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On equivalence of infinitary formulas
under the stable model semantics

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Abstract

Propositional formulas that are equivalent in intuitionistic logic, or in its extension known as the logic of here-and-there, have the same stable models. We extend this theorem to propositional formulas with infinitely long conjunctions and disjunctions and show how to apply this generalization to proving properties of aggregates in answer set programming.

KEYWORDS: answer set programming, semantics of aggregates, intuitionistic logic, strong equivalence

1 Introduction

This paper is about the extension of the stable model semantics to infinitary propositional formulas defined by Truszczynski (2012). This extension, introduced originally as a tool for proving a theorem about the logic FO(ID), has been used also to prove a new generalization of Fages' theorem (Lifschitz and Yang 2012).

One of the reasons why stable models of infinitary formulas are important is that they are closely related to aggregates in answer set programming (ASP). The semantics of aggregates proposed by Ferraris (2005, Section 4.1) treats a ground aggregate as shorthand for a propositional formula. An aggregate with variables has to be grounded before that semantics can be applied to it. For instance, to explain the precise meaning of the expression $1\{p(X)\}$ ("there exists at least one object with the property p") in the body of an ASP rule, we first rewrite it as

$$1\{p(t_1), \ldots, p(t_n)\},$$

where $t_1, \ldots, t_n$ are all ground terms in the language of the program, and then turn it into the propositional formula

$$p(t_1) \vee \cdots \vee p(t_n). \tag{1}$$

But this description of the meaning of $1\{p(X)\}$ implicitly assumes that the Herbrand universe of the program is finite. If the program contains function symbols, then an infinite disjunction has to be used instead of (1).

There is nothing exotic or non-computable about ASP programs containing both aggregates and function symbols. For instance, the program

$$p(f(a))$$
$$q \leftarrow 1\{p(X)\}$$

has simple intuitive meaning, and its stable model $\{p(f(a)), q\}$ can be computed by Version 3 of the answer set solver CLINGO.[1] More generally, stable models of infinitary propositional formulas in the sense of Truszczynski (2012) can be used to define the semantics of aggregates in the input language of CLINGO (Harrison 2013); this is our main motivation for studying their properties.

Remark 1
Attempts to define the semantics of aggregates for other ASP languages encounter similar difficulties if the Herbrand universe is infinite. For instance, the definition of a ground instance of a rule in Section 2.2 of the ASP Core document (`https://www.mat.unical.it /aspcomp2013/files/ASP-CORE-2.0.pdf`, Version 2.02) talks about replacing the expression $\{e_1; \ldots; e_n\}$ in a rule with a set denoted by inst($\{e_1; \ldots; e_n\}$). But that set can be infinite and then it cannot be included in a rule.

Our goal here is to develop methods for proving that pairs F, G of infinitary formulas have the same stable models. From the results of Pearce (1997) and Ferraris (2005) we know that in the case of grounded logic programs in the sense of Gelfond and Lifschitz (1988) and, more generally, sets of finite propositional formulas, it is sufficient to check that the equivalence $F \leftrightarrow G$ is provable intuitionistically. Some extensions of intuitionistic propositional logic, including the logic of here-and-there, can be used as well. In this paper, we extend these results to deductive systems of infinitary propositional logic.

This goal is closely related to the idea of strong equivalence (Lifschitz *et al.*, 2001). The provability of $F \leftrightarrow G$ in the deductive systems of infinitary logic described below guarantees not only that F and G have the same stable models but also that for any set $\mathscr{H}$ of infinitary formulas, $\mathscr{H} \cup \{F\}$ and $\mathscr{H} \cup \{G\}$ have the same stable models.

We review the stable model semantics of infinitary propositional formulas in Section 2. Then we define a basic infinitary system of natural deduction, similar to propositional intuitionistic logic (Section 3), and study its properties (Section 4). The main theorem is stated and proved in Section 5, and applied to examples involving aggregates in Section 6. A useful extension of the basic system is discussed in Section 7.

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[1] `http://potassco.sourceforge.net`.

2 Stable models of infinitary propositional formulas

The definitions of infinitary formulas and their stable models given below are equivalent to the definitions proposed by Truszczynski (2012).

Let σ be a propositional signature, that is, a set of propositional atoms. The sets $\mathscr{F}_0^\sigma, \mathscr{F}_1^\sigma, \ldots$ are defined as follows:

- $\mathscr{F}_0^\sigma = \sigma$,
- $\mathscr{F}_{i+1}^\sigma$ is obtained from $\mathscr{F}_i^\sigma$ by adding expressions $\mathscr{H}^\vee$ and $\mathscr{H}^\wedge$ for all subsets $\mathscr{H}$ of $\mathscr{F}_i^\sigma$, and expressions $F \to G$ for all $F, G \in \mathscr{F}_i^\sigma$.

The elements of $\bigcup_{i=0}^\infty \mathscr{F}_i^\sigma$ are called *(infinitary) formulas* over σ.

