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Reconstruction of a stationary flow from incomplete boundary data using iterative methods

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In this paper, three iterative procedures (Landweber-Fridman, conjugate gradient and minimal error methods) for obtaining a stable solution to the Cauchy problem in slow viscous flows are presented and compared. A section is devoted to the numerical investigations of these algorithms. There, we use the boundary element method together with efficient stopping criteria for ceasing the iteration process in order to obtain stable solutions.

1 Introduction

Much of the literature on the solution of inverse boundary value problems has been devoted to inverse transient heat transfer, Beck *et al.* [3], inverse elasticity, Yakhno [21], inverse steady-state heat conduction, Ingham & Yuan [12], and inverse scattering, Sabatier and Pike [20], whilst research in the field of inverse fluid flow problems has been limited; see for example Zeb *et al.* [23, 24]. In the latter references, emphasis was made on the numerical solution of several inverse problems in Stokes flows of a viscous fluid, but with little theoretical justification. Whilst these inverse formulations are much more difficult to attack from the theoretical point of view, in this paper we consider one classical Cauchy formulation and propose some iterative procedures, for which one can show convergence and stability. Let us just mention that a quite different alternating iterative procedure valid for the Stokes system, but not for the generalized Stokes system, with Cauchy data in fractional Sobolev spaces, has been recently proposed in Bastay *et al.* [2]. However, our methods work with L^2 boundary data (a very desirable feature from both the practical and computational points of view), as well as for the more general generalized Stokes system. Preliminary accepted papers by the authors, Johansson & Lesnic [13, 14], proposed a Landweber-Fridman and a conjugate gradient method, respectively, for the numerical solution of the Cauchy problem for the Stokes equations. In this paper, yet another iterative method, namely the minimal error method, is numerically implemented for the first time and furthermore, all these three iterative methods are compared in terms of the rate of convergence, accuracy and stability.

2 Cauchy problem in slow viscous flow

Consider slow viscous flow occupying a bounded region $D \subset \mathbb{R}^n$, where $n \geqslant 2$, between two infinitely long cylinders in 2-D, or between two spheres in 3-D, having outer boundary

Γ_0 and inner boundary Γ_1 such that $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$ is of class C^2. As a possible practical motivation, one can imagine that the inner wall Γ_1 of a pipe is moving with an unknown velocity and measurements of velocity and traction are taken on the exposed outer wall Γ_0 of the pipe. Alternatively, an inner sphere may rotate and oscillate with an unknown velocity whilst the outer sphere is kept fixed.

In non-dimensional form the generalized Stokes viscous fluid flow equations can be written as

$$\begin{cases} L\boldsymbol{u} - \nabla p = 0 & \text{in } \Omega, \\ \text{div}\boldsymbol{u} = 0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where

$$L\boldsymbol{u} = \left(\Delta u_i + \sum_{j=1}^{n} b_j(\boldsymbol{x}) \frac{\partial u_i}{\partial x_j} \right)_{1 \leqslant i \leqslant n}.$$

Here, $\boldsymbol{u} = (u_1, \ldots, u_n)$ is the fluid velocity, p is the pressure and $\boldsymbol{b}$ is a coefficient function which is assumed of class $C^1(\overline{\Omega})$. Letting $\boldsymbol{v} = (v_1, \ldots, v_n)$ be the outward unit normal to the boundary of Ω, the stress force $\boldsymbol{T}$ can be written as

$$\boldsymbol{T} = p\boldsymbol{v} - N\boldsymbol{u},$$

where

$$N\boldsymbol{u} = \left(\sum_{j=1}^{n} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) v_j \right)_{1 \leqslant i \leqslant n}.$$

In a direct problem formulation, the knowledge of the fluid velocity $\boldsymbol{u}$ and/or the stress force $\boldsymbol{T}$, on the whole boundary, give rise to well-posed boundary value problems [7]. However, in our formulation we only assume that one can prescribe $\boldsymbol{u}$ and $\boldsymbol{T}$ on the boundary part Γ_0. Altogether, we have the following problem to study:

$$\begin{cases} L\boldsymbol{u} - \nabla p = 0 & \text{in } \Omega, \\ \text{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{\varphi} & \text{on } \Gamma_0, \\ p\boldsymbol{v} - N\boldsymbol{u} = \boldsymbol{\psi} & \text{on } \Gamma_0. \end{cases} \tag{2.2}$$

This is termed as the Cauchy problem for slow viscous flow and it is ill-posed since although the uniqueness of the solution is ensured, the solution usually does not (globally) exist, or even if it exists it does not depend continuously on the Cauchy data $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$, see later the compactness of the operator defined in § 4.1.

2.1 Notations and preliminaries

Let $L^2(\Omega)$ be the space of square integrable real-valued functions on Ω with the usual norm. The space $H^k(\Omega)$, where $k = 1, 2, \ldots$, denotes the standard Sobolev space on Ω, i.e.,

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the space of functions with generalized derivatives of order $\leqslant k$ in $L^2(\Omega)$. The dual space of $H^k(\Omega)$ with respect to the L^2-inner product is denoted by $(H^k(\Omega))^*$. We let the product of n samples of a space X be denoted by X^n.

