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Addendum to 'Periodic orbits and dynamical spectra'

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Abstract. We survey some recent progress in the theory of dynamical zeta functions and explain its implications for counting problems.

0. *The results*

One particularly elegant aspect of dynamical zeta functions, particularly in the context of hyperbolic flows, is the analogy with the Riemann zeta function in number theory.

Following different earlier definitions by Selberg and Smale (1967), Ruelle (1976b) proposed a formal definition of a dynamical zeta function for such a flow of the following form

$$\zeta^*(s) = \prod_\tau (1 - e^{-sh \cdot l(\tau)})^{-1}, \quad s \in \mathbb{C} \tag{0.1}$$

where τ denotes a closed orbit of least period $l(\tau)$ (and the extra factor $h > 0$, denoting the topological entropy of the flow, has been introduced for our convenience. Cf. Baladi (1998), equation (2.14)). This definition should be compared with that of the more familiar Riemann zeta function

$$\zeta(s) = \prod_{n=1}^{\infty} (1 - p_n^{-s})^{-1}, \quad s \in \mathbb{C} \tag{0.2}$$

where $\{p_n\}_{n=1}^{\infty} = \{2, 3, 5, 7, 11, \ldots\}$ is the enumeration of the prime numbers.

In the case of both zeta functions, the formal products converge to analytic functions in the region $\mathrm{Re}(s) > 1$ and have extensions with simple poles at $s = 1$. However, this correspondence remains a formal one and there are no examples of hyperbolic flows for which $\zeta^*(s) = \zeta(s)$. The Riemann zeta function is an object of profound study in prime number theory and its analytic features hold the key to many important results on the distribution of prime numbers. Probably the best known statement is the following.

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PRIME NUMBER THEOREM. *$\zeta(s)$ has no zeros on the line* $\mathrm{Re}(s) = 1$. *Equivalently, the counting function* $N(x) = \mathrm{Card}\{p_n \leq x\}$ *is asymptotic to* $x/\log x$ *(i.e.* $\lim_{x\to+\infty} N(x)/(x/\log x) = 1$*).*

Undoubtedly the best known conjecture is the following.

RIEMANN HYPOTHESIS (OR CONJECTURE). *$\zeta(s)$ has no zeros in the half-plane* $\mathrm{Re}(s) > 1/2$*: equivalently, for any* $1/2 < \theta < 1$ *we can write* $\pi(x) = \mathrm{li}(x)(1 + O(x^\theta))$ *(i.e.* $\limsup_{x\to+\infty} |\pi(x) - \mathrm{li}(x)|/x^\theta < +\infty$*).*

We recall that $li(x) = \int_2^\infty du/\log u$ is asymptotic to $x/\log x$ as $x \to +\infty$.

The correspondence between the analytic properties of $\zeta(s)$ and the estimates of $N(x)$ is based on the following simple identity for the related quantity $\psi_1(T) = \int_1^T \psi(x)\,dx$, where $\psi(T) = \sum_{p_n^k \leq T} \log p_n$: for any $c > 1$

$$\psi_1(T) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{T^{s+1}}{s(s+1)}\,ds. \tag{0.3}$$

To estimate this quantity one moves the curve of integration past the line $\mathrm{Re}(s) = 1$ to a curve Γ in the pole free region for $(\zeta'/\zeta)(s)$. The pole at $s = 1$ then contributes the principal term and the error term comes from dominating the remaining integral over Γ. (We shall return to this point, in the context of the dynamical zeta functions, in §2.)

By additional features of the Riemann zeta function $\zeta(s)$ (in particular, the existence of functional equations) the Riemann hypothesis is equivalent to the zeros for $\zeta(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$ lying on the line $\mathrm{Re}(s) = 1/2$.