Remark 2

This definition differs from the syntax introduced in early work on infinitary propositional formulas (Scott and Tarski 1958; Karp 1964) in several ways. It treats the collection $\mathscr{H}$ of conjunctive or disjunctive terms as a set, rather than a family indexed by ordinals. Thus, there is no order among conjunctive or disjunctive terms in this framework, and there can be no repetitions among them. More importantly, there is no restriction here on the cardinality of the set of conjunctive or disjunctive terms. On the other hand, in the hierarchy $\mathscr{F}_i^\sigma$ of sets of formulas, i is a natural number; transfinite levels are not allowed.

A set $\mathscr{H}$ of formulas is *bounded* if it is contained in one of the sets $\mathscr{F}_i^\sigma$. For a bounded set $\mathscr{H}$ of formulas, $\mathscr{H}^\wedge$ and $\mathscr{H}^\vee$ are infinitary formulas.

The symbol $\bot$ will be understood as an abbreviation for $\emptyset^\vee$; $\neg F$ stands for $F \to \bot$, and $F \leftrightarrow G$ stands for $(F \to G) \wedge (G \to F)$.

We will write $\{F, G\}^\wedge$ as $F \wedge G$, and $\{F, G\}^\vee$ as $F \vee G$. This convention allows us to view finite propositional formulas over σ as a special case of infinitary formulas. For any bounded family $\{F_\alpha\}_{\alpha \in A}$ of formulas, we denote the formula $\{F_\alpha : \alpha \in A\}^\wedge$ by $\bigwedge_{\alpha \in A} F_\alpha$, and similarly for disjunctions.

Subsets of a signature σ will be also called its *interpretations*. The satisfaction relation between an interpretation I and a formula F is defined as follows:

- For every $p \in \sigma$, $I \models p$ if $p \in I$.
- $I \models \mathscr{H}^\wedge$ if for every formula $F \in \mathscr{H}$, $I \models F$.
- $I \models \mathscr{H}^\vee$ if there is a formula $F \in \mathscr{H}$ such that $I \models F$.
- $I \models F \to G$ if $I \not\models F$ or $I \models G$.

We say that I *satisfies* a set $\mathscr{H}$ of formulas if I satisfies all elements of $\mathscr{H}$. Two sets of formulas are *equivalent* to each other if they are satisfied by the same interpretations. A formula F is *tautological* if it is satisfied by all interpretations.

The *reduct* F^I of a formula F with respect to an interpretation I is defined as follows:

- For $p \in \sigma$, $p^I = \bot$ if $I \not\models p$; otherwise $p^I = p$.
- $(\mathscr{H}^\wedge)^I = \{G^I \mid G \in \mathscr{H}\}^\wedge$.
- $(\mathscr{H}^\vee)^I = \{G^I \mid G \in \mathscr{H}\}^\vee$.
- $(G \to H)^I = \bot$ if $I \not\models G \to H$; otherwise $(G \to H)^I = G^I \to H^I$.

The *reduct* $\mathscr{H}^I$ of a set $\mathscr{H}$ of formulas is the set consisting of the reducts of the elements of $\mathscr{H}$. An interpretation I is a *stable model* of a set $\mathscr{H}$ of formulas if it is minimal w.r.t. set inclusion among the interpretations satisfying $\mathscr{H}^I$; a stable model of a formula F is a stable model of singleton $\{F\}$. This is a straightforward extension of the definition of a stable model due to Ferraris (2005) to infinitary formulas.

It is easy to see that $I \models F^I$ iff $I \models F$. It follows that every stable model of $\mathscr{H}$ satisfies $\mathscr{H}$.

3 Basic infinitary system of natural deduction

Inference rules of the deductive system described below are similar to the standard natural deduction rules of propositional logic.[2] Its derivable objects are *(infinitary) sequents* – expressions of the form $\Gamma \Rightarrow F$, where F is an infinitary formula, and Γ is a *finite* set of infinitary formulas ("F under assumptions Γ"). To simplify notation, we will write Γ as a list. We will identify a sequent of the form $\Rightarrow F$ with the formula F.

There is one axiom schema $F \Rightarrow F$. The inference rules are the introduction and elimination rules for the propositional connectives

$$(\wedge I) \; \frac{\Gamma \Rightarrow H \quad \text{for all } H \in \mathscr{H}}{\Gamma \Rightarrow \mathscr{H}^\wedge} \qquad (\wedge E) \; \frac{\Gamma \Rightarrow \mathscr{H}^\wedge}{\Gamma \Rightarrow H} \quad (H \in \mathscr{H})$$

$$(\vee I) \; \frac{\Gamma \Rightarrow H}{\Gamma \Rightarrow \mathscr{H}^\vee} \quad (H \in \mathscr{H}) \qquad (\vee E) \; \frac{\Gamma \Rightarrow \mathscr{H}^\vee \quad \Delta, H \Rightarrow F \quad \text{for all } H \in \mathscr{H}}{\Gamma, \Delta \Rightarrow F}$$

$$(\to I) \; \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \to G} \qquad (\to E) \; \frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow F \to G}{\Gamma, \Delta \Rightarrow G},$$

where $\mathscr{H}$ is a bounded set of formulas, and the weakening rule

$$(W) \; \frac{\Gamma \Rightarrow F}{\Gamma, \Delta \Rightarrow F}.$$

Remark 3

The usual conjunction introduction rule is

$$\frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow G}{\Gamma, \Delta \Rightarrow F \wedge G};$$

the corresponding infinitary rule above is similar to the more restrictive version:

$$\frac{\Gamma \Rightarrow F \quad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \wedge G}.$$

In the presence of the weakening rule (W), the two versions are equivalent to each other. The situation with disjunction elimination is similar. The usual contradiction rule

$$(C) \; \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow F}$$

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[2] See, for instance, Lifschitz *et al.* (2008, Section 1.2.1).

is a special case of $(\vee E)$. We do not include the law of the excluded middle in the set of axioms so that this deductive system is similar to intuitionistic rather than classical, propositional logic.