We assume that the function $\boldsymbol{b}$ that appears in the operator L is chosen so that the problem (2.1), supplied with the boundary conditions $\boldsymbol{u} = 0$ on Γ_1 and $p\boldsymbol{v} - N\boldsymbol{u} = 0$ on Γ_0, has only the trivial solution in $H^2(\Omega)^n \times H^1(\Omega)$.

3 A regularizing procedure

Here, we present an iterative procedure for solving the Cauchy problem (2.2). Let us first introduce the problems

$$\begin{cases} L\boldsymbol{u} - \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{\eta} & \text{on } \Gamma_1, \\ p\boldsymbol{v} - N\boldsymbol{u} = \boldsymbol{\psi} & \text{on } \Gamma_0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} L^*\boldsymbol{v} - \nabla q = 0 & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} = 0 & \text{on } \Gamma_1, \\ q\boldsymbol{v} - N^*\boldsymbol{v} = \boldsymbol{\xi} & \text{on } \Gamma_0, \end{cases} \tag{3.2}$$

where

$$L^*\boldsymbol{v} = \left(\Delta v_i - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(b_j(\boldsymbol{x})v_i) \right)_{1 \leqslant i \leqslant n},$$

$$N^*\boldsymbol{v} = \left(\sum_{j=1}^{n} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - b_j(\boldsymbol{x})v_i \right) v_j \right)_{1 \leqslant i \leqslant n}.$$

The procedure is as follows.

- Choose a function $\boldsymbol{\eta}_0 \in L^2(\Gamma_1)^n$. The first approximation $\boldsymbol{u}_0$ and p_0 is obtained by solving (3.1) with $\boldsymbol{u} = \boldsymbol{\eta}_0$ on Γ_1.
- Then we find $\boldsymbol{v}_0$ and q_0 by solving (3.2) with $\boldsymbol{\xi} = \boldsymbol{u}_0 - \boldsymbol{\varphi}$ on Γ_0.
- Having constructed $\boldsymbol{u}_{k-1}$, p_{k-1}, $\boldsymbol{v}_{k-1}$, and q_{k-1}, we obtain $\boldsymbol{u}_k$ and p_k by solving problem (3.1) with $\boldsymbol{u} = \boldsymbol{\eta}_k$ on Γ_1, where

$$\boldsymbol{\eta}_k = \boldsymbol{\eta}_{k-1} + \gamma(q_{k-1}\boldsymbol{v} - N^*\boldsymbol{v}_{k-1})$$

  and γ is a constant.
- Finally, $\boldsymbol{v}_k$ and q_k are obtained by solving (3.2) with $\boldsymbol{\xi} = \boldsymbol{u}_k - \boldsymbol{\varphi}$ on Γ_0.

This procedure converges if the constant γ is chosen in a certain interval, see Theorem 4.1.

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4 Convergence of the procedure given in § 3

4.1 Definition of an operator

Let $\boldsymbol{u}$ and p_1 solve the problem

$$\begin{cases} L\boldsymbol{u} - \nabla p_1 = 0 & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{\eta} & \text{on } \Gamma_1, \\ p_1\boldsymbol{v} - N\boldsymbol{u} = 0 & \text{on } \Gamma_0, \end{cases} \tag{4.1}$$

and let $\boldsymbol{v}$ and p_2 solve

$$\begin{cases} L\boldsymbol{v} - \nabla p_2 = 0 & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} = 0 & \text{on } \Gamma_1, \\ p_2\boldsymbol{v} - N\boldsymbol{v} = \boldsymbol{\psi} & \text{on } \Gamma_0. \end{cases} \tag{4.2}$$

For the problem (4.1) let us introduce an operator $K : L^2(\Gamma_1)^n \to L^2(\Gamma_0)^n$ defined by

$$K\boldsymbol{\eta} = \boldsymbol{u}|_{\Gamma_0} \quad \text{for} \quad \boldsymbol{\eta} \in L^2(\Gamma_1)^n. \tag{4.3}$$

Similarly, for the problem (4.2) we introduce the operator $K_1 : L^2(\Gamma_0)^n \to L^2(\Gamma_0)^n$ defined by

$$K_1\boldsymbol{\psi} = \boldsymbol{v}|_{\Gamma_0} \quad \text{for} \quad \boldsymbol{\psi} \in L^2(\Gamma_0)^n. \tag{4.4}$$

It has been shown elsewhere [13] that these operators are well-defined, linear and bounded, and that K is an injective and compact operator.

Finding a solution to the Cauchy problem (2.2) is then equivalent to finding $\boldsymbol{\eta} \in L^2(\Gamma_1)^n$ such that

$$K\boldsymbol{\eta} = \boldsymbol{\varphi} - K_1\boldsymbol{\psi}. \tag{4.5}$$

If such an $\boldsymbol{\eta}$ exists, then by the construction of the operators K and K_1, we have

$$\boldsymbol{u}|_{\Gamma_0} = \boldsymbol{\varphi} - \boldsymbol{v}|_{\Gamma_0},$$

where $\boldsymbol{u}$ solves (4.1) and $\boldsymbol{v}$ solves (4.2). Hence, $\boldsymbol{u}+\boldsymbol{v}$ is a solution to problem (2.2). Since K is a compact operator this means in particular that K has no bounded inverse. Therefore, equation (4.5) is an ill-posed problem.