In order to formulate similar statements for the dynamical zeta function we need to first introduce a condition. We say that a flow is topologically weak-mixing if there are no non-trivial solutions to the identity $F \circ \phi_t = e^{iat}F$, where $F \in C^0(M, \mathbb{C})$. Parry and Pollicott (1983) showed that for a topologically weak-mixing hyperbolic flow $\phi_t : M \to M$ the analogue of the prime number theorem is true (earlier Margulis (1970) obtained this result in the context of Anosov flows). That is, that on the line $\mathrm{Re}(s) = 1$ there are no poles (or zeros) for $\zeta^*(s)$ except for the simple pole at $s = 1$. Using a similar analysis to that in the case of prime number theory, it follows that $\pi(x) = \{\tau : l(\tau) \leq x\}$ is asymptotic to e^{hx}/hx (cf. Baladi (1998), Theorem 2.8; see also Margulis (1969)). If ϕ_t is not weak-mixing, the situation can be reduced to the case of hyperbolic diffeomorphism (see Baladi (1998), Theorem 2.4).

Now we describe the case where an almost optimal result is known. Some partial results are discussed at the end of the paper. Let V denote a compact smooth surface of strictly negative, possibly variable, curvature. Set $M = \{(x, v) \in TV : \|v\| = 1\}$ to be the unit tangent bundle and define the geodesic flow $\phi_t : M \to M$ by $\phi_t(x, v) = (\gamma(t), \dot\gamma(t))$, where $\gamma : \mathbb{R} \to V$ is the unique unit speed geodesic $\gamma : \mathbb{R} \to V$ with $\gamma(0) = x$ and $\dot\gamma(0) = v$. ϕ_t is known to satisfy Axiom A. The following result describes the distribution of poles for the associated dynamical zeta function $\zeta^*(s)$.

THEOREM 1. (Dolgopyat) *Let $\phi_t : M \to M$ be a geodesic flow on a compact surface of negative curvature. There exists $\epsilon > 0$ such that $\zeta^*(s)$ has no zeros or poles in the half-plane* $\mathrm{Re}(s) > 1 - \epsilon$*, except for the simple pole at $s = 1$.*

Previously, it had been shown that $\zeta^*(s)$ has a meromorphic extension to a larger such half-plane $\mathrm{Re}(s) > 1 - \epsilon_0$ (where $0 < \epsilon \leq \epsilon_0$), but without any information on the location of poles (Pollicott 1986).

Note. The above theorem is true under a somewhat more general hypothesis. For example, the following hypotheses on the hyperbolic flow $\phi_t : M \to M$ suffice for the proof:

(1) the strong stable and strong unstable sub-bundles E^u and E^s are C^1;
(2) the splitting is not locally-integrable;
(3) ϕ is weak-mixing;
(4) the measure of maximal entropy m_0 satisfies the *Federer condition*: there is a constant C so that for any x in non-wandering set and any $r > 0$ $m_0(B(x, 2r)) \leq Cm_0(B(x, r))$.

These conditions hold automatically for a geodesic flow for a compact negatively curved surface V. In higher dimensions they can only be verified in very special cases. (See Hirsch and Pugh (1975) concerning hypothesis (1).)

Note. Local non-integrability means that the flow is not a suspension by a locally constant roof function. The necessity of this condition is shown by the following statement.

PROPOSITION 1.
(a) (Ruelle 1983, Pollicott 1985) *If a hyperbolic flow is locally integrable then* $\zeta^*(s)$ *has poles arbitrary close to the line* $\mathrm{Re}(s) = 1$*;*
(b) (Pollicott 1990) *For generic locally integrable hyperbolic flows the error term has polynomial decay.*

1. *The method of extending the zeta function*

There are two principal tools in extending the domain of the dynamical zeta function $\zeta^*(s)$. The first involves replacing the flow by a simplified model. The second involves analyzing the zeta function for this model using transfer operators.

