the sensitivity of investment to a marginal increase in risk (middle), and the volatility of investment (right), as a function of risk σ_t. Variables are shown with optimal contracting (black) and equity only (gray) and conditional on period $t-1$ capital equal to steady-state capital under autarky. Panel (b) shows aggregate investment (left), the sensitivity of investment to a marginal increase in productivity (middle), and the volatility of investment (right), as a function of productivity z_t. The volatility of investment $\sigma_{INV,t}$ is the standard deviation of the log-difference of aggregate investment between periods t and $t+1$.

FIGURE 3. Investment under optimal contracting and with equity only. (a) Variables as a function of risk σ_t. (b) Variables as a function of productivity z_t.

productivity if it leads to substitution toward a safer, less-productive technology as in Rampini (2004).

7. CONCLUSION

This paper studied the economy's response to shocks to idiosyncratic business risk under optimal contracting subject to limited commitment. I developed a business cycle model in which entrepreneurs are risk-averse and investment is subject to idiosyncratic risk. Financial markets are endogenously incomplete due to limited

https://doi.org/10.1017/S1365100519000865 Published online by Cambridge University Press

commitment. The idiosyncratic return to investment under optimal contracting is less risky than when firms are constrained to issue only standard equity. However, with optimal contracting, an increase in idiosyncratic business risk leads to deterioration in risk sharing. As a result, the effects of a large increase in idiosyncratic business risk can be amplified, rather than dampened, by optimal contracting. These results point to the importance of taking into account risk management and contracting frictions in models with risk shocks.

NOTES

1. The equilibrium contract can be implemented in a variety of ways, including through credit lines or the pooling of risk in conglomerates that own multiple businesses subject to idiosyncratic shocks (in which case the entrepreneurs can be thought of as senior managers). Thus, although markets are endogenously incomplete, the equilibrium contract can be readily connected to common instruments for managing risk. In practice, risk management—including through the use of credit lines—is an important part of firm behavior. In the USA, the total amount of unused commitments by commercial banks to fund loans to businesses was $2 trillion in 2010 [Bassett et al. (2012)]. Credit line drawdowns account for 75% of total bank lending to firms [Demiroglu and James (2011)].

2. For US public firms, the negative relationship between idiosyncratic risk and firm investment is stronger in firms in which managers hold a larger fraction of the firm's shares [Panousi and Papanikolaou (2012)] and with CEOs who are more risk averse [Roussanov and Savor (2014)]. At US public firms, managerial risk aversion is related to corporate policies [Graham et al. (2013)] and CEO incentives help explain the relationship between idiosyncratic risk and investment [Glover and Levine (2015)]. Managers of US public firms are exposed to significant firm-specific risk [Hall and Liebman (1998)]. Consistent with the equilibrium contract in the model, managers appear to be significantly insured against bad luck but benefit significantly from good luck [Murphy (1999) and Gopalan et al. (2010)]. US owners of private firms are exposed to considerable business income risk [DeBacker et al. (2015)] and ownership of private firms is associated with a positive risk premium [Kartashova (2014)].

3. Research using Bewley-style models of idiosyncratic business risk includes Angeletos and Calvet (2006), Angeletos and Panousi (2009), Luo et al. (2010), Angeletos and Panousi (2011), Benhabib et al. (2011), Braun and Nakajima (2012), Sandri (2014), and Panousi and Reis (2015). Chen et al. (2010) develop a partial-equilibrium model with exogenously incomplete markets including equity and risky debt.

4. In Christiano et al. (2014), Gilchrist et al. (2014), Arellano et al. (2018), and Letendre and Wagner (2018), default costs make firms effectively risk averse.

5. To simplify the notation, I suppress the dependence of endogenous variables on aggregate shocks.

6. The determination of the state-price density q_{t+1} is explained in Appendix B.

7. Similar limited commitment constraints appear, for example, in Lorenzoni (2008), Buera et al. (2011), and Jermann and Quadrini (2012).

8. See Section 4 for additional discussion.

9. In Sections 3.1 and 3.2, I study individual behavior assuming exogenous prices (q_{t+1}, ω_t). In general equilibrium, q_{t+1} and ω_t are endogenous objects.

10. Moreover, using the definitions of risk provided in Rasmusen (2007), a change in the distribution of s leads to a strict decrease in liquidity: (i) if s becomes "pointwise riskier"; and (ii) only if s becomes "extremum riskier." See the proof of Proposition 1 in Appendix A for further discussion.