The set of *theorems of the basic system* is the smallest set of sequents that includes the axioms of the system and is closed under the application of its inference rules. We say that formulas F and G are *equivalent in the basic system* if $F \leftrightarrow G$ is a theorem of the basic system. The reason why we are interested in this relation is that formulas equivalent in the basic system have the same stable models, as discussed in Section 5.

Example 1
Let $\{F_i\}_{i \in \mathbb{N}}$ be a bounded family of formulas. We will check that the formula

$$F_0 \wedge \bigwedge_{i \geqslant 0} (F_i \to F_{i+1}) \tag{2}$$

is equivalent in the basic system to the formula $\bigwedge_{i \geqslant 0} F_i$. The sequent

$$F_0 \wedge \bigwedge_{i \geqslant 0} (F_i \to F_{i+1}) \Rightarrow F_0 \wedge \bigwedge_{i \geqslant 0} (F_i \to F_{i+1})$$

belongs to the set of theorems of the basic system. Consequently, so do the sequents

$$F_0 \wedge \bigwedge_{i \geqslant 0} (F_i \to F_{i+1}) \Rightarrow F_0$$

and

$$F_0 \wedge \bigwedge_{i \geqslant 0} (F_i \to F_{i+1}) \Rightarrow F_j \to F_{j+1}$$

for all $j \geqslant 0$. Consequently the sequents

$$F_0 \wedge \bigwedge_{i \geqslant 0} (F_i \to F_{i+1}) \Rightarrow F_j$$

for all $j \geqslant 0$ belong to the set of theorems as well (by induction on j). Consequently, so does the sequent

$$F_0 \wedge \bigwedge_{i \geqslant 0} (F_i \to F_{i+1}) \Rightarrow \bigwedge_{i \geqslant 0} F_i.$$

A similar argument (except that induction is not needed) shows that the sequent

$$\bigwedge_{i \geqslant 0} F_i \Rightarrow F_0 \wedge \bigwedge_{i \geqslant 0} (F_i \to F_{i+1})$$

is a theorem of the basic system also. Consequently, so is the sequent

$$\Rightarrow F_0 \wedge \bigwedge_{i \geqslant 0} (F_i \to F_{i+1}) \leftrightarrow \bigwedge_{i \geqslant 0} F_i.$$

This argument could be expressed more concisely, without explicit references to the set of theorems of the basic system, as follows. Assume (2). Then F_0 and, for every $i \geqslant 0$, $F_i \to F_{i+1}$. Then, by induction, F_i for every i. And so forth. This style of presentation is used in the next example.

Example 2

Let $\{F_\alpha\}_{\alpha \in A}$ be a bounded family of formulas, and let G be a formula. Let us show that

$$\left(\bigvee_{\alpha \in A} F_\alpha \right) \to G \tag{3}$$

is equivalent in the basic system to the formula

$$\bigwedge_{\alpha \in A} (F_\alpha \to G). \tag{4}$$

Left-to-right: Assume (3) and F_α. Then $\bigvee_{\alpha \in A} F_\alpha$, and consequently G. Thus, we established $F_\alpha \to G$ under assumption (3) alone for every α, and consequently established (4) under this assumption as well. Right-to-left: Assume (4) and $\bigvee_{\alpha \in A} F_\alpha$, and consider the cases corresponding to the disjunctive terms of this disjunction. Assume F_α. From (4), $F_\alpha \to G$, and consequently G. Thus, we established G in each case so that (3) follows from (4) alone.

It is easy to see that the infinitary counterparts of the intuitionistically provable De Morgan's laws

$$\bigvee_{F \in \mathscr{H}} \neg F \to \neg \bigwedge_{F \in \mathscr{H}} F \tag{5}$$

and

$$\bigwedge_{F \in \mathscr{H}} \neg F \leftrightarrow \neg \bigvee_{F \in \mathscr{H}} F, \tag{6}$$

where $\mathscr{H}$ is a bounded set of formulas, are theorems of the basic system. So are the infinitary distributivity laws

$$\left(\bigvee_{\{F_i\}_{i \in I}} \bigwedge_{i \in I} F_i \right) \to \left(\bigwedge_{i \in I} \bigvee_{F \in \mathscr{H}_i} F \right) \tag{7}$$

and

$$\left(\bigvee_{i \in I} \bigwedge_{F \in \mathscr{H}_i} F \right) \to \left(\bigwedge_{\{F_i\}_{i \in I}} \bigvee_{i \in I} F_i \right) \tag{8}$$

for every non-empty family $\{\mathscr{H}_i\}_{i \in I}$ of sets of formulas such that its union is bounded. The disjunction in the antecedent of (7) and the conjunction in the consequent of (8) extend over all elements $\{F_i\}_{i \in I}$ of the Cartesian product of the family $\{\mathscr{H}_i\}_{i \in I}$. In Section 7, we discuss an extension of the basic system in which we postulate the converses of implications (5), (7), and (8).