4.2 The adjoint operator

The following lemma describes how problem (3.2) can be used to define the action of the adjoint operator K^*.

Lemma 4.1 [13] *Let $\boldsymbol{\xi} \in L^2(\Gamma_0)^n$. The adjoint operator $K^* : L^2(\Gamma_0)^n \to L^2(\Gamma_1)^n$ to the operator K defined in* (4.3) *is given by $K^*\boldsymbol{\xi} = -(q\boldsymbol{v} - N^*\boldsymbol{v})|_{\Gamma_1}$, where $\boldsymbol{v} \in L^2(\Omega)^n$ and $q \in (H^1(\Omega))^*$ solve the problem* (3.2).

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4.3 Theorem of convergence

We conclude with a theorem that the procedure given in § 3 is convergent.

Theorem 4.1 [13] *Let $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ be given in $L^2(\Gamma_0)^n$. Assume that the Cauchy problem* (2.2) *has a solution $\boldsymbol{u} \in L^2(\Omega)^n$ and $p \in (H^1(\Omega))^*$, and let $0 < \gamma < \|K\|^{-2}$. Let $\boldsymbol{u}_k$ and p_k be the k-th approximate solution in the procedure presented in* § 3. *Then*

$$\lim_{k\to\infty} \|\boldsymbol{u} - \boldsymbol{u}_k\|_{L^2(\Omega)^n} = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \|p - p_k\|_{(H^1(\Omega))^*} = 0$$

for any initial data element $\boldsymbol{\eta}_0 \in L^2(\Gamma_1)^n$.

We remark that the procedure described in § 3 is the Landweber-Fridman method for solving (4.5), Engl *et al.* [6]. From this it follows that the procedure presented above is a regularization method and that it therefore works with inexact data.

In what follows, we suppose that instead of $\boldsymbol{\varphi}$, we have only its approximation, say $\boldsymbol{\varphi}^\varepsilon \in L^2(\Gamma_0)^n$, where $\boldsymbol{\varphi}^\varepsilon$ is the measured data,

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}^\varepsilon\|_{L^2(\Gamma_0)^n} \leqslant \varepsilon \tag{4.6}$$

and $\varepsilon \geqslant 0$ is a known upper bound for the error in the measurements. No preassumption is stipulated on the nature of the noise and this, in turn, is very useful for applications wherein experimental (unknown) noise is inherently present in any practical measurement.

Then, we stop the iteration of the algorithm according to the discrepancy principle, namely at the smallest index $k = k(\varepsilon)$ for which

$$e_c(k) := \|\boldsymbol{u}_k^\varepsilon|_{\Gamma_0} - \boldsymbol{\varphi}^\varepsilon\|_{L^2(\Gamma_0)^n} \leqslant \tau\varepsilon \tag{4.7}$$

where $\tau > 1$ is some fixed constant and $\boldsymbol{u}_k^\varepsilon$ is obtained from the iterative procedure with Cauchy data $\boldsymbol{\varphi}^\varepsilon$ and $\boldsymbol{\psi}$. The Landweber-Fridman method (LFM) has been numerically implemented in Marin & Lesnic [18] and Johansson & Lesnic [13] for the Cauchy problem of the Lame system in elasticity and the Stokes system in hydrostatics, respectively.

From a numerical point of view, it might be difficult to choose the parameter γ in the right interval in order to ensure the convergence of the LFM, as given by Theorem 4.1. In addition, if $\gamma > 0$ is chosen too small then the LFM becomes computationally very slow. However, it is possible to propose parameter-free procedures, and two such methods are described in the next section.

5 Parameter free procedures

Let K and K_1 be the same operators as defined by equations (4.3) and (4.4), respectively. The Cauchy problem (2.2) is equivalent to the operator equation (4.5). Let us denote $\boldsymbol{y} = \boldsymbol{\varphi} - K_1\boldsymbol{\psi}$ and $\boldsymbol{y}^\varepsilon = \boldsymbol{\varphi}^\varepsilon - K_1\boldsymbol{\psi}$. Then equation (4.5) becomes $K\boldsymbol{\eta} = \boldsymbol{y}$ and the aim is to recover $\boldsymbol{\eta}$ from noisy approximations $\boldsymbol{y}^\varepsilon$ of $\boldsymbol{y}$, as given by (4.6), namely

$$\|\boldsymbol{y} - \boldsymbol{y}^\varepsilon\|_{L^2(\Gamma_0)^n} \leqslant \varepsilon. \tag{5.1}$$

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5.1 The conjugate gradient method (CGM)

The conjugate gradient method (CGM) for solving (2.2) is based on solving the normal equation

$$K^*(K\boldsymbol{\eta}) = K^*\boldsymbol{y}^\varepsilon,$$

which is equivalent to minimizing the residual functional

$$\|K\boldsymbol{\eta} - \boldsymbol{y}^\varepsilon\|^2_{L^2(\Gamma_0)^n}. \tag{5.2}$$

Using Lemma 4.1 for the definition of K^* and applying Algorithm 7.1 of Engl *et al.* [6] to our problem, produces the following convergent algorithm:

(a) Choose an arbitrary function $\boldsymbol{\eta}_0 \in L^2(\Gamma_1)^n$, and set $k = 0$.