11. As in Holmström and Tirole (1998), a key feature of credit lines is that drawdowns have negative net present value for the intermediary.

12. See also Dávila et al. (2012).

13. If the utility of a period-t worker is given by $c_t^w - \frac{\varsigma}{\varsigma+1}\omega_t \left(h_t^w\right)^{\frac{\varsigma+1}{\varsigma}}$, where c_t^w is the worker's consumption and h_t^w is labor, then the aggregate labor supply is $H_t = H\omega_t^{\varsigma}$, as in Section 2, and the indirect utility function is increasing in the period-t wage ω_t.

14. This is implied by $\lim_{s_H\to\infty} g_{t+1}(s_L) = \theta$ together with $\lim_{s_H\to\infty} p_H = 0$ and $g_{t+1}^{eq}(s_L) = \theta$.

15. In particular, $\lim_{s_H\to\infty} \mathbb{CE}(g_{t+1}(s)) = \lim_{s_H\to\infty} \mathbb{CE}(g_{t+1}^{eq}(s)) = \theta$.

16. See, for example, Angeletos (2007). With these assumptions, the capital-to-output ratio in the risky steady state is 11.1, and the investment-to-output ratio in the risky steady state is 0.22. These values are close to their empirical counterparts in the US data.

17. See, for example, Guvenen (2006), Gruber (2013), Bansal et al. (2016), and Albuquerque et al. (2016).

18. I match the standard deviation of Christiano et al. (2014)'s unanticipated risk shock.

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APPENDIX A: PROOFS

Proof of Lemma 2

The entrepreneur maximizes (8) subject to (6) and (7). Suppose that utility is given by $v_t(n) = \psi_t n$, as in (14). Then the first-order conditions of the entrepreneur's problem with respect to c_t^n, k_t^n, and $d_{t+1}^n(s)$ are, respectively:

$$\left(\frac{c_t^n}{v_t(n)}\right)^{-\frac{1}{\varepsilon}} = \lambda_t^n, \tag{A1}$$

$$v_t(n)^{\frac{1}{\varepsilon}}\beta \frac{E_t\left[\psi_{t+1}^{1-\gamma}(sR_{t+1}k_{t+1}^n - d_{t+1}^n(s))^{-\gamma} sR_{t+1}\right]}{\left(\mathbb{CE}_t\left[\psi_{t+1}(sR_{t+1}k_t^n - d_{t+1}^n(s))\right]\right)^{\frac{1}{\varepsilon}-\gamma}} = \lambda_t^n p_t^K - \theta E_t\left[\zeta_{t+1}^n(s)sR_{t+1}\right],$$

$$v_t(n)^{\frac{1}{\varepsilon}}\beta \frac{\psi_{t+1}^{1-\gamma}(sR_{t+1}k_{t+1}^n - d_{t+1}^n(s))^{-\gamma}}{\left(\mathbb{CE}_t\left[\psi_{t+1}(sR_{t+1}k_t^n - d_{t+1}^n(s))\right]\right)^{\frac{1}{\varepsilon}-\gamma}} = q_{t+1}\lambda_t^n - \zeta_{t+1}^n(s), \tag{A2}$$

where $\zeta_{t+1}^n(s)$ is the Lagrange multiplier on (7) and λ_t^n is the Lagrange multiplier on (6). Linear policies (19)–(22) solving (15)–(18) satisfy these first-order conditions. The envelope condition is $\lambda_t^n = \psi_t$. Substituting into (A1), one obtains

$$\psi_t = c_t^{\frac{1}{1-\varepsilon}}. \tag{A3}$$

Q.E.D.

Proof of Lemma 3

Equation (A2) implies that $a_{t+1}^n(s) - d_{t+1}^n(s)$ does not depend on s whenever the limited commitment constraint (17) is not binding. *Q.E.D.*

Proof of Proposition 1

Denote

$$G(l, \sigma_t) = \int_0^\infty \max\{l, (1-\theta)s\}\, p(s; \sigma_t)\, ds, \tag{A4}$$