4 Properties of the basic system

The following property of the basic system is easy to verify.

Proposition 1
If a sequent consisting of finite formulas is intuitionistically provable, then it is a theorem of the basic system.

Recall that we define the set of theorems of the basic system to be the smallest set of formulas that includes the axioms and is closed under the inference rules. When we want to prove that every theorem of the basic system has a certain property P, it is clearly sufficient to check that every axiom has the property P, and that the set of sequents that have the property P is closed under the application of the inference rules. In this way we can establish, in particular, the following fact.

Proposition 2
For any theorem $\Gamma \Rightarrow F$ of the basic system, the formula $\Gamma^\wedge \to F$ is tautological.

Remark 4
The assertion of Proposition 2 will remain true even if we extend the set of axioms to include the law of the excluded middle

$$F \vee \neg F. \tag{9}$$

The converse is not true, however, even in the presence of this axiom schema. This fact can be established by standard methods used to prove incompleteness in infinitary logic, which utilize the Downward Löwenheim–Skolem theorem and the Mostowski Collapsing lemma.[3] We can make the system complete by postulating the following infinitary version of the law of the excluded middle:

$$\bigvee_{J \subseteq I} \left(\bigwedge_{j \in J} F_j \wedge \bigwedge_{j \in I \setminus J} \neg F_j \right), \tag{10}$$

for any non-empty bounded family $\{F_i\}_{i \in I}$ of formulas.[4]

Let σ and σ' be disjoint signatures. In this section, a *substitution* is a bounded family of formulas over σ with index set σ'. For any substitution ϕ and any formula F over the signature $\sigma \cup \sigma'$, ϕF stands for the formula over σ formed as follows:

- If $F \in \sigma$ then $\phi F = F$.
- If $F \in \sigma'$ then $\phi F = \phi_F$.
- If F is $\mathscr{H}^\wedge$ then $\phi F = \{\phi G \mid G \in \mathscr{H}\}^\wedge$.
- If F is $\mathscr{H}^\vee$ then $\phi F = \{\phi G \mid G \in \mathscr{H}\}^\vee$.
- If F is $G \to H$ then $\phi F = \phi G \to \phi H$.

Formulas of the form ϕF will be called *instances* of F.

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[3] John Schlipf, personal communication.
[4] The proof of this fact is similar to the proof of completeness of classical propositional logic due to Kalmár (1935). For any interpretation I, let L_I denote the conjunction of the corresponding set of literals. It is easy to check by induction that for any formula F, $L_I \to F$ is a theorem of the basic system if I satisfies F, and $L_I \to \neg F$ is a theorem of the basic system otherwise. The completeness of the basic system with (10) added as an axiom schema easily follows.

Proposition 3

If F is a theorem of the basic system, then every instance of F is a theorem of the basic system also.

Proof

The notation ϕF extends to sequents in a natural way. The property "ϕS is a theorem of the basic system" holds for every axiom S of the basic system, and it is preserved by all inference rules. $\quad\square$

We will refer to Proposition 3 as the *substitution property* of the basic system.

Example 3

We will show that for any formulas F, G, the formula $\neg(F \vee G)$ is equivalent to $\neg F \wedge \neg G$ in the basic system. Note first that the formula

$$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q \tag{11}$$

is intuitionistically provable. By Proposition 1, it follows that it is a theorem of the basic system. The equivalence

$$\neg(F \vee G) \leftrightarrow \neg F \wedge \neg G$$

is an instance of (11): take $\phi_p = F$, $\phi_q = G$. By the substitution property, it follows that it is a theorem of the basic system as well.

Proposition 4

For any substitutions ϕ, ψ with the same index set, the implication

$$\bigwedge_p (\phi_p \leftrightarrow \psi_p) \rightarrow (\phi F \leftrightarrow \psi F)$$

(where p ranges over the indices) is a theorem of the basic system.

Proof

The proof is by induction on j such that $F \in \mathscr{F}_j^{\sigma \cup \sigma'}$, and it considers several cases, depending on the syntactic form of F. Assume, for instance, that F is $\mathscr{H}^\vee$. Then

$$\phi F = \{\phi G \mid G \in \mathscr{H}\}^\vee, \quad \psi F = \{\psi G \mid G \in \mathscr{H}\}^\vee.$$

By the induction hypothesis, for each G in $\mathscr{H}$, the implication

$$\bigwedge_p (\phi_p \leftrightarrow \psi_p) \rightarrow (\phi G \leftrightarrow \psi G) \tag{12}$$

is a theorem of the basic system. We need to show that

$$\bigwedge_p (\phi_p \leftrightarrow \psi_p) \rightarrow \left(\{\phi G \mid G \in \mathscr{H}\}^\vee \leftrightarrow \{\psi G \mid G \in \mathscr{H}\}^\vee\right) \tag{13}$$

is a theorem of the basic system also. Assume

$$\bigwedge_p (\phi_p \leftrightarrow \psi_p) \tag{14}$$

and $\{\phi G \mid G \in \mathscr{H}\}^\vee$, and consider the cases corresponding to the terms of this disjunction. Assume ϕG. Then, by (12) and (14), ψG. We can conclude $\{\psi G \mid G \in \mathscr{H}\}^\vee$,

that is, ψF. So we established the implication $\phi F \to \psi F$. The implication in the other direction is proved in a similar way. $\square$

Corollary 1
If for every index p, ϕ_p is equivalent to ψ_p in the basic system, then ϕF is equivalent to ψF in the basic system.

