Note on the pedal locus. By Mr J. P. GABBATT, Peterhouse.

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While the paper "On the pedal locus in non-euclidean hyperspace"* was in the press, Professor H. F. Baker kindly directed the writer's attention to a reference† from which it appeared that the euclidean case had first been studied by Beltrami‡. After publication, it was discovered that the main subject of Beltrami's paper was the non-euclidean case§. He proves, by analytical methods, theorems which may be stated as follows: Let $A_0, A_1, \ldots, A_n$ denote the vertices of a simplex, $[A]$, in non-euclidean space of n dimensions, and P a point such that the orthogonal projections of P on the walls of $[A]$ lie in a flat, p; then the locus of P is an $(n-1)$-fold, W, of order $n+1$, which is anallagmatic for the isogonal transformation q. $[A]$; the isogonal conjugate of P is the absolute pole of p, and the envelope, w, of p is therefore the absolute reciprocal of W. Beltrami does not note the theorem, fundamental for the geometrical treatment of the subject, that W is the Jacobian of a certain group of $n+1$ point-hyperspheres. He goes on to show that in the euclidean case the locus W breaks up into an n-ic $(n-1)$-fold and the flat at infinity, while the envelope w does not, in general, break up. Finally, he notes that in two dimensions there is also a special non-euclidean case, in which W breaks up into an order-conic and a line, and w into a class-conic and a point; and that the appropriate condition is fundamentally that which is necessary for the degeneration of a certain class-conic into a pair of points. "But what are the points? And what is the corresponding condition satisfied by the absolute conic? This is a question which it would be interesting to resolve." It does not appear that the subject has been pursued further; and in the present note an attempt is made to discuss fully the analogous case of degeneracy in n dimensions.

Let $[A]$ denote a simplex in non-euclidean space of n dimensions; let A_i denote a vertex of $[A]$, and a_i that wall of $[A]$ which does not contain A_i; let Ω denote the Absolute, B_i, b_i the absolute pole, polar of a_i, A_i respectively, and $[B]$ the simplex determined by the points B_i. Now (for each of the n possible values of j) the point B_j and the join of a_i, a_j determine a flat. If B_i' denote the meet of the n flats so determined, then (2·22) B_i' is the isogonal conjugate, q. $[A]$, of B_i. Let $[B']$ denote the simplex determined

* Gabbatt, *Proc. Camb. Phil. Soc.* XXI (1923), 763–771. References in the text, thus: (2·22), are to this paper.

† *Encyk. d. math. Wiss.* III, C 7, 806–7, footnote 121.

‡ *Mem. Acc. Bologna* (3) VII (1876), 241–262 = *Op. mat.* III, 53–72.

§ See *Encyk. d. math. Wiss.* III, C 7, 962.

by the points B_i', and b_i' that wall of $[B']$ which does not contain B_i'; then (2·31) there is a quadric, Ω', the *Secondary* q. $[A]$, such that $[A]$, $[B']$ are polar q. Ω' ($i = 0, 1, \ldots, n$).

Let the locus, W, of a point, P, which is such that the orthogonal projections of P on the walls of $[A]$ lie in a flat, p, be termed the *pedal locus*, and the envelope of p the *pedal envelope*, q. $[A]$*. Then (3·11, 3·22) W is the Jacobian of the point-hyperspheres determined by the $n+1$ points B_i, and (3·1) contains those points and the points B_i'. The locus W also contains all the $(n-2)$-folds, $a_i a_j$, of $[A]$, and the $n+1$ joins, $a_i b_i$, of corresponding walls of $[A]$, $[B]$.

If the $n+1$ lines $A_i B_i$ meet at a point, C, so that $[A]$ is orthocentric, and C the orthocentre of $[A]$; then the $n+1$ linear $(n-2)$-folds $a_i b_i$ lie in a flat, c, the *orthaxial* of $[A]$; the $n+1$ lines $A_i B_i'$ meet (2·4) at a point, C', the isogonal conjugate, q. $[A]$, of C; and the flat b_i' contains $a_i b_i$ ($i = 0, 1, \ldots, n$). The locus W is now (5·13) the Hessian of a cubic $(n-1)$-fold, U†; isogonally conjugate points of W are conjugate poles q. U; and (1·2, 3·12, 5) c is the mixed polar flat, q. U, of any other pair of isogonally conjugate points. We shall need three further theorems on this case (the *orthocentric case*).