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for any l. Then (27) can be written as $G(l_{t+1}, \sigma_t) = 1$. Note that $G(\theta, \sigma_t) < 1$ and $G(1, \sigma_t) > 1$. Since $\log(s) \sim N(-\frac{\sigma^2}{2}, \sigma^2)$, one can write

$$G(l, \sigma_t) = l\Phi\left(\frac{\sigma}{2} + \frac{a}{\sigma}\right) + (1-\theta)\Phi\left(\frac{\sigma}{2} - \frac{a}{\sigma}\right), \qquad \textbf{(A5)}$$

where $a = log\left(\frac{l}{1-\theta}\right)$ and Φ is the cumulative distribution function of the standard normal distribution. $G(l, \sigma_t)$ is continuous and increasing in l. Thus, for any σ_t, there is a unique solution $l_{t+1} \in (\theta, 1)$ to (27). Next, differentiate (A5) with respect to σ_t. With some algebra, one obtains

$$\frac{\partial G(l, \sigma_t)}{\partial \sigma_t} = \Phi'\left(\frac{\sigma}{2} + \frac{a}{\sigma}\right) l > 0,$$

implying that l_{t+1} is strictly decreasing in σ_t. Differentiating (A4) with respect to θ, one obtains:

$$\frac{\partial G(l, \sigma_t)}{\partial \theta} = \left(\Phi\left(\frac{l}{1-\theta}\right) - 1\right) l < 0,$$

implying that l_{t+1} is strictly increasing in θ.

Regarding endnote 10, observe that $G(l, \sigma_t) - l$ is proportional to the current value to a risk-neutral owner of a call option on a stock with terminal price $(1-\theta)\, s$ and strike price l; the proportionality reflects time discounting. Rasmusen (2007) provides definitions of "pointwise riskier" and "extremum riskier" such that this value strictly increases (i) if the terminal price of the stock becomes "pointwise riskier"; and (ii) only if the terminal price of the stock becomes "extremum riskier." *Q.E.D.*

Proof of Proposition 2

Substituting $d_{t+1}(s)$ from (24) and noting that $g_{t+1}(s)$ is independent of z_{t+1} and σ_{t+1} conditional on σ_t, the Bellman equation (15) can be simplified to:

$$\begin{aligned}\psi_t &= \max_{c_t, k_t} U^{-1}\left[U(c_t) + \beta U\left(\mathbb{CE}[\max\{l_{t+1}, (1-\theta)\, s\} \,|\sigma_t]\right) U\left(\mathbb{CE}_t\left[\psi_{t+1} R_{t+1}\right] k_t\right)\right],\\ &= \max_{c_t, k_t} U^{-1}\left[U(c_t) + \beta(\sigma_t) U\left(\mathbb{CE}_t\left[\psi_{t+1} R_{t+1}\right] k_t\right)\right].\end{aligned}$$

This is the same Bellman equation that would obtain if there were a representative entrepreneur with time-varying discount factor $\beta(\sigma_t)$. *Q.E.D.*

Proof of Proposition 3

Substitute $c_{t+1} = c_t = c_{ss}$ and $p^K_{ss} = 1$ into (29)–(30) to obtain

$$1 = \beta^{-\varepsilon} R_{ss}^{1-\varepsilon} \left(\mathbb{CE}[\max\{l_{ss}, (1-\theta)\, s\} \,|\bar{\sigma}]\right)^{1-\varepsilon} (1 - c_{ss}). \qquad \textbf{(A6)}$$

Substituting $c_{ss} + k_{ss} = 1$ and $R_{ss} k_{ss} = 1$, one obtains

$$R_{ss} = \frac{1}{\beta} \frac{1}{U\left(\mathbb{CE}[\max\{l_{ss}, (1-\theta)\, s\} \,|\bar{\sigma}]\right)}. \qquad \textbf{(A7)}$$

Lemma 1 and the assumption that F has constant returns to scale in effective capital and labor imply

$$R_{ss} = F_K\left(\frac{K_{ss}}{H_{ss}}, 1, z_{ss}\right) + (1-\delta). \qquad \textbf{(A8)}$$

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F has a positive and strictly diminishing marginal product of capital and the standard Inada conditions hold, implying that if $\varepsilon > 1$, then there is a unique value $\frac{K_{ss}}{H_{ss}} > 0$ that satisfies (A7)–(A8). Moreover, under these assumptions, (A7)–(A8) imply that $\frac{K_{ss}}{H_{ss}}$ is strictly decreasing in $\bar{\sigma}$. From Lemma 1 and labor market clearing,

$$F_L\left(1, \frac{H_{ss}}{K_{ss}}, z_{ss}\right) = \omega_{ss} = H_{ss}^{\frac{1}{\varsigma}},$$

implying that steady-state values for aggregate capital, investment, labor, and consumption are decreasing in $\bar{\sigma}$. *Q.E.D.*