(b) Solve for $\boldsymbol{u}_k$ and p_k the problem (3.1) with $\boldsymbol{\eta} = \boldsymbol{\eta}_k$ and determine the residual

$$\tilde{\boldsymbol{r}}_k = \boldsymbol{\varphi}^\varepsilon - \boldsymbol{u}_k|_{\Gamma_0}.$$

(c) Solve the adjoint problem (3.2) with $\boldsymbol{\xi} = \tilde{\boldsymbol{r}}_k$ on Γ_0. Determine

$$\boldsymbol{r}_k = (q_k\boldsymbol{v} - N^*\boldsymbol{v}_k)|_{\Gamma_1},$$

then calculate $\boldsymbol{d}_k = -\boldsymbol{r}_k + \beta_{k-1}\boldsymbol{d}_{k-1}$ on Γ_1, with the convention that $\beta_{-1} = 0$ and

$$\beta_{k-1} = \frac{\|\boldsymbol{r}_k\|^2_{L^2(\Gamma_1)^n}}{\|\boldsymbol{r}_{k-1}\|^2_{L^2(\Gamma_1)^n}}, \quad k \geqslant 1.$$

(d) Solve for $\boldsymbol{u}_k$ and p_k the problem (3.1) with $\boldsymbol{\eta} = \boldsymbol{d}_k$ on Γ_1 and $\boldsymbol{\psi} = 0$ on Γ_0. Set

$$\alpha_k = \frac{\|\boldsymbol{r}_k\|^2_{L^2(\Gamma_1)^n}}{\|\boldsymbol{u}_k|_{\Gamma_0}\|^2_{L^2(\Gamma_0)^n}},$$

and let $\boldsymbol{\eta}_{k+1} = \boldsymbol{\eta}_k + \alpha_k\boldsymbol{d}_k$ on Γ_1.

(e) Increase k by one and go to step (b).

If $\boldsymbol{\eta}_0 = 0$, the k—th iterate $\boldsymbol{\eta}_k$ minimizes the residual norm (5.2) among all elements $\boldsymbol{\eta}$ in the k—th Krylov subspace

$$K_k = \text{span}\{K^*\boldsymbol{y}^\varepsilon, (K^*K)K^*\boldsymbol{y}^\varepsilon, (K^*K)^2K^*\boldsymbol{y}^\varepsilon, ..., (K^*K)^{k-1}K^*y^\varepsilon\}, \tag{5.3}$$

see Groetsch [8, Lemma 2.5.2]. Moreover, the residuals

$$\|K\boldsymbol{\eta}_k - \boldsymbol{y}^\varepsilon\|_{L^2(\Gamma_0)^n} \text{ and } \|\boldsymbol{\eta}_k - K^{-1}\boldsymbol{y}^\varepsilon\|_{L^2(\Gamma_1)^n} \tag{5.4}$$

are strictly decreasing if $\boldsymbol{y}^\varepsilon \in R(K)$, where $R(K)$ denotes the range of K and K^{-1} : $R(K) \to L^2(\Gamma_1)^n$ is the inverse of the injective operator K. We have convergence in the sense that

$$\boldsymbol{\eta}_k \to K^{-1}\boldsymbol{y}^\epsilon, \quad \text{as } k \to \infty \tag{5.5}$$

for $\boldsymbol{y} \in R(K)$.

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As a stopping criterion, we choose the first $k = k(\varepsilon)$ such that

$$\|\tilde{\boldsymbol{r}}_k\|_{L^2(\Gamma_0)^n} \leqslant \tau\varepsilon, \tag{5.6}$$

where $\tau > 1$ is some fixed constant. It follows from Nemirovskii [19] that (5.6) is an order optimal stopping rule if $\boldsymbol{y}^\varepsilon \in R(K)$.

The CGM has been numerically implemented for the Cauchy problem: for the Laplace equation in Dinh Nho Hào & Lesnic [5], for the heat equation in Dinh Nho Hào & Reinhardt [4] and Bastay *et al.* [1], for the Lame system in elasticity in Marin *et al.* [17], and for the Stokes system in hydrostatics in Johansson & Lesnic [14].

5.2 The minimal error method (MEM)

The minimal error method (MEM) [15] is based on a variant of CGM, which, for exactly given data, i.e. $\varepsilon = 0$, minimizes the iteration error

$$\|\boldsymbol{\eta} - K^{-1}(\boldsymbol{\varphi}^\varepsilon - K_1\boldsymbol{\psi})\|^2_{L^2(\Gamma_1)^n}, \tag{5.7}$$

rather than the data fit (5.2) in the same Krylov subspace.

Using Lemma 4.1 for the definition of K^* and applying the algorithm of Hanke [9] to our problem, produces the following convergent algorithm:

(a) Choose an arbitrary function $\boldsymbol{\eta}_0 \in L^2(\Gamma_1)^n$ and set $k = 0$.