First, it is clear from the specification of B_0' that the polar flat of C q. the n-ple of flats (a_0^{n-1}, b_0) is the flat, d_0, determined by B_0' and $a_0 b_0$. Now the isogonal transform, c_0' (q. $[A]$), of the flat b_0' is an n-ic $(n-1)$-fold containing each of the points $A_1, A_2, \ldots, A_n$ $(n-1)$-ply and each of the points $B_1, B_2, \ldots, B_n$ simply; thus d_0 is the polar flat of C q. c_0'. Hence, and similarly: *If the simplex* $[A]$ *be orthocentric, and if* C *denote the orthocentre of* $[A]$, *and* c_i' *the isogonal transform, q.* $[A]$, *of the flat* b_i'; *then the polar flat, q.* c_i', *of* C *is the flat determined by the point* B_i' *and the join of the flats* a_i, b_i ($i = 0, 1, \ldots, n$). (I)

Again, the polar, q. Ω', of the line $A_i B_i'$ is the $(n-2)$-fold $a_i b_i'$, which in the present (orthocentric) case is $a_i b_i$. Moreover, if S denote the polar quadric, q. U, of C, then (5·12) the polar flat, q. S, of A_i contains $a_i b_i$; and since C, C' are isogonal conjugates, therefore the polar flat, q. S, of C' is the orthaxial, c, which also contains $a_i b_i$. Thus $A_i B_i'$, $a_i b_i$ are polar q. both the quadrics Ω', S ($i = 0, 1, \ldots, n$); whence: *In the orthocentric case, if* S *denote the polar quadric of the orthocentre q. the cubic* $(n-1)$*-fold of which the pedal locus is the Hessian; then* S *and the Secondary,* Ω', *touch at all their common points, and those points are in the orthaxial.* (II)

* Beltrami terms W, w respectively the *hypersteinerian* locus and envelope q. $[A]$. This term is used in a different sense by Brambilla, *Rend. Acc. Napoli*, XXXVIII (1899), 144–5.

† Bauer, *Sitz. Akad. München*, XVIII (1888), 337–354, proves this theorem for three dimensions; see Jessop, *Quartic Surfaces*, Cambridge (1916), 189–190.

Lastly, let Σ' denote the common section, by the orthaxial, c, of the quadrics S, Ω'. Then since S, which is the polar quadric, q. U, of C, contains every point, M, of Σ'; therefore the polar flat, m, of M contains C. Moreover, since M is a point of c, therefore m touches the polo-n-ic, c' (q. U), of c; also c' is the isogonal transform, q. $[A]$, of c; and the point of contact of m, c' is the isogonal conjugate of M. Thus: *If* Σ' *denote the section of the Secondary by the orthaxial, and* c' *the isogonal transform, q.* $[A]$, *of the orthaxial; then the isogonal transform of* Σ' *is the locus of the points of contact of the tangent flats to* c' *from the orthocentre.* (III)

We now proceed to examine the assumption that, in non-euclidean space, a flat, g, can form part of the locus W. Remembering that the section of W by a wall, a_i, of $[A]$ consists of $n+1$ linear $(n-2)$-folds (viz. the joins of a_i with the remaining walls of $[A]$, and with the corresponding wall, b_i, of $[B]$), and having regard to the section of the walls of $[A]$ by the assumed flat, g; we see that g must contain *either* (i) at least two of the joins $a_i a_j$ *or* (ii) at least n of the joins $a_i b_i$. In case (i), g must be a wall, a_i, of $[A]$; this occurs when a_i either touches the Absolute, or is the absolute polar of A_i. The case is not analogous to that of Beltrami, and will not be further discussed.