Proof of Proposition 4

With autarky, $g_{t+1}^{aut}(s) = s$. With equity-only, $g_{t+1}^{eq}(s) = \theta + (1-\theta)s$, implying $g_{t+1}^{eq}(s) > g_{t+1}^{aut}(s)$ if and only if $s < 1$. The *ex-post* idiosyncratic return under optimal contracting, $g_{t+1}(s)$, satisfies $g_{t+1}(s) > g_{t+1}^{eq}(s)$ if and only if $s < \frac{l_{t+1}-\theta}{1-\theta}$. With complete markets, $g_{t+1}^{cm}(s) = 1$, implying $g_{t+1}^{cm}(s) > g_{t+1}(s)$ if and only if $s < \frac{1}{1-\theta}$. *Q.E.D.*

Proof of Proposition 6

Note that $E_t\left[g_{t+1}(s)^{1-\gamma}\right] = l_{t+1}^{1-\gamma}\Pr(s < \frac{l_{t+1}}{1-\theta}) + E_t[g_{t+1}(s)^{1-\gamma}|s > \frac{l_{t+1}}{1-\theta}]\Pr(s > \frac{l_{t+1}}{1-\theta})$. From (27) and (A5), $l_{t+1} \in (\theta, 1)$, $\lim_{\bar{\sigma}\to\infty} l_{ss} = \theta$, and $\lim_{\bar{\sigma}\to 0} l_{ss} = 1$. Moreover, $\Pr(s < \frac{l_{t+1}}{1-\theta}) = \Phi(\frac{\ln(\frac{l_{t+1}}{1-\theta})}{\sigma_t} + \frac{\sigma_t}{2})$. Also, $E_t[g_{t+1}(s)^{1-\gamma}|s > \frac{l_{t+1}}{1-\theta}] \in (0, l_{t+1}^{1-\gamma})$ if $\gamma > 1$. Thus, $\lim_{\bar{\sigma}\to\infty} \mathbb{CE}[\max\{l_{ss}, (1-\theta)s\}\,|\bar{\sigma}] = \theta$ and $\lim_{\bar{\sigma}\to 0} \mathbb{CE}[\max\{l_{ss}, (1-\theta)s\}\,|\bar{\sigma}] = 1$. To see that $\lim_{\bar{\sigma}\to 0} K_{ss}^{aut} = K_{ss}^{cm}$, note that $s^{1-\gamma}$ is log-normally distributed, with $\log(s^{1-\gamma}) \sim N(-(1-\gamma)\frac{\bar{\sigma}^2}{2}, (1-\gamma)^2\bar{\sigma}^2)$, so that $\lim_{\bar{\sigma}\to 0} \mathbb{CE}[g_{t+1}^{aut}(s)] = 1$. *Q.E.D.*

Proof of Proposition 7

Linearity of the constraints (6) and (7) and linearity of returns in each entrepreneur's capital (13), together with homotheticity of preferences (8), imply that the constrained-efficient policies are linear in net worth. Therefore, the social planner solves (15)–(18), subject to (13) and the constraint $\omega_{t+1}^* \geq (U^w)^{-1}(\bar{U}_{t+1})$. Thus, the social planner's financial contract $d_{t+1}^*(s)$ satisfies (24) and (27) after substituting $k_t^* = k_t$ and $R_{t+1}^* = R_{t+1}$. For part (c), note that the constraint $U_{t+1}^w \geq \bar{U}_{t+1}^{w,CE}$ can be rewritten $K_t^* \geq K_t^{CE}$. The first-order condition of the planner's problem with respect to c_t^* is: $\left(c_t^*\right)^{-\frac{1}{\varepsilon}} = \beta U(\rho_t^*)\left(1-c_t^*\right)^{-\frac{1}{\varepsilon}} - z_t^*\frac{\partial R_{t+1}^*}{\partial c_t} - \lambda_t^*\left(\psi_t^*\right)^{\varepsilon}\frac{\partial K_t^*}{\partial c_t^*}$, where $z_t^* = \beta U(1 - c_t^*)\left(\rho_t^*\right)^{-\frac{1}{\varepsilon}} CE[\psi_{t+1}^* R_{t+1}^* g_{t+1}(s)]^{\gamma} E[(\psi_{t+1}^* g_{t+1}(s))^{1-\gamma}\left(R_{t+1}^*\right)^{-\gamma}] > 0$ and λ_t^* is the Lagrange multiplier associated with the constraint $K_t^* \geq K_t^{CE}$. This is identical to the individual entrepreneur's Euler equation (29) except that the social planner's Euler equation includes two extra terms: $z_t^*\frac{\partial R_{t+1}^*}{\partial c_t}$ and the Lagrange multiplier term. From (13), one obtains $\frac{\partial R_{t+1}^*}{\partial c_t^*} = -N_t^*\left((F_L(1, h_{t+1}^*, z) - \omega_{t+1}^*)\frac{\partial h_{t+1}^*}{\partial K_t^*} - \frac{\partial \omega_{t+1}^*}{\partial K_t^*}\right)$. Imposing labor-market optimization by the entrepreneurs, $\frac{\partial R_{t+1}^*}{\partial c_t^*} = N_t^*\frac{\partial \omega_{t+1}^*}{\partial K_t^*} > 0$. Thus, the term $z_t^*\frac{\partial R_{t+1}^*}{\partial c_t}$ reflects that if the constraint on workers' utility is not binding, the social planner will have the entrepreneurs act as monoposonists to reduce labor costs. Thus, the constraint $K_t^* \geq K_t^{CE}$ binds and the social planner's solution coincides with the competitive equilibrium. *Q.E.D.*