1.1. *The simplified model.*

Given the flow $\phi_t : M \to M$ we can choose a finite number of co-dimension one transverse sections $\mathcal{T}_1, \ldots, \mathcal{T}_N$. Let $\mathcal{T} = \cup_i \mathcal{T}_i$, then we can consider the Poincaré map $P : \mathcal{T} \to \mathcal{T}$ and the return time $f : \mathcal{T} \to \mathbb{R}$ (i.e. $\phi_{f(x)}(x) = P(x)$ for $x \in \mathcal{T}$).

We can introduce a new zeta function $\zeta_0^*(s)$ for the map $P : \mathcal{T} \to \mathcal{T}$ and the function $f : \mathcal{T} \to \mathbb{R}$ which is defined formally by

$$\zeta_0^*(s) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{T^n x = x} \exp\left(-sh \sum_{k=0}^{n-1} f(T^k x) \right) \tag{1.1}$$

(compare with Baladi (1998), equation (2.16)). Although at first sight this appears to be of a different form from the zeta function $\zeta^*(s)$ given in (0.2), they are intimately related. In particular, a periodic orbit $\{x, Px, \ldots, P^{n-1}x\}$ gives rise to a closed orbit τ of period $l(\tau) = f(x) + f(Px) + \cdots + f(P^{n-1}x)$. By also arranging for these sections to have an appropriate Markov property we have that $\zeta^*(s) = \zeta_0^*(s)\eta(s)$, where $\eta(s)$ is analytic for $\mathrm{Re}(s) > 1 - \epsilon_2$, for some $\epsilon_2 > 0$ (cf. Bowen (1973) §5). The function $\eta(s)$

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is a correction for the overcounting of closed orbits which pass through the boundaries of the sections $\mathcal{T}_1, \ldots, \mathcal{T}_N$.

To introduce the transfer operator, we need one further reduction in our model for the flow ϕ. In essence, the additional Markov property of the sections allows us to identify (or 'collapse') each of the sections $\mathcal{T}_i$ along the 'stable direction' (in a way that can be made completely rigorous) and so replace

(a) $\mathcal{T}$ by $X \subset \mathbb{R}^n$ (with dense interior);

(b) $P : T \to T$ by an expanding map $T : X \to X$;

(c) $f : T \to \mathbb{R}^+$ by a continuous function $r : X \to \mathbb{R}^+$.

Here n is the dimension of the 'unstable direction', which in the present example is one. Moreover, in this case we can identify X with a finite disjoint union of intervals.

These reductions do not effect the zeta function $\zeta_0^*(s)$.

1.2. *Transfer operators.* The transfer operator (associated to complex number s) $\mathcal{L}_{e^{-sr}} : C^0(X, \mathbb{C}) \to C^0(X, \mathbb{C})$ is defined by

$$\mathcal{L}_{e^{-sr}} k(x) = \sum_{Ty=x} e^{-sr(y)} k(y)$$

where $k \in C^0(X, \mathbb{C})$ (cf. Baladi (1998), equation (2.3)). For the proof of Theorem 1, we want to consider the operator acting on the smaller space of C^1 functions, i.e. $\mathcal{L}_{e^{-sr}} : C^1(X, \mathbb{C}) \to C^1(X, \mathbb{C})$. The appropriate norm on the Banach space $C^1(X, \mathbb{R})$ is $\|k\|_1 = \|k\|_\infty + \|Dk\|_\infty$, where $\|\cdot\|_\infty$ denotes the usual supremum norm and Dk denotes the derivative of $k \in C^1(X, \mathbb{C})$.

A very simple estimate on the spectral radius $\rho(\mathcal{L}_{e^{-sr}})$ of $\mathcal{L}_{e^{-sr}}$ is that $\rho(\mathcal{L}_{e^{-sr}}) \le e^{P(-\sigma r)}$, where $s = \sigma + it$ and $P(-\sigma r)$ denotes the pressure of the function $-\sigma r : X \to \mathbb{R}$ relative to $T : X \to X$ (cf. Baladi (1998), Theorem 2.1). In particular, when $\sigma = h$ then $P(-hr) = 0$ and $\rho(\mathcal{L}_{e^{-sr}}) \le 1$. The following result gives stronger estimates on this spectral radius when $t \neq 0$.