We will refer to this corollary as the replacement property of the basic system.

Example 4
The formula

$$\bigwedge_{k \geqslant 1} (p_k \to \neg p_k) \to p_0 \tag{15}$$

is equivalent to

$$\bigwedge_{k \geqslant 1} \neg p_k \to p_0 \tag{16}$$

in the basic system because (16) can be obtained from (15) by replacing $p_k \to \neg p_k$ with the intuitionistically equivalent $\neg p_k$. More formally, let q_k ($k \geqslant 1$) be the indices and let F be $\bigwedge_{k \geqslant 1} q_k \to p_0$. For the substitutions

$$\phi_{q_k} = p_k \to \neg p_k, \quad \psi_{q_k} = \neg p_k,$$

ϕF is (15), and ψF is (16). By the replacement property, (15) is equivalent to (16).

5 Relation of the basic system to stable models

Main Theorem
For any set $\mathscr{H}$ of formulas,

  (a) if a formula F is a theorem of the basic system, then $\mathscr{H} \cup \{F\}$ has the same stable models as $\mathscr{H}$;
  (b) if F is equivalent to G in the basic system, then $\mathscr{H} \cup \{F\}$ and $\mathscr{H} \cup \{G\}$ have the same stable models.

Lemma 1
For any formula F and interpretation I, if I does not satisfy F then $F^I \Rightarrow \bot$ is a theorem of the basic system.

  The proof is straightforward by induction on i such that $F \in \mathscr{F}_i^\sigma$.

  By Γ^I we denote the set $\{G^I \mid G \in \Gamma\}$; $(\Gamma \Rightarrow F)^I$ stands for $\Gamma^I \Rightarrow F^I$.

Lemma 2
For any sequent S and any interpretation I, if S is a theorem of the basic system, so is S^I.

Proof

Consider the property of sequents: "S^I is a theorem of the basic system." To prove the lemma, it suffices to show that all theorems of the basic system have that property. It is clear that the reduct of every axiom of the basic system is a theorem (of the basic system). Verifying that the set of sequents with that property is closed under inference rules follows the same pattern for all inference rules but those involving implication. Consider, for instance, disjunction elimination:

$$\frac{\Gamma \Rightarrow \mathcal{H}^\vee \qquad \Delta, H \Rightarrow F \quad \text{for all } H \in \mathcal{H}}{\Gamma, \Delta \Rightarrow F} \tag{17}$$

and assume that the reducts of all sequents that are premises of that rule are theorems. Because $(\mathcal{H}^\vee)^I$ is $(\mathcal{H}^I)^\vee$, all premises of the disjunction elimination rule:

$$\frac{\Gamma^I \Rightarrow (\mathcal{H}^I)^\vee \qquad \Delta^I, H^I \Rightarrow F^I \quad \text{for all } H \in \mathcal{H}}{\Gamma^I, \Delta^I \Rightarrow F^I}$$

are theorems. Therefore, so is the sequent $\Gamma^I, \Delta^I \Rightarrow F^I$ and consequently, also the sequent $(\Gamma, \Delta \Rightarrow F)^I$.

Consider now the implication introduction rule:

$$\frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \to G}$$

and assume that the reduct $(\Gamma, F \Rightarrow G)^I$ is a theorem. To show that $(\Gamma \Rightarrow F \to G)^I$ is a theorem, it suffices to show that $\Gamma^I \Rightarrow (F \to G)^I$ is a theorem.

Case 1: I satisfies Γ. Since the sequent $(\Gamma, F \Rightarrow G)^I$ is a theorem, so is the sequent $\Gamma^I, F^I \Rightarrow G^I$. Thus, $\Gamma^I \Rightarrow F^I \to G^I$ is a theorem and so $(\Gamma^I)^\wedge \to (F^I \to G^I)$ is tautological. Since I satisfies Γ, the comment at the end of Section 2 implies that I satisfies Γ^I. Consequently, I satisfies $F^I \to G^I$ and, by the same comment again, also $F \to G$. It follows that $(F \to G)^I$ is $F^I \to G^I$. Since the sequent $(\Gamma, F \Rightarrow G)^I$ or, equivalently, the sequent $\Gamma^I, F^I \Rightarrow G^I$ is a theorem, applying the rule

$$\frac{\Gamma^I, F^I \Rightarrow G^I}{\Gamma^I \Rightarrow F^I \to G^I}$$

we obtain that $\Gamma^I \Rightarrow F^I \to G^I$ is a theorem. Thus, $\Gamma^I \Rightarrow (F \to G)^I$ is a theorem too.