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Proof of Proposition 8

From (A6) and (A8), one obtains

$$c^j_{ss} = 1 - \beta U(\mathbb{CE}[g^j_{ss}]), \tag{A9}$$

where g^j_{ss} is the *ex-post* idiosyncratic return under contracting environment j, with $j \in \{aut, eq, cm\}$. Equation (A6) also holds for optimal contracting. Hence, if $\varepsilon > 1$, then $c^{cm}_{ss} < c^{opt}_{ss} < c^{eq}_{ss} < c^{aut}_{ss}$. If $\varepsilon < 1$, then $c^{cm}_{ss} > c^{opt}_{ss} > c^{eq}_{ss} > c^{aut}_{ss}$. The result then follows from (14), (A3), and Proposition 4. *Q.E.D.*

APPENDIX B: STATE-PRICE DENSITY

In a typical representative agent model with complete markets, the state price density is equal to the intertemporal marginal rate of substitution (IMRS) of the representative agent. In this paper, with endogenously incomplete markets, the state price density is equal to the IMRS of entrepreneurs experiencing idiosyncratic shocks such that the limited commitment constraint (17) is not binding. This result follows from a simple optimality condition. In any period t, if the IMRS for a given realization of the period $t+1$ aggregate shock and idiosyncratic shock were greater than the state-price density q_{t+1} associated with those aggregate shocks, an entrepreneur could improve its welfare by increasing its planned consumption conditional on those period $t+1$ aggregate and idiosyncratic shocks being realized, provided the limited commitment constraint is not binding.

In partial equilibrium, the state-price density q_{t+1} is exogenous. In general equilibrium, q_{t+1} is an equilibrium object.

Determination of q_{t+1} in general equilibrium. The IMRS for an n-type entrepreneur is

$$\frac{\partial v_t(n)/\partial dc^n_{t+1}}{\partial v_t(n)/\partial dc^n_t} = \beta \left(\frac{c^n_{t+1}}{c^n_t}\right)^{-\frac{1}{\varepsilon}} \left(\frac{v_{t+1}(a^n_{t+1}(s) - d^n_{t+1}(s))}{\mathbb{CE}_t\left[v_{t+1}(a^n_{t+1}(s) - d^n_{t+1}(s))\right]}\right)^{\frac{1}{\varepsilon}-\gamma}. \tag{B1}$$

Equation (B1) follows from Epstein–Zin preferences (8)–(9), together with the definition of the entrepreneur's net worth in period $t+1$. Conditional on the period-t aggregate state of the economy (K_{t-1}, z_t, σ_t) and the realization of productivity z_{t+1} and risk σ_{t+1}, the IMRS of an n-type entrepreneur varies with s. However, due to the linearity of optimal policies, the IMRS does not vary with n. That is, the IMRS

$$\frac{\partial v_t(n)/\partial dc^n_{t+1}}{\partial v_t(n)/\partial dc^n_t} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\frac{1}{\varepsilon}} \left(\frac{\psi_{t+1}(a_{t+1}(s) - d_{t+1}(s))}{\mathbb{CE}_t\left[\psi_{t+1}(a_{t+1}(s) - d_{t+1}(s))\right]}\right)^{\frac{1}{\varepsilon}-\gamma}$$

is a function of the optimal policies $(c_t, c_{t+1}, a_{t+1}(s), d_{t+1}(s))$, which do not depend on net worth n (Section 3.1).