PROPOSITION 2. (Dolgopyat 1996a) *There exists $C > 0$, $0 < \theta < 1$, $D > 1$ and $\epsilon > 0$ such that*

$$\|\mathcal{L}^n_{e^{-sr}}\|_1 \le C \cdot e^{mP(-\sigma r)} \cdot \theta^l$$

where $s = \sigma + it$, with $|t| \ge 2$ and $\sigma > h - \epsilon$, and $n = l[D \log|t|] + m$

In particular, we have the following elegant estimate.

COROLLARY 2.1. *For $s = \sigma + it$, with $\sigma > h - \epsilon$ and $|t| \ge 2$, we have that* $\rho(\mathcal{L}_{e^{-sr}}) \le \theta < 1$.

The proof of Proposition 2 involves a number of steps. One of the more familiar ingredients is the following simple inequality

$$\|D(\mathcal{L}_{e^{-sr}} k)\|_\infty \le C \cdot |t| \cdot \|k\|_\infty + \theta_0^n \|Dk\|_\infty \tag{1.2}$$

for some $C > 0$ and $0 < \theta_0 < 1$ and all $k \in C^1(X, \mathbb{C})$ (where we assume for convenience that $\mathcal{L}_{-\sigma r} 1 = 1$, by the simple device of modifying r up to the addition of a coboundary $u \circ T - u$ and a constant).

The most important step is to show that there exists $C_1 > 0$ and $0 < \theta_1 < 1$ such that given any function $k \in C^1(X, \mathbb{R})$ such that $\|k\|_0 = 1$ and $\|Dk\|_0 \leq 2C|t|$ then we can bound

$$\int |\mathcal{L}^n_{e^{-sr}} k| \, d\mu \leq C_1 \cdot \theta_1^n, \quad \forall n \geq 0 \tag{1.3}$$

where μ is a convenient measure. (In fact, μ is simply the Gibbs measure associated to the potential $-\sigma r : X \to \mathbb{R}$.)

This L^1 convergence is then converted into uniform convergence using the following identity

$$|\mathcal{L}^{2n}_{e^{-sr}} k(x)| \leq \left\| \mathcal{L}^n_{-\sigma r} \left| \mathcal{L}^n_{e^{-sr}} k \right| \right\|_\infty = \int \left| L^n_{e^{-sr}} k \right| \, d\mu + O(\theta_2^{2n} |t|), \tag{1.4}$$

for some $0 < \theta_2 < 1$, where the last inequality follows from the well-known convergence estimate $L^n_{-\sigma r} k = \int k \, d\mu + O(\theta_2^{2n} \|k\|_1)$, say, for the transfer operator with a real weighting (cf. Baladi (1998), Theorem 3.2(1)).

Comparing (1.3) and (1.4) we see that $\|L^n_{e^{-sr}} k\|_0 = O(\theta_1^n, \theta_2^n |t|)$.

Finally, by substituting $L^n_{e^{-sr}} k$ for k in (1.2) we can bound

$$\|L^{2n} k\| = O(\theta_0^n, |t|\theta_1^n, |t|\theta^n),$$

where all of the implied constants are independent of k and depend only on $C > 0$ in (1.3). In the event that $\|k\|_0 = 1$ and $\|Dk\|_0 > 2C$, then norm contraction follows directly from (1.2). In either case, the estimate in Proposition 2 can be easily deduced.