Case 2: I does not satisfy Γ. Then I does not satisfy one of the elements H of Γ. By Lemma 1, $H^I \Rightarrow \perp$ is a theorem, and $\Gamma^I \Rightarrow (F \to G)^I$ can be derived from $H^I \Rightarrow \perp$ by rules (C) and (W). Thus, it is a theorem.

Next, consider the implication elimination rule:

$$\frac{\Gamma \Rightarrow F \qquad \Delta \Rightarrow F \to G}{\Gamma, \Delta \Rightarrow G}$$

and assume that the sequents $(\Gamma \Rightarrow F)^I$ and $(\Delta \Rightarrow F \to G)^I$ are theorems. We will show that $(\Gamma, \Delta \Rightarrow G)^I$, or equivalently $\Gamma^I, \Delta^I \Rightarrow G^I$ is a theorem too.

Case 1: I satisfies $F \to G$. Then $(F \to G)^I$ is $F^I \to G^I$. Thus, the sequents $\Gamma^I \Rightarrow F^I$ and $\Delta^I \Rightarrow F^I \to G^I$ are theorems, and the claim follows by applying the rule

$$\frac{\Gamma^I \Rightarrow F^I \qquad \Delta^I \Rightarrow F^I \to G^I}{\Gamma^I, \Delta^I \Rightarrow G^I}.$$

Case 2: I does not satisfy $F \to G$. Then $(F \to G)^I$ is $\bot$ and so $\Delta^I \Rightarrow \bot$ is a theorem. Moreover, $\Gamma^I, \Delta^I \Rightarrow G^I$ can be derived from $\Delta^I \Rightarrow \bot$ by rules (C) and (W). Thus, $\Gamma^I, \Delta^I \Rightarrow G^I$ is a theorem too.   □

Proof of Main Theorem

(a) Assume that F is a theorem of the basic system. By Lemma 2, for any interpretation I, F^I is a theorem of the basic system, and consequently is tautological, by Proposition 2. It follows that $\mathcal{H}^I$ and $(\mathcal{H} \cup F)^I$ are satisfied by the same interpretations.

(b) Assume that F is equivalent to G in the basic system, that is, $F \leftrightarrow G$ is a theorem of the basic system. By Lemma 2, for every interpretation I, $(F \leftrightarrow G)^I$ is a theorem of the basic system. Moreover, by Proposition 2, $F \leftrightarrow G$ is tautological. Thus, $(F \leftrightarrow G)^I = F^I \leftrightarrow G^I$ and so $F^I \leftrightarrow G^I$ is a theorem of the basic system. Consequently, $F^I \leftrightarrow G^I$ is tautological, that is, F^I and G^I are equivalent. It follows that $(\mathcal{H} \cup F)^I$ and $(\mathcal{H} \cup G)^I$ are satisfied by the same interpretations.   □

6 Examples involving aggregates

As discussed in the Introduction, infinitary formulas can be used to precisely define the semantics of aggregates in ASP when the Herbrand universe is infinite. In this section, we give two examples demonstrating how the theory described above can be applied to prove equivalences between programs involving aggregates.

Example 5

Intuitively, the rule

$$q(X) \leftarrow 1\{p(X, Y)\} \tag{18}$$

has the same meaning as the rule

$$q(X) \leftarrow p(X, Y). \tag{19}$$

To make this claim precise, consider first the result of grounding rule (18) under the assumption that the Herbrand universe C is finite. In accordance with the standard practice in ASP, we treat variable X as global and variable Y as local. Then the result of grounding (18) is the set of ground rules

$$q(a) \leftarrow 1\{p(a, b) \mid b \in C\}$$

for all $a \in C$. In the spirit of the semantics for aggregates proposed by Ferraris (2005, Section 4.1) these rules have the same meaning as the propositional formulas

$$\left(\bigvee_{b \in C} p(a, b) \right) \to q(a). \tag{20}$$

Likewise, rule (19) can be viewed as shorthand for the set of formulas

$$p(a, b) \to q(a) \tag{21}$$

for all $a, b \in C$. It easy to see that these sets of formulas are intuitionistically equivalent.

How can we lift the assumption that the Herbrand universe is finite? We can treat (20) as an infinitary formula, and show that the conjunction of formulas (20) is equivalent to the conjunction of formulas (21) in the basic system. The fact that the conjunction of formulas (21) for all $b \in C$ is equivalent to (20) in the basic system follows from Example 2 (Section 3).

Example 6
Intuitively,

$$q(X) \leftarrow 2\{p(X, Y)\} \tag{22}$$

has the same meaning as the rule

$$q(X) \leftarrow p(X, Y1), \ p(X, Y2), \ Y1 \neq Y2. \tag{23}$$

To make this claim precise, consider the infinitary formulas corresponding to (22):

$$\left(\bigvee_{b \in C} p(a, b) \wedge \bigwedge_{b \in C} \left(p(a, b) \rightarrow \bigvee_{\substack{c \in C \\ c \neq b}} p(a, c) \right) \right) \rightarrow q(a) \tag{24}$$

($a \in C$); see Ferraris (2005, Section 4.1) for details on representing aggregates with propositional formulas. The formulas corresponding to (23) are

$$(p(a, b) \wedge p(a, c)) \rightarrow q(a) \tag{25}$$

($a, b, c \in C, \ b \neq c$). We will show that the conjunction of formulas (24) is equivalent to the conjunction of formulas (25) in the basic system.