In case (ii), if the Absolute be non-degenerate, then at least n of the lines $A_i B_i$ meet at a point. The $n+1$ lines $A_i B_i$ therefore meet at a point*; the simplex $[A]$ is orthocentric; and the flat g is the orthaxial, c. Now if c form part of the locus W, then the remainder of the locus is the isogonal transform, c', of c. Also W contains all the points B_i and their isogonal conjugates B_i'. Thus *either* (α) c contains all the points B_i *or* (β) c contains at least one of the points B_i'. In case (α), the Absolute is a flat quadric in c; this is the euclidean case.

We now consider case (β). Since, whenever $[A]$ is orthocentric, the flat determined by B_0' and $a_i b_i$ (which lies in the orthaxial c) contains all the remaining vertices of $[B']$ except B_i'; therefore if c contain (e.g.) B_0', then c also contains all the n points B_i' $(i=1, 2, \ldots, n)$. The quadric Ω' then degenerates into a flat quadric in c, and is therefore identical with Σ' (see III); also B_i' is the pole, q. Σ', of $a_i b_i$. Thus we have the following: *In non-euclidean space of n dimensions, the pedal locus, q. a simplex* $[A]$, *includes a flat if* $[A]$ *is orthocentric, and one of the points* B_i' *lies in the orthaxial, c, of* $[A]$. *Then all the* $n+1$ *points* B_i' *lie in c; the* $n+1$ *points* B_i *lie in the isogonal transform,* c', *(q.* $[A]$*) of c; the pedal locus consists of c and* c'*; and the pedal envelope consists of the orthocentre, C, of* $[A]$ *and the absolute reciprocal of* c'. (IV)

Referring to the case of theorem IV as *Beltrami's case*, we have from (I): *In Beltrami's case, the polar flat, q.* c', *of C is c.* (V)

* See Schläfli, *J. f. Math.* LXV (1866), 189–197.

Also from (II, III): *In Beltrami's case, the* $n+1$ *points* B_i' *are the poles (q. a flat quadric,* Σ'*, in c) of the* $n+1$ *joins of corresponding walls of* $[A]$ *and the absolute polar simplex,* $[B]$*; and* Σ' *is the isogonal transform, q.* $[A]$*, of the locus of the points of contact of the tangent flats from* C *to* c'*.* (VI)

It has not yet been formally proved that, in the orthocentric case, incidence of one of the points B_i' on the orthaxial is possible. It may therefore be noted that, if the restriction that the quadric Ω is the Absolute be removed, and if the terms required (perpendicular, isogonal conjugate, etc.) be defined projectively q. Ω, then (2·33) the substitution of Ω' for Ω results in the interchange of the simplices $[B]$, $[B']$ and of the quadrics Ω, Ω'; and the isogonal transformation, q. $[A]$, remains unaltered. Moreover (2·42) the simplex $[A]$ is either orthocentric q. both the quadrics Ω, Ω' or q. neither; and (5·14) in the orthocentric case (but only in that case) the pedal locus q. $[A]$ is unaffected by the substitution. Beltrami's case is therefore projectively identical with the orthocentric case in euclidean geometry; the apparent difference (as regards degeneracy) in the behaviour of the pedal envelope in the two cases being due to the circumstance that, in the former case, that one of the two quadrics Ω, Ω' which is non-degenerate is regarded as Absolute, while in the latter, that which is degenerate is so regarded.

It is interesting to remark that the euclidean expression of theorem V is as follows: If c' denote the pedal (n-ic) locus, q. an orthocentric simplex in euclidean space of n dimensions, and C' the isogonal conjugate of the orthocentre; then the polar flat of C' q. c' is the flat at infinity. Now since any line parallel to an edge of the simplex meets c' $(n-2)$-ply at infinity, it follows that: *In euclidean space of* n *dimensions, if* $[A]$ *denote any orthocentric simplex, and* C' *the isogonal conjugate, q.* $[A]$*, of the orthocentre: then any line containing* C' *and parallel to an edge of* $[A]$ *meets the pedal locus (q.* $[A]$*) at two actual points only; and the segment determined by the two points is bisected at* C'***. In a euclidean plane, this reduces to the theorem that the circumcentre of a triangle is the isogonal conjugate of the orthocentre. (VII)