Note. The hypothesis that the sub-bundles E^s and E^u are C^1 manifests itself in the functions T and r being C^1 and allows us to work in the Banach space $C^1(X, \mathbb{R})$. Although the proof is 'symbolic' in essence, the differential structure is important in the details of the proofs. The hypothesis that the splitting is not integrable is crucial to the proof of (1.3). If we knew that for each $x \in X$ there exists a continuous choice of two distinct pre-images $y_1 = y_1(x)$ and $y_2 = y_2(x)$ with the property that $x \mapsto y_1(x) - y_2(x)$ has a gradient large enough in comparison with $\|r\|_1$ and $\|\sigma^{-1}\|_1$ then we would essentially be able to 'integrate by parts' the expression $|\mathcal{L}^n_{e^{-sr}} k|^2(x)$ (and use the Hölder inequality) to obtain (1.3). For geodesic flows on negatively curved surfaces the last assumption follows from the contact structure of the flow (that is horocycles are orthogonal to geodesics) (cf. Plante 1972). In general, one can show (cf. Sinai 1972) that local non-integrability is equivalent to the existence of $x_0 \in X$, $\delta > 0$ such that for all large enough n there are two branches $y_{n,1}(x)$ and $y_{n,2}(x)$ of σ^{-n} so that

$$\left| \nabla_x (r_n(y_{n_1}) - r_n(y_{n,2})) \right| (x_0) > \delta$$

which still suffices for the proof.

1.3. *Applying the transfer operators.* It only remains to use the spectral estimate on the transfer operators in Proposition 2 to deduce the analytic extension in Theorem 1.

Let $X = \cup_i X_i$ be the partition (into intervals) corresponding to the union $\mathcal{T} = \cup_i \mathcal{T}_i$. The key result relating the transfer operators to the zeta function $\zeta_0^*(s)$ as follows.

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PROPOSITION 3. (Ruelle 1990) *There exists* $0 < \theta < 1$ *such that for any fixed choice of points* $x_i \in X_i$ *uniformly as* s *varies over a compact set*

$$\sum_{T^n x = x} \exp\bigg(- s \sum_{j=0}^{n-1} r(T^j x)\bigg) = \sum_{i=1}^{N} (\mathcal{L}^n_{e^{-sr}}(\chi_{X_i}))(x_i) + O(\theta^n), \quad n \geq 0 \tag{1.5}$$

where χ_{X_i} *denotes the characteristic function for* X_i.

Comparing (1.5) and Theorem 1 we see that the series in (1.1) is uniformly convergent for $s = \sigma + it$ satisfying $\sigma > 1 - \epsilon$ and $|t| > 2$. In particular, we see that $\zeta_0^*(s)$ is analytic in this domain and we can make the same deduction for $\zeta^*(s)$, completing the outline of the proof of Theorem 1.

2. *Applications and other results*

2.1. *Counting closed orbits.* By analogy with the Riemann hypothesis in number theory, one would expect that Theorem 1 would give rise to error terms in the counting of closed orbits for geodesic flows. The appropriate statement turns out to be as follows.

THEOREM 2. *Let* $\phi_t : M \to M$ *be the geodesic flow on a surface of variable negative curvature. The number of closed orbits is given by* $\pi(x) = \mathrm{li}(e^{hx})(1 + O(e^{-\epsilon_* x}))$, *for some* $\epsilon_* > 0$.

If we define $\psi^*(T) = \sum_{e^{nhl(\gamma)} \leq T} hl(\gamma)$ and $\psi_1^*(T) = \int_1^T \psi^*(x)\, dx$ then for any $c > 1$ we have a formula analogous to (0.3):

$$\psi_1^*(T) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta^{*\prime}(s)}{\zeta^*(s)}\right) \frac{T^{s+1}}{s(s+1)}\, ds.$$

If we move the line of integration to $\mathrm{Re}(s) = 1 - \epsilon^*/2$, say, then we can write

$$\psi_1^*(T) = \frac{T^2}{2} + \frac{1}{2\pi i} \int_{(1-\epsilon^*/2)-i\infty}^{(1-\epsilon^*/2)+i\infty} \left(-\frac{\zeta^{*\prime}(s)}{\zeta^*(s)}\right) \frac{T^{s+1}}{s(s+1)}\, ds \tag{2.1}$$

and the integral in (2.1) grows with a smaller exponent then the principal term $T^2/2$.