It is sufficient to check that for every $a \in C$, (24) is equivalent to the conjunction of formulas (25) over all $b, c \in C$ such that $b \neq c$. By Example 2, this conjunction is intuitionistically equivalent to

$$\left(\bigvee_{\substack{b, c \in C \\ b \neq c}} (p(a, b) \wedge p(a, c)) \right) \rightarrow q(a). \tag{26}$$

By the replacement property of infinitary formulas, it suffices to check that the antecedents of (24) and (26) are equivalent to each other.

Left-to-right: Assume

$$\bigvee_{b \in C} p(a, b) \wedge \bigwedge_{b \in C} \left(p(a, b) \rightarrow \bigvee_{\substack{c \in C \\ c \neq b}} p(a, c) \right). \tag{27}$$

Then $\bigvee_{b \in C} p(a, b)$. We will reason by cases, with one case corresponding to each possible value b_0 of b. Case $p(a, b_0)$: By the second conjunctive term of (27),

$$p(a, b_0) \rightarrow \bigvee_{\substack{c \in C \\ c \neq b_0}} p(a, c).$$

Then the consequent of this implication follows. Again, we will reason by cases, with one case for each value c_0 of c where $c_0 \neq b_0$. Case $p(a, c_0)$: then $p(a, b_0) \wedge p(a, c_0)$.

Consequently,

$$\bigvee_{\substack{b,c \in C \\ b \neq c}} p(a,b) \wedge p(a,c). \tag{28}$$

Right-to-left: Assume (28). We reason by cases, with one case for each pair b_0, c_0, where $b_0 \neq c_0$. Case $p(a,b_0) \wedge p(a,c_0)$: from $p(a,b_0)$ we derive the first conjunctive term of (27); from $p(a,c_0)$ we derive

$$\bigvee_{\substack{c \in C, \\ c \neq b}} p(a,c),$$

and consequently the implication

$$p(a,b) \to \bigvee_{\substack{c \in C \\ c \neq b}} p(a,c).$$

The conjunction of these implications for all $b \in C$ is the second conjunctive term of (27).

7 The extended system of natural deduction

In this section, we show that the assertion of the main theorem will remain true if we extend the basic system by adding the axiom schema

$$F \vee (F \to G) \vee \neg G \tag{29}$$

characterizing (in the finite case) the logic of here-and-there (Hosoi 1966), and the converses to the implications discussed at the end of Section 3:

$$\neg \bigwedge_{F \in \mathscr{H}} F \to \bigvee_{F \in \mathscr{H}} \neg F, \tag{30}$$

$$\left(\bigwedge_{i \in I} \bigvee_{F \in \mathscr{H}_i} F \right) \to \left(\bigvee_{\{F_i\}_{i \in I}} \bigwedge_{i \in I} F_i \right), \tag{31}$$

and

$$\left(\bigwedge_{\{F_i\}_{i \in I}} \bigvee_{i \in I} F_i \right) \to \left(\bigvee_{i \in I} \bigwedge_{F \in \mathscr{H}_i} F \right). \tag{32}$$

When all conjunctions and disjunctions are finite, formula (30) can be derived intuitionistically from (29), and (31) and (32) are intuitionistically provable. We do not know to what extent the additional axiom schemas postulated here are independent when infinite conjunctions and disjunctions are allowed.

In the extended system, we can derive the theorem

$$\bigvee_{J \subseteq I} \left(\neg \bigvee_{j \in I \setminus J} F_j \wedge \neg\neg \bigwedge_{j \in J} F_j \right) \tag{33}$$

for any non-empty bounded family $\{F_i\}_{i \in I}$ of formulas. (This is a generalization of the weak law of the excluded middle $\neg F \vee \neg\neg F$ to sets of infinitary formulas, similar

to the generalization of the law of the excluded middle given in Remark 2. It is equivalent in the basic system to the special case of (33) corresponding to a family with a single element.) Indeed

$$\bigwedge_{i \in I} (\neg F_i \vee \neg\neg F_i)$$

is a theorem of the extended system because $\neg F_i \wedge \neg\neg F_i$ can be intuitionistically derived from (29) with F_i as F and $\neg F_i$ as G. Using (31) we obtain

$$\bigvee_{J \subseteq I} \left(\bigwedge_{j \in I \setminus J} \neg F_j \wedge \bigwedge_{j \in J} \neg\neg F_j \right),$$

and (33) follows by De Morgan's laws.

In the extended system, we can also derive the theorem

$$\left(F \to \bigvee_{i \in I} G_i \right) \to \bigvee_{i \in I} (F \to G_i) \tag{34}$$

for any formula F and non-empty family $\{G_i\}_{i \in I}$ of formulas. We use instantiations of (29) for all G_i to obtain

$$\bigwedge_{i \in I} F \vee (F \to G_i) \vee \neg G_i.$$

By (31) we obtain

$$\bigvee_{\{F_i\}_{i \in I}} \bigwedge_{i \in I} F_i \tag{35}$$

where the disjunction extends over all elements $\{F_i\}_{i \in I}$ of the Cartesian product of the family $\{F, F \to G_i, \neg G_i\}_{i \in I}$. We reason by cases, with one case corresponding to each disjunctive term $\bigwedge_{i \in I} F_i$ of (35). If at least one of the formulas F_i is F then from the antecedent of (34) we can derive $\bigvee_{i \in I} G_i$, and the consequent of (34) follows immediately. If at least one of the formulas F_i is $F \to G_i$ then the consequent of (34) is immediate as well. Otherwise, $\bigwedge_{i \in I} F_i$ is $\bigwedge_{i \in I} \neg G_i$. Then from the antecedent of (34) we can derive $\neg F$ and every disjunctive term of the consequent follows.