(b) Solve for $\boldsymbol{u}_k$ and p_k the problem (3.1) with $\boldsymbol{\eta} = \boldsymbol{\eta}_k$ and determine the residual

$$\tilde{\boldsymbol{r}}_k = \boldsymbol{\varphi}^\varepsilon - \boldsymbol{u}_k|_{\Gamma_0}.$$

(c) Solve the adjoint problem (3.2) with $\boldsymbol{\xi} = \tilde{\boldsymbol{r}}_k$ on Γ_0. Determine

$$\boldsymbol{r}_k = (q_k\boldsymbol{v} - N^*\boldsymbol{v}_k)|_{\Gamma_1},$$

then calculate $\boldsymbol{\xi}_k = -\boldsymbol{r}_k + \gamma_{k-1}\boldsymbol{\xi}_{k-1}$ on Γ_1, with the convention that $\gamma_{-1} = 0$ and

$$\gamma_{k-1} = \frac{\|\tilde{\boldsymbol{r}}_k\|^2_{L^2(\Gamma_0)^n}}{\|\tilde{\boldsymbol{r}}_{k-1}\|^2_{L^2(\Gamma_0)^n}}, \quad k \geqslant 1.$$

(d) Solve for $\boldsymbol{u}_k$ and p_k in (3.1) with $\boldsymbol{\eta} = \boldsymbol{\eta}_{k+1}$ on Γ_1, where

$$\boldsymbol{\eta}_{k+1} = \boldsymbol{\eta}_k + \chi_k\boldsymbol{\xi}_k \qquad \text{on } \Gamma_1$$

and

$$\chi_k = \frac{\|\tilde{\boldsymbol{r}}_k\|^2_{L^2(\Gamma_0)^n}}{\|\boldsymbol{\xi}_k\|^2_{L^2(\Gamma_1)^n}}.$$

(e) Increase k by one and go to step (b).

The kth iterate $\boldsymbol{\eta}_k$ minimizes the error norm (5.7) at each iteration among all elements $\boldsymbol{\eta}$ in the kth Krylov subspace K_k defined by (5.3). From this it follows readily the convergence of the MEM iterates $\boldsymbol{\eta}_k$ to $K^{-1}\boldsymbol{y}^\varepsilon$ in the case $\boldsymbol{y}^\varepsilon \in R(K)$. However, unlike in

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the CGM, the errors (5.7) need not decrease monotonically. Moreover, if $\boldsymbol{y}^{\varepsilon} \notin R(K)$, then the MEM iteration either breaks down after a finite number of steps, or $\|\boldsymbol{\eta}_k\|_{L^2(\Gamma_1)^n} \to \infty$, as $k \to \infty$ [9].

It was previously shown [9] that the discrepancy principle (5.1) is no regularization stopping rule for the MEM; instead, as a stopping criterion, we choose the first $k = k(\varepsilon)$ such that

$$\left(\sum_{j=0}^{k} \| K\boldsymbol{\eta}_j + K_1\boldsymbol{\psi} - \boldsymbol{\varphi}^{\varepsilon} \|^{-2}_{L^2(\Gamma_0)^n} \right)^{-1/2} \leqslant \tau\varepsilon, \tag{5.8}$$

where $\tau > 1$ is some fixed constant. To the authors' knowledge the MEM has never been previously numerically implemented.

6 Numerical investigations

6.1 The boundary element method

In practice, the direct, well-posed, mixed problems (3.1) and (3.2) of the iterative procedures described in the previous sections can seldom be solved analytically and therefore some form of numerical approximation is necessary. Since only boundary data is needed, this can most advantageously be performed using the boundary element method (BEM), and in this subsection we briefly describe the BEM for the Stokes equations, i.e. $\boldsymbol{b} \equiv 0$ in the definition of the operator L, namely

$$\begin{cases} \Delta\boldsymbol{u} - \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega. \end{cases} \tag{6.1}$$

Using Green's formula we can recast the Stokes equations (6.1) into a boundary integral form [16]

$$c(\mathbf{x})\boldsymbol{u}(\mathbf{x}) = \int_{\Gamma} \left[\underline{\underline{K}}(\mathbf{x},\mathbf{y})\boldsymbol{u}(\mathbf{y}) - \underline{\underline{U}}(\mathbf{x},\mathbf{y})\boldsymbol{t}(\mathbf{y}) \right] dS(\mathbf{y}), \quad \mathbf{x} \in \overline{\Omega} \tag{6.2}$$

where $c(\mathbf{x})$ is a coefficient function which is equal to 1 if $\mathbf{x} \in \Omega$, and 0.5 if $\mathbf{x} \in \partial\Omega$, $\boldsymbol{t} = -p\boldsymbol{v} + N\boldsymbol{u}$ is the fluid traction and, in two-dimensions, the tensors $\underline{\underline{K}}$ and $\underline{\underline{U}}$ are given by

$$K_{kl}(\mathbf{x},\mathbf{y}) = -\frac{(x_k - y_k)(x_l - y_l)}{\pi \mid \mathbf{x} - \mathbf{y} \mid^4} \sum_{m=1}^{2} (x_m - y_m)v_m, \quad k,l = \overline{1,2},$$

$$U_{kl} = -\frac{1}{4\pi} \left[-\delta_{kl} ln \mid \mathbf{x} - \mathbf{y} \mid + \frac{(x_k - y_k)(x_l - y_l)}{\mid \mathbf{x} - \mathbf{y} \mid^2} \right], \quad k,l = \overline{1,2}.$$

Here δ_{kl} is the Kronecker tensor, $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$ and $\boldsymbol{v} = (v_1, v_2)$. Nevertheless, solving the boundary integral equation (6.2) analytically can be performed only in very simple cases and therefore numerical methods seem more appropriate. The discretisation

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of the boundary integral equation (6.2) applied at $\boldsymbol{x} \in \Gamma$ is performed using the BEM [22] for solving the direct mixed well-posed problems of the iterative algorithms described in § 3 and § 5.