The details of Theorem 2 appear in Pollicott and Sharp (1997).

Note. Passing from ψ^* to ψ_1^* depends on our knowledge of the behaviour of $\zeta^{*\prime}/\zeta^*$ for large $\mathrm{Im}(s)$. It follows from Dolgopyat (1996b) that under rather general circumstances $(\zeta^{*\prime}/\zeta^*)(1 + it) = O(|t|^N)$ for some N. In this case one has to convolve ψ^* with a rapidly decreasing function in order to get integrability. However, due to the lack of suitable analytic continuation one is only able to estimate the rate of convergences of ψ^* in the space of distributions (cf. Fried 1986a).

2.2. *Decay of correlations.* The prime orbit theorem is closely related to the problem of correlation decay (or rate of mixing) for hyperbolic flows. We briefly recall the statement. Let m_F be a Gibbs measure for a Hölder potential F on the unit tangent bundle M for the geodesic flow.

Given two smooth functions $A, B : M \to \mathbb{R}$ we denote

$$\rho^F_{A,B}(t) = \int A \circ \phi_t B\, dm_F - \int A\, dm_F \int B\, dm_F, \quad t \in \mathbb{R}.$$

THEOREM 3. (Dolgopyat 1996a) *The correlation function* $\rho^F_{A,B}(t)$ *tends to zero exponentially fast as* $|t| \to +\infty$, *i.e.* $|\rho^F_{A,B}(t)| \le e^{-\epsilon_*|t|}$ *for all* t, *for some* $\epsilon_* > 0$.

(See Baladi (1998) Subsection 2.2 for a list of earlier results.) More generally, one can obtain asymptotic estimates for integrals of the form

$$I_G(D,t) = \int D(\phi_t x, x) \exp\left[\int_x^{\phi_t x} G\right] dx,$$

where D is a function on $M \times M$ and G is a potential. The correlation decay problem corresponds to taking $D(x,y) = A(x)B(y)$, whereas if we take $D = \delta_{\text{diag}}$ we get prime orbit theorems. (Of course, in the later case we have to integrate in the time variable, since the graph of ϕ_t is not transversal to the diagonal and so there is no asymptotic for individual values of t (see Margulis (1970) for details).) Similarly, taking other submanifolds instead of the diagonal, one can obtain other types of counting theorems.

Theorem 3 remains true in the broader setting of Axiom A flows satisfying hypothesis (1)–(4) (with m_0 replaced by m_F).

2.3. *More general zeta functions.* Given a Hölder function $F: M \to \mathbb{R}$ one can weight closed orbits τ for the geodesic flow by the real numbers $\int_0^{l(\tau)} F(\phi_t x_\tau)\,dt$, for any choice $x_\tau \in \tau$. A natural generalization of the zeta function (0.2) which takes account of this weighting is the following:

$$\zeta_F^*(s) = \prod_\tau \left(1 - \exp\left[\int_0^{l(\tau)} (G(\phi_t x_\tau) - s)\,dt\right]\right)^{-1}, \quad s \in \mathbb{C}.$$

This formal product converges to an analytic function in the half-plane $\text{Re}(s) > P(F)$, where $P(F)$ is the topological pressure of the function $F: M \to \mathbb{R}$. The importance of generalized zeta function is clear from the following result. Let

$$\hat{\rho}^F_{A,B}(s) = \int_{\mathbb{R}_+} e^{-st} \rho^F_{A,B}(t)\,dt$$

be the Laplace transform of the correlation function.

PROPOSITION 4. (Pollicott 1985) *There is a number* ϵ *such that* $\hat{\rho}^F_{A,B}(s)/\zeta_F^*(s - P(F))$ *has analytic continuation to* $\text{Re}(s) > -\varepsilon$. *More precisely,* $\hat{\rho}^F_{A,B}$ *is meromorphic in this domain, its only possible poles are the poles of* $\zeta_F^*(s)$ *and the corresponding residues are given by a non-degenerate bilinear form of* A, B.