Moreover, all entrepreneurs for whom the limited commitment constraint is not binding ($s \le s^*_{t+1}$) have the same IMRS.[19] The first-order condition of the entrepreneur's problem (15)–(18) with respect to $d^n_{t+1}(s)$ is (A2). With (A3), this first-order condition implies that for any $s \le s^*$, the following optimality conditions hold:

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$$q_{t+1} = \frac{\partial v_t(n)/\partial dc^n_{t+1}}{\partial v_t(n)/\partial dc^n_t}. \tag{B2}$$

The state-price density q_{t+1} can be obtained by substituting the solution to (15)–(18) into (B2).

Relation of q_{t+1} to optimal policies in partial equilibrium. In partial equilibrium, the state-price density q_{t+1} is exogenous. Lemma 2 is a partial equilibrium result and hence the optimality condition (B2) also holds in partial equilibrium for any $s \leq s^*$.

APPENDIX C: CAPITAL-ADJUSTMENT COSTS

In the numerical simulation in Section 6, aggregate-level capital-adjustment costs are added for tractability.

To do so, capital-goods firms are introduced to the model. Capital-goods firms use consumption goods to produce capital goods, which they sell in a perfectly competitive market at price p^K_t. In period t, capital-goods firms purchase an aggregate quantity I_t of consumption goods. For an individual capital-goods firm, the technology for producing capital goods is linear: producing each capital good in period t requires $\phi_t = \phi(\frac{I_t}{K_{t-1}})$ consumption goods. Profit maximizing by capital-goods firms implies $p^K_t = \phi_t$, with capital-goods firms earning zero profits. I assume that $\phi(x) = \left(\frac{x}{\delta}\right)^\chi$, with $\chi \in [0, 1)$.

With the introduction of capital-adjustment costs, the budget constraint becomes $c^n_t + p^K_t k^n_t \leq n + E_t\left[q_{t+1} d^n_{t+1}(s)\right]$ and post-production assets are $a^n_{t+1}(s) = \pi^n_{t+1}(s) + (1 - \delta) s p^K_{t+1} k^n_t$. The net payment from an n-type entrepreneur to capital producers (in terms of consumption goods) in period t is $i^n_t = p^K_t(k_{t+1} - (1-\delta) s k_t)$. The market-clearing condition for consumption goods becomes $C_t + (K_{t+1} - (1-\delta)K_t)\,\phi_t = \Pi_t$. I set the capital-adjustment parameter $\chi = 0.1$. The value chosen for the capital-adjustment parameter implies that the increase in capital-goods production from increasing investment by 2%, conditional on being in steady state, is about equal to that in Christiano et al. (2014).

APPENDIX D: SENSITIVITY ANALYSIS

Table D.1 shows the differential response of investment to a marginal rise in risk under optimal contracting, relative to under equity only, conditional on period $t-1$ capital equal to steady-state capital under autarky. The response of investment to a marginal rise in risk is denoted by $\frac{\partial I_t}{\partial \varepsilon^\sigma_t}$ in the case of optimal contracting and $\frac{\partial I^{eq}_t}{\partial \varepsilon^\sigma_t}$ in the case of equity only. The differential response of investment is shown conditional on $\sigma_t \in \{0.2, 0.35, 0.5\}$. The first row shows the differential response using the baseline parametrization. In each of the other rows, the differential response is shown when one parameter differs from the baseline parametrization. For additional discussion, see Section 6.

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TABLE D.1. Sensitivity analysis

Parameters						$\frac{\partial I_t}{\partial \varepsilon_t^\sigma} - \frac{\partial I_t^{eq}}{\partial \varepsilon_t^\sigma}$		
γ	ε	θ	μ_σ	ρ_σ	σ_σ	$\sigma_t = 0.2$	$\sigma_t = 0.35$	$\sigma_t = 0.5$
2.5	2	0.4	0.26	0.97	0.07	−0.108	−0.043	0.021
2						−0.103	−0.054	0.002
3						−0.110	−0.029	0.039
	1.5					−0.068	−0.016	0.021
	2.5					−0.138	−0.072	0.016
		0.35				−0.105	−0.014	0.047
		0.45				−0.104	−0.066	−0.005
			0.24			−0.118	−0.057	0.010
			0.28			−0.099	−0.032	0.028
				0.965		−0.103	−0.047	0.009
				0.975		−0.113	−0.037	0.035
					0.06	−0.114	−0.049	0.017
					0.08	−0.102	−0.038	0.022