In the actual case discussed by Beltrami, the theorems IV–VI take the following form: *If* A_0, A_1, A_2 *denote the vertices of a non-euclidean plane triangle,* $[A]$*, and* B_0, B_1, B_2 *the corresponding vertices of the absolute polar triangle,* $[B]$*, and if the six points* A_i, B_i *be on a conic,* c'*; then the six sides* a_i, b_i *of* $[A]$, $[B]$ *touch a conic,* K, *the absolute reciprocal of* c'*. The pedal locus, q.* $[A]$*, breaks up into the conic* c' *and the orthaxis,* c*, of* $[A]$*; the pedal envelope breaks up into the conic* K *and the orthocentre,* C*, of* $[A]$*;* c' *is the isogonal transform, q.* $[A]$*, of* c*; and* c *is the polar, q.* c'*, of* C*. If* B_i' *denote*

* Cf. Gabbatt, *Proc. Lond. Math. Soc.* (2), xxiv (1925), 173.

the isogonal conjugate, q. [A], of B_i, and C_i the meet of a_i, b_i; then the two points B_i', C_i are conjugates in an involution range, on c, of which the double points, D, D', are the meets of c, c', and constitute the isogonally conjugate point-pair on c ($i = 0, 1, 2$)*. (VIII)

If the conditions of theorem VIII be satisfied, and if any line containing C meet c' at the points X, Y; then it is easy to show that $A_i\,(A_jA_kXY) \barwedge B_i\,(B_jB_kYX)$. In particular, the line XY may be the absolute polar of B_i'; and in that case it may readily be proved that the four lines A_iX, A_iY, B_iX, B_iY all touch the Absolute. The relation $A_i\,(A_jA_kXY) \barwedge B_i\,(B_jB_kYX)$ then expresses the congruence of one of the supplementary angles a_ja_k with one of the supplementary angles b_jb_k ($i, j, k = 0, 1, 2$). Thus: *In a non-euclidean plane, if $[\bar{A}]$ denote any triangle of which the angles are, independently, either congruent or supplementary to the angles of the triangle [A]; then the congruence of one of the triangles $[\bar{A}]$ with the absolute polar triangle of [A] is a necessary condition for degeneracy (of the type considered) of the pedal locus.* It may be shown† that the congruence, for a *single* pair of values of j, k, of one of the supplementary angles a_ja_k with one of the supplementary angles b_jb_k is in general a sufficient condition for degeneracy. (IX)

In the special case of theorem VIII, the point-pair D, D' replaces the class-conic, Ω', q. which, in general, the triangles $A_0A_1A_2$, $B_0'B_1'B_2'$ are mutually polar. It is to be observed that Ω' is not the conic of which the degeneracy in the same case is noted by Beltrami. If the Absolute be expressed by the line-equation

$$\Omega \equiv (\alpha_{00}, \ldots, \ldots, \alpha_{12}, \ldots, \ldots \between \xi_0, \xi_1, \xi_2)^2 = 0,$$

then

$$\Omega' \equiv (\alpha_{00}^{-1}, \ldots, \ldots, \alpha_{12}^{-1}, \ldots, \ldots \between \alpha_{00}\xi_0, \alpha_{11}\xi_1, \alpha_{22}\xi_2)^2 = 0,$$

whereas Beltrami's class-conic is expressed by the equation

$$(\alpha_{00}^{-1}, \ldots, \ldots, \alpha_{12}^{-1}, \ldots, \ldots \between \xi_0, \xi_1, \xi_2)^2 = 0;$$

the condition for the case of theorem VIII being the vanishing of the determinant

$$\begin{vmatrix} \alpha_{00}^{-1} & \alpha_{01}^{-1} & \alpha_{02}^{-1} \\ \alpha_{10}^{-1} & \alpha_{11}^{-1} & \alpha_{12}^{-1} \\ \alpha_{20}^{-1} & \alpha_{21}^{-1} & \alpha_{22}^{-1} \end{vmatrix} \quad (\alpha_{ij} = \alpha_{ji}).$$

* Cf. Neuberg, *Mathesis*, (3) VIII (1908), 159–160; Juhel-Rénoy, *ibid.* 257–258.
† E.g. from Gabbatt, *Proc. Camb. Phil. Soc.* XXI (1923), 297–362, §§ 9·12, 15·1.