Note. The similar statement holds for the Laplace transform of $I_G(D,t)$ but the corresponding formulas become more complicated. We refer the reader to Dolgopyat (1997) for details.

The following generalization of Theorem 1 is an immediate consequence of Proposition 4 and Theorem 3.

COROLLARY 4.1. *Given a geodesic flow on a negatively curved surface and a Hölder continuous potential* $F: M \to \mathbb{R}$ *there exists* $\epsilon > 0$ *such that the zeta function* $\zeta_F^*(s)$ *has an analytic extension to the half-plane* $\text{Re}(s) > P(F) - \epsilon$, *except for a simple pole at* $s = P(F)$.

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The value of this ϵ is clearly related to ϵ_* in Theorem 2 and is therefore an important value. However, from the outline of the proof of Theorem 1 one can see that it is difficult to estimate the value ϵ. It is known from the analysis of surfaces of constant negative curvature (using very different techniques) that there are examples for which ϵ_0 may be arbitrarily small. On the other hand, if for some more accessible value $0 < \tilde{\epsilon} < \epsilon_0$ we know that $\zeta_F^*(s)$ has a meromorphic extension to $\mathrm{Re}(s) > P(F) - \tilde{\epsilon}$ with only a finite number of poles, one can still hope that $\rho_{A,B}^F(t)$ can be approximated to a generalized trigonometric polynomial with error $O(e^{-\tilde{\epsilon}t})$. Denote by $\tilde{\varepsilon}$ the supremum of all $\tilde{\epsilon}$ with this property. The following procedure (diagonal approximation) to bound $\tilde{\varepsilon}$ is used in physics literature. Consider

$$|\mathcal{L}_{e^{F-sr}}^n k|^2(x) = \sum_{T^n y_1 = T^n y_2 = x} \exp[(F_n + \sigma r_n)(y_1) + (F_n + \sigma r_n)(y_2) - it(r_n(y_1) - r_n(y_2))] k(y_1)\overline{k(y_2)}.$$

When t is large one can argue that the main contribution comes from the non-oscillatory diagonal term (cf. the note at the end of §§1.2) which suggests the following estimate

$$\tilde{\varepsilon} \approx \bar{\varepsilon} = P(F) - \tfrac{1}{2} P(2F).$$

Unfortunately, there are few (if any) rigorous results about this approximation. We would therefore like to pose the following question.

Problem. Give a formula (or, at least, a reasonable estimate) for $\tilde{\varepsilon}$. In particular, is it true that $\tilde{\varepsilon} = \bar{\varepsilon}$ for geodesic flows on manifolds of constant negative curvature and C^∞ potentials?

2.4. *L-functions.* So far F has been considered to be a real-valued potential, but there is one important case when we have to deal with complex-valued functions. Namely, let $\vec{e}$ be the vectorfield generating our flow and let ω denote a closed form. If we set $F = 2\pi i \omega(\vec{e})$ then the Euler product

$$L(\omega, s) = \prod_\tau \left(1 - \exp\left[2\pi i \left(\int_0^{\lambda(\tau)} \omega(\vec{e})(\phi_t x_\tau) - s\right) dt\right]\right)^{-1}$$

is called a *dynamical L-function*. Clearly, it only depends on the cohomology class of ω so expanding $\omega = \sum_{j=1}^d \theta_j \omega_j$ in an appropriate basis $\omega_1, \ldots, \omega_d \in H^1(M, \mathbb{R})$ we obtain a function of two variables $L(\vec{\theta}, s)$, where we denote $\vec{\theta} = (\theta_1, \ldots, \theta_d)$. Since $L(\vec{\theta}, s)$ is periodic in $\vec{\theta}$ we can view it as a compact family of functions, each analytic in s for $\mathrm{Re}(s) > h$. The methods described above give the following result.