The iterative processes are ceased according to the stopping criteria (4.7), (5.6) and (5.8).

6.2 Numerical results and discussion

In this section we discuss the numerical results obtained using the iterative algorithms proposed in § 3 and § 5 combined with the BEM described in § 6.1 for solving a Cauchy problem in a two-dimensional bounded domain Ω, although the same conclusions hold in higher dimensions. Let Γ_0 and Γ_1 be two infinitely long circular cylinders of radii R_0 and R_1 satisfying $0 < R_1 < R_0$. Let $\Omega = \{(x, y) \mid R_1^2 < x^2 + y^2 < R_0^2\}$ be the annular domain between these cylinders which is filled with viscous fluid flowing at low Reynolds numbers governed by the Stokes equations (6.1). The inverse Cauchy problem is to determine the fluid velocity and the stress force (fluid traction) at the inner cylinder $\Gamma_1 = \{(x, y) \mid x^2 + y^2 = R_1^2\}$ by taking measurements at the outer cylinder $\Gamma_0 = \{(x, y) \mid x^2 + y^2 = R_0^2\}$. Let us consider the following benchmark test example

$$\boldsymbol{u}(x, y) = (4y^3 - x^2, 4x^3 + 2xy - 1), \quad p(x, y) = 24xy - 2x, \quad (x, y) \in \Omega \tag{6.3}$$

which satisfies the Stokes equations (6.1) and has been previously considered by Zeb *et al.* [23] in another type of inverse problem. Of course, in the practical fluid dynamics context the viscous flow assumption implies that the velocity at the boundary should equal the wall velocity, but in our benchmark test example we investigate a more general situation. Furthermore, although example (6.3) has no physical meaning, because its solution is explicitly known, the accuracy of the numerical results can easily be assessed. The analytical example (6.3) generates the Cauchy data on Γ_0 as given by

$$\boldsymbol{\varphi}(x, y) = (4y^3 - x^2, 4x^3 + 2xy - 1), \quad (x, y) \in \Gamma_0 \tag{6.4}$$

$$\boldsymbol{\psi}(x, y) = (-12y^3 - 2y^2 + 2x^2 + 12x^2y, -12x^3 + 12xy^2 - 8xy)/R_0, \quad (x, y) \in \Gamma_0. \tag{6.5}$$

For simplicity, we take $R_1 = 1$ and $R_0 = 2$ and employ $M = 128$ boundary elements to discretise uniformly the boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ in the BEM discretisation of (6.2). An arbitrary initial guess such as $\boldsymbol{\eta}_0 = \mathbf{0}$ has been chosen to initiate the iterative algorithms described in § 3 and § 5. To test the stability of the methods we add, in the input boundary velocity data $\boldsymbol{\varphi}$, some noise of the form

$$\boldsymbol{\varphi}^{\varepsilon} = \boldsymbol{\varphi}(1 + \alpha\delta) \tag{6.6}$$

where $\alpha\%$ represents the percentage of noise and, for each component of $\boldsymbol{\varphi}$, and δ is a random real number taken from the uniform distribution over the interval $[-1, 1]$.

Let us introduce the convergence and accuracy errors define by

$$e_c(k) = \|\boldsymbol{u}_k\|_{\Gamma_0} - \boldsymbol{\varphi}^{\varepsilon}\|_{L^2(\Gamma_0)^n} = \|\tilde{\boldsymbol{r}}_k\|_{L^2(\Gamma_0)^n}, \quad e_a(k) = \|\boldsymbol{\eta}_k - \boldsymbol{u}\|_{\Gamma_1}\|_{L^2(\Gamma_1)^n}, \tag{6.7}$$

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Table 1. *The optimal iteration numbers $k(\epsilon)$ for the CGM, LFM and MEM methods obtained according to the stopping criteria (6.8) with $\tau \approx 1$ and (6.9) with $\tau \approx 1.35$, and the convergence and accuracy errors e_c and e_a, respectively, for various percentages of noise $\alpha \in \{1, 3, 5\}\%$*