It is easy to check that the properties of the basic system proved in Section 4 hold for the extended system as well.

To show that the assertion of the main theorem applies to the extended system, we will prove the modification of Lemma 2 stated below. The *classical extended system* is obtained from the extended system by replacing the axiom schema (29) with the law of the excluded middle (9).

Lemma 3
For any sequent S and any interpretation I, if S is a theorem of the extended system, then S^I is a theorem of the classical extended system.

Proof
It suffices to show that every theorem S of the extended has this property: "S^I is a theorem of the classical extended system." We only need to check that the reducts of axioms (29)–(32) have this property; the fact that the set of sequents with that

property is closed under the inference rules is checked in the same way as in the proof of Lemma 2.

Let S be (29). Then S^I is

$$F^I \vee (F \to G)^I \vee (\neg G)^I.$$

If $I \models G$, then the second disjunctive term is $F^I \to G^I$, and the disjunction can be derived from $F^I \vee \neg F^I$. If $I \not\models G$, then the third disjunctive term is equivalent to $\neg\bot$.

Let S be (30). Since S is tautological, S^I is

$$\left(\neg \bigwedge_{F \in \mathcal{H}} F \right)^I \to \bigvee_{F \in \mathcal{H}} (\neg F)^I.$$

If I satisfies the conjunction in the antecedent, then the antecedent is $\bot$. Otherwise, at least one disjunctive term in the consequent is equivalent to $\neg\bot$.

Let S be (31). Since S is tautological, S^I is

$$\left(\bigwedge_{i \in I} \bigvee_{F \in \mathcal{H}_i} F^I \right) \to \left(\bigvee_{\{F_i\}_{i \in I}} \bigwedge_{i \in I} F_i^I \right),$$

which is an instantiation of the same axiom schema. The reasoning for sequents of the form (32) is similar.   □

Main Theorem for the Extended System
For any set $\mathcal{H}$ of formulas,

  (a) if a formula F is a theorem of the extended system, then $\mathcal{H} \cup \{F\}$ has the same stable models as $\mathcal{H}$;
  (b) if F is equivalent to G in the extended system, then $\mathcal{H} \cup \{F\}$ and $\mathcal{H} \cup \{G\}$ have the same stable models.

This assertion is derived from Lemma 3 in the same way as the Main Theorem was derived from Lemma 2, using the fact that all theorems of the classical extended system are tautological.

Example 7
Intuitively, the cardinality constraint $\{p(X)\}0$ ("the set of true atoms with form $p(X)$ has cardinality at most 0") has the same meaning as the conditional literal $\bot : p(X)$ ("for all X, $p(X)$ is false"). If we represent this conditional literal by the infinitary formula

$$\bigwedge_{a \in C} \neg p(a) \tag{36}$$

then this claim can be made precise by showing that (36) is equivalent in the extended system to the infinitary formula corresponding to $\{p(X)\}0$ in the sense of Ferraris (2005):

$$\bigwedge_{\substack{A \subseteq C \\ A \neq \emptyset}} \left(\bigwedge_{a \in A} p(a) \to \bigvee_{a \in C \setminus A} p(a) \right) \tag{37}$$

(where C is the Herbrand universe).

It is easy to derive (37) from (36) in the basic system. The derivation of (36) from (37) will use the following instance of (33):

$$\bigvee_{A \subseteq C} \left(\neg \bigvee_{a \in C \setminus A} p(a) \wedge \neg\neg \bigwedge_{a \in A} p(a) \right). \tag{38}$$

We will reason by cases, with one case corresponding to each disjunctive term D_A in (38). In the case that A is empty, (36) follows from the first conjunctive term of D_A by De Morgan's law. Otherwise, assume $\bigwedge_{a \in A} p(a)$. Then by (37), $\bigvee_{a \in C \setminus A} p(a)$, which contradicts the first conjunctive term of D_A. We conclude $\neg \bigwedge_{a \in A} p(a)$, which contradicts the second conjunctive term of D_A. So the assumptions D_A and (37) are contradictory. Consequently, they imply (36).

8 Conclusions

Two finite propositional formulas are strongly equivalent if and only if they are equivalent in the logic of here-and-there (Ferraris 2005, Proposition 2). The results of this paper show that, at least in one direction, an analogous statement holds for infinitary propositional formulas: If two infinitary propositional formulas are equivalent in the logic of here-and-there then they are strongly equivalent. We don't know how to extend the other direction of Ferraris's theorem to infinitary formulas. It is not clear whether or not any axioms or inference rules not included in the extended system will be required. However, as we illustrated with several examples, the results in this paper allow us to verify the equivalence of formulas involving aggregates.

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