THEOREM 4. *There are constants $R, \epsilon > 0$ such that for any $\vec{\theta} \in \mathbb{R}^d$ $L(\vec{\theta}, s)$ has analytic continuation into* $\mathrm{Re}(s) > h - \epsilon$, $|\mathrm{Im}(s)| > R$.

For a fixed homology class γ let $\pi_\gamma(t)$ be the number of closed orbits of period less than t in this class. Combining Theorem 4 with the analysis of $L(\vec{\theta}, s)$ for small s given by Adachi and Sunada (1987) (cf. also Parry and Pollicott (1986)) we get the following orbit distribution theorem.

THEOREM 5. *Under the conditions of the previous theorem $\pi_\gamma(t)$ has asymptotic series*

$$\pi_\gamma(t) \sim c_d \frac{e^{ht}}{t^{d/2+1}} \left(\sum_{j=0}^{\infty} \frac{c_j}{t^j} \right),$$

where $d = \dim H_1(M)$.

Note. In the constant curvature case, Phillips and Sarnak (1987) gave geometric interpretations of the first few coefficients of this series. It would be nice to do this for variable curvature.

3. *Concluding remarks*

We hope that the technique of twisted transfer operators described here can be useful in some other situations. Of course, there is no hope of obtaining any satisfactory results about either zeta functions or periodic orbit asymptotics without some hyperbolicity assumptions. However, even in the later case there are many open problems. The references to some partial result could be found in the following papers.

(1) Non-uniformly hyperbolic systems: Baladi (1998) §5 and §6, Young (1996).

(2) Non-compact negatively curved manifolds: Pollicott and Sharp (1994), Dolgopyat (1997).

(3) Anosov flows in higher dimensions: Dolgopyat (1996a).

We hope that these cases will be treated in the near future.

REFERENCES

The references given below are additional to those in the survey of Baladi (1998).

Adachi, T. and T. Sunada. Homology of closed geodesics in a negatively curved manifold. *J. Diff. Geom.* **26** (1987), 81–99.

Baladi, V. Periodic orbits and dynamical spectra. *Ergod. Th. & Dynam. Sys.* **18** (1998), 255–292.

Bowen, R. Symbolic dynamics for hyperbolic flows. *Amer. J. Math.* **95** (1973), 429–460.

Dolgopyat, D. On statistical properties of geodesic flows on negatively curved surfaces. *PhD Thesis*, Princeton University, 1997.

Hirsch, M. and C. Pugh. Smoothness of horocycle foliations. *J. Diff. Geom.* **10** (1975), 225–238.

Margulis, G. On some applications of ergodic theory to the study of manifolds of negative curvature. *Funct. Anal. Appl.* **3** (1969), 89–90.

—— On some problems in the theory of U-systems. *PhD Thesis*, Moscow State University, 1970.

Parry, W. and M. Pollicott. The Chebotarev theorem for Galois covering of Axiom A flows. *Ergod. Th. & Dynam. Sys.* **6** (1986), 133–148.

Phillips, R. and P. Sarnak. Geodesics in homology classes. *Duke Math. J.* **55** (1987), 287–297.

Plante, J. Anosov flows. *Amer. J. Math.* **94** (1972), 729–754.

Pollicott, M. A complex Ruelle–Perron–Frobenius theorem and two counterexamples. *Ergod. Th. & Dynam. Sys.* **4** (1984), 135–146.

—— Error terms in 'Prime Orbit Theorems' for locally constant suspended flows. *Quart. J. Math.* **41** (1990), 313–323.

Pollicott, M. and R. Sharp. Orbit counting for some discrete groups acting on simply connected manifolds with negative curvature. *Invent. Math.* **117** (1994), 275–302.

—— Exponential error terms for growth functions on negatively curved surfaces. *Preprint*, 1997.

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