	$\varepsilon = 0.4900$ $\tau\varepsilon = 0.6615$ $\alpha = 1\%$	$\varepsilon = 1.4702$ $\tau\varepsilon = 1.9845$ $\alpha = 3\%$	$\varepsilon = 2.4504$ $\tau\varepsilon = 3.3075$ $\alpha = 5\%$
k_{CGM}	9	7	5
k_{LFM}	239	91	42
k_{MEM}	6	3	2
$e_c(k_{CGM})$	0.4794	1.4691	2.4460
$e_c(k_{LFM})$	0.4898	1.4699	2.4483
$e_c(k_{MEM})$	0.6608	1.6220	3.1763
$e_a(k_{CGM})$	0.2881	1.2626	2.1000
$e_a(k_{LFM})$	0.5334	1.4606	2.1455
$e_a(k_{MEM})$	1.5988	2.7082	3.1075

where k is the iteration number, $\boldsymbol{u}\,|_{\Gamma_1}$ is the exact fluid velocity on Γ_1 which can be obtained from (6.3) if the analytical solution is available, and $\boldsymbol{\eta}_k$ is obtained from the iteration procedures with the Cauchy data $\boldsymbol{\varphi} = \boldsymbol{\varphi}^e$ and $\boldsymbol{\psi}$.

Based on (6.7), the LFM and CGM are ceased according to the discrepancy stopping criteria (4.7) and (5.6), i.e. at the smallest index $k = k(\varepsilon)$ for which

$$e_c(k) \leqslant \tau\varepsilon, \tag{6.8}$$

whilst the MEM is ceased according to the stopping criterion (5.8), i.e. at the smallest index $k = k(\varepsilon)$ for which

$$E_c(k) := \left(\sum_{j=0}^{k} e_c(j)^{-2} \right)^{-1/2} \leqslant \tau\varepsilon. \tag{6.9}$$

The comparison between the LFM (with $\gamma = 0.1$) and CGM has recently been undertaken by Johansson & Lesnic [14], and below we extend the comparison with the MEM.

Using the stopping criteria (6.8) and (6.9), Table 1 shows the values of the noise level ε, the stopping index $k(\varepsilon)$, the convergence error $e_c(k(\varepsilon))$ on Γ_0 and the accuracy error $e_a(k(\varepsilon))$ on Γ_1 given by (6.7), for various amounts of noise $\alpha \in \{1, 3, 5\}\%$ added in $\boldsymbol{\varphi}$ as in (6.6). Whilst for the CGM and the LFM the constant τ can be chosen close to unity, for the MEM, values of τ between 1 and 1.1 produce numerical results which exhibit a slightly oscillatory unstable behaviour and they become inaccurate, especially for the fluid traction $\boldsymbol{t}$. This investigation shows that the MEM is more sensitive to the choice of τ than in the other iterative methods, as something expected from the preliminary investigation of King [15]. In Table 1, a value of $\tau \approx 1.35$ was found to be optimal for the MEM for all levels of noise. However, the values of $\tau > 1$ chosen are still in the range of suggestions made by Hansen & Hanke [11] and Hanke [10]. In any case, from

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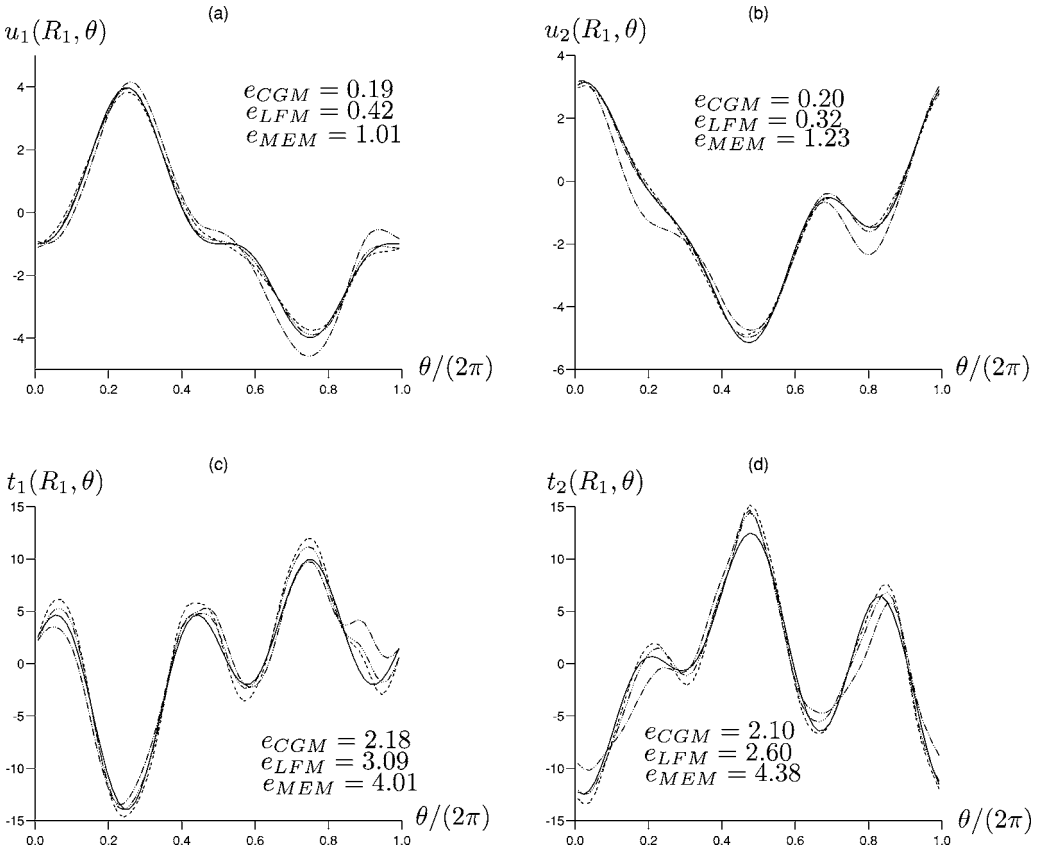

FIGURE 1. The numerical solution for (a) $u_1(R_1,\theta)$, (b) $u_2(R_1,\theta)$, (c) $t_1(R_1,\theta)$ and (d) $t_2(R_1,\theta)$ for $\alpha = 1\%$ obtained using the CGM (–....–), LFM (— — —) and MEM (–..–) methods in comparison with the corresponding analytical solutions (—) obtained from (6.3). The absolute errors (6.10) are also included.

Table 1 it can be clearly seen that the parameter free procedures described in § 5 are much faster (about 10–20 times) than the LFM described in § 3. In terms of accuracy, the CGM performs best, followed by the LFM and the MEM.

Figures 1(a)–(d) show the numerical solutions for $u_1(R_1,\theta)$, $u_2(R_1,\theta)$, $t_1(R_1,\theta)$ and $t_2(R_1,\theta)$, respectively, in comparison with the corresponding analytical solutions obtained from (6.3), when $\alpha = 1\%$. For this value of α, according to Table 1, the LFM, CGM and MEM are stopped after 239, 9 and 6 iterations, respectively. The corresponding absolute errors between the numerical and analytical solutions denoted by

$$e(u_i) = \|u_{ik}\,|_{\Gamma_1} - u_i\,|_{\Gamma_1}\,\|_{L^2(\Gamma_1)},\ e(t_i) = \|t_{ik}\,|_{\Gamma_1} - t_i\,|_{\Gamma_1}\,\|_{L^2(\Gamma_1)},\quad i = 1,2 \tag{6.10}$$

are also included in these figures. From Figures 1(a)–(d) it can be seen that the CGM outperforms the LFM followed by the MEM in terms of accuracy. However, all the methods produce stable and reasonably accurate numerical solutions. Similar results have been obtained for other amounts of noise $\alpha \in \{3,5\}\%$, and are therefore not presented.

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Once the boundary values for $\boldsymbol{u}$ and $\boldsymbol{t}$ are obtained accurately over the whole boundary Γ, then the fluid velocity $\boldsymbol{u}$ and the pressure p inside the domain Ω can easily be obtained from the integral equations [16]

$$\boldsymbol{u}(\boldsymbol{x}) = \int_{\Gamma} \left[\underline{\underline{K}}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}(\boldsymbol{y}) - \underline{\underline{U}}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{t}(\boldsymbol{y})\right] dS(\boldsymbol{y}), \quad \boldsymbol{x} \in \Omega$$

$$p(\boldsymbol{x}) = \int_{\Gamma} \left[\boldsymbol{L}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{q}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{t}(\boldsymbol{y})\right] dS(\boldsymbol{y}), \quad \boldsymbol{x} \in \Omega$$

where, in two-dimensions, the vectors $\boldsymbol{q}$ and $\boldsymbol{L}$ are given by

$$q_k(\boldsymbol{x},\boldsymbol{y}) = -\frac{(x_k - y_k)}{2\pi \mid \boldsymbol{x} - \boldsymbol{y} \mid^2}, \quad k = 1, 2,$$

$$L_k(\boldsymbol{x},\boldsymbol{y}) = \frac{n_k}{\pi \mid \boldsymbol{x} - \boldsymbol{y} \mid^2} - \frac{2(x_k - y_k)}{\pi \mid \boldsymbol{x} - \boldsymbol{y} \mid^2} \sum_{m=1}^{2} (x_m - y_m) n_m, \quad k = 1, 2.$$

7 Conclusions

In this paper, we studied numerically the Cauchy problem for viscous stationary linear generalized incompressible Stokes flow system with Cauchy data in L^2 using three iterative methods, namely the Landweber-Fridman (LFM), the conjugate gradient (CGM) and the minimal error (MEM) methods. This rather weak requirement for the Cauchy data offers practical applicability of the approach. The numerical implementation of the LFM, CGM and MEM is accomplished by using the boundary element method (BEM) for solving at each iteration two mixed direct well-posed boundary value problems for the Stokes system. The iterative methods are ceased according to optimal order stopping criteria. For a typical benchmark test example, a rapidly stable numerical solution is achieved using the parameter free procedures in less than 10 iterations. Furthermore, a comparison of the numerical results show that the CGM outperforms the LFM followed by the MEM in terms of accuracy. While the LFM is quite slow to be competitive, especially for large-scale problems, it requires further the choice of a parameter γ in a suitable range in order to achieve convergence of the iterative method. On the other hand, whilst the MEM whilst indicates a very fast iterative method, it is less stable and more inaccurate than the other methods investigated. Overall, from the three iterative methods investigated in this paper, the CGM is shown to perform the best in terms of rapidity of convergence, stability and accuracy.

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